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Trace principle for Riesz potentials on Herz-type spaces and applications

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Abstract

We establish trace inequalities for Riesz potentials on Herz-type spaces and examine the optimality of conditions imposed on specific parameters. We also present some applications in the form of Sobolev-type inequalities, including the Gagliardo–Nirenberg–Sobolev inequality and the fractional integration theorem in the Herz space setting. In addition, we obtain a Sobolev embedding theorem for Herz-type Sobolev spaces.

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1 Introduction and preliminaries

The Riesz potential operator I_γ is an integral operator defined by the convolution of a function f with the Riesz kernel $K_\gamma(x) := |x|^{\gamma-n}$. More precisely, for $n \in \mathbb{N}$ and $0 < \gamma < n$,

$$I_\gamma f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dm(y), \quad x \in \mathbb{R}^n,$$

where f is a suitable function, for example, a locally integrable function on $\mathbb{R}^n (L^1_{\mathrm{loc}}(\mathbb{R}^n))$ or a function with sufficiently rapid decay at infinity, particularly, if $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \frac{n}{\gamma}$, and m is the Lebesgue measure on $\mathbb{R}^n$. If $\gamma = 2 \neq n$, then this integral operator is called the Newtonian potential and is used to describe the potential energy distribution of a system of point masses in classical mechanics or the electrostatic potential created by a charge distribution in physics.

The trace problem for Riesz potentials deals with finding nonnegative (positive) Borel measures μ on $\mathbb{R}^n$ such that I_γ maps $\mathcal{F}(\mathbb{R}^n, m)$ boundedly into $\mathcal{F}'(\mathbb{R}^n, \mu)$, where $\mathcal{F}(\mathbb{R}^n, m)$ and $\mathcal{F}'(\mathbb{R}^n, \mu)$ are function spaces defined over $\mathbb{R}^n$ with respect to measures m and μ, respectively. Adams [1, 2] proved that for $1 < p_1 < p_2 < \infty$ and $0 < \gamma < \frac{n}{p_1}$,

$$\|I_\gamma f\|_{L^{p_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n,m)} \tag{1}$$

if and only if $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1}-\frac{\gamma}{n})}$ for every ball $B \subset \mathbb{R}^n$. Here we have used the standard notation $\zeta \lesssim \rho$ (or, equivalently, $\rho \gtrsim \zeta$) to express that there exists a positive constant c, independent of relevant variables, such that $\zeta \le c\rho$. Inequality (1) is not true when $p_1 = p_2$ (see, for example, [3]). Inequalities involving Riesz potentials often provide an important tool for estimating functions in terms of the norms of their derivatives. The wide-ranging applicability of trace inequalities for Riesz potentials has sparked significant interest in recent studies; see, for instance, [7, 9, 15, 24] and references therein. For Morrey–Lorentz spaces, the following theorem has been established in [7].

Theorem 1.1 *Let $1 < p_1 \le q_1 < \infty$ and $1 < p_2 \le q_2 < \infty$ satisfy $\frac{p_2}{q_2} \le \frac{p_1}{q_1}$ for all $1 < p_1 < p_2 < \infty$. Then the inequality*

$$\|I_\gamma f\|_{\mathcal{M}^{q_2}_{p_2,r_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\mathcal{M}^{q_1}_{p_1,r_1}(\mathbb{R}^n,m)}$$

holds if and only if the measure μ satisfies $\mu(B) \lesssim [m(B)]^{q_2(\frac{1}{q_1}-\frac{\gamma}{n})}$ for every ball $B \subset \mathbb{R}^n$, given that $n(\frac{1}{q_1} - \frac{1}{q_2}) \le \gamma < \frac{n}{q_1}$ and $1 \le r_1 < r_2 \le \infty$ (or $r_1 = r_2 = \infty$).

In particular, this yields the following outcome for Lorentz spaces (see Definition 1.3).

Corollary 1.2 *If $1 < p_1 < p_2 < \infty$, $n(\frac{1}{p_1} - \frac{1}{p_2}) \le \gamma < \frac{n}{p_1}$, and $1 \le r_1 < r_2 \le \infty$ (or $r_1 = r_2 = \infty$). Then*

$$\|I_\gamma f\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{L^{p_1,r_1}(\mathbb{R}^n,m)}$$

if and only if the measure μ satisfies $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1}-\frac{\gamma}{n})}$ for every ball $B \subset \mathbb{R}^n$.

1.1 Function spaces

In this subsection, we fix some notations and recall definitions of certain function spaces required for the subsequent discussion. We begin with Lorentz spaces.

Definition 1.3 A Lorentz space $L^{p,r}(\Omega,\nu)$ defined over a σ-finite measure space (Ω,Σ,ν) consists of all ν-measurable functions on Ω for which the functional $\|f\|_{L^{p,r}(\Omega,\nu)}$ is finite, where

$$\|f\|_{L^{p,r}(\Omega,\nu)} := \begin{cases} (\int_0^\infty (t^{\frac{1}{p}} f^*(t))^r \frac{dt}{t})^{1/r} & \text{if } 0 < p < \infty, 0 < r < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } 0 < p \le \infty, r = \infty, \end{cases}$$

and $f^*(t) := \inf\{s \ge 0 : \nu(\{x \in \Omega : |f| > s\}) \le t\}$, $t \ge 0$, is the decreasing (or nonincreasing) rearrangement of f.

