Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Unified interpolative of a Reich-Rus-Ćirić-type contraction in relational metric space with an application

Koti N.V.V. Vara Prasad[1], Vinay Mishra[1], Zoran D. Mitrović[2*], Dania Santina[3] and Nabil Mlaiki[3]

*Correspondence:
zoran.mitrovic@etf.unibl.org
[2]Faculty of Electrical Engineering, University of Banja Luka, Patre 5, Banja Luka, 78000, Bosnia and Herzegovina
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the notion of unified interpolative contractions of the Reich–Rus–Ćirić type and give some results about the fixed points for these mappings in the framework of relational metric spaces. We present examples where the results of some previous research are not relevant. Also, we apply our results to solving problems related to nonlinear matrix equations, emphasizing their practical importance.

Mathematics Subject Classification: 47H10; 54H25

Keywords: Unified interpolative Reich–Rus–Ćirić contraction; $\mathcal{R}$-admissible; Relational metric space

1 Introduction

The Banach contraction principle, a cornerstone of metric fixed-point theory, has found extensive applications across various disciplines, including physics, chemistry, economics, computer science, and biology. Consequently, the exploration and generalization of this principle have become focal points of research within nonlinear analysis [1–4].

The mappings that satisfy the Banach contraction principle are continuous. This prompts a natural question:

Can a discontinuous map in a complete metric space while satisfying analogous contractive conditions, possess a fixed point?

This intriguing question spurred investigation within the field, leading to an affirmative answer by Kannan [5]. Through the introduction of a novel form of contraction, Kannan [5] illuminated the possibility of fixed points within the realm of discontinuous maps and expanded the domain of inquiry within nonlinear analysis.

In 1972, Reich [6] extended the principles introduced by Banach and Kannan. For instance, a self-mapping $\mathcal{S} : X \to X$ is referred to as a Reich-contraction mapping if there exist values $\alpha, \beta, \gamma \in [0,1)$, where $\alpha + \beta + \gamma < 1$, such that

$$d(\mathcal{S}\nu, \mathcal{S}\mu) \leq \alpha d(\nu, \mathcal{S}\nu) + \beta d(\mu, \mathcal{S}\mu) + \gamma d(\nu, \mu) \tag{1.1}$$

for all $\nu, \mu \in X$.

Additional significant variations of the Banach contraction principle were explored independently by Ćirić, Reich, and Rus [7–9]. A collective outcome attributed to their work is presented below, recognized as the Ćirić–Reich–Rus contraction if there exits $\lambda \in [0, \frac{1}{3})$ such that

$$d(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda\bigl[d(\nu, \mu) + d(\nu, \mathcal{S}\nu) + d(\mu, \mathcal{S}\mu)\bigr] \tag{1.2}$$

for all $\nu, \mu \in X$.

In 2018, Karapınar [10] employed the interpolative method and converted the fundamental contraction concept of Kannan [5] into an interpolative form. Karapınar et al. [11] detected a deficiency in the analysis conducted by [10] concerning the assumption that the fixed point is unique. They accomplished this by presenting a counterexample and formulated an amended version, while also introducing the notions of interpolative Reich, Rus, and Ćirić type contractions, e.g., a mapping $\mathcal{S} : X \to X$ is called an interpolative Reich–Rus–Ćirić-type contraction, if there are constants $\lambda \in [0,1)$ and $\alpha, \beta \in (0,1)$ such that

$$d(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda\bigl(d(\nu, \mu)^{\alpha} \cdot d(\nu, \mathcal{S}\nu)^{\beta} \cdot d(\mu, \mathcal{S}\mu)^{1-\alpha-\beta}\bigr) \tag{1.3}$$

for all $\nu, \mu \in X \setminus F(\mathcal{S})$. They also proved that in the framework of partial metric space (X, d), a mapping $\mathcal{S}$, characterized as an interpolative Reich–Rus–Ćirić-type contraction, possesses a fixed point. Additionally, noteworthy contributions have been made by several authors [12–15], further enriching this area of study.

On the other hand, Gordji et al. [16] introduced the notion of orthogonal sets. It is imperative to note that we let X denote a nonempty set and $\perp$ represent a binary relation defined on it. The relation $\perp$ is termed orthogonal if there exists $\nu_0 \in X$ such that

$$(\forall \mu \in X)\nu_0 \perp \mu \vee (\forall \mu \in X)\mu \perp \nu_0, \tag{1.4}$$

where ν_0 is referred to as an orthogonal element, and the tuple $(X, \perp)$ is identified as an orthogonal set. An orthogonal set $(X, \perp)$ equipped with a metric d is denoted as an orthogonal-metric space, symbolized by $(X, \perp, d)$. In the framework of orthogonal metric spaces, Nazam et al. [17] have recently generalized condition (1.3) as follows.

Definition 1.1 [17] Let $(X, \perp, d)$ be an orthogonal metric space and

$$\psi, \phi : (0, +\infty) \to (-\infty, +\infty)$$

be two functions. The mapping $\mathcal{S} : X \to X$ is said to be a (ψ, ϕ)-orthogonal interpolative Reich–Rus–Ćirić-type contraction if there exists $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that

$$\psi\bigl(d(\mathcal{S}\nu, \mathcal{S}\mu)\bigr) \leq \phi\bigl(d(\nu, \mu)^{\alpha} d(\nu, \mathcal{S}\nu)^{\beta} d(\mu, \mathcal{S}\mu)^{1-\alpha-\beta}\bigr) \tag{11}$$

and

$$\min\left\{d(\mathcal{S}\nu,\mathcal{S}\mu),d(\nu,\mu),d(\mu,\mathcal{S}\mu),d(\nu,\mathcal{S}\nu)\right\}>0 \tag{1.5}$$

for all $(\nu,\mu)\in\{(\nu,\mu)\in X\times X:\nu\perp\mu\}$.

In recent years, the establishment of fixed-point results in metric spaces, characterized by various types of binary relations, has emerged as a significant area in fixed-point theory. Numerous types of binary relations, including partial orders, preorders, transitive relations, finitely transitive relations, locally $\mathcal{S}$-transitive relations, strict orders, and symmetric closures (see [18–26]), have been extensively employed in this endeavor.

Recently, Alam and Imdad [27] presented fixed-point results in metric spaces endowed with an arbitrary binary relation $\mathcal{R}$. Given the arbitrary nature of $\mathcal{R}$, it is notable that in specific cases, $\mathcal{R}$ can be construed as partial order [19, 20] (i.e., $\mathcal{R}$:= "$\preccurlyeq$"), orthogonal [16] (i.e., $\mathcal{R}$:= "$\perp$"), or similar instances. Due to its significance and wide applicability in the literature, numerous fixed-point results have been derived (see [28–34]). It is noteworthy that these results often pertain to weaker properties such as $\mathcal{R}$-continuity (not necessarily implying continuity) and $\mathcal{R}$-completeness (not necessarily implying completeness), among others. This context offers more flexibility as the contraction condition is not mandated for every element but only for those that are related. Importantly, these contraction conditions return to their usual forms when considering the universal relation.

In our current study, we introduce a broader idea called unified interpolative Reich–Rus–Ćirić-type contraction. This concept encompasses many existing findings, including those presented by [7–11, 17, 27]. We demonstrate several fixed-point results for such contractions within relational metric spaces.

