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New attitude on sequential Ψ-Caputo differential equations via concept of measures of noncompactness

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Abstract

In this paper, we have explored the existence and uniqueness of solutions for a pair of nonlinear fractional integro-differential equations comprising of the Ψ-Caputo fractional derivative and the Ψ-Riemann–Liouville fractional integral. These equations are subject to nonlocal boundary conditions and a variable coefficient. Our findings are drawn upon the Mittage–Leffler function, Babenko's attitude, and topological degree theory for condensing maps and the Banach contraction principle. To further elucidate our principal outcomes, we have presented two illustrative examples.

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1 Introduction

There exists a profound and extensive historical background pertaining to the subject matter of fractional calculus, which traces its origins back to the emergence of classical calculus. In previous times, certain scholars dedicated their efforts to the exploration of this particular field; nevertheless, contemporary researchers have displayed a heightened level of enthusiasm towards the study of the novel dynamic equations. Caputo fractional derivatives stand as prominent concepts commonly employed within various classes of fractional derivatives. The Riemann–Liouville derivative is accompanied by a certain degree of mathematical abstraction, whereas the Caputo fractional derivatives are predominantly favored by engineers [4, 15, 20, 25, 26, 36].

The domain of fractional differential equations has seemingly experienced significant growth, thereby serving as a testament to the prominent position and status that fractional calculus has attained within the realms of science and engineering. It is noteworthy to mention that fractional calculus finds wide-ranging applications in naturally occurring fields such as porous media, chemical physics, viscoelasticity, electrical networks and fluid dynamics. Consequently, scientists underscore the significance and relevance of this particular field [15–17].

The boundary value problems pertaining to fractional differential equations have attracted significant interest from numerous scholars as of late, emerging as an important field of research due to their wide range of uses in the fields of science. These applications encompass control theory, mechanics, biology and wave propagation, among others [1, 3–5, 8, 11–13, 22–24, 27–35].

Many effective theoretical studies have been published by several researchers focusing on the result of existence, uniqueness and the stability for differential equations involving a fractional derivative with various conditions, see [9, 10].

In [30], Tariboon and colleagues explored the existence and uniqueness of solutions for the subsequent FDE:

$$^{c}\mathfrak{D}^{\varphi}Z(\mathcal{Q}) = \digamma\left(\mathcal{Q}, Z(\mathcal{Q})\right), \quad 1 < \varphi \leq 2, \qquad \mathcal{Q} \in [0, \mathfrak{b}],$$

subject to

$$\sum_{i=1}^{m} \eta_i Z(\xi_i) = S_1, \qquad \sum_{j=1}^{\mathfrak{n}} \vartheta_j \left(\mathfrak{I}^{\beta_j} Z(\mathfrak{b}) - \mathfrak{I}^{\beta_j} Z(\ell_j)\right) = S_2,$$

as nonlocal fractional integral boundary conditions, where $S_1, S_2 \in \mathbb{R}$, $\digamma : [A, \mathfrak{b}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, for $i = 1, 2, \ldots, \mathfrak{m}$, $j = 1, 2, \ldots, \mathfrak{n}$ considering $\eta_i, \vartheta_j \in \mathbb{R}$, and using Banach's contractive principle, Krasnoselskii's fixed-point theorem and Leray–Schauder's nonlinear alternative [19].

In 2013, Yan et al. [32] conducted a study on the existence and uniqueness of solutions for the ensuing boundary value problems of fractional differential equations using multiple customary fixed point theorems:

$$^{c}\mathfrak{D}^{\varphi}Z(\mathcal{Q}) = \digamma\left(\mathcal{Q}, Z(\mathcal{Q})\right), \quad 1 < \varphi \leq 2, \qquad \mathcal{Q} \in [0, \mathfrak{b}],$$

with the nonlocal boundary condition:

$$Z(A) = f(Z), \qquad \int_0^{\mathfrak{b}} Z(\mathcal{Q}) d\mathcal{Q} = \eta,$$

where $\digamma : C^2[0, \mathfrak{b}] \to \mathbb{R}$ is a C^2 continuous functional.

Just recently, the existence and uniqueness of a nonlinear integral differential equation via a boundary condition was examined by Li et al. [21]. This study employed several fixed point theorems:

$$\begin{cases} {}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{p}} Z(\mathcal{Q}) + \mu \mathfrak{I}^{\kappa}_{\mathfrak{p}} Z(\mathcal{Q}) = \digamma\left(\mathcal{Q}, Z(\mathcal{Q})\right), & \mathcal{Q} \in [\mathfrak{p}, P], \quad l - 1 < \varphi \leq l, \quad \kappa \geq 0, \\ Z(A) = -f(Z) \qquad Z(\mathfrak{p}) = Z'(\mathfrak{p}) = \cdots = Z^{(l-1)}(P), & \end{cases}$$

in which $0 \leq \mathfrak{p} < P < +\infty$ and μ is a constant.

Let $f : C[A, \mathfrak{b}] \to \mathbb{R}$, $\digamma : [A, \mathfrak{b}] \times \mathbb{R} \to \mathbb{R}$ and $A(\mathcal{Q}) \in C[A, \mathfrak{b}]$.

We will examine the existence and uniqueness of solutions for the subsequent nonlinear Ψ- integral differential equation with nonlocal boundary condition and varying coefficients when $l < \varphi \leq l + 1$ and $\beta \geq 0$

$$\begin{cases} {}^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi} Z(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q}) = F(\mathcal{Q}, Z(\mathcal{Q})) \quad \mathcal{Q} \in [A, \mathfrak{b}], \\ Z(A) = -f(Z), \qquad Z''(A) = \cdots = Z^{(l)}(A) = 0, \\ \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q}) Z(\mathcal{Q}) d\mathcal{Q} = \eta, \end{cases} \tag{1.1}$$

where $A(\mathcal{Q})$ is a variable coefficient, η is constant, ${}^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi}$ and $\mathfrak{I}_{A^+}^{\kappa;\Psi}$ are considered the Ψ-Riemann–Liouville fractional integral operators and Ψ-Caputo fractional, in the state order. Babenko's attitude [11] and topological degree theory for condensing map are powerful tools for solving differential and integro-differential equations with initial conditions by treating bounded integral operators as normal variables. In particular, for $\Psi(\mathcal{Q}) = \mathcal{Q}$, problem (1.1) arises; as a result,

$$\begin{cases} {}^{c}\mathfrak{D}_{A^+}^{\varphi} Z(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa} Z(\mathcal{Q}) = F(\mathcal{Q}, Z(\mathcal{Q})), \quad \mathcal{Q} \in [A, \mathfrak{b}], \\ Z(A) = -f(Z), \qquad Z''(A) = \cdots = Z^{(l)}(A) = 0, \\ \int_A^{\mathfrak{b}} Z(\mathcal{Q}) d\mathcal{Q} = \eta, \end{cases} \tag{1.2}$$

and for $\Psi(\mathcal{Q}) = \mathcal{Q}$, $l = 1$, problem (1.1) turns out to be

$$\begin{cases} {}^{c}\mathfrak{D}_{A^+}^{\varphi} Z(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa} Z(\mathcal{Q}) = F(\mathcal{Q}, Z(\mathcal{Q})), \quad \mathcal{Q} \in [A, \mathfrak{b}], \\ Z(A) = -f(Z), \qquad \int_A^{\mathfrak{b}} Z(\mathcal{Q}) d\mathcal{Q} = \eta. \end{cases} \tag{1.3}$$

Very limited information is available in contemporary literature regarding the boundary value problem of Ψ-fractional integro-differential equations with integral boundary conditions and coefficients that vary. The paper is structured as follows. Section 2 encompasses several fundamental definitions, topological degree theory, an introductory overview of fractional calculus, and a collection of lemmas that are further elaborated upon in this article. In Sect. 3, the outcomes associated with the existence and uniqueness of solutions for Ψ-Caputo (1.1) are presented, employing the theory of topological degree coincidences for the contraction principle and the curtailing maps. Two specific examples of the study results are provided in Sect. 4 to illustrate its functionality and demonstrate its efficience.