Note that $L^{p,p} = L^p$. It is important to emphasize that $\|\cdot\|_{L^{p,r}(\Omega,\nu)}$ is not always a norm, but rather a quasi-norm (see [5, p. 216]). However, we can define a functional $\|\cdot\|_{L^{(p,r)}(\Omega,\nu)}$ on $L^{p,r}(\Omega,\nu)$ as follows:

$$\|f\|_{L^{(p,r)}(\Omega,\nu)} := \begin{cases} (\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^r \frac{dt}{t})^{1/r} & \text{if } 0 < p < \infty, 0 < r < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{if } 0 < p \le \infty, r = \infty, \end{cases}$$

where the function $f^{**}(t) := \frac{1}{t}\int_0^t f^*(t)\,dt$ is referred to as the maximal average function. Fortunately, this functional is subadditive. Consequently, $L^{(p,r)}(\Omega,\nu) := (L^{p,r}(\Omega,\nu), \|\cdot\|_{L^{(p,r)}(\Omega,\nu)})$ is a normed space for $1 < p < \infty$, $1 \le r \le \infty$, or $p = r = \infty$. Since $f^* \le f^{**}$, we have $L^{(p,r)} \hookrightarrow L^{p,r}$. Moreover, if $1 < p \le \infty$ and $1 \le r \le \infty$, then $L^{p,r} \hookrightarrow L^{(p,r)}$ (see [5, Lemma 4.5, p. 219]). The substitution of $L^{p,1}$ for L^{p_1} on the right-hand side of inequality (1) retains its validity in the limiting case $p_1 = p_2 = p$ (see [13, 14]).

Another important generalization of Lebesgue spaces is the classical Herz space, introduced by Herz [12] as a suitable environment for the action of Fourier transform on a Lipschitz class. Although the Herz spaces are defined in various equivalent ways, we adopt the formulation presented in [11, 19] with a slightly changed notation for our convenience.

Let $(\Omega_t)_{t\in\mathbb{Z}}$ be the dyadic decomposition of $\mathbb{R}^n$, i.e., $\Omega_t = \{x \in \mathbb{R}^n : 2^{t-1} \le |x| < 2^t\}$ for $t \in \mathbb{Z}$. We denote $\tilde{\chi}_{\Omega_t} = \chi_{\Omega_t}$ for $t \in \mathbb{Z}_+$, and $\tilde{\chi}_{\Omega_{-1}} = \chi_{B(0,\frac{1}{2})}$, where $B(0,\frac{1}{2})$ represents the ball centered at the origin with radius $\frac{1}{2}$ in $\mathbb{R}^n$.

Definition 1.4 Let $\lambda \in \mathbb{R}$ and $0 < p, q \le \infty$, and let ν be a positive measure on $\mathbb{R}^n$.

(i) The homogeneous Herz space $\dot{K}^p_{\lambda,q}(\mathbb{R}^n,\nu)$ is defined by

$$\dot{K}^p_{\lambda,q}(\mathbb{R}^n,\nu) := \{f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\},\nu) : \|f\|_{\dot{K}^p_{\lambda,q}(\mathbb{R}^n,\nu)} < \infty\},$$

where

$$\|f\|_{\dot{K}^p_{\lambda,q}(\mathbb{R}^n,\nu)} := \left(\sum_{t\in\mathbb{Z}} 2^{t\lambda q}\|f\chi_{\Omega_t}\|^q_{L^p(\mathbb{R}^n,\nu)}\right)^{\frac{1}{q}}.$$

(ii) The inhomogeneous Herz space $K^p_{\lambda,q}(\mathbb{R}^n,\nu)$ is defined by

$$K^p_{\lambda,q}(\mathbb{R}^n,\nu) := \{f \in L^p_{loc}(\mathbb{R}^n,\nu) : \|f\|_{K^p_{\lambda,q}(\mathbb{R}^n,\nu)} < \infty\},$$

where

$$\|f\|_{K^p_{\lambda,q}(\mathbb{R}^n)} := \left(\sum_{t=-1}^{\infty} 2^{t\lambda q}\|f\tilde{\chi}_{\Omega_t}\|^q_{L^p(\mathbb{R}^n,\nu)}\right)^{\frac{1}{q}}.$$

If p and/or q are infinite, then the usual modifications are made.

It is obvious that $\dot{K}^p_{0,p}(\mathbb{R}^n,\nu) = K^p_{0,p}(\mathbb{R}^n,\nu) = L^p(\mathbb{R}^n,\nu)$. In recent years, there has been substantial advancement in the development of Herz spaces, primarily driven by their wide range of applications (see, for instance, [4, 8, 10, 16, 22, 23] and references therein). However, Herz spaces alone are insufficient to describe some fine properties of functions and operators. Consequently, defining the Lorentz–Herz spaces $\dot{HL}^{p,r}_{\lambda,q}(\mathbb{R}^n,\nu)$ and $HL^{p,r}_{\lambda,q}(\mathbb{R}^n,\nu)$ emerges as a natural progression. These spaces are derived simply by amalgamating Lorentz spaces with Lebesgue sequence spaces, essentially replacing the functionals $\|\cdot\|_{L^p(\mathbb{R}^n,\nu)}$ with $\|\cdot\|_{L^{p,r}(\mathbb{R}^n,\nu)}$ in Definition 1.4. The properties of these spaces, even in more general settings, have been investigated in [6].

The trace principle for Riesz potentials on Herz spaces and their extensions remains absent from the academic literature. This absence is particularly worth noting, given the pivotal role that inequalities associated with Riesz potentials are indispensable tools for estimating functions in terms of their gradients, commonly referred to as Sobolev inequalities. These inequalities are considered as cornerstones of the Sobolev theory in partial differential equations.