2 Preliminaries

Before presenting our main results, it is important to introduce formal notations that will be used throughout this paper.

Let X be a nonempty set, with a binary relation $\mathcal{R}$. In this context, the pair $(X,\mathcal{R})$ is acknowledged as a relational set. Similarly, within the framework of a metric space (X,d), we designate that the triplet $(X,d,\mathcal{R})$ constitutes a relational metric space. The collection of fixed points of the self-mapping $\mathcal{S}$ is indicated by $F(\mathcal{S})$, and we let $X_{\mathcal{R}}$ denote the set defined by

$$X_{\mathcal{R}}=\left\{(\nu,\mu)\in X^2:(\nu,\mu)\in\mathcal{R}\text{ and }\nu,\mu\notin F(\mathcal{S})\right\}.$$

Furthermore, $X(\mathcal{S},\mathcal{R})$ is a subset of X, containing elements ν such that $(\nu,\mathcal{S}\nu)\in\mathcal{R}$. These formalized notations ensure precision and consistency throughout our subsequent analyses and discussions.

Definition 2.1 [27] In the context of a relational set $(X,\mathcal{R})$, and a self-map $\mathcal{S}$ defined on X:

(i) any two elements $\nu,\mu\in X$ are considered $\mathcal{R}$-comparative if $(\nu,\mu)\in\mathcal{R}$ or $(\mu,\nu)\in\mathcal{R}$. This relationship is symbolically represented as $[\nu,\mu]\in\mathcal{R}$;

(ii) a sequence $\{\nu_k\}\subset X$ satisfies the condition $(\nu_k,\nu_{k+1})\in\mathcal{R}$ for all $k\in\mathbb{N}_0$, is referred to as an $\mathcal{R}$-preserving sequence;

(iii) $\mathcal{R}$ is designated as $\mathcal{S}$-closed when it satisfies the condition that if (ν,μ) belongs to $\mathcal{R}$, then $(\mathcal{S}\nu,\mathcal{S}\mu)$ also belongs to $\mathcal{R}$, for any $\nu,\mu \in X$;

(iv) $\mathcal{R}$ is referred to as d-self-closed under the condition that whenever there exists a $\mathcal{R}$-preserving sequence $\{\nu_k\}$ such that $\nu_k \xrightarrow{d} \nu$, we can always find a subsequence $\{\nu_{k_n}\}$ of $\{\nu_k\}$ such that $[\nu_{k_n},\nu]$ belongs to $\mathcal{R}$ for all $n \in \mathbb{N}_0$.

Definition 2.2 [35] $(X,d,\mathcal{R})$ is considered as $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in X.

Definition 2.3 [35] A self-map $\mathcal{S}$ defined on X is termed $\mathcal{R}$-continuous at $\nu \in X$, if any $\mathcal{R}$-preserving sequence $\nu_k \xrightarrow{d} \nu$, implies $\mathcal{S}\nu_k \xrightarrow{d} \mathcal{S}\nu$. Furthermore, if $\mathcal{S}$ exhibits this behavior at every point in X, it is simply categorized as $\mathcal{R}$-continuous.

Definition 2.4 [36] Consider a self-map $\mathcal{S}$ defined on X. If for every $\mathcal{R}$-preserving sequence $\{\nu_n\} \subset \mathcal{S}(X)$, with a range denoted as $E = \{\nu_n : n \in \mathbb{N}\}$, $\mathcal{R}|_E$ is transitive, then $\mathcal{S}$ is designated as locally $\mathcal{S}$-transitive.

Samet et al. [37] introduced the concept of α-admissible mappings, which has been applied by various authors in numerous fixed-point theorems.

Definition 2.5 [37] Suppose $\mathcal{S}$ is a self-map on X, and $\alpha : X \times X \to \mathbb{R}^+$ is a function. Then, $\mathcal{S}$ is considered α-admissible if $\alpha(\nu,\mu) \geq 1 \Rightarrow \alpha(\mathcal{S}\nu,\mathcal{S}\mu) \geq 1$ for all $\nu,\mu \in X$.

In the following definition, we generalize this concept by incorporating certain relational metrical notions.

Definition 2.6 Let $(X,\mathcal{R})$ be a relational set. A self-map $\mathcal{S}$ defined on X is termed $\mathcal{R}$-admissible if there exists a function $\vartheta : X \times X \to [0,+\infty)$, satisfying the following conditions:

(r_1) $\vartheta(\nu,\mu) \geq 1$ for all $(\nu,\mu) \in \mathcal{R}$;

(r_2) $\mathcal{R}$ is $\mathcal{S}$-closed.

Remark 2.7 From the above two definitions, we can observe that if $\mathcal{S}$ is α-admissible, it also holds that $\mathcal{S}$ is $\mathcal{R}$-admissible when considering

$$\mathcal{R} = \left\{(\nu,\mu) \in X^2 : \vartheta(\nu,\mu) \geq 1\right\}.$$

However, it should be noted that the converse is not necessarily true, as illustrated in the following example.

Example 2.8 Let $X = \{0,1,2,3\}$, $\vartheta : X \times X \to \mathbb{R}^+$ by

$$\vartheta(\nu,\mu) = \begin{cases} 2, & (\nu,\mu) \in \{(0,1),(1,2),(2,3)\} \\ 1, & (\nu,\mu) \in \{(0,2),(1,1),(2,1),(2,2)\} \\ \frac{2}{\nu+5}, & \text{otherwise.} \end{cases}$$

Let $\mathcal{S} : X \to X$ be defined by

$$\mathcal{S}\nu = \begin{cases} 0, & \text{if } \nu = 0 \\ 2, & \text{if } \nu = 1 \\ 1, & \text{if } \nu = 2 \\ 3, & \text{if } \nu = 3. \end{cases}$$

In this example, it is evident that $\vartheta(2,3) \geq 1$, but $\vartheta(\mathcal{S}2, \mathcal{S}3) = \vartheta(1,3) \ngeq 1$, indicating that $\mathcal{S}$ is not α-admissible. Now, let us consider the binary relation $\mathcal{R}$ defined as

$$\mathcal{R} = \{(0,1), (0,2), (1,2), (2,1), (1,1), (2,2)\}.$$

It can be observed that for all $\nu, \mu \in X$ with $(\nu, \mu) \in \mathcal{R}$, we have $\vartheta(\nu, \mu) \geq 1$. Therefore, $\mathcal{S}$ satisfies condition (r_1). Furthermore, whenever $(\nu, \mu) \in \mathcal{R}$, we have $(\mathcal{S}\nu, \mathcal{S}\mu) \in \mathcal{R}$, indicating that $\mathcal{R}$ is $\mathcal{S}$-closed and satisfies condition (r_2). Hence, $\mathcal{S}$ is $\mathcal{R}$-admissible.

Let $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be two functions. Then, we consider the following conditions:

(C_1) ϕ is upper semicontinuous with $\phi(0) = 0$;
(C_2) ψ is lower semicontinuous;
(C_3) ψ, ϕ are nondecreasing;
(C_4) $\psi(t) > \phi(t)$, for all $t > 0$;
(C_5) $\limsup_{t \to c+} \phi(t) < \psi(c+)$, for all $c > 0$;
(C_6) $\limsup_{t \to e+} \phi(t) \leq \liminf_{t \to e+} \psi(t)$, for any $e > 0$.