2 Preliminaries

In this particular section, we shall revisit a few of the fundamental outcomes and concepts that will find applications within the context of this manuscript.

Definition 2.1 The Mittag–Leffler function with two parameters is stated as [26]

$$E_{\varphi,\kappa} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varphi k + \kappa)}.$$

Babenko's methodology [11] is an efficient instrument for resolving differential and integro-differential equations featuring initial conditions through the treatment of bounded integral operators as ordinary variables.

We hereby present the results that are furnished in the subsequent from [2, 14].

Definition 2.2 Authorizing θ represent the collection of bounded subsets of X which X denote a Banach space. The measure of noncompactness known as the Kuratowski measure, is a mapping $\ell : \theta \to [0, \infty)$, which is defined as follows:

$$\ell(\theta) = \inf\{\varepsilon > 0\,; \theta \subseteq \cup_{i=1}^{m} \theta_i \text{ and } diam(\theta_i) \leq \varepsilon\}.$$

Definition 2.3 Supposing $G : Z \to X$ and $Z \subset X$ is a map that are both continuous and bounded, it is possible to state that G is ℓ-Lipschitz if $\exists P \geq 0$ such that

$$\ell\,(G\,(\theta)) \leq P\ell\,(\theta)\,, \quad \forall\,\theta \subset Z \text{ bounded.}$$

In the event that $P < 1$, we classify G as a strict ℓ-contraction. It is possible to state that G is ℓ-condensing if

$$\ell\,(G\,(\theta)) \leq \ell\,(\theta)\,,$$

for every bounded and nonprecompact subset $\theta \subset Z$.

Definition 2.4 Assuming $Z \subset X$ and letting $G : Z \to X$, it is worth noting that G is said to be Lipschitz if $\exists P \geq 0$ such that

$$\|GZ - GW\| \leq P\|Z - W\|, \quad \forall Z, W \in Z,$$

and G is a strict contraction as $P > 1$.

We direct the interested reader for the subsequent results to reference [18].

Proposition 2.5 *If $F, G : A \to X$ represent mappings that are ℓ – Lipschitz and possess constants P_1 and P_2, for specified, it follows that the mapping $F + G : A \to X$ is also ℓ – Lipschitz containing $P_1 + P_2$.*

Proposition 2.6 *In case $G : A \to X$ represent, mappings that are compact, it follows that the mapping G is ℓ – Lipschitz featuring $P = 0$.*

Proposition 2.7 *In case $G : A \to X$ represent mappings that are Lipschitz and possess constant P, it follows that the mapping $G : A \to X$ is also ℓ – Lipschitz with a constant of P.*

Similarly, Isaia [18] derived the following findings using topological degree theory.

Theorem 2.8 *Assuming the mapping $\mathcal{K} : X \to X$ to be ℓ-condensing and considering the set*

$$\zeta = \{Z \in X : \ \exists k \in [0, 1] \text{ in a manner that } Z = k\mathcal{K}Z\}.$$

If ζ is a bounded set in X, whenever there exists a positive constant r such that ζ is contained in the ball $B_r(0)$ centered at the origin. In this case, for all $k \in [0, 1]$, there exist

$$deg\,(I - k\mathcal{K}, B_r(0), 0) = 1.$$

Afterwards, we can suggest that the mapping $\mathcal{K}$ has at least one fixed point and the set of fixed points of $\mathcal{K}$ is contained within the ball $B_r(0)$.

Later, we will provide a detailed explanation of the properties and conclusions related to the field of fractional calculus. This explanation will begin by introducing a definition of Ψ-Riemann–Liouville fractional integrals and derivatives. Furthermore, we will delve deeper into the subject matter.

Definition 2.9 [6] Regarding $\varphi > 0$, the left-sided Ψ-Riemann–Liouville fractional integral of variable order $l(\mathcal{Q})$ for a function $Z \in L(H,\mathbb{R})$ due to a different function $\Psi : H \to \mathbb{R}$, which is an increasing differentiable function in such a way that $\Psi'(\mathcal{Q}) \neq 0$, may be elucidated as follows:

$$\Im_{A^+}^{\mathcal{Q};\Psi} Z(\mathcal{Q}) = \frac{1}{\Gamma(\mathcal{Q})} \int_A^{\mathcal{Q}} (\Psi(\mathcal{Q}) - \Psi(F))^{F-1}\, \Psi'(F) Z(F) dF, \tag{2.1}$$

for all $\mathcal{Q} \in H$ in such a way that $l : [A, \mathfrak{b}] \to (0,1]$ is a continuous function.

It is imperative to acknowledge that the decline of (2.1) can be observed in relation to the Riemann–Liouville and Hadamard fractional integrals, provided that $\Psi(\mathcal{Q}) = \mathcal{Q}$ and $\Psi(\mathcal{Q}) = \ln \mathcal{Q}$, in the sequence offered.

Definition 2.10 [6] Considering n as a natural number and Z and Ψ as two functions belonging to $C^n(H;\mathbb{R})$, where Ψ is increasing and $\Psi'(\mathcal{Q})$ is not equal to zero for all $\mathcal{Q}$ in H, we can elaborate the left-sided Ψ-Caputo of Z of order φ

$$\mathfrak{D}_{A^+}^{\mathcal{Q};\Psi} Z(\mathcal{Q}) = \frac{1}{\Gamma(n-\mathcal{Q})} \int_A^{\mathcal{Q}} (\Psi(\mathcal{Q}) - \Psi(F))^{n-F-1}\, \Psi'(F) \left(\frac{1}{\Psi'(F)} \frac{d}{dF}\right)^n Z(F) dF, \tag{2.2}$$

$$^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi} Z(\mathcal{Q}) = \Im_{A^+}^{n-\varphi;\Psi} \left(\frac{1}{\Psi'(\mathcal{Q})} \frac{d}{d\mathcal{Q}}\right)^n Z(\mathcal{Q}).$$

From the equation (2.2), it is reduced to the CFD operator as long as $\Psi(\mathcal{Q}) = \mathcal{Q}$. Moreover, if $\Psi(\mathcal{Q}) = ln\mathcal{Q}$, therefore it gives rise to the Caputo-Hadamard fractional derivative.

Lemma 2.11 [7] *In the event that both φ and ϕ are greater than zero, and Z belongs to the space of integrable functions $L^1(H,\mathbb{R})$, with $\mathcal{Q}$ being an element of H, it follows that*

$$\Im_{A^+}^{\varphi;\Psi} \Im_{A^+}^{\phi;\Psi} Z(\mathcal{Q}) = \Im_{A^+}^{\varphi+\phi;\Psi} Z(\mathcal{Q}) \quad a.e.\ t \in H.$$

Especially, if Z belongs to $C(H,\mathbb{R})$, then $\Im_{A^+}^{\varphi;\Psi} \Im_{A^+}^{\phi;\Psi} Z(\mathcal{Q}) = \Im_{A^+}^{\varphi+\phi;\Psi} Z(\mathcal{Q})$, $\mathcal{Q} \in H$.