To establish such estimates within the Herz-type setting, the derivation of trace inequalities is of paramount importance. In the ensuing sections, we present rigorous proofs of trace inequalities for both Herz and Lorentz–Herz spaces. It is important to mention that our focus here is on homogeneous spaces; however, analogous proofs for nonhomogeneous spaces can be conducted similarly. Additionally, we engage in a comprehensive discussion on the optimality of specific parametric conditions inherent in these trace inequalities. The resulting trace theorems subsequently facilitate the proof of Sobolev inequalities within Herz space settings, providing succinct estimates for functions in relation to their gradients. As a consequential outcome, we establish a Sobolev embedding theorem for Herz-type Sobolev spaces.

2 Trace theorems

We begin this section by presenting the trace theorem for Herz spaces. To handle convolution operators with kernels having singularities at the origin, we adopt a conventional and widely used approach. It involves decomposing the summation into distinct components, systematically accounting for the presence of singularity. This well-established technique has found pervasive application in several research papers. Hereafter, if the measure associated with a particular norm is not explicitly mentioned, then it is to be understood as the Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.1 *Assume that* $1 < p_1 < p_2 < \infty$, $1 \le q_1 \le q_2 < \infty$, $0 < \gamma < \frac{n}{p_1}$, *and* $\gamma - \frac{n}{p_1} < \lambda < n - \frac{n}{p_1}$. *If* $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1} - \frac{\gamma}{n})}$ *for every ball* $B \subset \mathbb{R}^n$, *then*

$$\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}. \tag{2}$$

Proof Let $f \in \dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n, m)$. Since $0 < \frac{q_1}{q_2} \le 1$, we have

$$\begin{aligned}
\|I_\gamma f\|^{q_1}_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} &= \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_2}\left(\int_{\Omega_t} \left|I_\gamma f(x)\right|^{p_2} d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{q_1}{q_2}} \\
&\le \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\int_{\Omega_t} \left|I_\gamma f(x)\right|^{p_2} d\mu(x)\right)^{\frac{q_1}{p_2}}\right].
\end{aligned}$$

By setting $f_s = f\chi_{\Omega_s}$ for $s \in \mathbb{Z}$ we have $f = \sum_{s\in\mathbb{Z}} f_s$. Using Minkowski's inequality, we get

$$\begin{aligned}
\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} &\le \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\in\mathbb{Z}}\left(\int_{\Omega_t} \left|I_\gamma f_s(x)\right|^{p_2} d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}} \\
&\le \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\le t-2}\left(\int_{\Omega_t} \left|I_\gamma f_s(x)\right|^{p_2} d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}} \\
&\quad + \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{t-1\le s\le t+1}\left(\int_{\Omega_t} \left|I_\gamma f_s(x)\right|^{p_2} d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}}
\end{aligned}$$

$$+\left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left(\sum_{s\geq t+2}\left(\int_{\Omega_t}\left|I_\gamma f_s(x)\right|^{p_2}d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}}$$
$$:=E_1+E_2+E_3. \tag{3}$$

Now we estimate the terms E_1, E_2, and E_3 one by one.

Estimation of E_1: For $s\leq t-2$ and a.e. $x\in\Omega_t$, we have

$$\begin{aligned}\left|I_\gamma f_s(x)\right| &\lesssim 2^{-t(n-\gamma)}\left|\int_{\mathbb{R}^n}f_s(y)\,dm(y)\right|\\ &\leq 2^{-t(n-\gamma)}\|f_s\|_{L^{p_1}}\|\chi_{\Omega_s}\|_{L^{p_1'}}.\end{aligned}$$

Thus

$$\begin{aligned}E_1 &= \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left(\sum_{s\leq t-2}\left(\int_{\Omega_t}\left|I_\gamma f_s(x)\right|^{p_2}d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}}\\
&\lesssim \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left(\sum_{s\leq t-2}2^{-t(n-\gamma)}\|f_s\|_{L^{p_1}}\|\chi_{\Omega_s}\|_{L^{p_1'}}\|\chi_{\Omega_t}\|_{L^{p_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}\\
&\lesssim \left[\sum_{t\in\mathbb{Z}}\left(\sum_{s\leq t-2}2^{s\lambda}\|f_s\|_{L^{p_1}}\cdot 2^{\alpha(t-s)}\right)^{q_1}\right]^{\frac{1}{q_1}},\end{aligned}$$

where $\alpha=\frac{n}{p_1}-n+\lambda<0$. Using Hölder's inequality for inner sum and changing order of summations, we get

$$\begin{aligned}E_1 &\lesssim \left[\sum_{t\in\mathbb{Z}}\left(\sum_{s\leq t-2}2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\cdot 2^{\frac{\alpha q_1}{2}(t-s)}\left\{\sum_{s\leq t-2}2^{\frac{\alpha q_1'}{2}(t-s)}\right\}^{\frac{q_1}{q_1'}}\right)\right]^{\frac{1}{q_1}}\\
&\lesssim \left[\sum_{t\in\mathbb{Z}}\sum_{s\leq t-2}2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\cdot 2^{\frac{\alpha q_1}{2}(t-s)}\right]^{\frac{1}{q_1}}\\
&= \left[\sum_{s\in\mathbb{Z}}2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\sum_{t\geq s+2}2^{\frac{\alpha q_1}{2}(t-s)}\right]^{\frac{1}{q_1}}\\
&\lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}.\end{aligned}$$