In the next section, we will introduce a novel concept termed as the unified interpolative Reich–Rus–Ćirić-type contraction condition and establish several fixed-point results for such contractions.

3 Main results

First, we give a definition of a unified interpolative Reich–Rus–Ćirić-type contraction.

Definition 3.1 Let $(X, d, \mathcal{R})$ be a relational metric space. A self-mapping $\mathcal{S}$ defined on X is termed a unified interpolative Reich–Rus–Ćirić-type contraction, if there exist the functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$, and a function $\vartheta : X \times X \to \mathbb{R}^+$, along with the parameters $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that

$$\vartheta(\nu, \mu)\psi\big(d(\mathcal{S}\nu, \mathcal{S}\mu)\big) \leq \phi\Big(\Omega\big(d(\nu, \mu), d(\nu, \mathcal{S}\nu), d(\mu, \mathcal{S}\mu)\big)\Big) \tag{3.1}$$

for all $\nu, \mu \in X_{\mathcal{R}}$, where $\Omega : \mathbb{R}^3 \to \mathbb{R}$ is a mapping such that

$$\Omega(u, v, w) \leq \max\{u, v, w, u^\alpha v^\beta w^{1-\alpha-\beta}\}.$$

Remark 3.2 By giving the precise definitions of the functions ψ, ψ, ϑ, and Ω, it becomes evident that we can draw the following conclusions, underscoring the extensive applicability and versatility of Definition 3.1.

(i) When we consider $\vartheta(\nu,\mu) = 1$, and $\Omega(u,v,w) = u^{\alpha} \cdot v^{\beta} \cdot w^{1-\alpha-\beta}$, where $\alpha + \beta < 1$ in equation (3.1), also consider the binary relation $\mathcal{R}$ as

$$\mathcal{R} = \{(\nu,\mu) \in X^2 : \nu \perp \mu\},$$

we obtain the (ψ,ϕ)-orthogonal interpolative Ćirić–Reich–Rus-type contraction [17]

$$\psi\big(d(\mathcal{S}\nu,\mathcal{S}\mu)\big) \le \phi\big(d(\nu,\mu)^{\alpha} \cdot d(\nu,\mathcal{S}\nu)^{\beta} \cdot d(\mu,\mathcal{S}\mu)^{1-\alpha-\beta}\big) \tag{3.2}$$

for all $\nu,\mu \in X_{\mathcal{R}}$.

(ii) By taking $\alpha = 0$ in (3.2) we obtain the (ψ,ϕ)-orthogonal Kannan contraction [17],

$$\psi\big(d(\mathcal{S}\nu,\mathcal{S}\mu)\big) \le \phi\big(d(\nu,\mathcal{S}\nu)^{\beta} \cdot d(\mu,\mathcal{S}\mu)^{1-\beta}\big) \tag{3.3}$$

for all $\nu,\mu \in X_{\mathcal{R}}$.

(iii) By taking $\psi(t) = t$, and $\psi(t) = \lambda t$, $\lambda < 1$, and considering $\mathcal{R}$ as a universal relation in (3.2) and (3.3) we obtain the interpolative Reich–Rus–Ćirić-type contraction [11] and interpolative Kannan contraction [10], respectively.

(iv) By considering $\psi(t) = t$, $\phi(t) = \lambda t$, $\lambda < 1$, and $\Omega(u,v,w) = \frac{u+v+w}{3}$, we obtain the combined result of Ćirić, Reich, and Rus [7–9]:

$$d(\mathcal{S}\nu,\mathcal{S}\mu) \le \lambda\big(d(\nu,\mu) + d(\nu,\mathcal{S}\nu) + d(\mu,\mathcal{S}\mu)\big) \tag{3.4}$$

for all $\nu,\mu \in X_{\mathcal{R}}$.

(v) By considering $\psi(t) = t$ and $\phi(t) = \lambda t$, $\lambda < 1$, and $\Omega(u,v,w) = u$, we obtain the relational theoretic version of the famous Banach contraction that is introduced by Aftab Alam and Mohammad Imdad [27].

(vi) By considering $\psi(t) = t$, $\phi(t) = \lambda t$ and $\Omega(u,v,w) = \frac{v+w}{2}$, we obtain the Kannan contraction with the constant $\lambda \in [0,\frac{1}{2})$,

$$d(\mathcal{S}\nu,\mathcal{S}\mu) \le \lambda\big(d(\nu,\mathcal{S}\nu) + d(\mu,\mathcal{S}\mu)\big) \tag{3.5}$$

for all $\nu,\mu \in X_{\mathcal{R}}$.

Now, we will proceed to establish our main results concerning the unified interpolative Reich–Rus–Ćirić contraction maps.

Theorem 3.3 *Consider the relational metric space $(X,d,\mathcal{R})$, where $\mathcal{R}$ is a locally $\mathcal{S}$-transitive binary relation. Suppose that $\mathcal{S}$ is a unified interpolative Reich–Rus–Ćirić-type contraction, and there exist functions $\psi,\phi : [0,+\infty) \to [0,+\infty)$ satisfying conditions C_i, $(i = 1,2,3,4)$. Under the following conditions*:

(D_1) *$\mathcal{S}$ is $\mathcal{R}$-admissible*;

(D_2) *there exists $Y \subseteq X$ with $\mathcal{S}(X) \subseteq Y$, such that $(Y,d,\mathcal{R})$ is $\mathcal{R}$-complete*;

(D_3) *$X(\mathcal{S},\mathcal{R})$ is nonempty*;

(D_4) *either $\mathcal{S}$ is $\mathcal{R}|_Y$-continuous or $\mathcal{R}$ is d-self-closed*;

there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

Proof Under the assumption (D_3), suppose that $\nu_0 \in X(\mathcal{S}, \mathcal{R})$. Define the sequence $\{\nu_n\}$ of Picard iterates with initial point ν_0, i.e., $\nu_n = \mathcal{S}^n \nu_0$ for all $n \in \mathbb{N}_0$. As $(\nu_0, \mathcal{S}\nu_0) \in \mathcal{R}$ and $\mathcal{S}$ is $\mathcal{R}$-admissible, using (r_1) it follows that

$$\left(\mathcal{S}^n \nu_0, \mathcal{S}^{n+1}\nu_0\right) \in \mathcal{R}.$$

Consequently, $(\nu_n, \nu_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$, and this yields that the sequence $\{\nu_n\}$ is $\mathcal{R}$-preserving and from (r_2) we have $\vartheta(\nu_n, \nu_{n+1}) \geq 1$. Let $d_n = d(\nu_n, \nu_{n+1})$, and applying the contractive condition (3.1), we obtain that

$$\begin{aligned}
\psi(d_n) &\leq \vartheta(\nu_{n-1}, \nu_n)\psi\left(d(\mathcal{S}\nu_{n-1}, \mathcal{S}\nu_n)\right) \\
&\leq \phi\left(\Omega\left(d(\nu_{n-1}, \nu_n), d(\nu_{n-1}, \mathcal{S}\nu_{n-1}), d(\nu_n, \mathcal{S}\nu_n)\right)\right) \\
&\leq \phi\left(\max\left\{d_{n-1}, d_{n-1}, d_n, d_{n-1}^{\alpha} \cdot d_{n-1}^{\beta} \cdot d_n^{1-\alpha-\beta}\right\}\right) \\
&< \psi\left(\max\left\{d_{n-1}, d_n, d_{n-1}^{\alpha+\beta} \cdot d_n^{1-(\alpha+\beta)}\right\}\right).
\end{aligned} \tag{3.6}$$

