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Asymptotic properties of conditional value-at-risk estimate for asymptotic negatively associated samples

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Abstract

This article examines the strong consistency of the conditional value-at-risk (CVaR) estimate for asymptotic negatively associated (ANA or ρ^- , for short) random samples under mild conditions. It is demonstrated that the optimal rate can achieve nearly $O(n^{-1/2})$ under certain appropriate conditions. Furthermore, we present numerical simulations and a real data example to corroborate our theoretical results based on finite samples.

Mathematics Subject Classification: 60G05; 62G20

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1 Introduction

The introduction of the value-at-risk (VaR) model has revolutionised the investment, management and governance landscape. In the investment arena, VaR enables individuals to assess the risk associated with investment assets, enabling them to formulate investment strategies based on risk level and risk tolerance, thereby reducing investment uncertainty. At the operational level, VaR allows for the continuous monitoring of potential fluctuations in order to avoid significant losses due to adverse changes in certain factors. In management, the VaR model plays a key role in the internal management of institutions, including the development of investment strategies, the assessment and supervision of traders and the prudent allocation of resources, while also serving as a valuable tool for market regulators. Market regulators are tasked with preventing adverse effects on the overall market and economic system arising from excessive market risk accumulation and concentrated risk release, with the VaR model serving as the primary tool for quantifying market risk accumulation. This innovative VaR technology and risk management framework underpinned by the VaR model will improve the operations of financial institutions in China, promote more rational investment behaviour among investors and provide regulators with an effective mechanism for market supervision.

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Assume X to be a random cost variable with cumulative distribution function $F(u) = P(X \leq u)$. Let $F^{-1}(v)$ be its right continuous inverse, defined as $F^{-1}(v) = \inf\{u : F(u) > v\}$. For a fixed α , the value-at-risk VaR_α is defined as the α -quantile, represented as

$$\text{VaR}_\alpha(X) = F^{-1}(\alpha). \tag{1.1}$$

In order for financial institutions to quantify and mitigate financial risk, as required by the Basel III framework, VaR is commonly used. Despite its widespread adoption, VaR is considered insufficient as a comprehensive risk measure due to inherent mathematical limitations such as non-subadditivity and non-convexity. Additionally, optimising VaR based on scenarios presents challenges.

To address the limitations of VaR, some authors advocate the adoption of CVaR as a more comprehensive risk metric. Regarded as a coherent alternative, CVaR is gaining traction in the realm of financial risk management. For risk measurement, CVaR is proved to have better properties than VaR, where CVaR_α can be defined by

$$\text{CVaR}_\alpha(X) = E(X|X \geq \text{VaR}_\alpha(X)). \tag{1.2}$$

i.e., CVaR_α can be thought of as the conditional expectation of losses that exceed the $\text{CVaR}_\alpha(X)$ level. Pflug [1] put forward that the $\text{CVaR}_\alpha(X)$ can be viewed as the solution of an optimisation problem, namely,

$$\text{CVaR}_\alpha(X) = \inf_{x \in \mathbb{R}} \left\{ x + \frac{1}{1-\alpha} E[X - x]^+ \right\} := \theta^*, \tag{1.3}$$

where $[a]^+ := \max\{0, a\}$ denotes the positive part of $a \in \mathbb{R}$.

The CVaR model, a prevalent financial risk metric, enjoys broad support and acceptance within the international financial community. Its optimized version is recognised as a refined certainty equivalent risk measure, garnering increasing attention from practitioners and academics alike. Scholars have identified several key properties of CVaR as a coherent risk measure, including transition-equivariance, convexity, and positive homogeneity. For further insights, references such as Pflug [1], Artzner et al. [2], Embrechts et al. [3], Bodnar et al. [4], Pavlikov and Uryasev [5], Wang et al. [6], Luo [7], among others, offer detailed discussions on the subject.

In equation (1.3), consider $h_\alpha(X, x) = x + \frac{1}{1-\alpha} [X - x]^+$ and define $\theta^* = \text{CVaR}_\alpha(X)$. Therefore, $\text{CVaR}_\alpha(X) = \inf_{x \in \mathbb{R}} E h_\alpha(X, x)$. If X_1, \dots, X_n represent n realisations of the random variable X , then θ^* can be estimated as

$$\text{CVaR}_{n,\alpha}(X) = \inf_{x \in \mathbb{R}} n^{-1} \sum_{i=1}^n h_\alpha(X_i, x) := \hat{\theta}_n. \tag{1.4}$$

Trindade et al. [8] examined the consistency of $\hat{\theta}_n$ for independently and identically distributed samples as well as for stationary processes. However, time series data in fields like finance and economics typically exhibit interdependencies, making sample dependence inherent. Recently, Xing et al. [9] demonstrated the strong consistency of conditional value-at-risk estimation for φ -mixing samples under mild assumptions. Luo and Ou [10] delved into exponential inequalities, the strong laws of large numbers, and convergence

rates for CVaR estimators in α -mixing sequences. Ding et al. [11] explored Berry-Esseen type bounds for conditional value-at-risk estimators in ψ -mixing sequences, among other contributions in this area.

Next, we will review several different dependent structures. The NA random variable was proposed by Joag Dev and Proschan [12], which has a wide range of applications in multivariate statistical analysis and system reliability. Another important dependent structure is the ρ^* -mixing random variable proposed by Bradley [13]. Because ρ^* -mixing random variables contain certain moving average processes and specific categories of Markov chains, their importance extends to different fields such as economics, finance, and other scientific disciplines. Zhang and Wang [14] introduced the concept of asymptotically negatively associated random variables based on NA sequences and ρ^* -mixing sequences. As is well known, ANA random variables encompass a mixture of ρ^* -mixing and NA random variables as special instances, as detailed in Zhang [15] examples 2.2 and 2.3. The definition of ANA random variable is as follows.

