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On solvability of a two-dimensional symmetric nonlinear system of difference equations

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Abstract

We show that the system of difference equations

$$x_{n+k}=\frac{x_{n+l}y_n-ef}{x_{n+l}+y_n-e-f},\qquad y_{n+k}=\frac{y_{n+l}x_n-ef}{y_{n+l}+x_n-e-f},\qquad n\in\mathbb{N}_0,$$

where $k\in\mathbb{N}$, $l\in\mathbb{N}_0$, $l<k$, $e,f\in\mathbb{C}$, and $x_j,y_j\in\mathbb{C}$, $j=\overline{0,k-1}$, is theoretically solvable and present some cases of the system when the general solutions can be found in a closed form.

Mathematics Subject Classification: 39A20; 39A45

Keywords: Symmetric system of difference equations; Solvable system; Solution in closed form

1 Introduction

Let $\mathbb{N}=\{1,2,\dots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{Z}$ be the set of integers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{C}$ be the set of complex numbers. Throughout the paper we also employ the notation $j=\overline{r,s}$ instead of $r\le j\le s$ in the case when $r,s,j\in\mathbb{N}_0$ and r and s satisfy the condition $r\le s$.

The problem of solvability of difference equations is quite old. Book [15] contains the majority of the solvability results up to 1800 (see also [11, 16]). Many results up to the end of the nineteenth century can be found in [8, 20, 40]. In some later books such as [10, 12, 21, 24] we can mostly find some old solvability methods and see how the theory of difference equations continued to develop during the first half of the twentieth century. Although some solvable nonlinear difference equations were known at the end of the eighteenth century and in the beginning of the nineteenth century [15–19], there has not been a considerable progress in the direction since that time.

It seems that the majority of solvable nonlinear difference equations are connected, in this or that way, with the solvability of some linear ones (see, for instance, [2–4, 13, 16, 36, 42–44, 46–54] as well as some of the references quoted therein). The linear difference equations, especially the ones with constant coefficients, are also useful in estimating solutions to some nonlinear difference equations (see, e.g., [5–7, 41]). Some of the solvable equations stem from the methods in numerical mathematics [9, 49], whereas some are connected with the trigonometric functions [8, 16, 18, 19, 37, 52]. The impossibility of finding closed-form formulas for solutions to nonlinear difference equations motivates some authors to find their invariants instead [26, 28, 29, 33, 38, 39, 45], from which some information on the long-term behavior of their solutions can be obtained. However, generally speaking, unlike the linear difference equations, the majority of solvable nonlinear difference equations are obtained and studied in some *ad hoc* ways without forming some unifying theories.

On the other hand, because of this, it is always of some interest, not only to the experts in the area of difference equations but also to a wide audience, to find a new class of solvable equations in this or that way and present a method for finding its general solution.

Since the mid of the 1990s there have been some investigations of concrete nonlinear systems of difference equations, some of which are symmetric or closely related to the symmetric ones (see, for instance, [25, 27, 30–32, 34, 35, 38, 39] and the literature quoted therein). The investigations have motivated us to investigate the problem of solvability of such type of systems of difference equations (see, for example, [42–44, 46–48, 50, 51, 54] and the related references therein).

The solvability of the difference equation

$$z_{n+k}=\frac{z_{n+l}z_n-ef}{z_{n+l}+z_n-e-f},\quad n\in\mathbb{N}_0,\tag{1}$$

where $k\in\mathbb{N}$, $l\in\mathbb{N}_0$, $l<k$, $e,f\in\mathbb{R}$ (or $\mathbb{C}$), and $z_j\in\mathbb{R}$ (or $\mathbb{C}$), $j=\overline{0,k-1}$, was recently studied in [53].

Our motivation for considering equation (1) stemmed, among other things, from some investigations of the so-called hyperbolic-cotangent class of difference equations

$$z_n=\frac{z_{n-k}z_{n-l}+f}{z_{n-k}+z_{n-l}},\quad n\in\mathbb{N}_0,\tag{2}$$

where $k,l\in\mathbb{N}$, $f\in\mathbb{R}$ (or $\mathbb{C}$) and $z_{-j}\in\mathbb{R}$ (or $\mathbb{C}$), $j=\overline{0,\max\{k,l\}}$ (see, for instance, [37, 52]).

Motivated by it, we started investigating solvability of some two-dimensional systems of difference equations that are obtained from equation (2) in some natural ways (see [43, 44, 46–48, 53]).

Bearing in mind the above-mentioned studies of concrete nonlinear systems of difference equations, it is also a natural problem to investigate the solvability of the systems obtained from equation (1). In [50] we dealt with the problem by studying a system of nonlinear difference equations related to equation (1). There we presented several interesting ideas connected with some classes of difference equations and systems of difference equations and employed some of them in the study of the system.