By the monotonicity of the function ψ we obtain

$$d_n < \max\left\{d_{n-1}, d_n, d_{n-1}^{\alpha+\beta} \cdot d_n^{1-(\alpha+\beta)}\right\}. \tag{3.7}$$

Now, suppose there exists $n \in \mathbb{N}$ for which $d_{n-1} \leq d_n$, then from (3.7) we obtain that $d_n < d_n$, which is a contradiction. Therefore, $d_n \leq d_{n-1}$, now we can conclude that $\{\nu_n\}$ is a nonincreasing sequence and thus a nonnegative constant C exists such that, $\lim_{n\to+\infty} d_n = C+$. Suppose, if possible, $C > 0$, then from (3.6), it can be deduced that

$$\psi(C+) \leq \liminf \psi(d_n) \leq \limsup \phi(d_{n-1}) \leq \phi(C+),$$

but, from (C_4) we have $\psi(\nu) > \phi(\nu)$ for all $\nu > 0$, therefore C must be 0, i.e., $\lim_{n\to+\infty} d_n = 0$. Our next objective is to establish that the sequence $\{\nu_n\}$ is Cauchy. For the sake of contradiction, suppose it is not, then there exists a positive real number $\epsilon > 0$ along with subsequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$, with $n_k > m_k \geq k$, such that

$$d(\nu_{m_k}, \nu_{n_k}) \geq \epsilon, \quad \text{for all } k \in \mathbb{N}. \tag{3.8}$$

Selecting n_k as the smallest integer exceeding m_k such that (3.8) holds, we deduce that

$$d(\nu_{m_k}, \nu_{n_k-1}) < \epsilon. \tag{3.9}$$

Using the triangular inequality and (3.8) and (3.9) we obtain that

$$\begin{aligned}
\epsilon &\leq d(\nu_{m_k}, \nu_{n_k}) \\
&\leq d(\nu_{m_k}, \nu_{n_k-1}) + d(\nu_{n_k-1}, \nu_{n_k}) \\
&< \epsilon + d(\nu_{n_k-1}, \nu_{n_k}).
\end{aligned}$$

On taking the limit $k \to +\infty$ and utilizing the fact that $\lim_{n\to+\infty} d_n = 0$, we obtain

$$\lim_{k\to+\infty} d(\nu_{m_k}, \nu_{n_k}) = \epsilon + . \tag{3.10}$$

By using the triangular inequality, we obtain that

$$\left| d(\nu_{m_k+1}, \nu_{n_k+1}) - d(\nu_{m_k}, \nu_{n_k}) \right| \leq d\nu_{m_k} + d\nu_{n_k},$$

letting limit $k \to +\infty$ in the above inequality and employing (3.10), we obtain the following:

$$\lim_{k\to+\infty} d(\nu_{m_k+1}, \nu_{n_k+1}) = \lim_{k\to+\infty} d(\nu_{m_k}, \nu_{n_k}) = \epsilon. \tag{3.11}$$

Since $\{\nu_n\} \subset \mathcal{S}(X)$ and $\{\nu_n\}$ is $\mathcal{R}$-preserving, the local $\mathcal{S}$-transitivity of $\mathcal{R}$ leads to the implication that $(\nu_{m_k}, \nu_{n_k}) \in \mathcal{R}$. Thus, we can deduce

$$\begin{aligned}
\psi\big(d(\nu_{m_{k+1}}, \nu_{n_{k+1}})\big) &\leq \vartheta(\nu_{m_k}, \nu_{n_k})\psi\big(d(\mathcal{S}\nu_{m_k}, \mathcal{S}\nu_{n_k})\big) \\
&\leq \phi\big(\Omega\big(d(\nu_{m_k}, \nu_{n_k}), d(\nu_{m_k}, \mathcal{S}\nu_{m_k}), d(\nu_{n_k}, \mathcal{S}\nu_{n_k})\big)\big) \\
&\leq \phi\big(\max\big\{d(\nu_{m_k}, \nu_{n_k}), d_{m_k}, d_{n_k}, d(\nu_{m_k}, \nu_{n_k})^{\alpha} \cdot d_{m_k}^{\beta} \cdot d_{n_k}^{1-\alpha-\beta}\big\}\big).
\end{aligned}$$

On taking the limit as $k \to +\infty$ in the aforementioned inequality, leads to the contradiction with (C_4). Hence, $\{\nu_n\}$ is the $\mathcal{R}$- preserving Cauchy sequence in Y. The $\mathcal{R}$-completeness of the metric space $(Y, d, \mathcal{R})$ now guarantees the existence of a point $\gamma \in Y$ such that, $\lim_{n\to+\infty} \nu_n = \gamma$. First, we assumed that $\mathcal{S}$ is $\mathcal{R}$-continuous, then we can deduce that $\lim_{n\to+\infty} \nu_{n+1} = \lim_{n\to+\infty} \mathcal{S}\nu_n = \mathcal{S}\gamma$. Applying the uniqueness of the limit, we consequently establish that $\mathcal{S}\gamma = \gamma$, indicating that $\gamma \in F(\mathcal{S})$.

Alternatively, let $\mathcal{R}|_Y$ be d-self-closed. We again utilize the fact that $\{\nu_n\}$ is $\mathcal{R}$-preserving and $\{\nu_n\} \to \gamma$. This implies the existence of a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \gamma] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. If $(\nu_{n_k}, \gamma) \in \mathcal{R}$, then since $\mathcal{S}$ is a unified interpolative Reich–Rus–Ćirić contraction, we have

$$\begin{aligned}
&\psi\big(d(\mathcal{S}\nu_{n_k}, \mathcal{S}\gamma)\big) \\
&\quad\leq \vartheta(\nu_{n_k}, \gamma)\psi\big(d(\mathcal{S}\nu_{n_k}, \mathcal{S}\gamma)\big) \\
&\quad\leq \phi\big(\Omega\big(d(\nu_{n_k}, \gamma), d(\nu_{n_k}, \mathcal{S}\nu_{n_k}), d(\gamma, \mathcal{S}\gamma)\big)\big) \\
&\quad\leq \phi\big(\max\big\{d(\nu_{n_k}, \gamma), d_{n_k}, d(\gamma, \mathcal{S}\gamma), d(\nu_{n_k}, \gamma)^{\alpha} \cdot d_{n_k}^{\beta} \cdot d(\gamma, \mathcal{S}\gamma)^{1-\alpha-\beta}\big\}\big).
\end{aligned} \tag{3.12}$$

On taking the limit $k \to +\infty$, in (3.12), we obtain

$$\psi\big(d(\gamma, \mathcal{S}\gamma)\big) \leq \phi\big(d(\gamma, \mathcal{S}\gamma)\big). \tag{3.13}$$