Definition 1.1 A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically negatively associated (ANA or ρ^- , for short) if

$$\rho^-(s) = \sup\{\rho^-(S, T) : S, T \subset \mathbb{N}, \text{dist}(S, T) \geq s\} \rightarrow 0, \text{ as } s \rightarrow \infty, \tag{1.5}$$

where

$$\rho^-(S, T) = 0 \vee \left\{ \frac{\text{Cov}(f_1(X_i, i \in S), f_2(X_j, j \in T))}{\sqrt{\text{Var}(f_1(X_i, i \in S)) \cdot \text{Var}(f_2(X_j, j \in T))}} : f_1, f_2 \in \mathcal{C} \right\}, \tag{1.6}$$

and \mathcal{C} is the set of nondecreasing functions.

Since the inception of ANA random variables by Zhang and Wang [14], a plethora of noteworthy theoretical findings has emerged. For instance, Zhang and Wang [14] explored moment inequalities and complete convergence for partial sums of ANA random fields, while Zhang [16] derived central limit theorems. Yuan and Wu [17] delved into the limiting behaviour of the maximum of partial sums under residual Cesaro alpha-integrability assumptions. Tang et al. [18] established a Berry–Esseen type bound for wavelet estimators in a nonparametric regression model with ANA errors. Wu et al. [19] established a general result on complete moment convergence and the Marcinkiewicz–Zygmund-type strong law of large numbers for weighted sums of asymptotic negatively associated random variables. Additionally, Ko Mi-Hwa [20] elucidated the limiting behaviour of the maximum of partial sums in Hilbert space, among other notable contributions in this domain.

Building upon the insights from the aforementioned article, this paper delves into exploring the strong consistency of estimate $\text{CVaR}_{n,\alpha}(X)$ when the sample follows an the ANA sequence, and provide its convergence rate. Furthermore, the theoretical findings derived from a restricted sample are validated through numerical simulations and illustrative examples.

The paper is structured as follows: Main results and their proofs are detailed in Sect. 2. Section 3 includes numerical simulations, while Sect. 4 presents a real data example. Finally, the Appendix contains the necessary lemmas.

Throughout this paper, the symbol C represents a positive constant, which may vary across different instances. Let $I(A)$ be the indicator function of the set A , $[x]$ be the integer part of x , and $a \ll b$ mean that $a \leq Cb$.

2 Main result and proof

In this section, we give our main theorem as follows.

Theorem 2.1 *Suppose that $\{X_i, 1 \leq i \leq n\}$ is a sequence of identically distributed ANA random variables with $E |X_i|^r < \infty$ for some $r > 1$. Then we have*

$$|CVaR_{n,\alpha}(X) - CVaR_\alpha(X)| = O(n^{-\mu}) \text{ a.s.}, \tag{2.1}$$

where (i) if $1 < r \leq 2$, then $0 \leq \mu < 1 - \frac{1}{r}$; (ii) if $r > 2$, then $0 \leq \mu < \frac{1}{2}$.

Remark 2.2 Let $\mu > 0$ in (2.1), then we know that $\hat{\theta}_n$ is the strong consistent estimator of θ^* . And by (2.1), the strong consistency rate is $n^{-\mu}$ with $0 < \mu < 1 - 1/r$. Specially select appropriate parameters, and the convergence speed is close to $n^{-1/2}$ in order.

Remark 2.3 Since ANA sequences include NA (in particular, independent) and ρ^* -mixing sequences, Theorem 2.1 also apply for NA and ρ^* -mixing sequences. Furthermore, Theorem 2.1 relaxes the constraint $1 < r \leq 2$ in Theorem 2.1 of Xing et al. [9] to $r > 1$, and expands the scope of the sample from φ -mixing sequences to ANA sequences.

To prove the main results of the paper, the following lemma play a crucial role

Lemma 2.4 *Suppose that $\{X_i, 1 \leq i \leq n\}$ is a sequence of identically distributed ANA random variables with $E |X_i|^r < \infty$ for some $r > 1$. Then we have*

$$\frac{1}{n^{1-\mu}} \sum_{i=1}^n [h_\alpha(X_i, x) - Eh_\alpha(X_i, x)] \rightarrow 0 \text{ a.s.}, \tag{2.2}$$

where (i) if $1 < r \leq 2$, then $0 \leq \mu < 1 - \frac{1}{r}$; (ii) if $r > 2$, then $0 \leq \mu < \frac{1}{2}$.

Proof (i) Since $0 < \mu < 1 - 1/r$ for $1 < r \leq 2$, we have $r(1 - \mu) > 1$, then $\mu/(r - 1) < 1 - \mu$. Hence, there exist $\delta > 0$ and $s > 0$ such that

$$\frac{s + \mu}{r - 1} < \delta < 1 - \mu. \tag{2.3}$$

Set

$$\begin{aligned} h_\alpha^{(1)}(X_i, x) &= -n^\delta I(h_\alpha(X_i, x) < -n^\delta) + h_\alpha(X_i, x) I(|h_\alpha(X_i, x)| \leq n^\delta) + n^\delta I(h_\alpha(X_i, x) > n^\delta), \\ h_\alpha^{(2)}(X_i, x) &= (h_\alpha(X_i, x) + n^\delta) I(h_\alpha(X_i, x) < -n^\delta) + (h_\alpha(X_i, x) - n^\delta) I(h_\alpha(X_i, x) > n^\delta), \end{aligned}$$

and

$$H_\alpha^{(1)}(X_i, x) = h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x), \quad H_\alpha^{(2)}(X_i, x) = h_\alpha^{(2)}(X_i, x) - Eh_\alpha^{(2)}(X_i, x)$$

Without loss of generality, assume that $Eh_\alpha(X_i, x) = 0$. We can get

$$\begin{aligned}
 & \frac{1}{n^{1-\mu}} \sum_{i=1}^n h_\alpha(X_i, x) \\
 &= \frac{1}{n^{1-\mu}} \sum_{i=1}^n (h_\alpha(X_i, x) - Eh_\alpha(X_i, x)) \\
 &= \frac{1}{n^{1-\mu}} \sum_{i=1}^n (h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x)) + \frac{1}{n^{1-\mu}} \sum_{i=1}^n (h_\alpha^{(2)}(X_i, x) - Eh_\alpha^{(2)}(X_i, x)) \\
 &= \frac{1}{n^{1-\mu}} \sum_{i=1}^n H_\alpha^{(1)}(X_i, x) + \frac{1}{n^{1-\mu}} \sum_{i=1}^n H_\alpha^{(2)}(X_i, x) \\
 &:= S_{n1} + S_{n2}.
 \end{aligned} \tag{2.4}$$

From (2.4) it follows that to prove that (2.2) holds, it is sufficient to prove $S_{n1} \xrightarrow{a.s.} 0$ and $S_{n2} \xrightarrow{a.s.} 0$ for $x \in \mathbb{R}$ to obtain the desired result, respectively.