Estimation of E_2: Applying Minkowski's inequality and (1), we have

$$\begin{aligned}E_2 &\leq \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left(\sum_{t-1\leq s\leq t+1}\|I_\gamma f_s\|_{L^{p_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}\\
&\leq \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\|I_\gamma f_{t-1}\|_{L^{p_2}(\mathbb{R}^n,\mu)}^{q_1}\right]^{\frac{1}{q_1}}+\left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\|I_\gamma f_t\|_{L^{p_2}(\mathbb{R}^n,\mu)}^{q_1}\right]^{\frac{1}{q_1}}\\
&\quad+\left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\|I_\gamma f_{t+1}\|_{L^{p_2}(\mathbb{R}^n,\mu)}^{q_1}\right]^{\frac{1}{q_1}}\\
&\lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}.\end{aligned}$$

Estimation of E_3: For $s \geq t+2$ and a.e $x \in \Omega_t$, by using a similar technique as in estimation of E_1, we get

$$\left|I_\gamma f_s(x)\right| \lesssim 2^{-s(n-\gamma)} \|f_s\|_{L^{p_1}} \|\chi_{\Omega_s}\|_{L^{p_1'}}.$$

Therefore

$$\begin{aligned} E_3 &= \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\geq t+2}\left(\int_{\Omega_t}\left|I_\gamma f_s(x)\right|^{p_2} d\mu(x)\right)^{\frac{1}{p_2}}\right)^{q_1}\right]^{\frac{1}{q_1}} \\ &\lesssim \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\geq t+2} 2^{-s(n-\gamma)}\|f_s\|_{L^{p_1}}\|\chi_{\Omega_s}\|_{L^{p_1'}}\|\chi_{\Omega_t}\|_{L^{p_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}} \\ &\lesssim \left[\sum_{t\in\mathbb{Z}}\left(\sum_{s\geq t+2} 2^{s\lambda}\|f_s\|_{L^{p_1}}\cdot 2^{\beta(t-s)}\right)^{q_1}\right]^{\frac{1}{q_1}}, \end{aligned}$$

where $\delta = \frac{n}{p_1} - \gamma + \lambda > 0$. Now using Hölder's inequality for the inner sum and then interchanging the order of summations, we obtain

$$\begin{aligned} E_3 &\lesssim \left[\sum_{t\in\mathbb{Z}}\left(\sum_{s\geq t+2} 2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\cdot 2^{\frac{\delta q_1}{2}(t-s)}\left\{\sum_{s\geq t+2} 2^{\frac{\delta q_1'}{2}(t-s)}\right\}^{\frac{q_1}{q_1'}}\right)\right]^{\frac{1}{q_1}} \\ &\lesssim \left[\sum_{t\in\mathbb{Z}}\sum_{s\geq t+2} 2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\cdot 2^{\frac{\delta q_1}{2}(t-s)}\right]^{\frac{1}{q_1}} \\ &= \left[\sum_{s\in\mathbb{Z}} 2^{s\lambda q_1}\|f_s\|_{L^{p_1}}^{q_1}\sum_{t\leq s-2} 2^{\frac{\delta q_1}{2}(t-s)}\right]^{\frac{1}{q_1}} \\ &\lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}. \end{aligned}$$

This completes the proof. □

In the case of $p_i = q_i$ for $i = 1,2$ and $\lambda = 0$, the converse of the above theorem holds (see (1)). However, in general, the question of its converse remains an open problem. Nevertheless, we establish a partial answer for a particular set of parameters.

Proposition 2.2 *Let p_1, p_2, q_1, q_2, and γ be as in Theorem 2.1. Suppose $p_1 \leq q_1 \leq q_2 \leq p_2$ and $\lambda = 0$. If $\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}$, then $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1}-\frac{\gamma}{n})}$ for every ball $B \subset \mathbb{R}^n$.*

Proof For a given ball $B \subset \mathbb{R}^n$, set $f(x) = \chi_B(x)$. Then

$$\begin{aligned} \|f\|_{\dot{K}^{p_1}_{0,q_1}(\mathbb{R}^n,m)} &= \left[\sum_{t\in\mathbb{Z}}\left(\int_{\Omega_t}\left|\chi_B(x)\right|^{p_1} dm(x)\right)^{\frac{q_1}{p_1}}\right]^{\frac{1}{q_1}} \\ &\leq \left[\sum_{t\in\mathbb{Z}}\left[m(B\cap\Omega_t)\right]\right]^{\frac{1}{p_1}} \\ &= \left[m(B)\right]^{\frac{1}{p_1}}. \end{aligned} \tag{4}$$

Moreover,

$$\|I_\gamma f\|_{\dot{K}^{p_2}_{0,q_2}(\mathbb{R}^n,\mu)} = \left[\sum_{t\in\mathbb{Z}}\left(\int_{\Omega_t}\left|\int_{\mathbb{R}^n}\frac{\chi_B(y)}{|x-y|^{n-\gamma}}\,dm(y)\right|^{p_2}d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}}$$
$$\gtrsim \left[\sum_{t\in\mathbb{Z}}\left(\int_{\Omega_t\cap B}\left|\int_{B}\frac{1}{|x-y|^{n-\gamma}}\,dm(y)\right|^{p_2}d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}}.$$

Since $x,y\in B$, we have $|x-y|\le 2r$, where r is the radius of the ball. Thus

$$\frac{1}{|x-y|^{n-\gamma}} \gtrsim \frac{1}{(r^n)^{1-\frac{\gamma}{n}}} \gtrsim \left[m(B)\right]^{\frac{\gamma}{n}-1}.$$