It is important to note that in equation (3.13), if $d(\gamma, \mathcal{S}\gamma) \neq 0$, then we face a contradiction with (C_4). Similarly, if $(\gamma, \nu_{n_k}) \in \mathcal{R}$, then by utilizing the symmetry of d, we once again encounter a contradiction with (C_4). Therefore, $d(\gamma, \mathcal{S}\gamma) = 0$, implying $\gamma \in F(\mathcal{S})$. □

Theorem 3.4 *Consider the relational metric space $(X,d,\mathcal{R})$, where $\mathcal{R}$ is a locally $\mathcal{S}$-transitive binary relation. Suppose that $\mathcal{S}$ is a unified interpolative Reich–Rus–Ćirić-type contraction and there exist functions $\psi,\phi:[0,+\infty)\to[0,+\infty)$ satisfying conditions C_i, $(i=3,4,5,6)$ and D_j, $(j=1,2,3,4)$ holds. Then, there exists at least one $\gamma\in X$ such that $\gamma\in F(\mathcal{S})$.*

Proof Following the steps of the previous theorem we can obtain an $\mathcal{R}$-preserving and nonincreasing sequence $\{\nu_n\}$ such that there exists some $C\geq 0$ and d_n converges to $C+$ as $n\to+\infty$. Suppose $C>0$, then (3.6) implies that

$$\begin{aligned}\psi(C+)&=\lim_{n\to+\infty}\psi(d_n)\\&\leq\limsup_{n\to+\infty}\phi\big(\max\big\{d_{n-1},d_{n-1},d_n,d_{n-1}^{\alpha}\cdot d_{n-1}^{\beta}\cdot d_n^{1-\alpha-\beta}\big\}\big)\\&\leq\limsup_{k\to C+}\phi(k),\end{aligned}$$

a contradiction with (C_5), thus $C=0$, i.e., $\lim_{n\to+\infty}d_n=0$. Now, to establish that the sequence $\{\nu_n\}$ is Cauchy, we make a counter assumption. Suppose it is not Cauchy, then following the steps outlined in the previous theorem, there exists a positive real number $\epsilon>0$, along with subsequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$, where $n_k>m_k\geq k$, satisfying condition (3.11). Since $\{\nu_n\}\subset\mathcal{S}(X)$ and $\{\nu_n\}$ is $\mathcal{R}$-preserving, the local $\mathcal{S}$-transitivity of $\mathcal{R}$ leads to the implication that $(\nu_{m_k},\nu_{n_k})\in\mathcal{R}$. Thus, we can deduce

$$\begin{aligned}\psi\big(d(\nu_{m_{k+1}},\nu_{n_{k+1}})\big)&\leq\vartheta(\nu_{m_k},\nu_{n_k})\psi\big(d(\mathcal{S}\nu_{m_k},\mathcal{S}\nu_{n_k})\big)\\&\leq\phi\big(\max\big\{d(\nu_{m_k},\nu_{n_k}),d_{m_k},d_{n_k},d(\nu_{m_k},\nu_{n_k})^{\alpha}\cdot d_{m_k}^{\beta}\cdot d_{n_k}^{1-\alpha-\beta}\big\}\big).\end{aligned}$$

On taking the limit $k\to+\infty$ in the above equation, this implies that

$$\begin{aligned}\liminf_{a\to\epsilon+}\psi(a)&\leq\liminf_{k\to+\infty}\psi\big(d(\nu_{m_{k+1}},\nu_{n_{k+1}})\big)\\&\leq\limsup_{k\to+\infty}\phi\big(\max\big\{d(\nu_{m_k},\nu_{n_k}),d_{m_k},d_{n_k},d(\nu_{m_k},\nu_{n_k})^{\alpha}\cdot d_{m_k}^{\beta}\cdot d_{n_k}^{1-\alpha-\beta}\big\}\big)\\&\leq\limsup_{a\to\epsilon+}\phi(a).\end{aligned}$$

This results in a contradiction with (C_6), thus establishing that the $\{\nu_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence is in Y. Given that $(Y,d,\mathcal{R})$ is an $\mathcal{R}$-complete metric space, there exists $\gamma\in Y$ such that $\lim_{n\to+\infty}\nu_n=\gamma$. If the self-mapping $\mathcal{S}$ is $\mathcal{R}$-continuous, we can derive the desired conclusion, as demonstrated in the previous theorem.

Alternatively, let $\mathcal{R}|_Y$ be d-self-closed then by utilizing the fact that $\{\nu_n\}$ is $\mathcal{R}$-preserving and $\{\nu_n\}\to\gamma$, which implies the existence of a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k},\gamma]\in\mathcal{R}$, for all $k\in\mathbb{N}_0$. We claim that $d(\gamma,\mathcal{S}\gamma)=0$. Let us assume that $d(\gamma,\mathcal{S}\gamma)>0$, if $(\nu_{n_k},\gamma)\in\mathcal{R}$, then since $\mathcal{S}$ is a unified interpolative Reich–Rus–Ćirić contraction, we have

$$\begin{aligned}&\psi\big(d(\nu_{n_k+1},\mathcal{S}\gamma)\big)\\&\quad\leq\vartheta(\nu_{n_k},\gamma)\psi\big(d(\mathcal{S}\nu_{n_k},\mathcal{S}\gamma)\big)\\&\quad\leq\phi\big(\Omega\big(d(\nu_{n_k},\gamma),d(\nu_{n_k},\mathcal{S}\nu_{n_k}),d(\gamma,\mathcal{S}\gamma)\big)\big)\end{aligned}$$

$$\leq \phi\big(\max\big\{d(\nu_{n_k},\gamma), d_{n_k}, d(\gamma,\mathcal{S}\gamma), d(\nu_{n_k},\gamma)^{\alpha}\cdot d_{n_k}^{\beta}\cdot d(\gamma,\mathcal{S}\gamma)^{1-\alpha-\beta}\big\}\big)$$
$$< \psi\big(\max\big\{d(\nu_{n_k},\gamma), d_{n_k}, d(\gamma,\mathcal{S}\gamma), d(\nu_{n_k},\gamma)^{\alpha}\cdot d_{n_k}^{\beta}\cdot d(\gamma,\mathcal{S}\gamma)^{1-\alpha-\beta}\big\}\big)$$

and by using (C_3) and taking the limit as $k \to +\infty$, we deduce

$$d(\gamma,\mathcal{S}\gamma) < d(\gamma,\mathcal{S}\gamma), \tag{3.14}$$

which leads to a contradiction. Furthermore, if $(\gamma,\nu_{n_k}) \in \mathcal{R}$, then by utilizing the symmetry of d, we encounter again a contradiction. Hence, $d(\gamma,\mathcal{S}\gamma) = 0$, implying $\gamma \in F(\mathcal{S})$. □