All the above-mentioned motivates us to study solvability of the following symmetrization of equation (1):

$$x_{n+k}=\frac{x_{n+l}y_n-ef}{x_{n+l}+y_n-e-f},\quad y_{n+k}=\frac{y_{n+l}x_n-ef}{y_{n+l}+x_n-e-f},\quad n\in\mathbb{N}_0,\tag{3}$$

where $k\in\mathbb{N}$, $l\in\mathbb{N}_0$, $l<k$, $e,f\in\mathbb{C}$, and $x_j,y_j\in\mathbb{C}$, $j=\overline{0,k-1}$.

Our aim is to show that the system in (3) is theoretically solvable for any $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ satisfying the condition $l < k$. Beside this, we also present several cases of the system of difference equations when the general solution can be found in a closed form, extending and complementing some results in the literature (see, for instance, [50, 53]).

2 Solvability of system (3) in a theoretical sense

This section considers the solvability of system (3) in a theoretical sense. Namely, we find a connection of the system with a homogeneous linear difference equation with constant coefficients, as well as with a product-type difference equation with integer powers, which both are theoretically solvable. Recall that the basic result on solvability of homogeneous linear difference equations (see, e.g., [10, 16, 23]) says that these equations are solvable in a closed form if we know the roots of the associated characteristic polynomials, which is, as is well known, not always the case [1]. However, the form of the general solution of the linear equation is known, so we can speak about its theoretical solvability.

Before we start with our analysis, we quote a useful auxiliary result, which can be found, for example, in [43] (see also [54]).

Lemma 1 *Let $m \in \mathbb{N}$, $l \in \mathbb{Z}$, and $(x_n)_{n\ge l-m}$ be the solution to*

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_mx_{n-m} \tag{4}$$

for $n \ge l$ such that

$$x_{j-m} = 0,\ j = \overline{l, l+m-2}, \quad \text{and} \quad x_{l-1} = 1,$$

where $a_j \in \mathbb{C}$, $j = \overline{1,m}$, $a_m \ne 0$.

Suppose that r_k, $k = \overline{1,m}$, are the zeros of the polynomial

$$q_m(r) = r^m - a_1r^{m-1} - a_2r^{m-2} - \cdots - a_m,$$

such that $r_i \ne r_j$ when $i \ne j$.

Then

$$x_n = \sum_{k=1}^{m} \frac{r_k^{n+m-l}}{q_m'(r_k)}$$

for $n \ge l - m$.

Remark 1 Note that the Fibonacci sequence (see, e.g., [14, 22, 55]) is a solution of a special case of equation (4) satisfying the above initial conditions. Recall that it satisfies the second order linear difference equation

$$x_n = x_{n-1} + x_{n-2}, \quad n \ge 2, \tag{5}$$

with the initial values

$$x_0 = 0 \quad \text{and} \quad x_1 = 1.$$

Note that these initial conditions produce the same solution $(x_n)_{n\in\mathbb{N}}$ to equation (5) as the initial conditions $x_1 = 1$ and $x_2 = 1$, that is, when the domain of indices is $\mathbb{N}$.

Now we conduct an analysis of the solvability of system (3) in a theoretical way. We would like to say that from now on we will ignore not well-defined solutions to the system. The following result is the main one in this direction.

Theorem 1 *Suppose $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, and $e,f \in \mathbb{C}$. Then system* (3) *is theoretically solvable.*

Proof If $e = f$, then we have

$$x_{n+k} - e = \frac{(x_{n+l} - e)(y_n - e)}{x_{n+l} + y_n - 2e}, \tag{6}$$

$$y_{n+k} - e = \frac{(y_{n+l} - e)(x_n - e)}{y_{n+l} + x_n - 2e} \tag{7}$$

for $n \in \mathbb{N}_0$.

Let

$$\mu_n = \frac{1}{x_n - e}, \quad \nu_n = \frac{1}{y_n - e} \tag{8}$$

for $n \in \mathbb{N}_0$.

Then from (6), (7), and (8) it follows that

$$\mu_{n+k} = \mu_{n+l} + \nu_n, \quad \nu_{n+k} = \nu_{n+l} + \mu_n \tag{9}$$

for $n \in \mathbb{N}_0$.

It is not difficult to see that the relations in (9) imply that μ_n and ν_n are two solutions to the equation

$$\omega_{n+2k} - 2\omega_{n+k+l} + \omega_{n+2l} - \omega_n = 0 \tag{10}$$

for $n \in \mathbb{N}_0$.

According to the basic results in the theory of linear difference equations with constant coefficients, we get the theoretical solvability of equation (10) and consequently of (9), which together with (8) implies the theoretical solvability of system (3) in this case.

