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RESEARCH

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Existence of solution for some nonlinear g-Caputo fractional-order differential equations based on Wardowski–Mizoguchi–Takahashi contractions

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Abstract

In this study, we prove the existence and uniqueness of a solution to a g-Caputo fractional differential equation with new boundary value conditions utilizing the combined Wardowski–Mizoguchi–Takahashi contractions via reduction of this equation to a fractional integral equation. We provide an example to demonstrate our findings.

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1 Introduction

Many academics have recently looked at the mathematical modeling of some physical processes using fractional integro-differential operators (see, for instance, [1–5]). The most well-known and often utilized fractional operators are the Riemann–Liouville and Caputo integro-differential operators. A new fractional integro-differential operator, known as the g-Caputo fractional derivative (g-C.f.d.), which is the fractional derivative with respect to another strictly increasing differentiable function, was introduced in [6] and used in [7] to have a broad scope of investigations of mathematical models. Later, this operator has been employed by various mathematicians in a variety of fields (see, for instance, [8–13]). The following fractional differential inclusion with respect to a strictly increasing function g has been recently studied by Belmor et al. [7]:

$$ {}^{c}D^{\eta}_{0^+;g}\omega(y) \in \Upsilon(y,\omega(y)), \quad y \in [0,\ell], 1 < \eta \le 2, $$

with boundary value conditions

$$\omega(0) - \delta_g\omega(0) = \frac{a}{\Gamma(\theta)}\int_0^p g'(\xi)(g(p)-g(\xi))^{\theta-1}k(\xi,\omega(\xi))d\xi = a\mathcal{I}^{\theta}_{0^+;g}k(p,\omega(p)),$$
$$\omega(\ell) + \delta_g\omega(\ell) = \frac{b}{\Gamma(\mu)}\int_0^q g'(\xi)(g(q)-g(\xi))^{\mu-1}\chi(\xi,\omega(\xi))d\xi = b\mathcal{I}^{\mu}_{0^+;g}\chi(q,\omega(q)),$$

where ${}^cD^{\eta}_{0^+;g}$ is the g-C.f.d. introduced by Jarad et al. [6], $\Upsilon : [0,\ell] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of nonempty subsets of $\mathbb{R}$, $\mathcal{I}^{\kappa}_{0^+;g}$ stands for the g-Riemann–Liouville fractional integral (g-R-L.f.i.) of order κ on $[0,\ell]$, $0 < p,q < \ell$, $k,\chi,$: $[0,\ell] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\delta_g = \frac{1}{g'(y)}\frac{d}{dy}$, and a and b are suitably chosen constants. Belmor et al. used the fixed point result via φ-weak contractions provided by Moradi and Khojasteh [14] to evaluate whether the aforementioned problem might be solved. Etemad et al. [15] presented a fractional boundary value inclusion problem and looked for sufficient and necessary criteria for the intended existence results. In fact, they developed a class of inclusions for fractional multiterm Caputo–Hadamard differential inclusions. Our result is more general than those discussed above.

Using Wardowski-type Mizoguchi–Takahashi contractions, we look for the existence and uniqueness of a solution to the g-Caputo fractional differential equation with arbitrary coefficients under new boundary value conditions. We specifically consider the solvability of the following problem:

$$\begin{cases} {}^cD^{r}_{a^+;g}\omega(y) = \Upsilon(y,\omega(y)), \quad y \in [a,b], 2 < r \leq 3, \\ \varsigma_1\omega(a) + \varsigma_2\omega(b) = \mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p_1,\omega(p_1)), \\ \varsigma_3\delta_g\omega(a) + \varsigma_4\delta_g\omega(b) = \mathcal{I}^{\mu}_{a^+;g}\chi(p_2,\omega(p_2)), \\ \varsigma_5\delta_g^2\omega(a) + \varsigma_6\delta_g^2\omega(b) = \mathcal{I}^{\lambda}_{a^+;g}\Psi(p_3,\omega(p_3)), \end{cases} \tag{1.1}$$

where $\Upsilon,\mathcal{K},\chi,\Psi : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous, $a < p_1,p_2,p_3 < b$, $\theta,\mu,\lambda > 0$, and ς_i, $i = 1,2,\ldots,6$, are arbitrary coefficients.

We organize the paper as follows. In Sect. 2, we give some known definitions, notations, and results, which form the background of the remaining sections. Section 3 contains the main results on the existence and uniqueness of a solution to (1.1) supported by an example. Section 4 is the conclusion.

2 Preliminaries and auxiliary notions

Let Θ be a nonempty set endowed with metric $\mathcal{D}$. Similarly to [16], let $\mathcal{CB}(\Theta)$ be the set of nonempty bounded closed subsets of Θ. Let $\mathcal{H}$ be the Hausdorff–Pompieu metric on $\mathcal{CB}(\Theta)$ generated by the metric $\mathcal{D}$, that is,

$$\mathcal{H}(\nabla_1,\nabla_2) = \max\left\{\sup_{\hbar_1\in\nabla_1}\mathcal{D}(\hbar_1,\nabla_2), \sup_{\hbar_2\in\nabla_2}\mathcal{D}(\hbar_2,\nabla_1)\right\}$$

for $\nabla_1,\nabla_2 \in \mathcal{CB}(\Theta)$.

If $\theta \in \mathcal{U}\theta$ for some element $\theta \in \Theta$, then θ is called a fixed point of a multivalued mapping $\mathcal{U} : \Theta \to \mathcal{P}(\Theta)$.

The theorem established by Mizoguchi and Takahashi [17] is as follows.

Theorem 2.1 ([17]) *Let Θ be endowed with a complete metric $\mathcal{D}$. Let*

$$\mathcal{H}(\mathcal{U}\hbar,\mathcal{U}\hbar') \leq \mathcal{G}(\mathcal{D}(\hbar,\hbar'))\mathcal{D}(\hbar,\hbar')$$

for all $\hbar,\hbar' \in \Theta$, where $\mathcal{U} : \Theta \to \mathcal{CB}(\Theta)$, and $\mathcal{G} : [0,\infty) \to [0,1)$ is such that $\limsup_{t\to r^+} \mathcal{G}(t) < 1$ for each $r \geq 0$. Then $\mathcal{U}$ possesses a fixed point.