Theorem 3.5 *Consider the relational metric space $(X,d,\mathcal{R})$, where $\mathcal{R}$ is locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed. Suppose conditions D_j, $(j = 1,2,3,4)$ hold and there exist functions $\psi,\phi : [0,+\infty) \to [0,+\infty)$ satisfying conditions C_i, $(i = 1,2,3,4)$ or $(i = 3,4,5,6)$, such that,*

$$\psi\big(d(\mathcal{S}\nu,\mathcal{S}\mu)\big) \leq \phi\big(\Omega\big(d(\nu,\mathcal{S}\nu), d(\mu,\mathcal{S}\mu)\big)\big), \quad \textit{for all } \nu,\mu \in X_{\mathcal{R}}. \tag{3.15}$$

Then, there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

Example 3.6 Let (X,d) be a metric space with $X = [0,+\infty)$ and d is the usual metric, define the self-map $\mathcal{S}$ on X by

$$\mathcal{S}\nu = \begin{cases} \frac{5\nu^2}{6}, & \text{if } \nu \leq 1, \\ \frac{1}{2}, & \text{if } \nu > 1. \end{cases}$$

Then, it is important to note that $\mathcal{S}$ is not a Ćirić–Reich–Rus-type contraction [7–9]. It is evident that when considering $\nu = 1$ and $\mu = \frac{1}{2}$, there does not exist any constant $\lambda \in (0,\frac{1}{3}]$ for which condition (1.2) holds. Additionally, for the same values of $\nu = 1$ and $\mu = \frac{1}{2}$, there is no pair of $\lambda \in [0,1)$ and $\alpha,\beta \in [0,1)$ satisfying $\alpha + \beta < 1$ for which (3.2) holds. Consequently, $\mathcal{S}$ is not an interpolative Reich–Rus–Ćirić-type contraction [10]. Now, let us define the binary relation $\mathcal{R}$ on X as

$$\mathcal{R} = \left\{(\nu,\mu) \in X^2 : \max\{\nu,\mu\} \leq \frac{1}{2}\right\}.$$

This relation $\mathcal{R}$ exhibits the property of being locally $\mathcal{S}$-transitive, and $\mathcal{S}$ is $\mathcal{R}$-continuous. It can also be observed that $\mathcal{R}$ is $\mathcal{S}$-closed. Moreover, the set $X(\mathcal{S},\mathcal{R})$ is nonempty, and there exists a subset $Y = [0,1]$ of X such that $\mathcal{S}(X) \subseteq Y$ and (Y,d), is $\mathcal{R}$-complete.

Observing the definition of $\mathcal{R}$, it is clear that $\mathcal{R}$ is not an orthogonal relation, as there does not exist any $\nu_0 \in X$ that satisfies condition (1.4). As a consequence, the function $\mathcal{S}$ is not a (ψ,ϕ)-orthogonal interpolative Reich–Rus–Ćirić-type contraction [17]. However, we will now demonstrate that $\mathcal{S}$ is indeed a unified interpolative Reich–Rus–Ćirić-type contraction. Consider $\vartheta : X \times X \to [0,+\infty)$ defined by

$$\vartheta(\nu,\mu) = \begin{cases} 1, & \text{if } \nu,\mu \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

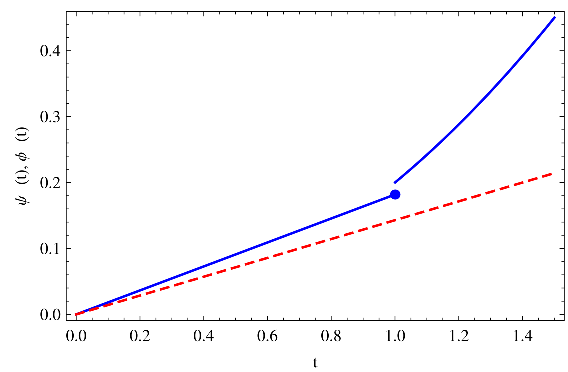

Figure 1 Graphs of the function ψ and ϕ in Example 3.6

Observing that $\vartheta(\nu,\mu) \geq 1$ for all $\nu,\mu \in X$ with $(\nu,\mu) \in \mathcal{R}$, and that $(\nu,\mu) \in \mathcal{R}$ implies $(\mathcal{S}\nu,\mathcal{S}\mu) \in \mathcal{R}$, it follows that $\mathcal{S}$ is $\mathcal{R}$-admissible. Suppose there exist functions ψ,ϕ : $[0,+\infty) \to [0,+\infty)$ defined by $\phi(t) = \frac{t}{7}$,

$$\text{and, } \psi(t) = \begin{cases} \frac{2t}{11}, & \text{if } t \leq 1, \\ \frac{t^2}{5}, & \text{if } t > 1. \end{cases}$$

In Fig. 1, the red (dashed) line represents $\phi(t)$, while the blue line denotes $\psi(t)$. It is evident that ϕ is upper semicontinuous, $\phi(0) = 0$, and ψ is lower semicontinuous, such that $\psi(t) > \phi(t)$, and both ψ, ϕ are nondecreasing.

We now aim to show that $\mathcal{S}$ satisfies (3.1). Consider the function $\Omega : X \times X \to [0,+\infty)$ defined as $\Omega(u,v,w) = \frac{4}{5}\max\{u,v,w,u^\alpha v^\beta w^{1-\alpha-\beta}\}$. Then, for every $\nu,\mu \in X_{\mathcal{R}}$, we can observe that

$$\vartheta(\nu,\mu)\psi\big(d(\mathcal{S}\nu,\mathcal{S}\mu)\big) = \frac{5|\nu^2-\mu^2|}{33} \tag{3.16}$$

and

$$\begin{aligned}
&\phi\big(\Omega\big(d(\nu,\mu),d(\nu,T\nu),d(\mu,T\mu)\big)\big) \\
&\quad = \frac{4}{35}\Bigg(\max\Bigg\{|\nu-\mu|,\frac{|6\nu-5\nu^2|}{6}, \\
&\qquad \frac{|6\mu-5\mu^2|}{6},|\nu-\mu|^\alpha\cdot\bigg(\frac{|6\nu-5\nu^2|}{6}\bigg)^\beta\cdot\bigg(\frac{|6\mu-5\mu^2|}{6}\bigg)^{1-\alpha-\beta}\Bigg\}\Bigg).
\end{aligned} \tag{3.17}$$

In Fig. 2, for each point $\nu,\mu \in X$ such that $(\nu,\mu) \in \mathcal{R}$, corresponds to a three-dimensional representation of equation (3.16) (illustrated by the red plane) and equation (3.17) (depicted by the blue plane), with the given parameters $\alpha = 0.1$ and $\beta = 0.2$. It is evident from the observation that the red plane remains consistently below or coincident with the blue plane. Consequently, we can deduce that equation (3.16), representing the left-hand side of (3.1), consistently maintains a value that is less than or equal to equation (3.17), representing the right-hand side of (3.1). Hence, it follows that Equation (3.1) holds true for all $\nu,\mu \in X$ with $(\nu,\mu) \in \mathcal{R}$.

Consequently, we deduce that $\mathcal{S}$ is a unified interpolative Reich–Rus–Ćirić contraction.