Lemma 2.12 [7] *Assuming that φ is greater than zero, if Z belongs to $C(H,\mathbb{R})$, then for $\mathcal{Q}$ belongs to H*

$$^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi} \Im_{A^+}^{\phi;\Psi} = Z(\mathcal{Q}).$$

For $n-1 < \varphi < n$, if $Z \in C^n(H,\mathbb{R})$, then

$$\Im_{A^+}^{\phi;\Psi}\, {}^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi} Z(\mathcal{Q}) = Z(\mathcal{Q}) - \sum_{k=0}^{n-1} \frac{Z_{\Psi}^{[k]}(A)}{k!} (\Psi(\mathcal{Q}) - \Psi(A))^k, \quad \textit{for all } \mathcal{Q} \in H. \tag{2.3}$$

Notation For the remaining stages, in order to streamline and enhance the ease of computation, it is necessary that the subsequent notations be posited

$$T = \left(3t_1 + \frac{2t\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi}}{\Gamma(\varphi+2)} + \frac{2M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right), \tag{2.4}$$

$$\mu^* = \left(\frac{2\mu\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi}}{\Gamma(\varphi+2)} + \frac{2M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right), \tag{2.5}$$

$$\nu^* = \left(3\|f\| + \frac{2|\eta|}{\Psi(\mathfrak{b}) - \Psi(A)} + \frac{2\nu\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\right), \tag{2.6}$$

$$k_1 = (\Psi(\mathfrak{b}) - \Psi(A))^{\varphi}\, E_{\varphi+\kappa,\varphi+1}\left(M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}\right), \tag{2.7}$$

$$k_2 = TE_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}\right). \tag{2.8}$$

3 Main results

Here, we obtain our primary findings concerning the existence and uniqueness for the given problem (1.1).

Lemma 3.1 *Given a postulated function* $\mathbf{H} \in C(H,\mathbb{R})$, *the solution to the fractional BVP*

$$\begin{cases} {}^{c}\mathfrak{D}^{\varphi;\Psi}_{A^+} Z(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+} Z(\mathcal{Q}) = H(\mathcal{Q}), \quad \mathcal{Q} \in [A,\mathfrak{b}], \quad \mathfrak{k} > 0, \\ Z(A) = -f(Z), \qquad Z''(A) = \cdots = Z^{(l)}(A) = 0, \\ \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})Z(\mathcal{Q})d\mathcal{Q} = \eta, \end{cases} \tag{3.1}$$

is determined by

$$\begin{aligned} Z(\mathcal{Q}) ={}& \sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j \mathfrak{I}^{\varphi;\Psi}_{A^+} H(\mathcal{Q}) - f(Z)\sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j \\ &+ \frac{2\eta}{(\Psi(\mathfrak{b}) - \Psi(A))^2}\sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j (\Psi(\mathcal{Q}) - \Psi(A)) \\ &+ \frac{2f(Z)}{\Psi(\mathfrak{b}) - \Psi(A)}\sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j (\Psi(\mathcal{Q}) - \Psi(A)) \\ &+ \frac{2}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)}\int_A^{\mathfrak{b}} \Psi'(F)\,(\Psi(\mathfrak{b}) - \Psi(F))^{\varphi} H(F)dF) \\ &\times \sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j (\Psi(\mathcal{Q}) - \Psi(A)) \\ &- \left(\frac{2\,(\Psi(\mathcal{Q}) - \Psi(A))}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)\Gamma(\kappa)}\int_A^{\mathfrak{b}} \Psi'(X_1)\,(\Psi(\mathcal{Q}) - \Psi(X_1))^{\varphi} A(X_1)\right. \\ &\left.\times\left(\int_A^{X_1} \Psi'(F)\,(\Psi(X_1) - \Psi(F))^{\kappa-1} Z(F)dF\right) ds_1\right) \\ &\times \sum_{j=0}^{\infty}(-1)^j \left(\mathfrak{I}^{\varphi;\Psi}_{A^+} A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j (\Psi(\mathcal{Q}) - \Psi(A)). \end{aligned} \tag{3.2}$$

Proof Utilizing Lemma 2.12, the comprehensive solution of the Ψ-fractional differential equation

$$ {}^{c}\mathfrak{D}_{A^+}^{\varphi;\Psi} Z(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q}) = H(\mathcal{Q}), $$

may be expressed as follows

$$ Z(\mathcal{Q}) + f(Z) + \mathfrak{c}_1 (\Psi(\mathcal{Q}) - \Psi(A)) + \mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q}) = \mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}), $$

or

$$ \left(1 + \mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right) Z(\mathcal{Q}) = -f(Z) - \mathfrak{c}_1 (\Psi(\mathcal{Q}) - \Psi(A)) + \mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}), \tag{3.3} $$

where $\mathfrak{c}_1 \in \mathbb{R}$. Obviously,

$$ \begin{aligned} &\int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}) d\mathcal{Q} \\ &= \frac{1}{\Gamma(\varphi)} \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q}) \left(\int_A^{\mathcal{Q}} \Psi'(F) (\Psi(\mathcal{Q}) - \Psi(F))^{\varphi-1} H(F) dF \right) d\mathcal{Q} \\ &= \frac{1}{\Gamma(\varphi)} \int_A^{\mathfrak{b}} \Psi'(F) \left(\int_F^{\mathfrak{b}} \Psi'(F) (\Psi(\mathcal{Q}) - \Psi(F))^{\varphi-1} d\mathcal{Q} \right) H(F) dF \\ &= \frac{1}{\Gamma(\varphi+1)} \int_A^{\mathfrak{b}} \Psi'(F) (\Psi(\mathcal{Q}) - \Psi(F))^{\varphi} H(F) dF. \end{aligned} $$

Similarly,

$$ \begin{aligned} &\int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} \left(A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q})\right) d\mathcal{Q} \\ &= \frac{1}{\Gamma(\varphi+1)\Gamma(\kappa)} \int_A^{\mathfrak{b}} \Psi'(X_1) (\Psi(\mathfrak{b}) - \Psi(X_1))^{\varphi} A(X_1) \\ &\quad \times \left(\int_A^{X_1} \Psi'(F) (\Psi(X_1) - \Psi(F))^{\kappa-1} Z(F) dF \right) ds_1. \end{aligned} $$

Thus,

$$ \begin{aligned} &\int_A^{\mathfrak{b}} \Psi'(\mathcal{Q}) Z(\mathcal{Q}) d\mathcal{Q} + \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q}) f(Z) d\mathcal{Q} + \mathfrak{c}_1 \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q}) (\Psi(\mathcal{Q}) - \Psi(A)) d\mathcal{Q} \\ &\quad + \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} \left(A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q})\right) d\mathcal{Q} = \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}) d\mathcal{Q}, \end{aligned} $$

Now, from conditions in (3.1), we get

$$ \begin{aligned} &\eta + f(Z) (\Psi(\mathfrak{b}) - \Psi(A)) + \mathfrak{c}_1 \frac{(\Psi(\mathfrak{b}) - \Psi(A))^2}{2} \\ &\quad + \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} \left(A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi} Z(\mathcal{Q})\right) d\mathcal{Q} = \int_A^{\mathfrak{b}} \Psi'(\mathcal{Q})\mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}) d\mathcal{Q}. \end{aligned} $$