We denote by $\mho$ the set of maps $\aleph : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(1) $s = 0 \quad \Leftrightarrow \quad \aleph(s) = 0$;

(2) $\aleph$ is lower semicontinuous and nondecreasing;

(3) $\limsup_{s\longrightarrow 0^+} \dfrac{s}{\aleph(s)} < \infty$.

Consider the following condition:

(H): If $\hbar_n \to \hbar$ as $n \to \infty$, then $\hbar_n \preceq \hbar$ for each $n \geq 0$, where $\{\hbar_n\} \subseteq \Theta$ is an increasing sequence.

For single-valued maps, Gordji and Ramezani [18] explored the following variation of Theorem 2.1.

Theorem 2.2 ([18]) *Let Θ be endowed with complete metric $\mathcal{D}$ and partially ordered relation $\preceq$. Suppose that for an increasing mapping $\mathcal{U} : \Theta \longrightarrow \Theta$, there exists $\hbar_0 \in \Theta$ such that $\hbar_0 \preceq \mathcal{U}(\hbar_0)$. Suppose that for some $\aleph \in \mho$, we have*

$$\aleph(\mathcal{D}(\mathcal{U}\hbar,\mathcal{U}\hbar')) \leq \mathcal{G}(\aleph(\mathcal{D}(\hbar,\hbar')))\aleph(\mathcal{D}(\hbar,\hbar'))$$

for all comparable $\hbar,\hbar' \in \Theta$, where $\mathcal{G} : [0,\infty) \longrightarrow [0,1)$ satisfies $\limsup_{s\longrightarrow t^+} \mathcal{G}(s) < 1$ for any $t > 0$. There is a fixed point for $\mathcal{U}$ if either (H) holds or $\mathcal{U}$ is continuous.

Definition 2.3 ([19]) Having a self-mapping $\mathcal{U}$ on Θ and $\nu : \Theta^2 \to [0,\infty)$ such $\mathcal{U}$ is triangular ν-admissible if

(T1) $\nu(\hbar,\hbar') \geq 1 \quad \text{implies} \quad \nu(\mathcal{U}\hbar,\mathcal{U}\hbar') \geq 1, \quad \hbar,\hbar' \in \Theta$,

(T2) $\begin{cases} \nu(\hbar,\zeta) \geq 1 \\ \nu(\zeta,\hbar') \geq 1 \end{cases} \quad \text{implies} \quad \nu(\hbar,\hbar') \geq 1, \quad \hbar,\hbar',\zeta \in \Theta.$

Recently, Mohammadi et al. discovered the following fixed point theorems for ν-admissible Wardowski type contractions by Mizoguchi-Takahashi approach:

Let $\mathbb{B}$ be the set of all functions $\mathcal{B} : (0,\infty) \longrightarrow [0,1)$ such that

$$\limsup_{x\longrightarrow t^+} \mathcal{B}(x) < 1$$

for any $t > 0$.

Let $\mathbb{Q}$ be the set of all functions $\mathcal{Q} : (0,\infty) \to \mathbb{R}$ such that

(δ_1) $\mathcal{Q}$ is continuous and strictly increasing,

(δ_2) $s = 1 \quad \Leftrightarrow \quad \mathcal{Q}(s) = 0$.

Some examples of elements of $\mathbb{Q}$:

(i) $\mathcal{Q}_1(t) = \ln(t)$,

(ii) $\mathcal{Q}_2(t) = \ln(t) + t$,

(iii) $\mathcal{Q}_3(t) = -\frac{1}{\sqrt{t}} + 1$,

(iv) $\mathcal{Q}_4(t) = -\frac{1}{t} + 1$.

We denote by $\mho'$ the set of maps $\aleph : [0,\infty) \longrightarrow [0,\infty)$ such that

(1) $s = 0 \quad \Leftrightarrow \quad \aleph(s) = 0$;

(2) $\aleph$ is continuous and nondecreasing.

For $\hbar, \hbar' \in \Theta$, set

$$M(\hbar,\hbar') = \max\{\mathcal{D}(\hbar,\hbar'), \mathcal{D}(\hbar,\mathcal{U}\hbar), \mathcal{D}(\hbar',\mathcal{U}\hbar)\}.$$

Consider the following condition:

(K): If $\nu(\hbar_n,\hbar_{n+1}) \geq 1$ for each integer $n \geq 0$ and $\hbar_n \to \hbar$ as $n \to +\infty$, then $\nu(\hbar_n,\hbar) \geq 1$ for each $n \geq 0$, where $\{\hbar_n\}$ is a sequence in Θ.

Theorem 2.4 ([20]) *Let $\mathcal{U}$ be a self-mapping on the complete metric space $(\Theta,\mathcal{D})$ such that for some function $\nu : \Theta^2 \to [0,\infty)$, we have*

$$\mathcal{Q}(\nu(\hbar,\hbar')\aleph(\mathcal{D}(\mathcal{U}\hbar,\mathcal{U}\hbar'))) \leq \mathcal{Q}(\mathcal{B}(\aleph(\mathcal{D}(\hbar,\hbar')))) + \mathcal{Q}(\aleph(M(\hbar,\hbar'))) \tag{2.1}$$

for all $\hbar,\hbar' \in \Theta$ with $\mathcal{U}\hbar \neq \mathcal{U}\hbar'$, where $\mathcal{Q} \in \mathbb{Q}$, $\mathcal{B} \in \mathbb{B}$, and $\aleph \in \mho'$. If $\mathcal{U}$ is triangular ν-admissible and $\nu(\hbar_0,\mathcal{U}\hbar_0) \geq 1$ for some $\hbar_0 \in \Theta$, then $\mathcal{U}$ possesses a fixed point if either $\mathcal{U}$ is continuous, or (K) holds.

Furthermore, if $\nu(\hbar,\hbar') \geq 1$ for all fixed points $\hbar$, $\hbar'$ of $\mathcal{U}$, then we have the uniqueness of fixed point.

Theorem 2.4 can be reduced to the following conclusion if $\aleph$ is the identity function.