If $e \neq f$, then we have

$$x_{n+k} - f = \frac{(x_{n+l} - f)(y_n - f)}{x_{n+l} + y_n - e - f},$$

$$x_{n+k} - e = \frac{(x_{n+l} - e)(y_n - e)}{x_{n+l} + y_n - e - f},$$

$$y_{n+k} - f = \frac{(y_{n+l} - f)(x_n - f)}{y_{n+l} + x_n - e - f},$$

$$y_{n+k} - e = \frac{(y_{n+l} - e)(x_n - e)}{y_{n+l} + x_n - e - f}$$

for $n \in \mathbb{N}_0$ and consequently

$$\frac{x_{n+k}-f}{x_{n+k}-e} = \frac{(x_{n+l}-f)(y_n-f)}{(x_{n+l}-e)(y_n-e)}$$

and

$$\frac{y_{n+k}-f}{y_{n+k}-e} = \frac{(y_{n+l}-f)(x_n-f)}{(y_{n+l}-e)(x_n-e)}$$

for $n \in \mathbb{N}_0$.

Let

$$\mu_n = \frac{x_n-f}{x_n-e}, \quad \nu_n = \frac{y_n-f}{y_n-e} \tag{11}$$

for $n \in \mathbb{N}_0$.

Then we have

$$\mu_{n+k} = \mu_{n+l}\nu_n, \quad \nu_{n+k} = \nu_{n+l}\mu_n \tag{12}$$

for $n \in \mathbb{N}_0$.

Since

$$\mu_n = \frac{\nu_{n+k}}{\nu_{n+l}},$$

then from (12) we get

$$\nu_{n+2k} = \nu_{n+k+l}^2 \nu_{n+2l}^{-1} \nu_n \tag{13}$$

for $n \in \mathbb{N}_0$.

Due to the symmetry of (12), we also have

$$\mu_{n+2k} = \mu_{n+k+l}^2 \mu_{n+2l}^{-1} \mu_n \tag{14}$$

for $n \in \mathbb{N}_0$.

Since the product-type difference equation

$$\omega_{n+2k} = \omega_{n+k+l}^2 \omega_{n+2l}^{-1} \omega_n, \quad n \in \mathbb{N}_0, \tag{15}$$

is with integer exponents, it is theoretically solvable, from which together with the following consequences of (11)

$$x_n = \frac{e\mu_n - f}{\mu_n - 1}, \quad y_n = \frac{e\nu_n - f}{\nu_n - 1}, \quad n \in \mathbb{N}_0, \tag{16}$$

the theoretical solvability of system (3) in this case follows. □

3 Some special cases of system (3)

This section deals with the practical solvability of system (3). A natural problem is to find special cases of system (3) for which it is possible to find some closed-form formulas for their general solutions.

As we have already mentioned, such problems are frequently connected to the roots of some specific polynomials, which is also the case with system (3). Therefore, to do this, note that the characteristic polynomial associated with (10) is

$$q_{2k}(r) = r^{2k} - 2r^{k+l} + r^{2l} - 1.$$

We have

$$q_{2k}(r) = (r^k - r^l)^2 - 1 = (r^k - r^l - 1)(r^k - r^l + 1). \tag{17}$$

Hence, equation (10) is solvable in a closed form if we can find the roots of the polynomials

$$r^k - r^l - 1 \quad \text{and} \quad r^k - r^l + 1.$$

This can be certainly done when $k \le 4$.

Thus, if we assume that $k \le 4$, due to the assumption $0 \le l < k$, we see that one of the cases must hold: 1° $k = 1$, $l = 0$; 2° $k = 2$, $l = 0$; 3° $k = 2$, $l = 1$; 4° $k = 3$; $l = 0$; 5° $k = 3$; $l = 1$; 6° $k = 3$; $l = 2$; 7° $k = 4$; $l = 0$; 8° $k = 4$; $l = 1$; 9° $k = 4$; $l = 2$; 10° $k = 4$; $l = 3$.

We will consider some of the cases in detail and leave the other ones to the reader as some exercises.

The case $k = 1$ and $l = 0$ is known [50], because of which we give only a sketch of the proof of the following theorem on solvability.

Theorem 2 *Suppose $k = 1$, $l = 0$, $e, f \in \mathbb{C}$. Then the following statements hold*:

(a) *If $e = f$, then the general solution to system* (3) *is*

$$x_n = e + \frac{(x_0 - e)(y_0 - e)}{2^{n-1}(x_0 + y_0 - 2e)}, \tag{18}$$

$$y_n = e + \frac{(x_0 - e)(y_0 - e)}{2^{n-1}(x_0 + y_0 - 2e)} \tag{19}$$

for $n \in \mathbb{N}$.

(b) *If $e \ne f$, then the general solution to system* (3) *is*

$$x_n = \frac{e\left(\frac{(x_0-f)(y_0-f)}{(x_0-e)(y_0-e)}\right)^{2^{n-1}} - f}{\left(\frac{(x_0-f)(y_0-f)}{(x_0-e)(y_0-e)}\right)^{2^{n-1}} - 1}, \tag{20}$$

$$y_n = \frac{e\left(\frac{(x_0-f)(y_0-f)}{(x_0-e)(y_0-e)}\right)^{2^{n-1}} - f}{\left(\frac{(x_0-f)(y_0-f)}{(x_0-e)(y_0-e)}\right)^{2^{n-1}} - 1} \tag{21}$$

for $n \in \mathbb{N}$.