First, we prove $S_{n1} \xrightarrow{a.s.} 0$ for $x \in \mathbb{R}$. By C_r inequality and Lemma A.3, we have for $x \in \mathbb{R}$ and $p > 2$,

$$\begin{aligned}
 & E |h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x)|^p \\
 &\leq 2^{p-1} \left(E |h_\alpha^{(1)}(X_i, x)|^p + |Eh_\alpha^{(1)}(X_i, x)|^p \right) \\
 &\ll E |h_\alpha^{(1)}(X_i, x)|^p + (E |h_\alpha^{(1)}(X_i, x)|)^p \\
 &\ll E |h_\alpha^{(1)}(X_i, x)|^p \\
 &\ll E |-n^\delta I(h_\alpha(X_i, x) < -n^\delta) + h_\alpha(X_i, x) I(|h_\alpha(X_i, x)| \leq n^\delta) + n^\delta I(h_\alpha(X_i, x) > n^\delta)|^p \\
 &\ll n^{\delta(p-r)} E |h_\alpha(X_i, x)|^r I(|h_\alpha(X_i, x)| \leq n^\delta) + n^{\delta p} P(|h_\alpha(X_i, x)| > n^\delta) \\
 &\ll n^{\delta(p-r)}.
 \end{aligned} \tag{2.5}$$

Similarly, we can prove that

$$E (h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x))^2 \ll n^{\delta(2-r)}. \tag{2.6}$$

Thus, by Markov’s inequality Lemma A.1 and Lemma A.2, combined with (2.5) and (2.6), it follows that for any $\varepsilon > 0$,

$$\begin{aligned}
 P(|S_{n1}| > \varepsilon) &\ll n^{-(1-\mu)p} E \left| \sum_{i=1}^n H_\alpha^{(1)}(X_i, x) \right|^p \\
 &\ll n^{-(1-\mu)p} \left\{ \sum_{i=1}^n E |h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x)|^p \right. \\
 &\quad \left. + \left(\sum_{i=1}^n E |(h_\alpha^{(1)}(X_i, x) - Eh_\alpha^{(1)}(X_i, x))^2 \right)^{p/2} \right\}
 \end{aligned}$$

$$\begin{aligned} &\ll n^{-(1-\mu)p} \{n^{\delta(p-r)+1} + n^{p/2+\delta(2-r)p/2}\} \\ &= n^{-(1-\mu-\delta)p+1-r\delta} + n^{-[1-2\mu-\delta(2-r)]p/2+\delta}. \end{aligned} \tag{2.7}$$

Then, from $1 < r \leq 2$ and $\mu < 1 - \frac{1}{r}$ we have $1 - 2\mu - (1 - \mu)(2 - r) > 0$. Moreover, by $\delta < 1 - \mu$, we can get $1 - \mu - \delta > 0$ and $1 - 2\mu - \delta(2 - r) > 0$. Note that $1 - \delta r < 1 - \delta$ and $0 < \delta < 1$, we now choose p such that

$$(1 - \mu - \delta)p - 1 + \delta r > 1, \quad [1 - 2\mu - \delta(2 - r)]p/2 - \mu > 1. \tag{2.8}$$

Which together with (2.7) and (2.8), implies that $\sum P(|S_{n1}| > \varepsilon) < \infty$ for sufficiently large p . It follows by Borel–Cantelli lemma that

$$S_{n1} \xrightarrow{a.s.} 0, \quad \text{for } x \in \mathbb{R}. \tag{2.9}$$

Next, we prove $S_{n2} \xrightarrow{a.s.} 0$ for $x \in \mathbb{R}$. For this purpose, set $S_{n2}^{(2)} = \frac{1}{n^{1-\mu}} \sum_{i=1}^n h_{\alpha}^{(2)}(X_i, x)$. Then, $S_{n2} = S_{n2}^{(2)} - ES_{n2}^{(2)}$.

Noting that $\mu - \delta(r - 1) < -s < 0$ and (2.6) implies $\mu - \delta(r - 1) < 0$. Hence, applying Lemma A.3, we obtain for $x \in \mathbb{R}$,

$$\begin{aligned} &|ES_{n2}^{(2)}| \\ &\leq \frac{1}{n^{1-\mu}} \sum_{i=1}^n E \{ |h_{\alpha}(X_i, x)| I(|h_{\alpha}(X_i, x)| > n^{\delta}) + n^{\delta} I(|h_{\alpha}(X_i, x)| > n^{\delta}) \} \\ &\ll \frac{1}{n^{1-\mu}} \left\{ n^{\delta(1-r)} \sum_{i=1}^n [E|h_{\alpha}(X_i, x)|^r I(|h_{\alpha}(X_i, x)| > n^r)] + n^{\delta+1} P(|h_{\alpha}(X_i, x)| > n^r) \right\} \\ &\ll n^{\mu-\delta(r-1)} \rightarrow 0, \quad n \rightarrow \infty, \end{aligned} \tag{2.10}$$