Theorem 2.5 *Let $\mathcal{U}$ be a self-mapping on the complete metric space $(\Theta,\mathcal{D})$ such that*

$$\mathcal{Q}(\nu(\hbar,\hbar')\mathcal{D}(\mathcal{U}\hbar,\mathcal{U}\hbar')) \leq \mathcal{Q}(\mathcal{B}(\mathcal{D}(\hbar,\hbar'))) + \mathcal{Q}(M(\hbar,\hbar')) \tag{2.2}$$

for all $\hbar,\hbar' \in \Theta$ with $\nu(\hbar,\hbar') \geq 1$ and $\mathcal{U}\hbar \neq \mathcal{U}\hbar'$, where $\nu : \Theta^2 \to [0,\infty)$, $\mathcal{Q} \in \mathbb{Q}$, and $\mathcal{B} \in \mathbb{B}$. Suppose that $\mathcal{U}$ is triangular ν-admissible and $\nu(\hbar_0,\mathcal{U}\hbar_0) \geq 1$ for some $\hbar_0 \in \Theta$. Then $\mathcal{U}$ has a fixed point, provided that either $\mathcal{U}$ is continuous or (K) holds.

Furthermore, if $\nu(\hbar,\hbar') \geq 1$ for all fixed points $\hbar$, $\hbar'$ of $\mathcal{U}$, then we have the uniqueness of the fixed point.

Let us review the basic definitions of fractional differential equations from the beginning. The R-L.f.i. of order κ for a continuous function $\mathcal{U} : [0,\infty) \to \mathbb{R}$ is defined as

$$\mathcal{I}_a^{\kappa}\mathcal{U}(t) = \frac{1}{\Gamma(\kappa)}\int_0^t (t-\xi)^{\kappa-1}\mathcal{U}(\xi)d\xi. \tag{2.3}$$

The definition of the C.f.d. of order κ is

$${}^{c}D^{\kappa}\mathcal{U}(t) = \frac{1}{\Gamma(n-\kappa)}\int_0^t (t-\xi)^{n-\kappa-1}\mathcal{U}^{(n)}(\xi)d\xi \quad (n-1<\kappa<n, n=[\kappa]+1). \tag{2.4}$$

On the other hand, the fractional derivative of order κ in the sense of Reimann–Liouville is

$$D^{\kappa}\mathcal{U}(t) = \frac{1}{\Gamma(n-\kappa)}(\frac{d}{dt})^n\int_0^t (t-\xi)^{n-\kappa-1}\mathcal{U}(\xi)d\xi \quad (n-1<\kappa<n, n=[\kappa]+1). \tag{2.5}$$

Definition 2.6 Let g be an increasing map such that $g'(s) > 0$ for all $s \in [a,b]$. Then the g-R-L.f.i. of order κ of an integrable function $\mathcal{U} : [a,b] \to \mathbb{R}$ with respect to g is defined as

$$\mathcal{I}^{\kappa}_{a^+;g}\mathcal{U}(t) = \frac{1}{\Gamma(\kappa)} \int_a^t g'(\xi)(g(t) - g(\xi))^{\kappa-1}\mathcal{U}(\xi)d\xi \tag{2.6}$$

if the right-hand side of this equality is finite.

It should be observed that the g-R-L.f.i. (2.6) obviously reduces to the ordinary R-L.f.i. (2.3) if $g(t) = t$.

Definition 2.7 ([6]) Let $n = [\kappa] + 1$. The g-R-L.f.d. of order κ for a real mapping $\mathcal{U} \in C([a,b],\mathbb{R})$ is written as follows:

$$D^{\kappa}_{a^+;g}\mathcal{U}(t) = \frac{1}{\Gamma(n-\kappa)}(\frac{1}{g'(t)}\frac{d}{dt})^n \int_a^t g'(\xi)(g(t) - g(\xi))^{n-\kappa-1}\mathcal{U}(\xi)d\xi \tag{2.7}$$

if the right-hand side of this equality is finite.

In a similar way, it is evident that the standard R-L.f.d. (2.5) is a particular case of the g-R-L.f.d. of order κ if $g(t) = t$. Almeida provided a novel g-version of the C.f.d. in the formulation that follows, in which he is motivated by the above operators.

Definition 2.8 ([21]) Let $n = [\kappa] + 1$, and let $\mathcal{U} \in AC^n([a,b],\mathbb{R})$ be an increasing map with $g'(s) > 0$ for all $s \in [a,b]$. The g-C.f.d. of order κ of $\mathcal{U}$ with respect to g is

$${}^cD^{\kappa}_{a^+;g}\mathcal{U}(t) = \frac{1}{\Gamma(n-\kappa)} \int_a^t g'(\xi)(g(t) - g(\xi))^{n-\kappa-1}(\frac{1}{g'(\xi)}\frac{d}{d\xi})^n\mathcal{U}(\xi)d\xi, \tag{2.8}$$

assuming that the right-hand side of this equality is finite.

It can be observed that the g-C.f.d. (2.8) of order κ reduces to the conventional Caputo derivative (2.4) of order κ if $g(s) = s$. The g-Caputo and g-Riemann–Liouville integro-derivative operators are usefully specified in the sections that follow.

Let $AC([0,l],\mathbb{R})$ be the set of absolutely continuous functions from $[0,l]$ into $\mathbb{R}$. Define

$$AC^n_g([0,\ell],\mathbb{R}) = \left\{w : [0,\ell] \to \mathbb{R} | \delta^{n-1}_g w \in AC([0,\ell],\mathbb{R}), \delta_g = \frac{1}{g'(y)}\frac{d}{dy}\right\}.$$

Lemma 2.9 ([6]) *Let $n = [\kappa] + 1$. For a real mapping $\mathcal{U} \in AC^n([a,b],\mathbb{R})$,*

$$\mathcal{I}^{\kappa}_{a^+;g}{}^cD^{\kappa}_{a^+;g}\mathcal{U}(t) = \mathcal{U}(t) - \sum_{k=0}^{n-1} \frac{(\delta^k_g\mathcal{U})(a)}{k!}(g(t) - g(a))^k, \tag{2.9}$$

where $\delta^k_g = \underbrace{\delta_g\delta_g\cdots\delta_g}_{k \text{ times}}$.