Proof Note that $x_n = y_n$, $n \in \mathbb{N}$, which implies $\mu_n = \nu_n$, $n \in \mathbb{N}$. The relations in (9) become

$$\mu_{n+1} = \mu_n + \nu_n, \quad \nu_{n+1} = \nu_n + \mu_n$$

for $n \in \mathbb{N}_0$, so that $\mu_{n+1} = 2\mu_n$, $n \in \mathbb{N}$, and consequently

$$\mu_n = 2^{n-1}\mu_1 = \nu_n$$

for $n \in \mathbb{N}$. From this and by employing (8) we get formulas (18) and (19) when $e = f$.

If $e \neq f$, then (14) implies

$$\mu_{n+2} = \mu_{n+1}^2, \quad n \in \mathbb{N}_0,$$

so that $\mu_n = \mu_1^{2^{n-1}} = \nu_n$, $n \in \mathbb{N}$, from which together with (16) formulas (20) and (21) follow. □

Corollary 1 *Suppose* $e, f \in \mathbb{C}$, $l = 0$, $k \in \mathbb{N} \setminus \{1\}$. *Then system* (3) *is solvable in a closed form.*

Proof System (3) in this case becomes

$$x_{n+k} = \frac{x_n y_n - ef}{x_n + y_n - e - f}, \quad y_{n+k} = \frac{y_n x_n - ef}{y_n + x_n - e - f} \tag{22}$$

for $n \in \mathbb{N}_0$.

Now note that (22) is a system with interlacing indices (for the terminology see, for instance, [51]).

Let

$$x_m^{(j)} = x_{mk+j}, \quad y_m^{(j)} = y_{mk+j}$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, k-1}$.

Then $(x_m^{(j)}, y_m^{(j)})_{m \in \mathbb{N}_0}$, $j = \overline{0, k-1}$, are k solutions to the system

$$x_{m+1} = \frac{x_m y_m - ef}{x_m + y_m - e - f}, \quad y_{m+1} = \frac{y_m x_m - ef}{y_m + x_m - e - f}, \quad m \in \mathbb{N}_0,$$

which, in fact, is system (3) in the case $k = 1$ and $l = 0$.

Employing Theorem 2, we have that in the case $e = f$ the general solution to the system is given by

$$x_m^{(j)} = e + \frac{(x_0^{(j)} - e)(y_0^{(j)} - e)}{2^{m-1}(x_0^{(j)} + y_0^{(j)} - 2e)},$$

$$y_m^{(j)} = e + \frac{(x_0^{(j)} - e)(y_0^{(j)} - e)}{2^{m-1}(x_0^{(j)} + y_0^{(j)} - 2e)}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$, whereas in the case $e \neq f$ the general solution to the system is given by

$$x_m^{(j)} = \frac{e\left(\frac{(x_0^{(j)}-f)(y_0^{(j)}-f)}{(x_0^{(j)}-e)(y_0^{(j)}-e)}\right)^{2^{m-1}} - f}{\left(\frac{(x_0^{(j)}-f)(y_0^{(j)}-f)}{(x_0^{(j)}-e)(y_0^{(j)}-e)}\right)^{2^{m-1}} - 1},$$
$$y_m^{(j)} = \frac{e\left(\frac{(x_0^{(j)}-f)(y_0^{(j)}-f)}{(x_0^{(j)}-e)(y_0^{(j)}-e)}\right)^{2^{m-1}} - f}{\left(\frac{(x_0^{(j)}-f)(y_0^{(j)}-f)}{(x_0^{(j)}-e)(y_0^{(j)}-e)}\right)^{2^{m-1}} - 1}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$, that is, in the case $e = f$ we have

$$x_{mk+j} = e + \frac{(x_j - e)(y_j - e)}{2^{m-1}(x_j + y_j - 2e)},$$
$$y_{mk+j} = e + \frac{(x_j - e)(y_j - e)}{2^{m-1}(x_j + y_j - 2e)}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$, whereas in the case $e \neq f$ the general solution to the system is given by

$$x_{mk+j} = \frac{e\left(\frac{(x_j-f)(y_j-f)}{(x_j-e)(y_j-e)}\right)^{2^{m-1}} - f}{\left(\frac{(x_j-f)(y_j-f)}{(x_j-e)(y_j-e)}\right)^{2^{m-1}} - 1},$$
$$y_{mk+j} = \frac{e\left(\frac{(x_j-f)(y_j-f)}{(x_j-e)(y_j-e)}\right)^{2^{m-1}} - f}{\left(\frac{(x_j-f)(y_j-f)}{(x_j-e)(y_j-e)}\right)^{2^{m-1}} - 1},$$

finishing the proof of the corollary. □

The case $k = 2$, $l = 1$ considers the following result. This is the main example in this paper concerning the practical solvability of a special case of system (3). Namely, we prove in detail that for system (3) in this case its general solution in a closed form in all possible cases (the two cases $e = f$ and $e \neq f$ are considered separately) can be found.