which means that

$$ES_{n2}^{(2)} \rightarrow 0. \tag{2.11}$$

Moreover,

$$\begin{aligned} |S_{n2}^{(2)}| &\leq \frac{1}{n^{1-\mu}} \sum_{i=1}^n \{ |h_{\alpha}(X_i, x)| I(|h_{\alpha}(X_i, x)| > n^{\delta}) + n^{\delta} I(|h_{\alpha}(X_i, x)| > n^{\delta}) \} \\ &\ll \frac{1}{n^{1-\mu}} \sum_{i=1}^n |h_{\alpha}(X_i, x)| I(|h_{\alpha}(X_i, x)| > n^{\delta}) + \frac{1}{n^{1-\mu}} \sum_{i=1}^n n^{\delta} I(|h_{\alpha}(X_i, x)| > n^{\delta}) \\ &\ll \frac{1}{n^{1-\mu}} \sum_{i=1}^n |h_{\alpha}(X_i, x)| I(|h_{\alpha}(X_i, x)| > i^{\delta}) + \frac{1}{n^{1-\mu}} \sum_{i=1}^n n^{\delta} I(|h_{\alpha}(X_i, x)| > i^{\delta}) \\ &\ll \frac{1}{n^{1-\mu}} \sum_{i=1}^n \xi_i + \frac{n^{\delta}}{n^{1-\mu}} \sum_{i=1}^n \eta_i. \end{aligned} \tag{2.12}$$

where $\xi_i = |h_{\alpha}(X_i, x)| I(|h_{\alpha}(X_i, x)| > i^{\delta})$, $\eta_i = I(|h_{\alpha}(X_i, x)| > i^{\delta})$. It follows from (2.12) that to prove $S_{n2} \xrightarrow{a.s.} 0$, it is only necessary to prove $S_{n2}^{(2)} \xrightarrow{a.s.} 0$. Hence, to obtain $S_{n2}^{(2)} \xrightarrow{a.s.} 0$, it is

sufficient to show

$$n^{-(1-\mu)} \sum_{i=1}^n \xi_i \xrightarrow{a.s.} 0, \quad n^{-(1-\mu-\delta)} \sum_{i=1}^n \eta_i \xrightarrow{a.s.} 0.$$

Set

$$J_n^{(1)} = \sum_{i=1}^n i^{-(1-\mu)} \xi_i, \quad J_n^{(2)} = \sum_{i=1}^n i^{-(1-\mu-\delta)} \eta_i.$$

In what follows, we will prove $J_n^{(1)}$ and $J_n^{(2)}$ almost surely converges by subsequence method.

By (2.3), there exists $s > 0$ such that $(s + \mu)/(r - 1) < \delta < 1 - \mu$, we have for all $m \geq n \geq 1$,

$$\begin{aligned} E |J_n^{(1)} - J_m^{(1)}| &\leq \sum_{i=n+1}^m i^{-(1-\mu)} E \xi_i \\ &\ll \sum_{i=n+1}^m i^{-(1-\mu)} E |h_\alpha(X_i, \mathcal{X})| I(|h_\alpha(X_i, \mathcal{X})| > i^\delta) \\ &\ll \sum_{i=n+1}^m i^{\delta(1-r)+\mu-1} \ll \sum_{i=n+1}^\infty i^{-(1+s)} \\ &\ll n^{-s} \rightarrow 0, n \rightarrow \infty. \end{aligned} \tag{2.13}$$

By (2.13), the random sequence $\{J_n^{(1)}\}$ is a Cauchy sequence in L_1 . Hence, there exists a random variable $J^{(1)}$ such that $E|J^{(1)}| < \infty$ and $E |J_n^{(1)} - J^{(1)}| \rightarrow 0$. Then, for any $\epsilon > 0$,

$$\begin{aligned} P \left(|J_{2^k}^{(1)} - J^{(1)}| > \epsilon \right) &\leq E |J_{2^k}^{(1)} - J_n^{(1)}| + E |J_n^{(1)} - J^{(1)}| \\ &\leq \limsup_{n \rightarrow \infty} E |J_{2^k}^{(1)} - J_n^{(1)}| \\ &\ll \sum_{i=2^{k+1}}^\infty i^{-(1+s)} \ll 2^{-ks}. \end{aligned} \tag{2.14}$$

which implies that $\sum_{k=1}^\infty P \left(|J_{2^k}^{(1)} - J^{(1)}| > \epsilon \right) < \infty$, that is to say that $J_{2^k}^{(1)} \xrightarrow{a.s.} J^{(1)}$. Also,

$$P \left(\max_{2^{k-1} < n \leq 2^k} |J_n^{(1)} - J_{2^{k-1}}^{(1)}| > \epsilon \right) \ll \sum_{i=2^{k-1}+1}^{2^k} i^{-(1+s)} \ll 2^{-ks}, \tag{2.15}$$

which means that $\sum_{k=1}^\infty P \left(\max_{2^{k-1} < n \leq 2^k} |J_n^{(1)} - J_{2^k}^{(1)}| > \epsilon \right) < \infty$, then $\max_{2^{k-1} < n \leq 2^k} |J_n^{(1)} - J_{2^k}^{(1)}| \xrightarrow{a.s.} 0$ as $k \rightarrow \infty$. So we have

$$J_n^{(1)} \xrightarrow{a.s.} J^{(1)}. \tag{2.16}$$

By (2.16) and Kronecker lemma,

$$\frac{1}{n^{(1-\mu)}} \sum_{i=1}^n \xi_i \xrightarrow{a.s.} 0 \tag{2.17}$$

is obtained.