Proposition 2.10 ([6, 21]) *Let $n = [\kappa] + 1$. For a real mapping $\mathcal{U} \in AC^n([a,b],\mathbb{R})$,*

(i) ${}^cD^{\kappa}_{a^+;g}(g(t) - g(a))^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\kappa+1)}(g(t) - g(a))^{\sigma-\kappa}$, $\kappa > 0$, $\sigma > -1$,

(ii) $\mathcal{I}^{\kappa}_{a^+;g}(g(t) - g(a))^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\kappa+1)}(g(t) - g(a))^{\sigma+\kappa}$, $\kappa > 0$, $\sigma > -1$, *and*

(iii) ${}^cD^{\sigma}_{a^+;g}(\mathcal{I}^{\kappa}_{a,g}\mathcal{U})(t) = \mathcal{I}^{\kappa-\sigma}_{a^+;g}\mathcal{U}(t)$, $0 < \sigma \le \kappa$.

3 Main results

We are now prepared to disclose and support the key findings of this investigation. Take it for granted from this point that $\Theta = C([a,b],\mathbb{R})$ is the Banach space of continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the supremum norm

$$\|Ł\|_\infty = \sup\{|Ł(t)| : t \in [a,b]\}.$$

First, let us prove the following auxiliary lemma.

Lemma 3.1 *Let ϑ, ρ_1, ρ_2, ρ_3 be real continuous functions on $[a,b]$, $2 < r \leq 3$, $\theta,\mu,\lambda > 0$, $p_1,p_2,p_3 \in [a,b]$, and let ς_i $(i = 1,2,\dots,6)$ be arbitrary constants. Then $\hbar \in AC_g^3([a,b],R)$ is a solution of the following fractional boundary value problem*:

$$\begin{cases} {}^cD^{\kappa}_{a^+;g}\omega(y) = \vartheta(y), \quad y \in [a,b], 2 < \kappa \leq 3, \\ \varsigma_1\omega(a) + \varsigma_2\omega(b) = \mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1), \\ \varsigma_3\delta_g\omega(a) + \varsigma_4\delta_g\omega(b) = \mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2), \\ \varsigma_5\delta_g^2\omega(a) + \varsigma_6\delta_g^2\omega(b) = \mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) \end{cases} \tag{3.1}$$

if and only if ω is a solution of the fractional-order integral equation

$$\omega(y) = L_\omega(y) + \int_a^b G_g(y,\xi)\vartheta(\xi)d\xi, \tag{3.2}$$

where

$$G_g(y,\xi) = g'(\xi)\begin{cases} \frac{(g(y)-g(\xi))^{\kappa-1}}{\Gamma(\kappa)} - \frac{\varsigma_2}{\Gamma(\kappa)(\varsigma_1+\varsigma_2)}(g(b)-g(\xi))^{\kappa-1} \\ + \frac{-\varsigma_4(g(b)-g(a))+(\varsigma_1+\varsigma_2)(g(y)-g(a))}{\Gamma(\kappa-1)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}(g(b)-g(\xi))^{\kappa-2} \\ + \frac{\left\{\begin{array}{l} \varsigma_6(2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))) \\ -\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2 \\ -\varsigma_6(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(g(y)-g(a))^2 \end{array}\right\}}{2\Gamma(\kappa-2)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}(g(b)-g(\xi))^{\kappa-3}, \ a \leq \xi \leq y, \\ - \frac{\varsigma_2}{\Gamma(\kappa)(\varsigma_1+\varsigma_2)}(g(b)-g(\xi))^{\kappa-1} \\ + \frac{-\varsigma_4(g(b)-g(a))+(\varsigma_1+\varsigma_2)(g(y)-g(a))}{\Gamma(\kappa-1)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}(g(b)-g(\xi))^{\kappa-2} \\ + \frac{\left\{\begin{array}{l} \varsigma_6(2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))) \\ -\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2 \\ -\varsigma_6(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(g(y)-g(a))^2 \end{array}\right\}}{2\Gamma(\kappa-2)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}(g(b)-g(\xi))^{\kappa-3}, \ y \leq \xi \leq b, \end{cases}$$

and

$$\begin{aligned} L_\omega(y) ={}& \frac{1}{\varsigma_1+\varsigma_2}\mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1) + \frac{(\varsigma_1+\varsigma_2)(g(y)-g(a)) - (g(b)-g(a))}{(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) \\ &+ \frac{\left\{\begin{array}{l} \varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2 \\ -2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))(g(y)-g(a)) \\ +(g(y)-g(a))^2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4) \end{array}\right\}}{2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3). \end{aligned}$$

Proof Applying $\mathcal{I}^{\kappa}_{a^+;g}$ to both sides of (3.1) and using Lemma 2.9, we obtain

$$\omega(y) = k_1 + k_2(g(y)-g(a)) + k_3(g(y)-g(a))^2 + \mathcal{I}^{\kappa}_{a^+;g}\vartheta(y), \tag{3.3}$$

where k_1, k_2, k_3 are unknown constants. Now we will compute these constants in view of boundary value conditions (3.1). Applying δ_g to both sides of (3.3), we obtain

$$\delta_g \omega(y) = k_2 + 2k_3(g(y) - g(a)) + \mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(y). \tag{3.4}$$

Applying δ_g to both sides of (3.4), we obtain

$$\delta_g^2 \omega(y) = 2k_3 + \mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(y). \tag{3.5}$$

From (3.1), (3.3), (3.4), and (3.5) we get

$$\begin{cases} \varsigma_1 k_1 + \varsigma_2[k_1 + k_2(g(b) - g(a)) + k_3(g(b) - g(a))^2 + \mathcal{I}^{\kappa}_{a^+;g}\vartheta(b)] = \mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1), \\ \varsigma_3 k_2 + \varsigma_4[k_2 + 2k_3(g(b) - g(a)) + \mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)] = \mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2), \\ 2\varsigma_5 k_3 + \varsigma_6[2k_3 + \mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)] = \mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3). \end{cases}$$

Therefore

$$\begin{cases} (\varsigma_1 + \varsigma_2)k_1 + \varsigma_2(g(b) - g(a))k_2 + 2\varsigma_2(g(b) - g(a))^2 k_3 = \mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1) - \varsigma_2\mathcal{I}^{\kappa}_{a^+;g}\vartheta(b), \\ (\varsigma_3 + \varsigma_4)k_2 + 2\varsigma_4(g(b) - g(a))k_3 = \mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) - \varsigma_4\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b), \\ 2(\varsigma_5 + \varsigma_6)k_3 = \mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b). \end{cases} \tag{3.6}$$