Hence

$$\|I_\gamma f\|_{\dot{K}^{p_2}_{0,q_2}(\mathbb{R}^n,\mu)} \gtrsim \left[\sum_{t\in\mathbb{Z}}\left[m(B)\right]^{\frac{q_2\gamma}{n}}\left[\mu(\Omega_t\cap B)\right]^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}}$$
$$\gtrsim \left[m(B)\right]^{\frac{\gamma}{n}}\left[\sum_{t\in\mathbb{Z}}\left[\mu(\Omega_t\cap B)\right]\right]^{\frac{1}{p_2}}$$
$$= \left[m(B)\right]^{\frac{\gamma}{n}}\left[\mu(B)\right]^{\frac{1}{p_2}}. \tag{5}$$

From (4) and (5) we get $\mu(B)\lesssim [m(B)]^{p_2(\frac{1}{p_1}-\frac{\gamma}{n})}$. □

Next, we present the trace inequality for homogeneous Lorentz–Herz spaces. Since the proof is similar to that of Theorem 2.1, we only provide the necessary steps and point out the differences in the arguments.

Theorem 2.3 *Let p_1, p_2, q_1, q_2, μ be as in Theorem 2.1, and let $1\le r_1<r_2\le\infty$ (or $r_1=r_2=\infty$). Suppose $n(\frac{1}{p_1}-\frac{1}{p_2})\le\gamma<\frac{n}{p_1}$ and $\gamma-\frac{n}{p_1}<\lambda<n-\frac{n}{p_1}$. Then*

$$\|I_\gamma f\|_{\dot{H}L^{p_2,r_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{H}L^{p_1,r_1}_{\lambda,q_1}(\mathbb{R}^n,m)}.$$

Proof Let $f\in \dot{H}L^{p_1,r_1}_{\lambda,q_1}(\mathbb{R}^n,m)$. Then it is easy to see that

$$\|I_\gamma f\|_{\dot{H}L^{p_2,r_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \le \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\|I_\gamma f\cdot\chi_{\Omega_t}\|^{q_1}_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right]^{\frac{1}{q_1}}.$$

As before, setting $f_s=f\chi_{\Omega_s}$, $s\in\mathbb{Z}$, and using the triangle inequality of the Lorentz norm, we obtain

$$\|I_\gamma f\|_{\dot{H}L^{p_2,r_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \le \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left\|I_\gamma\left(\sum_{s\in\mathbb{Z}}f_s\right)\cdot\chi_{\Omega_t}\right\|^{q_1}_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right]^{\frac{1}{q_1}}$$
$$\le \left[\sum_{t\in\mathbb{Z}}2^{t\lambda q_1}\left\|\sum_{s\in\mathbb{Z}}I_\gamma f_s\cdot\chi_{\Omega_t}\right\|^{q_1}_{L^{(p_2,r_2)}(\mathbb{R}^n,\mu)}\right]^{\frac{1}{q_1}}$$

$$\lesssim \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\in\mathbb{Z}} \|I_\gamma f_s\cdot\chi_{\Omega_t}\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}.$$

The inner sum can be broken into three parts, and then by the application of Minkowski's inequality we may write

$$\begin{aligned}\|I_\gamma f\|_{\dot{H}L^{L^{p_2,r_2}(\mathbb{R}^n,\mu)}_{\lambda,q_2}(\mathbb{R}^n,\mu)} &\lesssim \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\le t-2} \|I_\gamma f_s\cdot\chi_{\Omega_t}\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}\\ &\quad+\left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{t-1\le s\le t+1} \|I_\gamma f_s\cdot\chi_{\Omega_t}\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}\\ &\quad+\left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1}\left(\sum_{s\ge t+2} \|I_\gamma f_s\cdot\chi_{\Omega_t}\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)}\right)^{q_1}\right]^{\frac{1}{q_1}}\\ &:= E_1+E_2+E_3.\end{aligned}$$

Now, we proceed as in Theorem 2.1, except that we use the Hölder inequality for Lorentz spaces and rely on the fact that $\|\chi_{\Omega_s}\|_{L^{p_1',r_1'}}\|\chi_{\Omega_t}\|_{L^{p_2,r_2}(\mathbb{R}^n,\mu)} \lesssim 2^{[t(n-\gamma)+\frac{n}{p_1'}(s-t)]}$ (which follows along lines similar to [6, Lemma 3.1.2.1]). Furthermore, the estimation of E_2 is based on Corollary 1.2. □

We wrap up this section with the following theorem addressing the limiting case $p_1 = p_2 = p$. By employing a simple modification of the proof of either Theorem 2.1 or Theorem 2.3, in combination with [14, Theorem 1.2] (see also [13, Theorem 3.1]), we get

Theorem 2.4 *Let $1 < p < \infty$, $1 \le q_1 \le q_2 < \infty$, $0 < \gamma < \frac{n}{p}$, and $\gamma - \frac{n}{p} < \lambda < n - \frac{n}{p}$. Suppose that for every ball $B \subset \mathbb{R}^n$, we have $\mu(B) \lesssim [m(B)]^{(1-\frac{\gamma p}{n})}$. Then*

$$\|I_\gamma f\|_{\dot{K}^p_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\dot{H}L^{p,1}_{\lambda,q_1}(\mathbb{R}^n,m)}.$$

The question whether Theorem 2.4 holds when replacing $\dot{H}L^{p,1}_{\lambda,q_1}$ with a space that is not as narrow as this (e.g., $\dot{H}L^{p,r}_{\lambda,q_1}$ for $1 < r < p$) remains open.