Similar to the proof of $J_n^{(1)} \xrightarrow{a.s.} J^{(1)}$. For $m' \geq n \geq 1$, which together with Lemma A.3 yields that

$$\begin{aligned}
 E \left| J_{m'}^{(2)} - J_n^{(2)} \right| &\leq \sum_{i=n+1}^{m'} i^{-(1-\mu-\delta)} E \eta_i \\
 &\ll \sum_{i=n+1}^{m'} i^{-(1-\mu-\delta)} P \left(|h_\alpha(X_i, x)| > i^\delta \right) \\
 &\ll \sum_{i=n+1}^{m'} i^{\delta(1-r)+\mu-1} \ll \sum_{i=n+1}^{\infty} i^{-(1+s)} \\
 &\ll n^{-s} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}
 \tag{2.18}$$

By (2.18), the random sequence $\{J_n^{(2)}\}$ is a Cauchy sequence in L_1 . Hence, there exists a random variable $J^{(2)}$ such that $E|J^{(2)}| < \infty$ and $E \left| J_n^{(2)} - J^{(2)} \right| \rightarrow 0$. Similarly to the derivation of (2.17)–(2.19), it is obtained that

$$\frac{1}{n^{(1-\mu-\delta)}} \sum_{i=1}^n \eta_i \xrightarrow{a.s.} 0.
 \tag{2.19}$$

From $S_{n2}^{(2)} \xrightarrow{a.s.} 0$, it implies that $S_{n2} \xrightarrow{a.s.} 0$ for $x \in \mathbb{R}$. Therefore, by combining equations (2.6) and (2.19), for $1 < r \leq 2$, consider $0 \leq \mu < 1 - 1/r$. It is demonstrated that equation (2.1) holds true.

(ii) When $r > 2$, it follows from $0 \leq \mu < 1/2$ that there exists $\delta > 0$ and $s > 0$ such that $(2s + \mu)/(r - 1) < \delta < 1 - \mu$. Similar to the proof of $1 < r \leq 2$, the equation (2.1) still holds.

The proof is completed. □

In the following, we will give the proofs of theorems.

Proof of Theorem 2.1 It is easy to observe that

$$\begin{aligned}
 &n^\mu \left| \text{CVaR}_{n,\alpha}(X) - \text{CVaR}_\alpha(X) \right| \\
 &= n^\mu \left| \left(\hat{\theta}_n - \theta^* \right) I \left(\hat{\theta}_n - \theta^* \geq 0 \right) + \left(\hat{\theta}_n - \theta^* \right) I \left(\hat{\theta}_n - \theta^* < 0 \right) \right| \\
 &\leq n^\mu \left| \left(\hat{\theta}_n - \theta^* \right) I \left(\hat{\theta}_n - \theta^* \geq 0 \right) \right| + n^\mu \left| \left(\hat{\theta}_n - \theta^* \right) I \left(\hat{\theta}_n - \theta^* < 0 \right) \right| \\
 &:= I_{n1} + I_{n2}.
 \end{aligned}
 \tag{2.20}$$

By Lemma 2.4, it follows that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$n^\mu \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - E h_\alpha(X_1, x) \right| \leq \frac{\varepsilon}{2} \quad \text{a.s.},
 \tag{2.21}$$

for $n > N$ and $x \in \mathbb{R}$. Hence, we have

$$\begin{aligned}
 I_{n1} &= n^\mu \left| \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| I(\hat{\theta}_n - \theta^* \geq 0) \\
 &= n^\mu \left| \inf_{x \in \mathbb{R}} \left\{ \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right\} \right| I(\hat{\theta}_n - \theta^* \geq 0) \\
 &= n^\mu \inf_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| I(\hat{\theta}_n - \theta^* \geq 0) \\
 &\leq n^\mu \inf_{x \in \mathbb{R}} \left\{ \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - E h_\alpha(X_1, x) \right| + \left| E h_\alpha(X_1, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| \right\} \\
 &\leq \inf_{x \in \mathbb{R}} \left\{ \frac{\varepsilon}{2} + n^\mu \left| E h_\alpha(X_1, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| \right\} \\
 &= \frac{\varepsilon}{2} + n^\mu \inf_{x \in \mathbb{R}} \left\{ \left| E h_\alpha(X_1, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| \right\} = \frac{\varepsilon}{2} \quad \text{a.s.,} \tag{2.22}
 \end{aligned}$$

as $n > N$. Similarly, we obtain

$$\begin{aligned}
 I_{n2} &= n^\mu \left| \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} E h_\alpha(X_1, x) \right| I(\hat{\theta}_n - \theta^* < 0) \\
 &= n^\mu \left| \inf_{x \in \mathbb{R}} \left\{ E h_\alpha(X_1, x) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right\} \right| I(\hat{\theta}_n - \theta^* < 0) \\
 &= n^\mu \inf_{x \in \mathbb{R}} \left| E h_\alpha(X_1, x) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right| I(\hat{\theta}_n - \theta^* < 0) \\
 &= n^\mu \inf_{x \in \mathbb{R}} \left\{ \left| E h_\alpha(X_1, x) - \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right| + \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right| \right\} \\
 &\leq \inf_{x \in \mathbb{R}} \left\{ \frac{\varepsilon}{2} + n^\mu \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right| \right\} \\
 &= \frac{\varepsilon}{2} + n^\mu \inf_{x \in \mathbb{R}} \left\{ \left| \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n h_\alpha(X_i, x) \right| \right\} = \frac{\varepsilon}{2} \quad \text{a.s.,} \tag{2.23}
 \end{aligned}$$

for $n > N$. A combination of (2.20)–(2.23) yields that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$n^\mu |CVaR_{n,\alpha}(X) - CVaR_\alpha(X)| = n^\mu |\hat{\theta}_n - \theta^*| \leq \varepsilon \text{ a.s.,} \tag{2.24}$$

for $n > N$ and $x \in \mathbb{R}$, which implies that Theorem 2.1 holds. The proof is complete. \square

3 Numerical simulation

In this section, we will carry out a simulation to study the numerical performance of the strong consistency for conditional value-at-risk estimators based on ANA sample. The simulations are conducted for the following two cases.