From (3.6) we obtain

$$k_3 = \frac{\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)}{2(\varsigma_5 + \varsigma_6)}, \tag{3.7}$$

$$k_2 = \frac{\begin{Bmatrix} 2(\varsigma_5 + \varsigma_6)[\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) - \varsigma_4\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)] \\ -2\varsigma_4(g(b) - g(a))[\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)] \end{Bmatrix}}{2(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)}, \tag{3.8}$$

and

$$k_1 = \frac{\begin{Bmatrix} 2(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)[\mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1) - \varsigma_2\mathcal{I}^{\kappa}_{a^+;g}\vartheta(b)] \\ -2(\varsigma_5 + \varsigma_6)(g(b) - g(a))[\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) - \varsigma_4\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)] \\ +\varsigma_2(\varsigma_4 - \varsigma_3)(g(b) - g(a))^2[\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)] \end{Bmatrix}}{2(\varsigma_1 + \varsigma_2)(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)}. \tag{3.9}$$

Substituting (3.7), (3.8), and (3.9) into (3.3), we obtain

$$\begin{aligned} \omega(y) = {} & \frac{\begin{Bmatrix} 2(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)[\mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1) - \varsigma_2\mathcal{I}^{\kappa}_{a^+;g}\vartheta(b)] \\ -2(\varsigma_5 + \varsigma_6)(g(b) - g(a))[\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) - \varsigma_4\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)] \\ +\varsigma_2(\varsigma_4 - \varsigma_3)(g(b) - g(a))^2[\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)] \end{Bmatrix}}{2(\varsigma_1 + \varsigma_2)(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)} \\ & + \frac{\begin{Bmatrix} 2(\varsigma_5 + \varsigma_6)[\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2) - \varsigma_4\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)] \\ -2\varsigma_4(g(b) - g(a))[\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3) - \varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)] \end{Bmatrix}}{2(\varsigma_3 + \varsigma_4)(\varsigma_5 + \varsigma_6)}(g(y) - g(a)) \end{aligned}$$

$$+\frac{\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3)-\varsigma_6\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)}{\varsigma_5+\varsigma_6}(g(y)-g(a))^2$$
$$+\mathcal{I}^{\kappa}_{a^+;g}\vartheta(y).$$

Therefore

$$\begin{aligned}\omega(y)&=\frac{1}{\varsigma_1+\varsigma_2}\mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1)+\frac{(g(y)-g(a))(\varsigma_1+\varsigma_2)-(g(b)-g(a))}{(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2)\\&+\frac{\begin{Bmatrix}\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2\\-2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))(g(y)-g(a))\\+(g(y)-g(a))^2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)\end{Bmatrix}}{2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3)\\&-\frac{\varsigma_2}{\varsigma_1+\varsigma_2}\mathcal{I}^{\kappa}_{a^+;g}\vartheta(b)+\left[\frac{-\varsigma_4(g(b)-g(a))+(\varsigma_1+\varsigma_2)(g(y)-g(a))}{(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}\right]\mathcal{I}^{\kappa-1}_{a^+;g}\vartheta(b)\\&+\frac{\begin{Bmatrix}\varsigma_6[2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))(g(y)-g(a))-\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2]\\-\varsigma_6(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(g(y)-g(a))^2\end{Bmatrix}}{2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}\\&\times\mathcal{I}^{\kappa-2}_{a^+;g}\vartheta(b)+\mathcal{I}^{\kappa}_{a^+;g}\vartheta(y).\end{aligned}$$

Thus

$$\omega(y)=L_\omega(y)+\int_a^b G_g(y,\xi)\vartheta(\xi)d\xi,$$

where

$$G_g(y,\xi)=g'(\xi)\begin{cases}\frac{(g(y)-g(\xi))^{\kappa-1}}{\Gamma(\kappa)}-\frac{\varsigma_2}{\Gamma(\kappa)(\varsigma_1+\varsigma_2)}(g(b)-g(\xi))^{\kappa-1}\\+\frac{-\varsigma_4(g(b)-g(a))+(\varsigma_1+\varsigma_2)(g(y)-g(a))}{\Gamma(\kappa-1)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}(g(b)-g(\xi))^{\kappa-2}\\+\frac{\begin{Bmatrix}\varsigma_6(2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a)))\\-\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2\\-\varsigma_6(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(g(y)-g(a))^2\end{Bmatrix}}{2\Gamma(\kappa-2)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}(g(b)-g(\xi))^{\kappa-3}, & a\le\xi\le y,\\-\frac{\varsigma_2}{\Gamma(\kappa)(\varsigma_1+\varsigma_2)}(g(b)-g(\xi))^{\kappa-1}\\+\frac{-\varsigma_4(g(b)-g(a))+(\varsigma_1+\varsigma_2)(g(y)-g(a))}{\Gamma(\kappa-1)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}(g(b)-g(\xi))^{\kappa-2}\\+\frac{\begin{Bmatrix}\varsigma_6(2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a)))\\-\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2\\-\varsigma_6(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(g(y)-g(a))^2\end{Bmatrix}}{2\Gamma(\kappa-2)(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}(g(b)-g(\xi))^{\kappa-3}, & y\le\xi\le b,\end{cases}$$

and

$$\begin{aligned}L_\omega(y)&=\frac{1}{\varsigma_1+\varsigma_2}\mathcal{I}^{\theta}_{a^+;g}\rho_1(p_1)+\frac{(\varsigma_1+\varsigma_2)(g(y)-g(a))-(g(b)-g(a))}{(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)}\mathcal{I}^{\mu}_{a^+;g}\rho_2(p_2)\\&+\frac{\begin{Bmatrix}\varsigma_2(\varsigma_4-\varsigma_3)(g(b)-g(a))^2\\-2\varsigma_4(\varsigma_1+\varsigma_2)(g(b)-g(a))(g(y)-g(a))\\+(g(y)-g(a))^2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)\end{Bmatrix}}{2(\varsigma_1+\varsigma_2)(\varsigma_3+\varsigma_4)(\varsigma_5+\varsigma_6)}\mathcal{I}^{\lambda}_{a^+;g}\rho_3(p_3),\end{aligned}$$

which is (3.2). Conversely, if ω is faitful in (3.2), then equation (3.1) clearly holds. □

Lemma 3.2 *Let*

$$\tilde{G}_g = \sup_{y\in[a,b]} \int_a^b |G_g(y,\xi)|d\xi$$

and

$$M_g = \frac{1}{\Gamma(\kappa+1)} \left\{ \begin{array}{l} 1 + \frac{|\varsigma_2|}{|\varsigma_1+\varsigma_2|} + \kappa \frac{|\varsigma_4|+(|\varsigma_1+\varsigma_2|)}{|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|} \\ +(\kappa-1)\kappa \frac{2|\varsigma_6||\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_2||\varsigma_4-\varsigma_3|+|\varsigma_6||\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|} \end{array} \right\} (g(b)-g(a))^\kappa .$$

Then $\tilde{G}_g \le M_g$.