Theorem 3 *Suppose $k = 2$, $l = 1$, $e, f \in \mathbb{C}$. Then system* (3) *is solvable in a closed form.*

Proof Case $e \neq f$. Relations (13) and (14) imply that $(\mu_n)_{n\in\mathbb{N}_0}$ and $(\nu_n)_{n\in\mathbb{N}_0}$ are two solutions to the equation

$$\omega_{n+4} = \omega_{n+3}^2 \omega_{n+2}^{-1} \omega_n \tag{23}$$

for $n \in \mathbb{N}_0$ with the following initial values:

$$\mu_0, \quad \mu_1, \quad \mu_2 = \mu_1 \nu_0, \quad \mu_3 = \mu_1 \nu_0 \nu_1, \tag{24}$$

$$\nu_0, \quad \nu_1, \quad \nu_2 = \nu_1 \mu_0, \quad \nu_3 = \nu_1 \mu_0 \mu_1, \tag{25}$$

respectively.

Let

$$a_1 := 2, \quad b_1 := -1, \quad c_1 := 0, \quad d_1 := 1. \tag{26}$$

Then we have

$$\omega_n = \omega_{n-1}^{a_1} \omega_{n-2}^{b_1} \omega_{n-3}^{c_1} \omega_{n-4}^{d_1} \tag{27}$$

for $n \geq 4$.

Using (23), where n is replaced by $n-5$ in (27), we have

$$\begin{aligned} \omega_n =& (\omega_{n-2}^2 \omega_{n-3}^{-1} \omega_{n-5})^{a_1} \omega_{n-2}^{b_1} \omega_{n-3}^{c_1} \omega_{n-4}^{d_1} \\ =& \omega_{n-2}^{2a_1+b_1} \omega_{n-3}^{-a_1+c_1} \omega_{n-4}^{d_1} \omega_{n-5}^{a_1} \\ =& \omega_{n-2}^{a_2} \omega_{n-3}^{b_2} \omega_{n-4}^{c_2} \omega_{n-5}^{d_2} \end{aligned} \tag{28}$$

for $n \geq 5$, where

$$a_2 = 2a_1 + b_1, \quad b_2 = -a_1 + c_1, \quad c_2 = d_1, \quad d_2 = a_1.$$

Assume that

$$\omega_n = \omega_{n-k}^{a_k} \omega_{n-k-1}^{b_k} \omega_{n-k-2}^{c_k} \omega_{n-k-3}^{d_k} \tag{29}$$

for $n \geq k+3$ and

$$a_k = 2a_{k-1} + b_{k-1}, \quad b_k = -a_{k-1} + c_{k-1}, \quad c_k = d_{k-1}, \quad d_k = a_{k-1} \tag{30}$$

for $k \geq 2$.

Using (23), where the index n is replaced by $n-k-4$, in (29), and the method of mathematical induction, it is not difficult to see that assumptions (29) and (30) are correct.

Take $k = n-3$ in (29). Then (30) yields

$$\begin{aligned} \omega_n =& \omega_3^{a_{n-3}} \omega_2^{b_{n-3}} \omega_1^{c_{n-3}} \omega_0^{d_{n-3}} \\ =& \omega_3^{a_{n-3}} \omega_2^{a_{n-2}-2a_{n-3}} \omega_1^{a_{n-5}} \omega_0^{a_{n-4}} \end{aligned} \tag{31}$$

for $n \geq 6$.

Using (30) we obtain

$$a_k = 2a_{k-1} - a_{k-2} + a_{k-4} \tag{32}$$

for $k \ge 5$, whereas the initial values are

$$a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 6. \tag{33}$$

The characteristic polynomial associated with (32) is

$$q_4(\lambda) = \lambda^4 - 2\lambda^3 + \lambda^2 - 1, \tag{34}$$

and its roots are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \quad \text{and} \quad \lambda_{3,4} = \frac{1 \pm i\sqrt{3}}{2} \tag{35}$$

(they are the roots of the polynomials $\lambda^2 - \lambda - 1$ and $\lambda^2 - \lambda + 1$, respectively).

From (32) we have

$$a_{k-4} = a_k - 2a_{k-1} + a_{k-2}, \tag{36}$$

from which together with (33) a_k for $k \le 0$ are calculated. From (33), (36), and some simple calculations, we get

$$a_{-3} = a_{-2} = a_{-1} = 0 \quad \text{and} \quad a_0 = 1. \tag{37}$$

From (37) we see that the solution of equation (32) satisfies the conditions of Lemma 1. Hence,

$$a_n = \frac{\lambda_1^{n+3}}{q_4'(\lambda_1)} + \frac{\lambda_2^{n+3}}{q_4'(\lambda_2)} + \frac{\lambda_3^{n+3}}{q_4'(\lambda_3)} + \frac{\lambda_4^{n+3}}{q_4'(\lambda_4)} \tag{38}$$

for $n \in \mathbb{Z}$.