Table 1 The estimated values of CVaR data under MA(1) process

α	CVaR	Estimated values of CVaR					
		$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 300$	$n = 500$
0.01	3.077524	2.215625	2.362955	2.855645	2.870715	2.943208	3.062084
0.02	2.795422	2.099229	2.304749	2.512123	2.583326	2.766623	2.785824
0.03	2.618936	2.263051	2.347108	2.430953	2.507920	2.555459	2.605924
0.04	2.487623	2.196140	2.317410	2.352794	2.403516	2.447980	2.479876
0.05	2.381816	2.005741	2.119395	2.219372	2.276383	2.346065	2.370149
0.06	2.292523	2.004113	2.059640	2.102880	2.150995	2.230583	2.290441
0.07	2.214846	2.061753	2.089570	2.108897	2.168029	2.190359	2.210211
0.08	2.145812	2.011019	2.072660	2.085980	2.126311	2.166322	2.191727
0.09	2.083473	1.988638	1.998568	2.005189	2.038770	2.060024	2.080606
0.1	2.026480	1.925144	1.944972	1.962678	1.977837	2.008739	2.024052

Case I MA(1) process.

The following MA(1) process is considered:

$$X_t = \varepsilon_t - 0.5\varepsilon_{t-1}, \tag{3.1}$$

where $\{\varepsilon_t\}$ is the white noise sequence, with the properties:

$$E(X_t) = 0, \quad D(X_t) = 1.25, \quad Cov(X_t, X_{t-1}) = -0.5.$$

For $k \geq 1$, $Cov(X_t, X_{t-k}) = -0.5$. Therefore, this MA(1) process forms a ρ^* -mixing sequence, consequently qualifying as an ANA sequence. Utilizing this MA(1) process to generate random numbers, we consider probability levels $\alpha = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1$. with sample sizes $n = 25, 50, 100, 200, 300, 500$. Initially, we compute the true value of the Conditional Value at Risk (CVaR) based on the generated ANA samples. Subsequently, we estimate the CVaR values using different probability levels α and sample sizes n as specified above. Finally, we derive the average of 1000 simulation repetitions to obtain the true and estimated CVaR values, resulting in the CVaR dataset (refer to Table 1).

Based on the aforementioned MA(1) model, simulations were conducted 1000 times for sample sizes $n = 25, 50, 100, 200, 300, 500$. The average of these simulations was taken to represent the true and estimated values of CVaR, resulting in the curves for the true and estimated values (refer to Fig. 1). In the graph, the estimated values of CVaR are depicted by the blue dashed line, while the true values of CVaR are represented by the red solid line.

In the MA(1) model, based on the real and estimated values in Table 1 and the curve in Fig. 1, it can be observed that as the sample size n increases, the estimated values of VaR and CVaR also approach the true values more closely.

Case II ARMA(1,1) process.

The following ARMA(1,1) process is selected

$$X_t = 0.3X_{t-1} + \varepsilon_t - 0.7\varepsilon_{t-1} \tag{3.2}$$

where $\{\varepsilon_t\}$ is the white noise sequence, noting that

$$E(X_t) = 0, \quad D(X_t) = 1.175824, \quad Cov(X_t, X_{t-1}) = -0.6196608$$

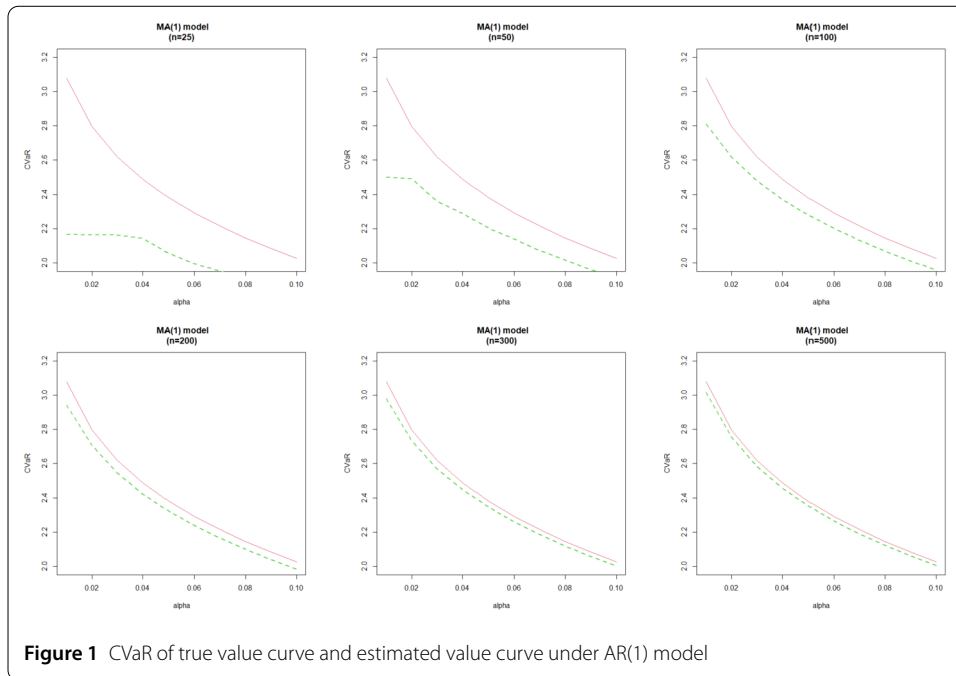


Table 2 The estimated values of CVaR data under ARMA(1,1) process

α	CVaR	Estimated values of CVaR					
		$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 300$	$n = 500$
0.01	2.890036	2.170455	2.474507	2.617419	2.772543	2.839317	2.865084
0.02	2.625126	2.150338	2.413390	2.509375	2.556982	2.594755	2.624383
0.03	2.459386	2.134721	2.294406	2.391566	2.404295	2.417741	2.435263
0.04	2.336072	2.100238	2.196019	2.210806	2.292701	2.308157	2.325940
0.05	2.236711	2.075447	2.139127	2.178833	2.183212	2.204913	2.215631
0.06	2.152858	1.985531	2.082286	2.123782	2.134671	2.147312	2.149778
0.07	2.079914	1.951368	1.990108	2.053718	2.096666	2.049605	2.063238
0.08	2.015085	1.907529	1.960907	1.981756	1.995446	2.004159	2.008428
0.09	1.956544	1.855084	1.873512	1.917522	1.939128	1.948190	1.951798
0.1	1.903023	1.801423	1.836655	1.860401	1.866697	1.878758	1.898029

when $k \geq 1$, $Cov(X_t, X_{t-k}) = 0.3Cov(X_t, X_{t-k+1})$. Thus this ARMA(1,1) process is an NA sequence, and it follows that it is also an ANA sequence. Other settings are the same as in Case I. Similar to the MA(1) model, simulations were conducted 1000 times for sample sizes $n = 25, 50, 100, 200, 300, 500$. The average of these simulations was taken as the true and estimated values of CVaR. The estimated values of CVaR under the ARMA(1,1) model are presented in Table 2. Furthermore, we obtained the corresponding true value curve and estimated value curve for CVaR under the ARMA(1,1) model (refer to Fig. 2). In the graph, the estimated values of CVaR are depicted by the blue dashed line, while the true values of CVaR are represented by the red solid line.