Proof For arbitrary $y \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |G_g(y,\xi)|d\xi \\
&\le \frac{(g(y)-g(a))^\kappa}{\Gamma(\kappa+1)} \\
&+ \frac{|\varsigma_2|}{\Gamma(\kappa+1)|\varsigma_1+\varsigma_2|}\Big[(g(b)-g(a))^\kappa - (g(b)-g(y))^\kappa\Big] \\
&+ \frac{(|\varsigma_4| + |\varsigma_1+\varsigma_2|)(g(b)-g(a))}{\Gamma(\kappa)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}\Big[g(b)-g(a))^{\kappa-1} - (g(b)-g(y))^{\kappa-1}\Big] \\
&+ \left\{ \frac{[2|\varsigma_6||\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_2||\varsigma_4-\varsigma_3|+|\varsigma_6||\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|](g(b)-g(a))^2}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|} \right\} \\
&\times \Big[g(b)-g(a))^{\kappa-2} - (g(b)-g(y))^{\kappa-2}\Big] \\
&+ \frac{|\varsigma_2|}{\Gamma(\kappa+1)|\varsigma_1+\varsigma_2|}\Big[(g(b)-g(y))^\kappa\Big] \\
&+ \frac{(|\varsigma_4| + |\varsigma_1+\varsigma_2|)(g(b)-g(a))}{\Gamma(\kappa)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}(g(b)-g(y))^{\kappa-1} \\
&+ \left\{ \frac{[2|\varsigma_6||\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_2||\varsigma_4-\varsigma_3|+|\varsigma_6||\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|](g(b)-g(a))^2}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|} \right\} (g(b)-g(y))^{\kappa-2} \\
&\le \frac{1}{\Gamma(\kappa+1)} \left\{ \begin{array}{l} 1 + \frac{|\varsigma_2|}{|\varsigma_1+\varsigma_2|} + \kappa \frac{|\varsigma_4|+(|\varsigma_1+\varsigma_2|)}{|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|} \\ +(\kappa-1)\kappa \frac{2|\varsigma_6||\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_2||\varsigma_4-\varsigma_3|+|\varsigma_6||\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|} \end{array} \right\} (g(b)-g(a))^\kappa \\
&= M_g.
\end{aligned}$$

Taking sup on $y \in [a,b]$ on both sides of the above inequality, we get the desired result. □

Theorem 3.3 *Suppose that*

(i) $\Upsilon, \mathcal{K}, \chi, \Psi : [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous functions*;

(ii) *there are functions* $\mathcal{Q} \in \mathbb{Q}$ *and* $\mathcal{B} \in \mathbb{B}$ *such that*

$$|\Upsilon(y,u) - \Upsilon(y,v)| \le \frac{\kappa}{M_g} \mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-v|)) + \mathcal{Q}(|u-v|)\Big),$$

$$|\mathcal{K}(y,u) - \mathcal{K}(y,v)| \le \frac{\sigma\Gamma(\theta+1)|\varsigma_1+\varsigma_2|}{(g(p_1)-g(a))^\theta} \mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-v|)) + \mathcal{Q}(|u-v|)\Big),$$

$$|\chi(y,u)-\chi(y,v)| \leq \frac{\gamma\Gamma(\mu+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-v|))+\mathcal{Q}(|u-v|)\Big)}{(|\varsigma_1+\varsigma_2|+1)(g(p_2)-g(a))^{\mu}(g(b)-g(a))},$$

$$|\Psi(y,u)-\Psi(y,v)| \leq \frac{2\xi\Gamma(\lambda+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-v|))+\mathcal{Q}(|u-v|)\Big)}{\left\{\begin{array}{l}|\varsigma_2||\varsigma_4-\varsigma_3|\\ +2|\varsigma_4||\varsigma_1+\varsigma_2|\\ +|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|\end{array}\right\}(g(p_3)-g(a))^{\lambda}(g(b)-g(a))^2},$$

for all $y \in [a,b]$ and $u,v \in \mathbb{R}$, where $\kappa,\sigma,\gamma,\xi \geq 0$ and $\kappa+\sigma+\gamma+\xi \leq 1$.
Then problem (1.1) has a unique solution.

Proof According to Lemma 3.1, we know that (1.1) possesses a unique solution if and only if (3.2) has a unique solution. Define $T: AC_g^3([a,b],\mathbb{R}) \to AC_g^3([a,b],\mathbb{R})$ by

$$(T\omega)(y) = L_{\omega}(y) + \int_a^b G_g(y,\xi)\Upsilon(y,\omega(\xi))d\xi$$

for $\omega \in AC_g^3([a,b],\mathbb{R})$ and $y \in [a,b]$.