Further, we have

$$q_4'(\lambda) = 4\lambda^3 - 6\lambda^2 + 2\lambda = 2\lambda(\lambda - 1)(2\lambda - 1),$$

from which along with some simple calculations we obtain the relations

$$q_4'(\lambda_1) = 2\frac{1+\sqrt{5}}{2}\left(\frac{\sqrt{5}-1}{2}\right)\sqrt{5} = 2\sqrt{5}, \tag{39}$$

$$q_4'(\lambda_2) = 2\frac{1-\sqrt{5}}{2}\left(\frac{\sqrt{5}+1}{2}\right)\sqrt{5} = -2\sqrt{5}, \tag{40}$$

$$q_4'(\lambda_3) = 2\frac{1+i\sqrt{3}}{2}\left(\frac{i\sqrt{3}-1}{2}\right)i\sqrt{3} = -2i\sqrt{3}, \tag{41}$$

$$q_4'(\lambda_4) = 2\frac{1-i\sqrt{3}}{2}\left(\frac{i\sqrt{3}+1}{2}\right)i\sqrt{3} = 2i\sqrt{3}. \tag{42}$$

Employing (39)–(42) in (38) it follows that

$$a_n = \frac{\lambda_1^{n+3} - \lambda_2^{n+3}}{2\sqrt{5}} - \frac{\lambda_3^{n+3} - \lambda_4^{n+3}}{2i\sqrt{3}} \tag{43}$$

for $n \in \mathbb{Z}$. From this it easily follows that formula (31) holds not only for $n \geq 6$ but also for $n = \overline{0,5}$.

From (24) and (31) we have

$$\begin{aligned}\mu_n &= \mu_3^{a_{n-3}} \mu_2^{a_{n-2}-2a_{n-3}} \mu_1^{a_{n-5}} \mu_0^{a_{n-4}}\\ &= (\mu_1 \nu_0 \nu_1)^{a_{n-3}} (\mu_1 \nu_0)^{a_{n-2}-2a_{n-3}} \mu_1^{a_{n-5}} \mu_0^{a_{n-4}}\\ &= \mu_0^{a_{n-4}} \mu_1^{a_{n-2}-a_{n-3}+a_{n-5}} \nu_0^{a_{n-2}-a_{n-3}} \nu_1^{a_{n-3}}\\ &= \mu_0^{a_{n-4}} \mu_1^{a_{n-1}-a_{n-2}} \nu_0^{a_{n-2}-a_{n-3}} \nu_1^{a_{n-3}} \end{aligned} \tag{44}$$

for $n \in \mathbb{N}_0$, whereas from (25) and (31) and because of the symmetry of the system we have

$$\nu_n = \mu_0^{a_{n-2}-a_{n-3}} \mu_1^{a_{n-3}} \nu_0^{a_{n-4}} \nu_1^{a_{n-1}-a_{n-2}} \tag{45}$$

for $n \in \mathbb{N}_0$.

We have

$$\begin{aligned} a_n - a_{n-1} &= \frac{(\lambda_1 - 1)\lambda_1^{n+2} - (\lambda_2 - 1)\lambda_2^{n+2}}{2\sqrt{5}} - \frac{(\lambda_3 - 1)\lambda_3^{n+2} - (\lambda_4 - 1)\lambda_4^{n+2}}{2i\sqrt{3}}\\ &= \frac{-\lambda_2\lambda_1^{n+2} + \lambda_1\lambda_2^{n+2}}{2\sqrt{5}} + \frac{\lambda_4\lambda_3^{n+2} - \lambda_3\lambda_4^{n+2}}{2i\sqrt{3}}\\ &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{2\sqrt{5}} + \frac{\lambda_3^{n+1} - \lambda_4^{n+1}}{2i\sqrt{3}} \end{aligned} \tag{46}$$

for $n \in \mathbb{Z}$.

Using (43) and (46) in (44) and (45) we obtain

$$\begin{aligned}\mu_n &= \mu_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \mu_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}\\ &\quad\times \nu_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \nu_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{47}$$

$$\begin{aligned}\nu_n &= \nu_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \nu_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}\\ &\quad\times \mu_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \mu_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \end{aligned} \tag{48}$$

for $n \in \mathbb{N}_0$.

From (47) and (48) and (11) with $n = 0, 1$, it follows that

$$\begin{aligned}\mu_n &= \Big(\frac{x_0 - f}{x_0 - e}\Big)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \Big(\frac{x_1 - f}{x_1 - e}\Big)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}\\ &\quad\times \Big(\frac{y_0 - f}{y_0 - e}\Big)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \Big(\frac{y_1 - f}{y_1 - e}\Big)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{49}$$

$$\nu_n = \Big(\frac{x_0 - f}{x_0 - e}\Big)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \Big(\frac{x_1 - f}{x_1 - e}\Big)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}$$

$$\times \left(\frac{y_0-f}{y_0-e}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}\left(\frac{y_1-f}{y_1-e}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \tag{50}$$

for $n \in \mathbb{N}_0$.