Thus,

$$\mathfrak{c}_1 = \frac{-2\eta}{(\Psi(\mathfrak{b}) - \Psi(A))^2} - \frac{2f(Z)}{\Psi(\mathfrak{b}) - \Psi(A)} + \frac{2}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)} \times$$
$$\int_A^{\mathfrak{b}} \Psi'(F)\,(\Psi(\mathfrak{b}) - \Psi(F))^{\varphi}\,H(F)dF - \frac{2}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)\Gamma(\kappa)} \times$$
$$\int_A^{\mathfrak{b}} \Psi'(X_1)\,(\Psi(\mathcal{Q}) - \Psi(X_1))^{\varphi}\,A(X_1)\left(\int_A^{X_1} \Psi'(F)\,(\Psi(X_1) - \Psi(F))^{\kappa-1}\,Z(F)dF\right) ds_1,$$

by noting that $f(\mathcal{Q}) \in \mathbb{R}$. So, from the relation (3.3),

$$\begin{aligned}
&\left(1 + \mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right) Z(\mathcal{Q}) \\
&= \mathfrak{I}_{A^+}^{\varphi;\Psi} H(\mathcal{Q}) - f(Z) \\
&\quad - (\Psi(\mathcal{Q}) - \Psi(A))\left(\frac{-2\eta}{(\Psi(\mathfrak{b}) - \Psi(A))^2} - \frac{2f(Z)}{\Psi(\mathfrak{b}) - \Psi(A)}\right. \\
&\quad \left. + \frac{2}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)} \int_A^{\mathfrak{b}} \Psi'(F)\,(\Psi(\mathfrak{b}) - \Psi(F))^{\varphi}\,H(F)dF\right) \\
&\quad - \frac{2\,(\Psi(\mathcal{Q}) - \Psi(A))}{(\Psi(\mathfrak{b}) - \Psi(A))^2\,\Gamma(\alpha+1)\Gamma(\kappa)} \int_A^{\mathfrak{b}} \Psi'(X_1)\,(\Psi(\mathcal{Q}) - \Psi(X_1))^{\varphi}\,A(X_1) \\
&\qquad \times \left(\int_A^{X_1} \Psi'(F)\,(\Psi(X_1) - \Psi(F))^{\kappa-1}\,Z(F)dF\right) ds_1.
\end{aligned}$$

Considering the factor $\left(1 + \mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)$ as a variable, we can infer that through Babenko's attitude (3.2), the inverse form of the lemma can be derived through a straightforward computation. Consequently, it is possible to consider the proof as completed. □

In the subsequent discussion, we shall expound upon the primary findings pertaining to the presence of resolutions for the aforementioned issue (1.1). In this section, it is appropriate to posit the following hypotheses: *(H1)* a constant $t > 0$ is posited such that for each $\mathcal{Q} \in H$ and for each $Z, Z^* \in \mathbb{R}$:

$$|\digamma(\mathcal{Q}, Z) - \digamma(\mathcal{Q}, Z^*)| \le t|Z - Z^*|.$$

(H2) There exists t_1 such that for each $\mathcal{Q} \in H$ and for each $Z, Z^* \in \mathbb{R}$:

$$|f(Z) - f(Z^*)| \le t_1\|Z - Z^*\|.$$

(H3) The functions $\digamma$ fulfills the next rising concessions for $\mu, \nu > 0$:

$$|\digamma(\mathcal{Q}, Z)| \le \mu\|Z\| + \nu \textit{For each } \mathcal{Q} \in H \textit{ and each } Z \in \mathbb{R}.$$

In view of Lemma 3.1, we assume two operators $G_1; G_2 : C(H, \mathbb{R}) \to C(H, \mathbb{R})$ as follows:

$$G_1 Z(\mathcal{Q}) = \sum_{j=0}^{\infty} (-1)^j \left(\mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j \mathfrak{I}_{A^+}^{\varphi;\Psi} \digamma(\mathcal{Q}, Z(\mathcal{Q})),$$

$$\begin{aligned}
G_2 Z(\mathcal{Q}) &= -f(Z)\sum_{j=0}^{\infty}(-1)^j\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j \\
&\quad+\frac{2\eta}{(\Psi(\mathfrak{b})-\Psi(A))^2}\sum_{j=0}^{\infty}(-1)^j\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j(\Psi(\mathcal{Q})-\Psi(A)) \\
&\quad+\frac{2f(Z)}{\Psi(\mathfrak{b})-\Psi(A)}\sum_{j=0}^{\infty}(-1)^j\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j(\Psi(\mathcal{Q})-\Psi(A)) \\
&\quad+\frac{2}{(\Psi(\mathfrak{b})-\Psi(A))^2\,\Gamma(\alpha+1)}\int_A^{\mathfrak{b}}\Psi'(F)\,(\Psi(\mathfrak{b})-\Psi(F))^{\varphi}H(F)dF \\
&\quad\times\sum_{j=0}^{\infty}(-1)^j\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j(\Psi(\mathcal{Q})-\Psi(A)) \\
&\quad-\left(\frac{2\,(\Psi(\mathcal{Q})-\Psi(A))}{(\Psi(\mathfrak{b})-\Psi(A))^2\,\Gamma(\alpha+1)\Gamma(\kappa)}\int_A^{\mathfrak{b}}\Psi'(X_1)\,(\Psi(\mathcal{Q})-\Psi(X_1))^{\varphi}A(X_1)\right. \\
&\quad\left.\times\left(\int_A^{X_1}\Psi'(F)\,(\Psi(X_1)-\Psi(F))^{\kappa-1}Z(F)dF\right)ds_1\right) \\
&\quad\times\sum_{j=0}^{\infty}(-1)^j\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j(\Psi(\mathcal{Q})-\Psi(A))\,.
\end{aligned}$$

Then, it is possible to rephrase the integral equation mentioned in reference (3.2) as stated in Lemma 3.1, $\mathcal{K}Z(\mathcal{Q}) = G_1 Z(\mathcal{Q}) + G_2 Z(\mathcal{Q})$ for $\mathcal{Q} \in H$.

The continuous $\mathcal{F}$ implies that the operator $\mathcal{K}$ is well-founded and fixed points of the operator equation can be considered as solutions of the integral equations (3.2) in Lemma 3.1.

Lemma 3.2 G_1 *is a continuous function that satisfies the growth condition mentioned below*:

$$\|G_1 Z\| \le k_1\,(\mu\|Z\| + \nu)\,.$$

Proof To establish the continuity of G_1, let us consider the scenario where $Z_n, Z \in C(H,\mathbb{R})$ and the $\lim_{n\to+\infty}\|Z_n - Z\| \to 0$ holds. It is evident that the collection $\{Z_n\}$ can be categorized as a bounded subset of $C(H,\mathbb{R})$. Consequently, there exists a constant $r > 0$ such that the norm of Z_n is bounded by r for all $n \ge 1$. Upon evaluating the limit, it becomes evident that $\|Z\| \le r$.

Subsequently, it is not difficult to observe that as n approaches infinity, the function $\mathcal{F}(F, Z_n(F))$ converges to $\mathcal{F}(F, Z(F))$, given the continuity of the function $\mathcal{F}$.