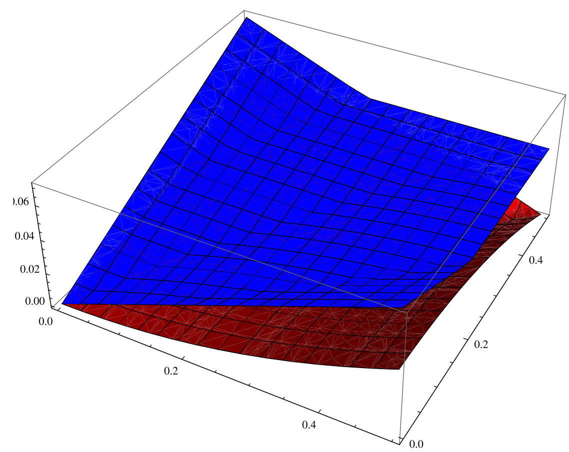

Figure 2 3D representation of equations (3.16) and (3.17) for values $\alpha = 0.1$ and $\beta = 0.2$

4 An application

In this section, we have applied our research findings to derive a result concerning the existence of solutions for a nonlinear matrix equation. In this context, let the set denoted as $\mathcal{M}(n)$ encompass all square matrices with dimensions of $n \times n$, while $\mathcal{H}(n)$, $\mathcal{P}(n)$, and $\mathcal{K}(n)$, respectively, represent the sets of Hermitian matrices, positive-definite matrices, and positive semidefinite matrices. When we have a matrix $\mathcal{C}$ from $\mathcal{H}(n)$, we use the notation $\|\mathcal{C}\|_{\mathrm{tr}}$ to refer to its trace norm, which is the sum of all its singular values. If we have matrices $\mathcal{P}$ and $\mathcal{Q}$ from $\mathcal{H}(n)$, the notation $\mathcal{P} \succeq \mathcal{Q}$ signifies that the matrix $\mathcal{P} - \mathcal{Q}$ is an element of the set $\mathcal{K}(n)$, while $\mathcal{P} \succ \mathcal{Q}$ indicates that $\mathcal{P} - \mathcal{Q}$ belongs to the set $\mathcal{P}(n)$. The upcoming discussion relies on the significance of the following lemmas.

Lemma 4.1 [38] *If $X \in \mathcal{H}(n)$ satisfies $X \prec \mathcal{I}_n$, then $\|X\| < 1$.*

Lemma 4.2 [38] *For $n \times n$ matrices $X \succeq O$ and $Y \succeq O$, the following inequalities hold*:

$$0 \leq \mathrm{tr}(XY) \leq \|X\| \, \mathrm{tr}(Y).$$

We shall now examine the following nonlinear matrix equation:

$$X = \mathcal{A} + \sum_{j=1}^{u} \sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X) \mathcal{C}_j. \tag{4.1}$$

In the above equation, $\mathcal{A}$ is defined as a Hermitian and positive-definite matrix. Additionally, the notation $\mathcal{C}_j^*$ refers to the conjugate transpose of a square matrix $\mathcal{C}_j$ of size $n \times n$. Furthermore, Υ_k represents continuous functions that preserve order, mapping from $\mathcal{H}(n)$ to $\mathcal{P}(n)$. It is noteworthy that $\Upsilon(O) = O$, where O represents a zero matrix.

Theorem 4.3 *Consider the nonlinear matrix equation expressed in* (4.1) *and assume the following*:

(H_1) *there exists $\mathcal{A} \in \mathcal{P}(n)$ with $\sum_{j=1}^{u} \sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(\mathcal{A}) \mathcal{C}_j \succ 0$;*

(H_2) *for every $X, Y \in \mathcal{P}(n)$, $X \preceq Y$ implies*

$$\sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X)\mathcal{C}_j \preceq \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(Y)\mathcal{C}_j;$$

(H_3) $\sum_{j=1}^{u}\mathcal{C}_j\mathcal{C}_j^* \prec N\mathcal{I}_n$, *for some positive number N, and for all $X, Y \in \mathcal{P}(n)$ with $X \preceq Y$, the following inequality holds*

$$\begin{aligned}
&\max_k(\mathrm{tr}(\Upsilon_k(Y) - \Upsilon_k(X))) \\
&\quad \le \frac{2}{3N\nu}\max\left\{\begin{array}{l} \mathrm{tr}(Y-X), \mathrm{tr}\Big(X - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X)\mathcal{C}_j\Big), \mathrm{tr}\Big(Y - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*, \Upsilon_k(Y)\mathcal{C}_j\Big), \\ \Big(\mathrm{tr}\Big(X - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X)\mathcal{C}_j\Big)\Big)^{\frac{1}{2}} \cdot \Big(\mathrm{tr}\Big(Y - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*, \Upsilon_k(Y)\mathcal{C}_j\Big)\Big)^{\frac{1}{2}} \end{array}\right\}.
\end{aligned}$$

Then, there exists at least one solution of the nonlinear matrix equation (4.1). *Moreover, the iteration*

$$X_r = \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X_{r-1})\mathcal{C}_j, \tag{4.2}$$

where $X_0 \in \mathcal{P}(n)$ satisfies $X_0 \preceq \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X_0)\mathcal{C}_j$and converges towards the solution of the matrix equation, in the context of trace norm $\|\cdot\|_{\mathrm{tr}}$.

Proof Let $\mathfrak{T} : \mathcal{P}(n) \to \mathcal{P}(n)$ be a mapping defined by

$$\mathfrak{T}(X) = \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X)\mathcal{C}_j, \quad \text{for all } X \in \mathcal{P}(n).$$

Consider $\mathcal{R} = \{(X, Y) \in \mathcal{P}(n) \times \mathcal{P}(n) : X \preceq Y\}$. Consequently, the fixed point of $\mathfrak{T}$ serves as a solution to the nonlinear matrix equation (4.1). It is pertinent to mention that $\mathcal{R}$ is $\mathfrak{T}$-closed and $\mathfrak{T}$ is well defined as well as $\mathcal{R}$-continuous. Form condition (H_1) we have $\sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\Upsilon_k(X)\mathcal{C}_j \succ 0$ for some $X \in \mathcal{P}(n)$, thus $(X, \mathfrak{T}(X)) \in \mathcal{R}$ and consequently $\mathcal{P}(n)(\mathfrak{T}, \mathcal{R})$ is nonempty.

Define $d : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}^+$ by

$$d(X, Y) = \|X - Y\|_{\mathrm{tr}}, \quad \text{for all } X, Y \in \mathcal{P}(n).$$