Employing (49) and (50) in (16) we got the following closed-form formulas for the general solutions to system (3) in this case:

$$x_n = \frac{e(\frac{x_0-f}{x_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{x_1-f}{x_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}(\frac{y_0-f}{y_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{y_1-f}{y_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}-f}{(\frac{x_0-f}{x_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{x_1-f}{x_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}(\frac{y_0-f}{y_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{y_1-f}{y_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}-1},$$

$$y_n = \frac{e(\frac{y_0-f}{y_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{y_1-f}{y_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}(\frac{x_0-f}{x_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{x_1-f}{x_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}-f}{(\frac{y_0-f}{y_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{y_1-f}{y_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}(\frac{x_0-f}{x_0-e})^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}(\frac{x_1-f}{x_1-e})^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}-1}$$

for $n \in \mathbb{N}_0$.

Case $e = f$. From the proof of Theorem 1 we see that in this case system (9) becomes

$$\mu_{n+2} = \mu_{n+1} + \nu_n, \quad \nu_{n+2} = \nu_{n+1} + \mu_n \tag{51}$$

for $n \in \mathbb{N}_0$.

Hence, the sequences $(\mu_n)_{n\in\mathbb{N}_0}$ and $(\nu_n)_{n\in\mathbb{N}_0}$ satisfy the following linear difference equation:

$$\omega_{n+4} - 2\omega_{n+3} + \omega_{n+2} - \omega_n = 0 \tag{52}$$

for $n \in \mathbb{N}_0$.

The general solution to equation (52) is

$$\omega_n = c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + c_4\lambda_4^n \tag{53}$$

for $n \in \mathbb{N}_0$, where c_j, $j = \overline{1,4}$, are some arbitrary constants, whereas λ_j, $j = \overline{1,4}$, are given in (35).

From (51) we have

$$\mu_2 = \mu_1 + \nu_0, \quad \mu_3 = \mu_1 + \nu_1 + \nu_0, \tag{54}$$

$$\nu_2 = \nu_1 + \mu_0, \quad \nu_3 = \mu_1 + \nu_1 + \mu_0. \tag{55}$$

From (53) and (54) a closed-form formula for the solution $(\mu_n)_{n\in\mathbb{N}_0}$ can be obtained, whereas from (53) and (55) a closed-form formula for the solution $(\nu_n)_{n\in\mathbb{N}_0}$ can be obtained.

The formulas can be obtained similar to the case $e \neq f$. Namely, we can iterate the relation

$$\begin{aligned}\omega_n &= 2\omega_{n-1} - \omega_{n-2} + \omega_{n-4}\\ &= a_1\omega_{n-1} + b_1\omega_{n-2} + c_1\omega_{n-3} + d_1\omega_{n-4}\end{aligned}$$

and show that

$$\omega_n = a_k\omega_{n-k} + b_k\omega_{n-k-1} + c_k\omega_{n-k-2} + d_k\omega_{n-k-3} \tag{56}$$

for $n \geq k+3$, where the sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, and $(d_k)_{k\in\mathbb{N}}$ satisfy the relations in (26) and (30). Hence, the sequence $(a_k)_{k\in\mathbb{N}}$ is given by formula (43).

Taking $k = n-3$, we get

$$\omega_n = a_{n-3}\omega_3 + (a_{n-2} - 2a_{n-3})\omega_2 + a_{n-5}\omega_1 + a_{n-4}\omega_0 \tag{57}$$

for $n \geq 6$.

From (54) and (57) we obtain

$$\begin{aligned}\mu_n =& a_{n-3}(\mu_1 + \nu_0 + \nu_1) + (a_{n-2} - 2a_{n-3})(\mu_1 + \nu_0) + a_{n-5}\mu_1 + a_{n-4}\mu_0\\ =& a_{n-4}\mu_0 + (a_{n-2} - a_{n-3} + a_{n-5})\mu_1 + (a_{n-2} - a_{n-3})\nu_0 + a_{n-3}\nu_1\\ =& a_{n-4}\mu_0 + (a_{n-1} - a_{n-2})\mu_1 + (a_{n-2} - a_{n-3})\nu_0 + a_{n-3}\nu_1\end{aligned} \tag{58}$$

for $n \in \mathbb{N}_0$.

Due to the symmetry we have

$$\nu_n = a_{n-4}\nu_0 + (a_{n-1} - a_{n-2})\nu_1 + (a_{n-2} - a_{n-3})\mu_0 + a_{n-3}\mu_1 \tag{59}$$

for $n \in \mathbb{N}_0$.

Using relation (8) with $n = 0, 1$ and formula (43) in (58), and finally relation (59), we obtain

$$\begin{aligned}\mu_n =& \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - e} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - e}\\ &+ \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - e} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - e},\end{aligned} \tag{60}$$

$$\begin{aligned}\nu_n =& \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - e} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - e}\\ &+ \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - e} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - e}\end{aligned} \tag{61}$$

for $n \in \mathbb{N}_0$.

From (8) we have

$$x_n = \frac{e\mu_n + 1}{\mu_n}, \quad y_n = \frac{f\nu_n + 1}{\nu_n} \tag{62}$$

for $n \in \mathbb{N}_0$.