3 Optimality conditions

In this section, we present some examples to illustrate the optimality of certain parametric conditions assumed in Theorem 2.1 or Theorem 2.3. To that end, we focus on the case where $n = 1$ and $\beta := p_2(\frac{1}{p_1} - \gamma) \le 1$. Suppose μ is the positive Borel measure generated by $g(x) = x^{\beta-1}\chi_{(0,\infty)}(x)$ on the the Borel sigma algebra $\mathcal{B}$ of $\mathbb{R}$, i.e., $\mu(S) = \int_S g\,dm$ for $S \in \mathcal{B}$. We will proceed by working out a few examples using this choice of μ.

Example 3.1 Consider the function $f = \chi_{\Omega_1}$ in $\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}, m)$. For $x \in \Omega_t$ and $y \in \Omega_1$, it is evident that

$$\frac{1}{|x-y|^{1-\gamma}} \gtrsim \frac{1}{(2+2^t)^{1-\gamma}}.$$

Consequently,

$$\begin{aligned}\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} &= \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_2}\left(\int_{\Omega_t}\left|\int_{\mathbb{R}} \frac{\chi_{\Omega_1}(y)}{|x-y|^{1-\gamma}}\,dm(y)\right|^{p_2} d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}} \\ &\gtrsim \left[\sum_{t\in\mathbb{Z}} \frac{2^{t\lambda q_2}[\mu(\Omega_t)]^{\frac{q_2}{p_2}}}{(2+2^t)^{q_2(1-\gamma)}}\right]^{\frac{1}{q_2}} \\ &\gtrsim \left[\sum_{t\le 0} 2^{tq_2\lambda}\big[\mu(\Omega_t)\big]^{\frac{q_2}{p_2}} + \sum_{t\ge 1} 2^{tq_2(\lambda-1+\gamma)}\big[\mu(\Omega_t)\big]^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}}.\end{aligned}$$

Therefore

$$\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} \gtrsim \left[\sum_{t\le 0} 2^{tq_2(\lambda+\frac{1}{p_1}-\gamma)} + \sum_{t\ge 1} 2^{tq_2(\lambda-1+\frac{1}{p_1})}\right]^{\frac{1}{q_2}}.$$

Thus, for the estimate $\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} \lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R},m)}$, it is necessary that $\gamma - \frac{1}{p_1} < \lambda < 1 - \frac{1}{p_1}$.

The subsequent example demonstrates the necessity of the condition $q_1 \le q_2$.

Example 3.2 Consider the function $f_k(x) = |x|^{-(\lambda+\frac{1}{p_1})}\chi_{\{1<|x|<2^k\}}(x)$, $k \in \mathbb{N}$. It is not difficult to see that $\|f_k\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R},m)} \lesssim k^{\frac{1}{q_1}}$. Moreover,

$$\begin{aligned}\|I_\gamma f_k\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} &= \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_2}\left(\int_{\Omega_t}\left|\int_{\mathbb{R}} \frac{f_k(y)}{|x-y|^{1-\gamma}}\,dm(y)\right|^{p_2} d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}} \\ &\gtrsim \left[\sum_{t\in\mathbb{Z}} 2^{t\lambda q_2}\left(\int_{\Omega_t}\left|\int_{\Omega_t} \frac{f_k(y)}{|x-y|^{1-\gamma}}\,dm(y)\right|^{p_2} d\mu(x)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}}.\end{aligned}$$

Notice that $\frac{1}{|x-y|^{1-\gamma}} \ge 2^{-(t+1)(1-\gamma)}$ and $f_k(y) \ge 2^{-t(\lambda+\frac{1}{p_1})}$ for $x, y \in \Omega_t$, $1 \le t \le k$. Consequently,

$$\begin{aligned}\|I_\gamma f_k\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} &\gtrsim \left[\sum_{1\le t\le k} 2^{q_2(t\lambda-(t+1)(1-\gamma)-t(\lambda+\frac{1}{p_1}))}\left(\big[m(\Omega_t)\big]^{p_2}\mu(\Omega_t)\right)^{\frac{q_2}{p_2}}\right]^{\frac{1}{q_2}} \\ &\gtrsim \left[\sum_{1\le t\le k} 1\right]^{\frac{1}{q_2}} = k^{\frac{1}{q_2}}.\end{aligned}$$

Using $\|I_\gamma f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R},\mu)} \lesssim \|f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R},m)}$, we deduce that $k^{\frac{1}{q_2}} \lesssim k^{\frac{1}{q_1}}$. As $k \in \mathbb{N}$ is arbitrary, we must have $q_1 \le q_2$.

The remaining conditions, $p_1 < p_2$ and $\gamma < \frac{n}{p_1}$, are known to be optimal in Lebesgue spaces, and, consequently, they stand as optimal conditions for Theorem 2.1 (or Theorem 2.3).