Taking into consideration another viewpoint, when we consider *(H3)*, we will encounter the subsequent inequality:

$$\begin{aligned}
&\frac{\Psi'(F)\,(\Psi(\mathcal{Q})-\Psi(F))^{\varphi-1}}{\Gamma(\varphi)}\|\mathcal{F}(F,Z_n(F))-\mathcal{F}(F,Z(F))\| \\
&\quad\le\frac{2\Psi'(F)\,(\Psi(\mathcal{Q})-\Psi(F))^{\varphi-1}}{\Gamma(\varphi)}(\mu r+\nu)\,.
\end{aligned}$$

It is observable that the function below

$$\mathcal{Q} \mapsto \frac{2\left(\Psi(\mathcal{Q}) - \Psi(F)\right)^{\varphi-1}}{\Gamma(\varphi)} \Psi'(F)\left(\mu r + \nu\right),$$

is integrated over $[\mathcal{Q}_{i-1}, \mathcal{Q}]$ using Lebesgue integration. As $n \to +\infty$, it can be inferred from the present argument along with the Lebesgue dominated convergence theorem.

$$\int_A^{\mathcal{Q}} \frac{\Psi'(F)\left(\Psi(\mathcal{Q}) - \Psi(F)\right)^{\varphi-1}}{\Gamma(\varphi)} \left\| \digamma(F, Z_n(F) - \digamma(F, Z(F)) \right\| dF \to 0.$$

As n approaches infinity, the expression $\|G_1 Z_n - G_1 Z\| \to 0$.

Which also shows that the operator G_1 possesses a continuous attribute. For the growth condition, utilizing the assumption *(H2)*, the result will be

$$\begin{aligned}
|G_1 Z(\mathcal{Q})| &= \left| \sum_{j=0}^{\infty} (-1)^j \left(\mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q}) \mathfrak{I}_{A^+}^{\kappa;\Psi} \right)^j \mathfrak{I}_{A^+}^{\varphi;\Psi} \digamma(\mathcal{Q}, Z(\mathcal{Q})) \right| \\
&\leq \sum_{j=0}^{\infty} \left\| \left(\mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q}) \mathfrak{I}_{A^+}^{\kappa;\Psi} \right)^j \mathfrak{I}_{A^+}^{\varphi;\Psi} \digamma(\mathcal{Q}, Z(\mathcal{Q})) \right\| \\
&\leq \|\digamma(\mathcal{Q}, Z(\mathcal{Q}))\| \sum_{j=0}^{\infty} \frac{M^j \left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{(\varphi+\kappa)j+\varphi}}{\Gamma\left(\varphi + \kappa)j + \varphi + 1\right)} \\
&\leq \left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi} E_{\varphi+\kappa, \varphi+1}\left(M\left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi+\kappa}\right)\left(\mu\|Z\| + \nu\right) \\
&= k_1\left(\mu\|Z\| + \nu\right).
\end{aligned}$$

Therefore,

$$\|G_1 Z\| \leq k_1\left(\mu\|Z\| + \nu\right). \tag{3.4}$$

This serves as proof of the fulfillment of the lemma (3.3). □

Lemma 3.3 *G_2 is Lipschitz via constant $k_2 = TE_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi+\kappa}\right)$. Plus, G_2 fulfills the growth condition stated below*

$$\|G_2 Z(\mathcal{Q})\| \leq E_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi+\kappa}\right)\left(\mu^*\|Z\| + \nu^*\right).$$

Proof In order to demonstrate that the operator G_2 is Lipschitz via constant $l_{\digamma} = t\eta_2$, and authorizing $Z, Z^* \in C(H, \mathbb{R})$, and $\mathfrak{M} = \sup_{A \leq \mathcal{Q} \leq \mathfrak{b}} |\frac{\mathfrak{k}S_2}{S_1} - e^{-\mathfrak{k}\Psi(\mathcal{Q})}|$, we will encounter for all $\mathcal{Q} \in H$:

$$\begin{aligned}
\left|G_2 Z(\mathcal{Q}) - G_2 Z^*(\mathcal{Q})\right| &\leq \|f(Z) - f(Z^*)\| \sum_{j=0}^{\infty} \left\| \left(\mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q}) \mathfrak{I}_{A^+}^{\kappa;\Psi} \right)^j \right\| \\
&\quad + \frac{2\|f(Z) - f(Z^*)\|}{\Psi(\mathfrak{b}) - \Psi(A)} |\Psi(\mathcal{Q}) - \Psi(A)| \sum_{j=0}^{\infty} \left\| \left(\mathfrak{I}_{A^+}^{\varphi;\Psi} A(\mathcal{Q}) \mathfrak{I}_{A^+}^{\kappa;\Psi} \right)^j \right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\left|\Psi(\mathcal{Q})-\Psi(A)\right|}{\left(\Psi(\mathfrak{b})-\Psi(A)\right)^2\Gamma(\varphi+1)}\sum_{j=0}^{\infty}\left\| \left(\mathfrak{I}^{\varphi;\Psi}_{A^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j\right\| \\
&\times \int_A^{\mathfrak{b}} \Psi'(F)\left(\Psi(\mathfrak{b})-\Psi(F)\right)^{\varphi}\left|\digamma(F,Z(F))-\digamma(F,Z^*(F))\right| dF \\
&+ \frac{2\left|\Psi(\mathcal{Q})-\Psi(A)\right|}{\left(\Psi(\mathfrak{b})-\Psi(A)\right)^2\Gamma(\varphi+1)\Gamma(\kappa)}\sum_{j=0}^{\infty}\left\| \left(\mathfrak{I}^{\varphi;\Psi}_{A^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j\right\| \\
&\times \int_A^{\mathfrak{b}} \Psi'(X_1)\left(\Psi(\mathfrak{b})-\Psi(X_1)\right)^{\varphi}A(X_1) \\
&\left(\int_A^{X_1}\Psi'(F)\left(\Psi(X_1)-\Psi(F)\right)^{\kappa-1}\left|Z(F))-Z^*(F)\right| dF\right) dX_1 \\
&\le 3\|f(Z)-f(Z^*)\|\sum_{j=0}^{\infty}\frac{M^j\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&+ \frac{2\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi}}{\Gamma(\varphi+2)}\left|\digamma(F,Z(F))-\digamma(F,Z^*(F))\right| \\
&\times \sum_{j=0}^{\infty}\frac{M^j\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&+ \frac{2M\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\left\|Z-Z^*\right\|\sum_{j=0}^{\infty}\frac{M^j\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&\le \left(3t_1+\frac{2t\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi}}{\Gamma(\varphi+2)}+\frac{2M\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right) \\
&\times E_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi+\kappa}\right)\|Z-Z^*\| \\
&= TE_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi+\kappa}\right)\|Z-Z^*\| \\
&= k_2\|Z-Z^*\|.
\end{aligned}
$$

In relation to the supremum of $\mathcal{Q}$, the next inequality will be accomplished.

$$
\left\|G_2Z(\mathcal{Q})-G_2Z^*(\mathcal{Q})\right\| \le k_2\|Z-Z^*\|.
$$

Hence, the operator G_2, which maps from $C(H,\mathbb{R})$ to $C(H,\mathbb{R})$, is a Lipschitzian operator on $C(H,\mathbb{R})$ with a Lpischitz constant $k_2 = TE_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b})-\Psi(A)\right)^{\varphi+\kappa}\right)$.