From the relations in (62) together with the formulas in (60) and (61) it follows that

$$x_n=\frac{e\Bigg(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1-e}+\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1-e}\Bigg)+1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1-e}+\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1-e}},$$

$$y_n=\frac{e\Bigg(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1-e}+\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1-e}\Bigg)+1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}-\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}+\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1-e}+\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}}+\frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0-e}+\frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}}-\frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1-e}}$$

for $n\in\mathbb{N}_0$.

These formulas are closed-form formulas for the general solution to system (3) under the condition $e=f$. □

Corollary 2 *Assume* $e,f\in\mathbb{C}$, $k=2s$, $l=s$ *for some* $s\in\mathbb{N}$. *Then system* (3) *is solvable in a closed form.*

Proof Since $k=2s$ and $l=s$ for an $s\in\mathbb{N}$, system (3) becomes

$$x_{n+2s}=\frac{x_{n+s}y_n-ef}{x_{n+s}+y_n-e-f},\qquad y_{n+2s}=\frac{y_{n+s}x_n-ef}{y_{n+s}+x_n-e-f}\tag{63}$$

for $n\in\mathbb{N}_0$.

Now note that system (63) is a system of difference equations with interlacing indices.

Let $x_m^{(j)}=x_{ms+j}$, $y_m^{(j)}=y_{ms+j}$ for $m\in\mathbb{N}_0$ and $j=\overline{0,s-1}$.

Then $(x_m^{(j)},y_m^{(j)})_{m\in\mathbb{N}_0}$, $j=\overline{0,s-1}$, are s solutions to the system

$$x_{m+2}=\frac{x_{m+1}y_m-ef}{x_{m+1}+y_m-e-f},\qquad y_{m+2}=\frac{y_{m+1}x_m-ef}{y_{m+1}+x_m-e-f}\tag{64}$$

for $m\in\mathbb{N}_0$.

Now note that system (64) is nothing but system (3) in the case $k=2$ and $l=1$.

Employing Theorem 3, if $e\neq f$, we have

$$x_m^{(j)}=\frac{e\big(\frac{x_0^{(j)}-f}{x_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{x_1^{(j)}-f}{x_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\big(\frac{y_0^{(j)}-f}{y_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{y_1^{(j)}-f}{y_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-f}{\big(\frac{x_0^{(j)}-f}{x_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{x_1^{(j)}-f}{x_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\big(\frac{y_0^{(j)}-f}{y_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{y_1^{(j)}-f}{y_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

$$y_m^{(j)}=\frac{e\big(\frac{y_0^{(j)}-f}{y_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{y_1^{(j)}-f}{y_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\big(\frac{x_0^{(j)}-f}{x_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{x_1^{(j)}-f}{x_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-f}{\big(\frac{y_0^{(j)}-f}{y_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{y_1^{(j)}-f}{y_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\big(\frac{x_0^{(j)}-f}{x_0^{(j)}-e}\big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\big(\frac{x_1^{(j)}-f}{x_1^{(j)}-e}\big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1}$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $e = f$, we get

$$x_m^{(j)} = \frac{e\Big(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-e}\Big) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-e}},$$

$$y_m^{(j)} = \frac{e\Big(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-e}\Big) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-e}}$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, that is, if $e \neq f$, we have

$$x_{ms+j} = \frac{e(\frac{x_j-f}{x_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{x_{s+j}-f}{x_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}(\frac{y_j-f}{y_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{y_{s+j}-f}{y_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}} - f}{(\frac{x_j-f}{x_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{x_{s+j}-f}{x_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}(\frac{y_j-f}{y_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{y_{s+j}-f}{y_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}} - 1},$$

$$y_{ms+j} = \frac{e(\frac{y_j-f}{y_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{y_{s+j}-f}{y_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}(\frac{x_j-f}{x_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{x_{s+j}-f}{x_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}} - f}{(\frac{y_j-f}{y_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{y_{s+j}-f}{y_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}(\frac{x_j-f}{x_j-e})^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}(\frac{x_{s+j}-f}{x_{s+j}-e})^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}} - 1}$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $e = f$, we get

$$x_{ms+j} = \frac{e\Big(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j}-e}\Big) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j}-e}},$$

$$y_{ms+j} = \frac{e\Big(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j}-e}\Big) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j}-e} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j-e} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j}-e}}$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$. □

Remark 2 The other cases, that is, the cases 5°, 6°, 8°, and 10°, are dealt with in a similar fashion. The only difference is that there are more technical details than in the above considered cases. We leave the details to the interested reader as some exercises.

Remark 3 Note that the above analyses and proofs show that the solvability of system (3) is also closely connected to the solvability of linear difference equations, as it was the case in many previous investigations in the area [2, 3, 16, 36, 37, 42–44, 46–54].

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Author contributions
SS initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. BI, WK and ZŠ analyzed the proposed ideas, made some calculations, and gave some ideas and comments. All authors read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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