In the ARMA(1,1) model, based on the real and estimated values in Table 2 and the curve in Fig. 2, furthermore, as the sample size n increases, the estimated values of CVaR also approach the true values more closely.

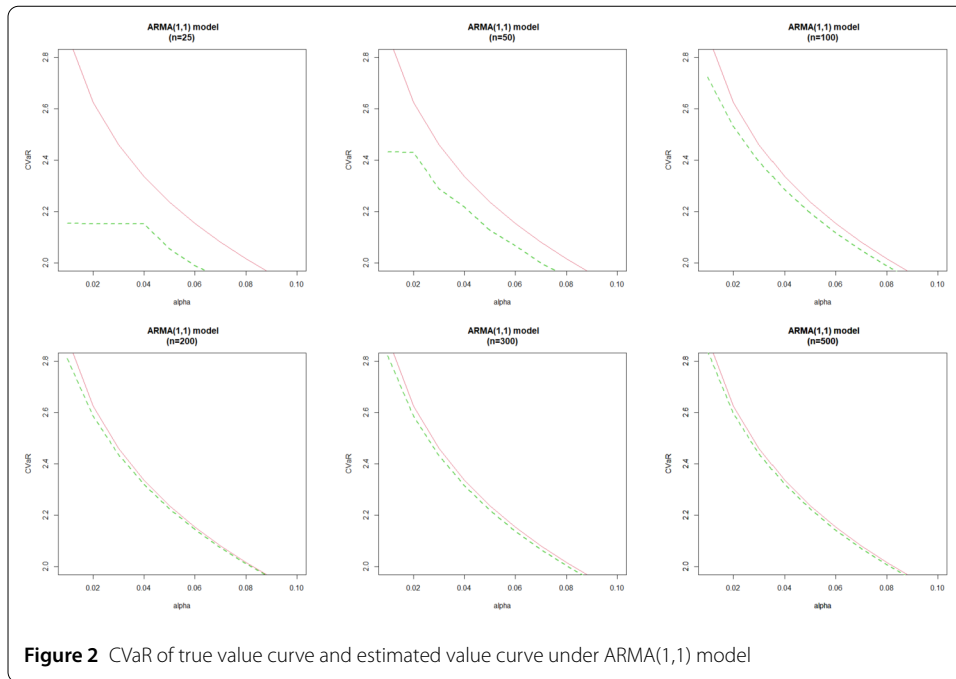


Figure 2 CVaR of true value curve and estimated value curve under ARMA(1,1) model

4 Real data exercises

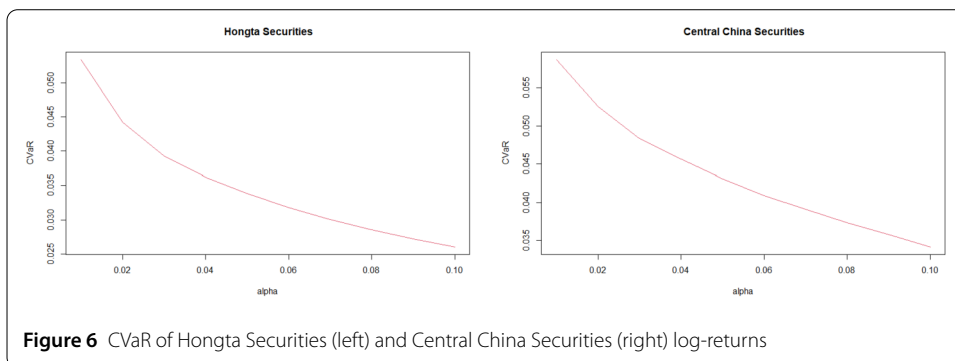
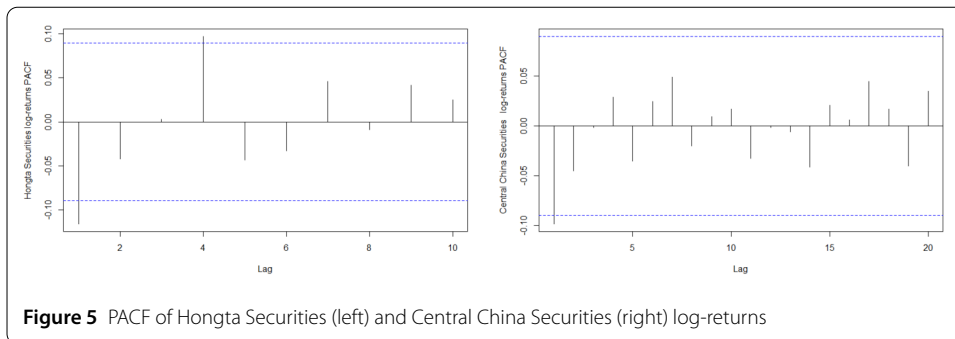
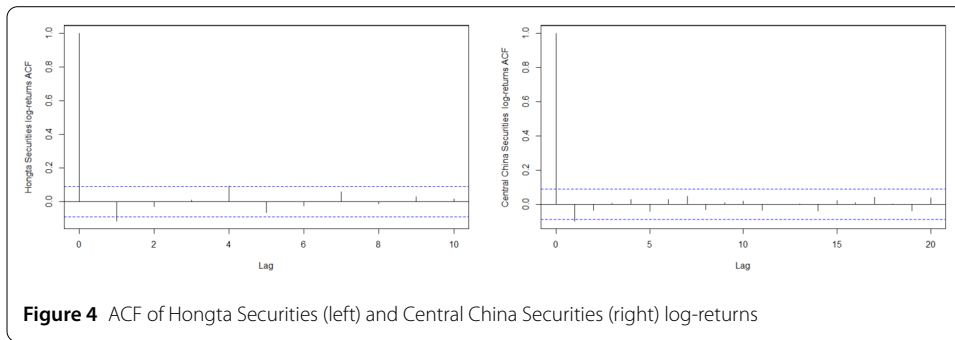
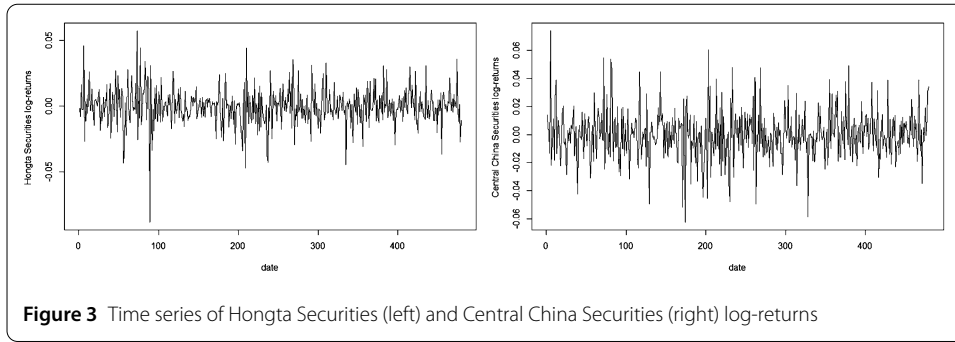
In this section, we apply the optimised CVaR estimation method discussed in this article to analyse a dataset consisting of the closing prices of Hongta Securities and Central China Securities on workdays from May 6, 2021, to May 5, 2023. To account for the absence of transactions on certain trading days, these data points have been excluded, resulting in a sample size of $n = 481$. Each time series comprises $n = 481$ data points obtained from the CSMAR database. Previous studies have utilized multivariate regression models or time-series models to analyse similar stock data. Refer to Wang et al. [6], Luo [7], and others for further details.