According to Proposition 2.7, G is ℓ-Lipschitz with constant k_2. Furthermore, considering the growth condition, we obtain

$$
\begin{aligned}
|G_2Z(\mathcal{Q})| \le |f(Z)|&\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}^{\varphi;\Psi}_{A^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j\right\| \\
&+ \frac{2|\eta|\left|\Psi(\mathcal{Q})-\Psi(A)\right|}{\left(\Psi(\mathfrak{b})-\Psi(A)\right)^2}\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}^{\varphi;\Psi}_{A^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j\right\| \\
&+ \frac{2f(Z)\left|\Psi(\mathcal{Q})-\Psi(A)\right|}{\Psi(\mathfrak{b})-\Psi(A)}\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}^{\varphi;\Psi}_{A^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\Psi}_{A^+}\right)^j\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2|\digamma(\mathcal{Q},Z(\mathcal{Q}))|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\sum_{j=0}^{\infty}\left\|\left(\Im_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\Im_{A^+}^{\kappa;\Psi}\right)^{j}\right\| \\
&+\frac{2M\|Z\|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\alpha+2)\Gamma(\kappa+1)}\sum_{j=0}^{\infty}\left\|\left(\Im_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\Im_{A^+}^{\kappa;\Psi}\right)^{j}\right\| \\
\le\;& 3|f(Z)|\sum_{j=0}^{\infty}\frac{M^{j}\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&+\frac{2|\eta|}{(\Psi(\mathfrak{b})-\Psi(A))}\sum_{j=0}^{\infty}\frac{M^{j}\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&+\frac{2\,(\mu\|Z\|+\nu)\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\sum_{j=0}^{\infty}\frac{M^{j}\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&+(\frac{2M\|Z\|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\alpha+2)\Gamma(\kappa+1)}\sum_{j=0}^{\infty}\frac{M^{j}\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
\le\;&\left(3\|f\|+\frac{2|\eta|}{(\Psi(\mathfrak{b})-\Psi(A))}+\frac{2\,(\mu\|Z\|+\nu)\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\right. \\
&\qquad\left.+\frac{2M\|Z\|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right)E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right) \\
=\;&\left(\frac{2\mu\|Z\|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}+\frac{2M\|Z\|\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right. \\
&\left.+3\|f\|+\frac{2|\eta|}{(\Psi(\mathfrak{b})-\Psi(A))}+\frac{2\nu\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\right) \\
&\quad\times E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right) \\
=\;&E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right)\left(\mu^{*}\|Z\|+\nu^{*}\right).
\end{aligned}
$$

Hence it follows that

$$
\|G_2Z(\mathcal{Q})\|\le E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right)\left(\mu^{*}\|Z\|+\nu^{*}\right).
$$

This serves as proof of the fulfillment of the lemma 3.3. □

Lemma 3.4 *The operator G_1 is compact, regarding $G_1 : C(H,\mathbb{R}) \to C(H,\mathbb{R})$. As a consequence, G_1 is ℓ-Lipschitz through zero consistent.*

Proof Consider a bounded category $\Omega \subset B_r = \{Z \in C(H,\mathbb{R}) : |Z| \le r\}$. We must show that $G_1(Z)$ is relatively compact in $C(H,\mathbb{R})$. For any $Z \in \Omega \subset B_r$, we will achieve this by using the estimations referred to in (3.4)

$$
\|G_1Z\|\le(\mu r+\nu)\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}\,E_{\varphi+\kappa,\varphi+1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right),
$$

and $G_1(Z)$ remains uniformly bounded. Moreover, for any $Z \in C(H,\mathbb{R})$ and $\mathcal{Q} \in H$, to demonstrate the equicontinuity of G_1, consider $\mathcal{Q}_1, \mathcal{Q}_2 \in H$ with $\mathcal{Q}_1 < \mathcal{Q}_2$, and let $Z \in \Omega$.

Subsequently, we will proceed

$$
\begin{aligned}
&|G_1Z(\mathcal{Q}_2) - G_1Z(\mathcal{Q}_1)| \\
&\quad \le 2M\,(\mu r + \nu) \sum_{j=0}^{\infty} \frac{(\Psi(\mathcal{Q}_2) - \Psi(A))^{(\varphi+\kappa)+\varphi} - (\Psi(\mathcal{Q}_1) - \Psi(A))^{(\varphi+\kappa)+\varphi}}{\Gamma\left((\varphi+\kappa)j + \varphi + 1\right)}.
\end{aligned}
$$

Based on our latest estimate, we can infer that as $\mathcal{Q}_2 \to \mathcal{Q}_1$ the expression $|G_1Z(\mathcal{Q}_2) - G_1Z(\mathcal{Q}_1)|$ goes to 0, which means that G_1 is equicontinuous. Thus, using the Ascoli-Arzela theorem, we conclude that the operator G_1 is compact. Also, from Proposition 2.6, it follows that G_1 is ℓ-Lipschitz via zero constant. □

Theorem 3.5 *Presuming that conditions* (H1)–(H3) *are satisfied, it follows that the BVP* (1.1) *possesses at least one solution denoted by Z and belonging to the set of continuous functions from H to $\mathbb{R}$, provided that the constant $k_2 < 1$. Moreover, the set of solutions is encompassed within the space $C(H, \mathbb{R})$.*

Proof Assume that the operators G_1, G_2, $\mathcal{K}$, and $\mathcal{K}$ have been introduced as described in the preceding section. These operators possess a continuous nature and are encompassed within their respective spaces. Plus

- operator G_2 is ℓ-Lipschitz via constant k_2, through Lemma 3.3,
- operator G_1 is ℓ-Lipschitz with constant 0 through Lemma 3.4. Thus, $\mathcal{K}$ is ℓ-Lipschitz with constant k_2, through Lemma 2.5.

Furthermore, the operator $\mathcal{K}$ can be characterized as a strict contraction with respect to the constant k_2. Given that k_2 is less than 1, it can be deduced that $\mathcal{K}$ is ℓ-condensing. The subsequent category is then considered

$$
\zeta = \{Z \in C(H, \mathbb{R}) : \ \exists k \in \mathfrak{I} \text{ such that } Z = k\mathcal{K}Z\}.
$$

The boundary of category ζ can be demonstrated. For $Z \in \zeta$, we have

$$
Z = k\mathcal{K}Z = k\left(\mathfrak{G}_1(Z) + \mathfrak{G}_2(Z)\right),
$$

which implies that

$$
\begin{aligned}
\|Z\| &\le k\left(\|G_1Z\| + \|G_2Z\|\right) \\
&\le (\Psi(\mathfrak{b}) - \Psi(A))^{\varphi}\, E_{\varphi+\kappa,\varphi+1}\left(M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}\right)(\mu\|Z\| + \nu) \\
&\quad + E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}\right)\left(\mu^*\|Z\| + \nu^*\right) \\
&= k_1\,(\mu\|Z\| + \nu) + E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b}) - \Psi(A))^{\varphi+\kappa}\right)\left(\mu^*\|Z\| + \nu^*\right).
\end{aligned}
$$

In this manner, the set ζ is bounded, and the operator $\mathcal{K}$ possesses at least one fixed point that corresponds to the solution of the BVP (1.1). □