Therefore the statement that there is a solution for (1.1) is equivalent to the fact that T has a fixed point. Now let $\omega_1,\omega_2 \in AC_g^3([a,b],\mathbb{R})$ be such that $T\omega_1 \neq T\omega_2$. Then for $t \in [a,b]$ such that $T\omega_1(y) \neq T\omega_2(y)$, we have $\omega_1(y) \neq \omega_2(y)$. According to our hypotheses, we have

$$\begin{aligned}
&\Big|T\omega_1(y)-T\omega_2(y)\Big| \leq \Big|L_{\omega_1}(y)-L_{\omega_2}(y)\Big| + \int_a^b |G_g(y,\xi)|\Big|\Upsilon(\xi,\omega_1(\xi))-\Upsilon(\xi,\omega_2(\xi))\Big|d\xi \\
&\leq \frac{1}{|\varsigma_1+\varsigma_2|}\Big|\mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p_1,\omega_1(p_1))-\mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p_1,\omega_2(p_1))\Big| \\
&+ \frac{(|\varsigma_1+\varsigma_2|+1)(g(b)-g(a))}{|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}\Big|\mathcal{I}^{\mu}_{a^+;g}\chi(p_2,\omega_1(p_2))-\mathcal{I}^{\mu}_{a^+;g}\chi(p_2,\omega_2(p_2))\Big| \\
&+ \frac{|\varsigma_2||\varsigma_4-\varsigma_3|+2|\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|} \\
&\times (g(b)-g(a))^2\Big|\mathcal{I}^{\lambda}_{a^+;g}\Psi(p_3,\omega_1(p_3))-\mathcal{I}^{\lambda}_{a^+;g}\Psi(p_3,\omega_2(p_3))\Big| \\
&+ \int_a^b |G_g(y,\xi)|\Big|\Upsilon(\xi,\omega_1(\xi))-\Upsilon(\xi,\omega_2(\xi))\Big|d\xi \\
&\leq \sigma\,\mathcal{Q}^{-1}\Bigg\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\Bigg\} \qquad (3.10) \\
&+ \gamma\,\mathcal{Q}^{-1}\Bigg\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\Bigg\} \\
&+ \xi\,\mathcal{Q}^{-1}\Bigg\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\kappa\mathcal{Q}^{-1}\left\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\right\}\\
&=(\kappa+\sigma+\gamma+\xi)\mathcal{Q}^{-1}\left\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\right\}\\
&\le\mathcal{Q}^{-1}\left\{\mathcal{Q}(\mathcal{B}(||\omega_1-\omega_2||))+\mathcal{Q}(||\omega_1-\omega_2||)\right\}.
\end{aligned}
$$

Therefore

$$\|T\omega_1-T\omega_2\|\le\mathcal{Q}^{-1}[\mathcal{Q}(\mathcal{B}(\|\omega_1-\omega_2\|)+\mathcal{Q}(\|\omega_1-\omega_2\|)],$$

and so

$$\mathcal{Q}(\|T\omega_1-T\omega_2\|)\le\mathcal{Q}(\mathcal{B}(\|\omega_1-\omega_2\|)+\mathcal{Q}(\|\omega_1-\omega_2\|).$$

As a result, according to Theorem 2.5, T has a unique fixed point, and therefore problem (1.1) has a unique solution in $AC_g^3([a,b],\mathbb{R})$. □

Example 3.4 Consider the differential equation of fractional order

$$
\begin{cases}
{}^cD^{\frac{5}{2}}_{2^+;g}\omega(y)=\frac{6\sqrt{\pi}}{1700\sqrt{5}(\xi+1)}\frac{|\omega(y)|}{2+|\omega(y)|}, \quad y\in[2,3], g(y)=y^2,\\
\omega(2)+2\omega(3)=\mathcal{I}^{\frac{1}{2}}_{2^+;g}\mathcal{K}(\frac{7}{3},\omega(\frac{7}{3})),\ \mathcal{K}(\xi,u)=\frac{27\sqrt{\pi}}{10\sqrt{13}}\frac{e^{-\xi}\sin(\xi^2+1)|u|}{1+\frac{1}{2}|u|},\\
2\delta_g\omega(2)-\delta_g\omega(3)=\mathcal{I}^{\frac{3}{2}}_{2^+;g}\chi(\frac{5}{2},\omega(\frac{5}{2})),\ \chi(\xi,u)=\frac{\sqrt{\pi}}{150}\frac{\cos(2\xi+1)|u|}{1+\frac{1}{2}|u|},\\
3\delta_g^2\omega(2)+2\delta_g^2\omega(3)=\mathcal{I}^{\frac{5}{2}}_{2^+;g}\Psi(\frac{9}{4},\omega(\frac{9}{4})),\ \Psi(\xi,u)=\frac{192\sqrt{16\pi}}{7225\sqrt{17}}\frac{\cos(2\xi+1)|u|}{1+\frac{1}{2}|u|}.
\end{cases}
\tag{3.11}
$$

Note that

$$\Upsilon(\xi,u)=\frac{6\sqrt{\pi}}{1700\sqrt{5}(\xi+1)}\frac{|u|}{2+|u|}.$$

Here $\kappa=\frac{5}{2}$, $\varsigma_1=1$, $\varsigma_2=2$, $\varsigma_3=2$, $\varsigma_4=-1$, $\varsigma_5=3$, $\varsigma_6=2$. $a=2$, $b=3$, $p_1=\frac{7}{3}$, $p_2=\frac{5}{2}$, $p_3=\frac{9}{4}$, $\theta=\frac{1}{2}$, $\mu=\frac{3}{2}$, $\lambda=\frac{5}{2}$;

$$
\begin{aligned}
M_g&=\frac{1}{\Gamma(r+1)}\left\{\begin{array}{l}1+\frac{|\varsigma_2|}{|\varsigma_1+\varsigma_2|}+\kappa\frac{|\varsigma_4|+(|\varsigma_1+\varsigma_2|)}{|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}\\ +(\kappa-1)\kappa\frac{2|\varsigma_6||\varsigma_4||\varsigma_1+\varsigma_2|+|\varsigma_2||\varsigma_4-\varsigma_3|+|\varsigma_6||\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}{2|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||c_5+\varsigma_6|}\end{array}\right\}(g(b)-g(a))^{\kappa}\\
&=\frac{1}{\Gamma(\frac{7}{2})}\left\{1+\frac{2}{3}+\frac{5}{2}\frac{1+3}{3}+\frac{3}{2}\frac{5}{2}\frac{[12+6+6]}{30}\right\}(3^3-2^3)^{\frac{5}{2}}=\frac{680}{3}\sqrt{\frac{5}{\pi}}.
\end{aligned}
$$