4 Sobolev inequalities

The Riesz potential operator on $\mathbb{R}^n$ classically arises from the $\frac{\gamma}{2}$th-order fractional Laplace equation $(-\Delta)^{\frac{\gamma}{2}}(u) = f$. For $0 < \gamma < n$, the function $\mathcal{G}(\gamma)(2\pi|x|)^{-\gamma}$ is the Fourier transform of the function $|x|^{\gamma-n}$ [20, p. 66]. Here the constant $\mathcal{G}(\gamma)$, known as the normalized constant, is given by

$$\mathcal{G}(\gamma) = \frac{\pi^{\frac{n}{2}} 2^{\gamma} \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\alpha}{2})},$$

where Γ is the Euler gamma function. Based on this, it can be readily inferred that the equation $u = \frac{I_\gamma(f)}{\mathcal{G}(\gamma)}$ solves the aforementioned equation. Thus the results in Sect. 2 indicate that under the conditions of Theorem 2.1 (or Theorem 2.3), if f belongs to $\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n, m)$ [or $\dot{HL}^{p,r}_{\lambda,q}(\mathbb{R}^n, m)$], then the solution of fractional-order equation $(-\Delta)^{\frac{\gamma}{2}}(u) = f$ belongs to $\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n, \mu)$ [resp., $\dot{HL}^{p,r}_{\lambda,q}(\mathbb{R}^n, \mu)$].

Another important observation is that if μ is the Lebesgue measure restricted to Borel sets in $\mathbb{R}^n$ and $\frac{1}{p_1} - \frac{1}{p_2} = \frac{\gamma}{n}$, then we can immediately deduce the following Hardy–Littlewood–Sobolev theorem of fractional integration in the context of Lorentz–Herz spaces.

Corollary 4.1 *Let* $1 < p_1 < p_2 < \infty$, $1 \leq q_1 \leq q_2 < \infty$, *and* $1 \leq r_1 < r_2 \leq \infty$ (*or* $r_1, r_2 = \infty$). *Then*

$$\|I_\gamma f\|_{\dot{HL}^{p_2,r_2}_{\lambda,q_2}(\mathbb{R}^n,m)} \lesssim \|f\|_{\dot{HL}^{p_1,r_1}_{\lambda,q_1}(\mathbb{R}^n,m)},$$

provided that $\frac{1}{p_1} - \frac{1}{p_2} = \frac{\gamma}{n}$ *and* $\gamma - \frac{n}{p_1} < \lambda < n - \frac{n}{p_1}$.

In particular, if $r_i = p_i$, $i = 1, 2$, then we get the corresponding theorem for Herz spaces (cf. [17]).

Let us recall the definition of homogeneous Herz-type Sobolev spaces from [18]. For consistency, we make a slight adjustment to the notation.

Definition 4.2 Let $1 < p < \infty$, $0 < \lambda < n(1 - \frac{1}{p})$, $0 < q < \infty$, and $k \in \mathbb{Z}_+$. The homogeneous Herz-type Sobolev space $\dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n) := \left\{ f \in \dot{K}^{p}_{\lambda,q}(\mathbb{R}^n) : \text{for } |\beta| \leq k, \frac{\partial^\beta f}{\partial f^\beta} \text{ exists on } \mathcal{D}'(\mathbb{R}^n), \text{and } \frac{\partial^\beta f}{\partial f^\beta} \in \dot{K}^{p}_{\lambda,q}(\mathbb{R}^n) \right\},$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n_+$, $\frac{\partial^0 f}{\partial f^0} = f$, and the space is equipped with the functional

$$\|f\|_{\dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n)} := \sum_{|\beta| \leq k} \left\| \frac{\partial^\beta f}{\partial f^\beta} \right\|_{\dot{K}^{p}_{\lambda,q}(\mathbb{R}^n)}.$$

The parameters in this definition are subjected to specific conditions to ensure the reasonableness of definition, as outlined in [18].

Theorem 4.3 *Let* $1 < p_1 < n$, $p_2 < \infty$, $1 \leq q_1 \leq q_2 < \infty$, *and* $0 < \lambda < n - \frac{n}{p_1}$. *Suppose that for every ball* $B \subset \mathbb{R}^n$, *we have* $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1} - \frac{1}{n})}$. *Then*

$$\|f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \|\nabla f\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}$$

for every $f \in \dot{K}^{p_1,1}_{\lambda,q_1}(\mathbb{R}^n)$.

Proof First, assume that $g \in \mathcal{D}(\mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Then it is well known that $|g(x)| \lesssim I_1(|\nabla g|)(x)$ for every $x \in \mathbb{R}^n$. Therefore by Theorem 2.1 and the ideal property of Herz spaces ([6, Proposition 3.6]) it follows that

$$\begin{aligned} \|g\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} &\lesssim \big\|I_1\big(|\nabla g|\big)\big\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \\ &\lesssim \|\nabla g\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}, \end{aligned} \tag{6}$$

where $\|\nabla g\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)} = \|(|\nabla g|)\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}$ and $|\nabla g| = \sum_{j=1}^n |\frac{\partial g}{\partial x_j}|$. Now let $f \in \dot{K}^{p_1,1}_{\lambda,q_1}(\mathbb{R}^n)$. Then $f \in \dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} \in \dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n)$, $j = 1, 2, \dots, n$. Moreover, there exists a sequence $\{f_k\}$ in $\mathcal{D}(\mathbb{R}^n)$ such that $f_k \to f$ in $\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n)$ and $\frac{\partial f_k}{\partial x_j} \to \frac{\partial f}{\partial x_j}$ in $\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n)$ [18, Proposition 2.1]. Therefore by equation (6) we get

$$\|f_k - f_l\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \left\| \sum_{j=1}^n \left| \frac{\partial f_k}{\partial x_j} - \frac{\partial f_l}{\partial x_j} \right| \right\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)},$$

from which it follows that the sequence $\{f_k\}$ converges to f in $\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)$. This completes the proof. □

The repeated application of the pointwise estimate $|g(x)| \lesssim I_1(|\nabla g|)(x)$, in combination with semigroup property $I_\alpha I_\beta = I_{\alpha+\beta}$, ensures the above inequality for higher-order Sobolev-type Herz spaces as well.