Theorem 3.6 *In the context of the assumption labeled as* (H1)–(H2), *the BVP cited as* (1.1) *has a unique solution given that the prescribed condition is satisfied*:

$$
P = tk_1 + k_2 < 1.
$$

Proof Assuming that both Z and Z^* belong to $C(H,\mathbb{R})$, and that $\mathcal{Q}$ is an element of H, we will encounter

$$
\begin{aligned}
|\mathcal{K}Z(\mathcal{Q})-\mathcal{K}Z^*(\mathcal{Q})| \leq {} & \sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j\mathfrak{I}_{A^+}^{\varphi;\Psi}\right\|\left\|\digamma(\mathcal{Q},Z(\mathcal{Q}))-\digamma(\mathcal{Q},Z^*(\mathcal{Q}))\right\| \\
& \times\int_A^{\mathcal{Q}}\Psi'(F)\,(\Psi(\mathcal{Q})-\Psi(F))^{\varphi-1}\,dF \\
& +\|f(Z)-f(Z^*)\|\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j\right\| \\
& +\frac{2\|f(Z)-f(Z^*)\|}{\Psi(\mathfrak{b})-\Psi(A)}\,|\Psi(\mathcal{Q})-\Psi(A)|\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j\right\| \\
& +\frac{2\,|\Psi(\mathcal{Q})-\Psi(A)|}{(\Psi(\mathfrak{b})-\Psi(A))^2\,\Gamma(\varphi+1)} \\
& \sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j\right\| \\
& \times\int_A^{\mathfrak{b}}\Psi'(F)\,(\Psi(\mathfrak{b})-\Psi(F))^{\varphi}\left|\digamma(F,Z(F))-\digamma(F,Z^*(F))\right|dF \\
& +\frac{2\,|\Psi(\mathcal{Q})-\Psi(A)|}{(\Psi(\mathfrak{b})-\Psi(A))^2\,\Gamma(\varphi+1)\Gamma(\kappa)}\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{A^+}^{\varphi;\Psi}A(\mathcal{Q})\mathfrak{I}_{A^+}^{\kappa;\Psi}\right)^j\right\| \\
& \times\int_A^{\mathfrak{b}}\Psi'(X_1)\,(\Psi(\mathfrak{b})-\Psi(X_1))^{\varphi}A(X_1) \\
& \left(\int_A^{X_1}\Psi'(F)\,(\Psi(X_1)-\Psi(F))^{\kappa-1}\left|Z(F))-Z^*(F)\right|dF\right)dX_1 \\
\leq {} & \left\|\digamma(\mathcal{Q},Z(\mathcal{Q}))-\digamma(\mathcal{Q},Z^*(\mathcal{Q}))\right\|\sum_{j=0}^{\infty}\frac{M^j\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j+\varphi}}{\Gamma\left((\varphi+\kappa)j+\varphi+1\right)} \\
& +3\|f(Z)-f(Z^*)\|\sum_{j=0}^{\infty}\frac{M^j\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
& +\frac{2\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}\left|\digamma(F,Z(F))-\digamma(F,Z^*(F))\right| \\
& \times\sum_{j=0}^{\infty}\frac{M^j\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
& +\frac{2M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\left\|Z-Z^*\right\|\sum_{j=0}^{\infty}\frac{M^j\,(\Psi(\mathfrak{b})-\Psi(A))^{(\varphi+\kappa)j+\varphi}}{\Gamma\left(\varphi+\kappa)j+\varphi+1\right)} \\
\leq {} & t\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}E_{\varphi+\kappa,\varphi+1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right) \\
& +\left(3t_1+\frac{2t\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi}}{\Gamma(\varphi+2)}+\frac{2M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right) \\
& \times E_{\varphi+\kappa,1}\left(M\,(\Psi(\mathfrak{b})-\Psi(A))^{\varphi+\kappa}\right)\Big]\,\|Z-Z^*\|
\end{aligned}
$$

$$= (tk_1 + k_2)\,\|Z - Z^*\|$$
$$= P\|Z - Z^*\|.$$

In light of the aforementioned concession $P < 1$, it can be comprehended that the mapping denoted by $\mathcal{K}$ exhibits the property of contraction. Consequently, by virtue of the Banach fixed-point theorem, $\mathcal{K}$ possesses a distinct fixed point, which can be regarded as the unique solution to problem (1.1). □

4 Illustrative examples

In this section, certain issues are approached as a means of illustrating our findings:

Test example 4.1 Examine the problem below:

$$\begin{cases} {}^{c}\mathfrak{D}^{3.3}_{0^+}Z(\mathcal{Q}) + \frac{4\sin\mathcal{Q}}{\mathcal{Q}^2+100}\mathfrak{I}^{1.2}_{0^+}Z(\mathcal{Q}) = \frac{1}{10e^{\mathcal{Q}}+90}\left(1 + \frac{|Z(\mathcal{Q})|}{1+|Z(\mathcal{Q})|}\right), \quad \mathcal{Q} \in \mathfrak{I} = [0,1], \\ Z(A) = 0.01\sin Z(.5), \qquad Z''(0) = Z'''(0) = 0, \\ \int_A^{\mathfrak{b}} Z(\mathcal{Q})d\mathcal{Q} = 1. \end{cases} \tag{4.1}$$

It is important to acknowledge that the issue at hand represents a particular instance of problem (1.1), encompassing the subsequent information

$$\Psi(\mathcal{Q}) = \mathcal{Q},\ \varphi = 3.3,\ \kappa = 1.2,\ f(Z) = -0.01\sin Z(.5),\ \mathbf{A} = 0,\ \mathfrak{b} = 1,$$
$$A(\mathcal{Q}) = \frac{4\sin\mathcal{Q}}{\mathcal{Q}^2 + 100},\ \eta = 1,$$

the continuous function $F : \mathbf{J} \times \mathbb{R} \to \mathbb{R}$ is shown below

$$F(\mathcal{Q}, Z(\mathcal{Q})) = \frac{1}{10e^{\mathcal{Q}} + 90}\left(1 + \frac{|Z(\mathcal{Q})|}{1 + |Z(\mathcal{Q})|}\right) \quad \mathcal{Q} \in \mathbf{J}, Z \in \mathbb{R}.$$

Point 1: Hypothesis *(H1)* is confirmed:

For every $\mathcal{Q} \in \mathfrak{I}$ and $Z, W \in \mathbb{R}$, we obtain the following

$$\begin{aligned} |F(\mathcal{Q}, Z(\mathcal{Q})) - F(\mathcal{Q}, W(\mathcal{Q}))| &= \frac{1}{10e^{\mathcal{Q}} + 90}\left|\left(1 + \frac{|Z(\mathcal{Q})|}{1 + |Z(\mathcal{Q})|}\right) - \left(1 + \frac{|W(\mathcal{Q})|}{1 + |W(\mathcal{Q})|}\right)\right| \\ &\le \frac{1}{10e^{\mathcal{Q}} + 90}\,\frac{|Z(\mathcal{Q}) - W(\mathcal{Q})|}{1 + |Z(\mathcal{Q})|)(1 + |Z(\mathcal{Q})|)} \\ &\le \underbrace{0.01}_{t}\,|Z(\mathcal{Q}) - W(\mathcal{Q})|\,. \end{aligned}$$

Point 2: Hypothesis *(H2)* holds:

For each $Z, W \in \mathbb{R}$, we gain

$$\begin{aligned} \left|f(Z) - f(Z^*)\right| &= 0.01\,|\sin Z(0.5) - \sin W(0.5)| \\ &= 0.01\,\left|Z(.5) - Z^*(0.5)\right| \\ &= \underbrace{0.01}_{t_1}\,\|Z - W\|\,. \end{aligned}$$