Take $\kappa=\frac{2}{5}$, $\sigma=\frac{1}{5}$, $\gamma=\frac{1}{5}$, $\xi=\frac{1}{5}$. Then

$$
\begin{aligned}
&\frac{\sigma\Gamma(\theta+1)|\varsigma_1+\varsigma_2|}{(g(p_1)-g(a))^{\theta}}=\frac{\frac{1}{5}\Gamma(\frac{3}{2})(3)}{((\frac{7}{3})^2-2^2)^{\frac{1}{2}}}=\frac{27\sqrt{\pi}}{10\sqrt{13}},\\
&\frac{\gamma\Gamma(\mu+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|}{(|\varsigma_1+\varsigma_2|+1)(g(p_2)-g(a))^{\mu}(g(b)-g(a))}=\frac{\frac{1}{5}(3)\Gamma(\frac{5}{2})}{4((\frac{5}{2})^2-2^2)^{\frac{3}{2}}(3^2-2^2)}=\frac{\sqrt{\pi}}{150},
\end{aligned}
$$

and

$$\frac{2\xi\Gamma(\lambda+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|}{\left\{\begin{array}{l}|\varsigma_2||\varsigma_4-\varsigma_3|\\ +2|\varsigma_4||\varsigma_1+\varsigma_2|\\ +|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|\end{array}\right\}(g(p_3)-g(a))^{\lambda}(g(b)-g(a))^2}=\frac{192\sqrt{16\pi}}{7225\sqrt{17}}.$$

Now for all $\xi\in[a,b]=[2,3]$ and $u,\nu\in\mathbb{R}$, we have

$$\begin{aligned}\Big|\Upsilon(\xi,u)-\Upsilon(\xi,\nu)\Big| &= \frac{6\sqrt{\pi}}{1700\sqrt{5}}\left|\frac{|u|}{2+|u|}-\frac{|\nu|}{2+|\nu|}\right| \\ &= \frac{3\sqrt{\pi}}{1700\sqrt{5}}\left|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|\nu|}{1+\frac{1}{2}|\nu|}\right| \\ &\le \frac{3\sqrt{\pi}}{1700\sqrt{5}}\frac{\Big||u|-|\nu|\Big|}{(1+\frac{1}{2}|u|)(1+\frac{1}{2}|\nu|)} \\ &\le \frac{3\sqrt{\pi}}{1700\sqrt{5}}\frac{\Big||u|-|\nu|\Big|}{1+\frac{1}{2}(\Big||u|-|\nu|\Big|)} \\ &\le \frac{3\sqrt{\pi}}{1700\sqrt{5}}\frac{|u-\nu|}{1+\frac{1}{2}|u-\nu|} \\ &= \frac{\kappa}{M_g}\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-\nu|))+\mathcal{Q}(|u-\nu|)\Big),\end{aligned}$$

where $\mathcal{B}(t)=\frac{2}{3}$ and $\mathcal{Q}(t)=\frac{-1}{t}+1$.

On the other hand,

$$\begin{aligned}\Big|\mathcal{K}(\xi,u)-\mathcal{K}(\xi,\nu)\Big| &\le \frac{27\sqrt{\pi}}{10\sqrt{13}}\left|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|\nu|}{1+\frac{1}{2}|\nu|}\right| \\ &\le \frac{27\sqrt{\pi}}{10\sqrt{13}}\frac{\Big||u|-|\nu|\Big|}{1+\frac{1}{2}(\Big||u|-|\nu|\Big|)} \\ &\le \frac{27\sqrt{\pi}}{10\sqrt{13}}\frac{|u-\nu|}{1+\frac{1}{2}|u-\nu|} \\ &= \frac{\sigma\Gamma(\theta+1)|\varsigma_1+\varsigma_2|}{(g(p)-g(a))^{\theta}}\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-\nu|))+\mathcal{Q}(|u-\nu|)\Big), \\ \Big|\chi(\xi,u)-\chi(\xi,\nu)\Big| &\le \frac{\sqrt{\pi}}{150}\left|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|\nu|}{1+\frac{1}{2}|\nu|}\right| \\ &\le \frac{\sqrt{\pi}}{150}\frac{\Big||u|-|\nu|\Big|}{1+\frac{1}{2}(\Big||u|-|\nu|\Big|)} \\ &\le \frac{\sqrt{\pi}}{150}\frac{|u-\nu|}{1+\frac{1}{2}|u-\nu|} \\ &= \frac{[\gamma\Gamma(\mu+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|]\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-\nu|))+\mathcal{Q}(|u-\nu|)\Big)}{(|\varsigma_1|+|\varsigma_2|)(g(q)-g(a))^{\mu}(g(b)-g(a))},\end{aligned}$$

and

$$\begin{aligned}
&\Big|\Psi(\xi,u)-\Psi(\xi,v)\Big| \\
&\leq \frac{192\sqrt{16\pi}}{7225\sqrt{17}}\left|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|v|}{1+\frac{1}{2}|v|}\right| \\
&\leq \frac{192\sqrt{16\pi}}{7225\sqrt{17}}\frac{\Big||u|-|v|\Big|}{1+\frac{1}{2}\left(\Big||u|-|v|\Big|\right)} \\
&\leq \frac{192\sqrt{16\pi}}{7225\sqrt{17}}\frac{|u-v|}{1+\frac{1}{2}|u-v|} \\
&= \frac{[2\xi\Gamma(\lambda+1)|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4||\varsigma_5+\varsigma_6|]\mathcal{Q}^{-1}\Big(\mathcal{Q}(\mathcal{B}(|u-v|))+\mathcal{Q}(|u-v|)\Big)}{\left\{\begin{array}{l}|\varsigma_2||\varsigma_4-\varsigma_3|\\ +2|\varsigma_4||\varsigma_1+\varsigma_2|\\ +|\varsigma_1+\varsigma_2||\varsigma_3+\varsigma_4|\end{array}\right\}(g(p_3)-g(a))^{\lambda}(g(b)-g(a))^2}.
\end{aligned}$$

Also, $\kappa+\sigma+\gamma+\xi=1$. Thus all the conditions of Theorem 3.3 are satisfied. Thus problem (3.11) has a unique solution according to this theorem.

4 Conclusions

By applying the Wardowsky–Mizoguchi–Takahashi attractive fixed point theorem we investigated the existence of a solution of a fractional differential equation of finite order between 2 and 3 and with new boundary value conditions. To use fixed point theorems to check the solvability of such differential equations, we first transformed them into integral equations. We have also provided an example in support of our findings.

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Data availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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