Theorem 4.4 *Let $k \in \mathbb{N}$, $1 < p_1 < \frac{n}{k}$, $p_2 < \infty$, $1 \le q_1 \le q_2 < \infty$, and $0 < \lambda < n - \frac{n}{p_1}$. If $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_1} - \frac{k}{n})}$ for every ball $B \subset \mathbb{R}^n$, then*

$$\|f\|_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\mu)} \lesssim \big\|\nabla^k f\big\|_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,m)}$$

for every $f \in \dot{K}^{p_1,k}_{\lambda,q_1}(\mathbb{R}^n)$.

We say that p^* is the k-Sobolev conjugate of p if $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$, where k is a positive integer. We simply write Sobolev conjugate for 1-Sobolev conjugate. Putting $\mu = m$ in the above theorem, we get the following Sobolev embedding theorem for Herz-type Sobolev spaces.

Corollary 4.5 *Let $k \in \mathbb{N}$, $1 < p < \frac{n}{k}$, $1 \le q < \infty$, and $0 < \lambda < n - \frac{n}{p}$, and let p^* be the k-Sobolev conjuage of p. Then*

$$\|f\|_{\dot{K}^{p^*}_{\lambda,q}(\mathbb{R}^n,m)} \lesssim \big\|\nabla^k f\big\|_{\dot{K}^{p}_{\lambda,q}(\mathbb{R}^n,m)}$$

for every $f \in \dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n)$. In particular, $\dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n) \hookrightarrow \dot{K}^{p^*}_{\lambda,q}(\mathbb{R}^n)$.

Finally, we prove the following Gagliardo–Nirenberg–Sobolev (GNS) inequality in the setting of Herz spaces.

Theorem 4.6 *Let $0 \leq \theta \leq 1$, $1 < p_0 < n$, $1 \leq p_0, p_1 < p_2 < \infty$, $1 \leq q_0, q_1 < q_2 < \infty$, and $0 < \lambda < n - \frac{n}{p_0}$. Suppose that $\mu(B) \lesssim [m(B)]^{p_2(\frac{1}{p_0} - \frac{1}{n})}$ for every ball $B \subset \mathbb{R}^n$. Then*

$$\|f\|_{\dot{K}^{p}_{\lambda,q}(\mathbb{R}^n,\mu)} \leq \|f\|^{1-\theta}_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,\mu)} \|\nabla f\|^{\theta}_{\dot{K}^{p_0}_{\lambda,q_0}(\mathbb{R}^n,m)}$$

for every $f \in \dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,\mu) \cap \dot{K}^{p_0,1}_{\lambda,q_0}(\mathbb{R}^n,m)$, provided that $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$.

Proof Let $0 \leq \theta \leq 1$ and $1 \leq p_1 < p_2 \leq \infty$. Using the interpolation inequality $\|f\|_{L^p(\mathbb{R}^n,\nu)} \leq \|f\|^{1-\theta}_{L^{p_1}(\mathbb{R}^n,\nu)} \|f\|^{\theta}_{L^{p_2}(\mathbb{R}^n,\nu)}$, which holds for any positive measure ν on $\mathbb{R}^n$, provided that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, and the Hölder inequality, we obtain

$$\sum_{t\in\mathbb{Z}} 2^{t\lambda q} \|f\chi_{\Omega_t}\|^{q}_{L^p(\mathbb{R}^n,\nu)} \leq \left(\sum_{t\in\mathbb{Z}} 2^{t\lambda q_1} \|f\chi_{\Omega_t}\|^{q_1}_{L^{p_1}(\mathbb{R}^n,\nu)}\right)^{\frac{q(1-\theta)}{q_1}} \left(\sum_{t\in\mathbb{Z}} 2^{t\lambda q_2} \|f\chi_{\Omega_t}\|^{q_2}_{L^{p_2}(\mathbb{R}^n,\nu)}\right)^{\frac{q\theta}{q_2}}.$$

Consequently,

$$\|f\|_{\dot{K}^{p}_{\lambda,q}(\mathbb{R}^n,\nu)} \leq \|f\|^{1-\theta}_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,\nu)} \|f\|^{\theta}_{\dot{K}^{p_2}_{\lambda,q_2}(\mathbb{R}^n,\nu)}.$$

Using Theorem 4.3, it follows that

$$\|f\|_{\dot{K}^{p}_{\lambda,q}(\mathbb{R}^n,\mu)} \leq \|f\|^{1-\theta}_{\dot{K}^{p_1}_{\lambda,q_1}(\mathbb{R}^n,\mu)} \|\nabla f\|^{\theta}_{\dot{K}^{p_0}_{\lambda,q_0}(\mathbb{R}^n,m)}.$$ □

Remark 4.7

(i) If $q \leq p$, then the definition of $\dot{K}^{p,k}_{\lambda,q}(\mathbb{R}^n)$ remains reasonable even when $\lambda = 0$. Evidently, all the aforementioned results (Theorem 4.3 onwards) are still true when $\lambda = 0$ and $q_i \leq p_i$ $(i = 0, 1, 2)$.

(ii) If $\lambda = 0$, $\mu = m$, $p_i = q_i$ for $i = 0, 1, 2$, and p_2 is the Sobolev conjugate of p_0, we obtain the GNS inequality for the Lebesgue spaces (see [21, Sect. 1]).

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M.A.B. formulated the results and wrote the manuscript. G.S.R.K. supervised the research, provided critical feedback, and revised the manuscript.

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