Point 3: Hypothesis *(H3)* is confirmed:

For each $\mathcal{Q} \in \mathfrak{I}$ and $Z \in \mathbb{R}$, we obtain the following equation

$$F(\mathcal{Q}, Z(\mathcal{Q})) = \frac{1}{10e^{\mathcal{Q}} + 90}\left(1 + \frac{|Z(\mathcal{Q})|}{1 + |Z(\mathcal{Q})|}\right) \leq 0.01\left(|Z| + 1\right),$$

that shows here $\mu = \nu = 0.01$, $\mathfrak{p} = 1$. Due to the theorem 3.5

$$\zeta = \{Z \in C(H, \mathbb{R}) : \exists k \in \mathfrak{I} \textit{ such that } Z = k\mathcal{K}Z\}$$

is the solution set; then, we gain

$$\|Z\| \leq \frac{k_1\nu + \nu^* E_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi+\kappa}\right)}{1 - \left(k_1\mu + \mu^* E_{\varphi+\kappa,1}\left(M\left(\Psi(\mathfrak{b}) - \Psi(A)\right)^{\varphi+\kappa}\right)\right)} = 0.0076,$$

where $\mu^* = 0.050525$, $\nu^* = 0.007217$, $k_1 = 0.001129$, $E_{4.5,1}(0.04) = 1.00076$. And so, theorem 3.5 confirms that the system is resolved. Furthermore

$$tk_1 + k_2 = 0.01 \times 0.001129 + 0.032456 = 0.032467 < 1.$$

Theorem 3.6 proves that system (4.1) possesses a unique solution.

Test example 4.2 Allow us to examine the next problem:

$$\begin{cases} {}^{cH}\mathfrak{D}_{1^+}^{4.7} Z(\mathcal{Q}) + \frac{\sqrt{8\mathcal{Q}}}{\mathcal{Q}^2+1}\, {}^{H}\mathfrak{J}_{1^+}^{1.2} Z(\mathcal{Q}) = \frac{1}{5(1+t)^4}\left(Z(\mathcal{Q}) + \sqrt{1 + Z^2(\mathcal{Q})}\right), \quad \mathcal{Q} \in \mathfrak{I} = [1,2], \\ Z(1) = \frac{1}{20}\sin^2 Z(1.5), \qquad Z''(1) = Z'''(1) = Z^{(4)}(1) = 0, \\ \int_1^2 Z(\mathcal{Q})\frac{1}{\mathcal{Q}}\,d\mathcal{Q} = 1. \end{cases} \tag{4.2}$$

It is important to acknowledge that the issue at hand represents a particular instance of problem (1.1), encompassing the subsequent information

$$\Psi(\mathcal{Q}) = \ln \mathcal{Q}, \ \varphi = 4.7, \ \ \kappa = 1.2, \ \ f(Z) = \frac{1}{20}\sin^2 Z(1.5), \ \ \mathbf{A} = 1, \ \ \mathfrak{b} = 2,$$

$$A(\mathcal{Q}) = \frac{\sqrt{8\mathcal{Q}}}{\mathcal{Q}^2 + 1}, \ \ \eta = 1,$$

the operators denoted by ${}^{cH}\mathfrak{D}1^{+4.7}$ and ${}^{H}\mathfrak{J}1^{+1.2}$ represent the Caputo–Hadamard derivative and Hadamard integral, respectively, and continuous function $F : \mathbf{J} \times \mathbb{R} \to \mathbb{R}$ is stated as

$$F(\mathcal{Q}, Z(\mathcal{Q})) = \frac{1}{10(1+t)^4}\left(Z(\mathcal{Q}) + \sqrt{1 + Z^2(\mathcal{Q})}\right), \quad \mathcal{Q} \in \mathbf{J}, \ \omega \in \mathbb{R}.$$

Point 1: Hypothesis *(H1)* is confirmed: $\forall \mathcal{Q} \in J$ and $Z, W \in \mathbb{R}$, we gain

$$\begin{aligned} &|F(\mathcal{Q}, Z(\mathcal{Q})) - F(\mathcal{Q}, W(\mathcal{Q}))| \\ &\quad = \frac{1}{5(1+t)^4}\left|Z(\mathcal{Q}) - W(\mathcal{Q}) + \sqrt{1 + Z^2(\mathcal{Q})} - \sqrt{1 + (W^2(\mathcal{Q})}\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| (Z(\mathcal{Q}) - W(\mathcal{Q})) \left(1 + \frac{Z(\mathcal{Q}) - W(\mathcal{Q})}{\sqrt{1 + Z^2(\mathcal{Q})} - \sqrt{1 + (W^2(\mathcal{Q})}} \right) \right| \\
&\leq \underbrace{0.05}_{t} |Z(\mathcal{Q}) - W(\mathcal{Q})| .
\end{aligned}
$$

Point 2: Hypothesis *(H2)* hold: For each $Z, Z^* \in \mathbb{R}$, we gain

$$
\begin{aligned}
\left| f(Z) - f(Z^*) \right| &= \frac{1}{20} \left| \sin^2 Z(1.5) - \sin^2 Z^*(1.5) \right| \\
&\leq \frac{1}{20} \left| \sin Z(1.5) + \sin Z^*(1.5) \right| \left| \sin Z(1.5) - \sin Z^*(1.5) \right| \\
&\leq \frac{1}{10} \left| \sin Z(1.5) - \sin Z^*(1.5) \right| \\
&\leq \underbrace{0.1}_{t_1} \left\| Z - Z^* \right\| .
\end{aligned}
$$

Point 3: Condition $P < 1$ is confirmed:

$$
\begin{aligned}
P &= tk_1 + k_2 \\
&= \frac{1}{5} (\ln 2)^{4.7} E_{5.9,5.7} \left(2 (\ln 2)^{5.9} \right) \\
&\quad + \left(3 \times 0.1 + \frac{\frac{2}{5} (\ln 2)^{4.7}}{\Gamma(6.7)} + \frac{4 (\ln 2)^{5.9}}{\Gamma(2.2)\Gamma(6.7)} \right) E_{5.9,1} \left(2 (\ln 2)^{5.9} \right) \\
&= 0.301286 < 1.
\end{aligned}
$$

Theorem 3.6 guarantees that system (4.2) possesses a unique solution.

5 Conclusions

We have investigated the existence and uniqueness of solutions to the nonlinear Ψ fractional integral differential equation (1.1), where nonlocal boundary conditions and variable coefficients are present. To achieve this, we used the Mittag–Leffler function, Babenko's approach, and Banach's contraction principle and topological degree theory. The proof of existence results is based on the fixed point theorem due to Isaia [18], who pretty recently obtained such a fixed-point theorem via coincidence degree theory for condensing maps and that of uniqueness of the solution is proven by applying the Banach contraction principle. Of course, two representative examples have been given to illustrate the efficiency and performance of the results of the present study.

The above technique, obviously, opens up the doors for further study involving various other types of boundary conditions or with different fractional derivatives. The scope of this study encompasses the investigation of the BVP related to the nonlinear fractional partial Integral-Differential Equations with varying coefficients, along with the scrutiny of nonlinear integro-differential equations that incorporate the Hilfer fractional derivatives.

Author contributions

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Declarations

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This article does not contain any studies with human participants or animals performed by any of the authors.

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