By taking the logarithm of the closing price sequence of the stocks, we consider the yield variable Y and the loss variable X , where $Y = -X$. Consequently, CVaR is computed using the yield variable. In this context, the logarithmic yield is adopted as the yield variable. The formula for calculating the logarithmic return of stocks is given by:

$$R_t = \ln \left(\frac{P_t}{P_{t-1}} \right)$$

Here, P_t and P_{t-1} denote the closing prices of the stock on days $t - 1$ and t , respectively.

Subsequently, we depict the scatter plot of the time series of log-returns in Fig. 3. Following this, the autocorrelation function (ACF) and partial autocorrelation function (PACF) of the two samples are illustrated in Figs. 4 and 5, respectively. The ACF and PACF results depicted in Figs. 4 and 5 indicate that both time series sets exhibit a stationary process. We then proceed to conduct the Augmented Dickey-Fuller (ADF) test, where the null hypothesis assumes non-stationarity in the time series. The α value obtained less than 0.05, signifying that the two time series of log-returns exhibit weak stationarity. By considering the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), we determine that both sequences conform to ARMA (1,1) models. Consequently, these two sets of log-return time series data can be regarded as ANA random samples.



Finally, employ optimisation techniques to estimate the Conditional Value at Risk (CVaR) of log-returns. Illustrated in Fig. 6, at equivalent probability levels, the CVaR of Hongta Securities is observed to be lower than that of Central China Securities. This sug-

gests that the risk associated with Hongta Securities is comparatively lower than that of Central China Securities.

Appendix

To prove our main results, the following lemmas are indispensable. The first one can be found in Zhang and Wang [15].

Lemma A.1 *Let $\{X_n; n \geq 1\}$ be a sequence of ANA random variables. If $\{g_n(\cdot); n \geq 1\}$ are all increasing or all decreasing functions, then $\{g_n(X_n); n \geq 1\}$ is still a sequence of ANA random variables.*

The next one is the Rosenthal type inequality for ANA random variables, which was established by Zhang [16].

Lemma A.2 *Suppose that $\{X_k; k \in N^d\}$ is a ANA random variables fields with $EX_k = 0$ and $\|X_k\|_p < \infty$ for some $p \geq 2$ and all k . Then there exists a positive constant C_p depending only on p and $\rho^-(\cdot)$ such that for any finite set $S \subset N^d$*

$$E \left| \sum_{k \in S} X_k \right|^p \leq C_p \left\{ \sum_{k \in S} E|X_k|^p + \left(\sum_{k \in S} E|X_k|^2 \right)^{p/2} \right\}.$$

When $d = 1$, the Rosenthal-type inequality remains true for the maximal partial sums, if some conditions on $\rho^-(\cdot)$ are added.

The third lemma is derived from Xing et al. [9].

Lemma A.3 *If $E|X|^r < \infty$ for $r > 0$, then there exists a positive constant C such that for $x \in \mathbb{R}$,*

$$E |h_\alpha(X, x) - Eh_\alpha(X, x)|^r \leq CE|X|^r, \tag{A.1}$$

where $h_\alpha(X, x) = x + \frac{1}{1-\alpha}[X - x]^+$.

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Author contributions

RJ derived and proved the inequality and was a major contributor in writing the manuscript. XFT did numerical simulation and real data analysis. KC made significant contributions to data acquisition, analysis, and English writing. All authors read and approved the final manuscript.

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Data Availability

The actual data used in this article comes from the CSMAR database dataset, which includes the closing prices of Hongta Securities and Huazhong Securities from May 6, 2021 to May 5, 2023. Simulation data was produced by R software.

Declarations

Competing interests

The authors declare no competing interests.

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