Then, $(\mathcal{P}(n), d, \mathcal{R})$ is an $\mathcal{R}$-complete relational metric space. Then,

$$\begin{aligned}
\big\|\mathfrak{T}(Y) - \mathfrak{T}(X)\big\|_{\mathrm{tr}} &= \mathrm{tr}\big(\mathfrak{T}(Y) - \mathfrak{T}(X)\big) \\
&= \mathrm{tr}\left(\sum_{j=1}^{u}\sum_{k=1}^{v}\mathcal{C}_j^*\big(\Upsilon_k(Y) - \Upsilon_k(X)\big)\mathcal{C}_j\right) \\
&= \sum_{j=1}^{u}\sum_{k=1}^{v}\mathrm{tr}\big(\mathcal{C}_j\mathcal{C}_j^*\big(\Upsilon_k(Y) - \Upsilon_k(X)\big)\big) \\
&= \mathrm{tr}\left(\left(\sum_{j=1}^{u}\mathcal{C}_j\mathcal{C}_j^*\right)\sum_{k=1}^{v}\big(\Upsilon_k(Y) - \Upsilon_k(X)\big)\right)
\end{aligned}$$

$$\begin{aligned}
&\le \left\| \sum_{j=1}^{u} \mathcal{C}_j \mathcal{C}_j^* \right\| \times \nu \times \max \left\| \left(\Upsilon_k(Y) - \Upsilon_k(X) \right) \right\|_{\mathrm{tr}} \\
&\le \frac{2}{3} \times \max \Big\{ \|X - Y\|_{\mathrm{tr}}, \|X - \mathfrak{T}X\|_{\mathrm{tr}}, \|Y - \mathfrak{T}Y\|_{\mathrm{tr}}, \\
&\quad \|X - \mathfrak{T}X\|_{\mathrm{tr}}^{\frac{1}{2}} \cdot \|Y - \mathfrak{T}Y\|_{\mathrm{tr}}^{\frac{1}{2}} \Big\} \\
&= \frac{2}{3} \Big(\Omega \big(\|X - Y\|_{\mathrm{tr}}, \|X - \mathfrak{T}X\|_{\mathrm{tr}}, \|Y - \mathfrak{T}Y\|_{\mathrm{tr}} \big) \Big). \qquad (4.3)
\end{aligned}$$

Now, we consider $\psi(t) = t$, $\phi(t) = \frac{2t}{3}$, $\alpha = 0$, and $\beta = \frac{1}{2}$, then equation (4.3) becomes

$$\psi \big(d(\mathfrak{T}X, \mathfrak{T}Y) \big) \le \phi \Big(\Omega \big(d(X, Y), d\big(X, \mathfrak{T}(X)\big), d\big(Y, \mathfrak{T}(Y)\big) \big) \Big).$$

Consequently, upon fulfilling all the hypotheses stated in Theorem 3.3, it can be deduced that there exists an element $X^* \in \mathcal{P}(n)$ for which $\mathfrak{T}(X^*) = X^*$ holds good. As a result, the matrix equation (4.1) is guaranteed to possess a solution within the set $\mathcal{P}(n)$. □

Example 4.4 Consider the nonlinear matrix equation (4.1) for $u = \nu = 2$, and $n = 3$, with $\Upsilon_1(X) = X^{\frac{1}{4}}$, $\Upsilon_2(X) = X^{\frac{1}{5}}$, i.e.,

$$X = \mathcal{A} + \mathcal{C}_1^* X^{\frac{1}{4}} \mathcal{C}_1 + \mathcal{C}_1^* X^{\frac{1}{5}} \mathcal{C}_1 + \mathcal{C}_2^* X^{\frac{1}{4}} \mathcal{C}_2 + \mathcal{C}_2^* X^{\frac{1}{5}} \mathcal{C}_2, \qquad (4.4)$$

where

$$\mathcal{A} = \begin{bmatrix} 0.177855454222667 & 0.001123654123643 & 0.144563214565439 \\ 0.001123532012243 & 0.177856213654500 & 0.133214521452362 \\ 0.144562121365390 & 0.133214526352116 & 0.266521364125960 \end{bmatrix}$$

$$\mathcal{C}_1 = \begin{bmatrix} 0.213588080307819 & 0.166601444550695 & 0.127622658649550 \\ 0.116601444550695 & 0.113891601170827 & 0.022954463850304 \\ 0.127622658649550 & 0.122954463850304 & 0.307677316314136 \end{bmatrix}$$

$$\mathcal{C}_2 = \begin{bmatrix} 1.835353913428885 & 0.533419165306540 & 0.639329778947828 \\ 0.533419165306540 & 0.334906218729761 & 0.379215073620121 \\ 0.639329778947828 & 0.379215073620121 & 1.705203352336594 \end{bmatrix}.$$

By taking $N = 7$, the conditions specified in Theorem 4.3 can be validated numerically by evaluating various specific values for the matrices involved. For example, they can be tested (and verified to be true) for

$$X = \begin{bmatrix} 0.601344857294582 & 0.012123214144452 & 0.112254124523620 \\ 0.123121452122110 & 0.488267073906088 & 0.213214212362145 \\ 0.332112514256214 & 0.112451236214521 & 0.424257724862306 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1.000171251644134 & 0.123565455662234 & 0.231452114522455 \\ 0.234512141422554 & 1.213180056807297 & 0.365455111122332 \\ 0.551221455112244 & 0.231452334558489 & 1.113265841608538 \end{bmatrix}.$$

To ascertain the convergence of $\{X_n\}$ defined in (4.2), we commence with three distinct initial values:

$$U_0 = \begin{bmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{15} & 0 \\ 0 & 0 & \frac{1}{15} \end{bmatrix}$$

$$V_0 = \begin{bmatrix} 1.380006158840432 & 0.729620753048356 & 0.597565069778261 \\ 0.729620753048356 & 0.547945757472219 & 0.551317326877231 \\ 0.597565069778261 & 0.551317326877231 & 1.3012791227936167 \end{bmatrix}$$

$$W_0 = \begin{bmatrix} 2.592408887372435 & 1.027321364808460 & 0.873755458971548 \\ 1.027321364808460 & 0.593069924137297 & 0.762603684965625 \\ 0.873755458971548 & 0.762603684965625 & 1.252077566327681 \end{bmatrix}.$$

After conducting 15 iterations, the subsequent approximation of the positive-definite solution for the system presented in (4.1) is as follows:

$$\hat{U} \approx U_{15} = \begin{bmatrix} 17.163329497461348 & 6.253639655399002 & 12.195915736525205 \\ 6.253639507101289 & 2.692272944078430 & 5.374711308897728 \\ 12.195914405376412 & 5.374711264964739 & 14.480006392095785 \end{bmatrix}$$

with error 1.24906×10^{-7},

$$\hat{V} \approx V_{15} = \begin{bmatrix} 17.163329508247507 & 6.253639659448254 & 12.195915744696777 \\ 6.253639511150539 & 2.692272945689973 & 5.374711312279821 \\ 12.195914413547982 & 5.374711268346832 & 14.480006400501384 \end{bmatrix}$$

with error 5.28502×10^{-8},

$$\hat{W} \approx W_{15} = \begin{bmatrix} 17.16332950926931 & 6.253639659834532 & 12.195915745492652 \\ 6.253639511536818 & 2.692272945842136 & 5.374711312591148 \\ 12.19591441434386 & 5.374711268658160 & 14.480006401183488 \end{bmatrix}$$

with error 4.64279×10^{-8}.

In Fig. 3, we present a graphical depiction illustrating the convergence phenomenon.

5 Conclusion

In our current study, we introduce a broader idea called a unified interpolative Reich–Rus–Ćirić-type contraction. This concept encompasses many existing findings, including those presented by [7–11, 17, 27]. We demonstrate several fixed-point results for such contractions within relational metric spaces.

It is important to note that in relational metric spaces, we often deal with weaker properties like $\mathcal{R}$-continuity (not necessarily implying continuity), $\mathcal{R}$-completeness (not necessarily implying completeness), and so on. In this context, we have more flexibility since the contraction condition is not required for every element but only for related ones. Importantly, these contraction conditions return to their usual forms when considering the universal relation.

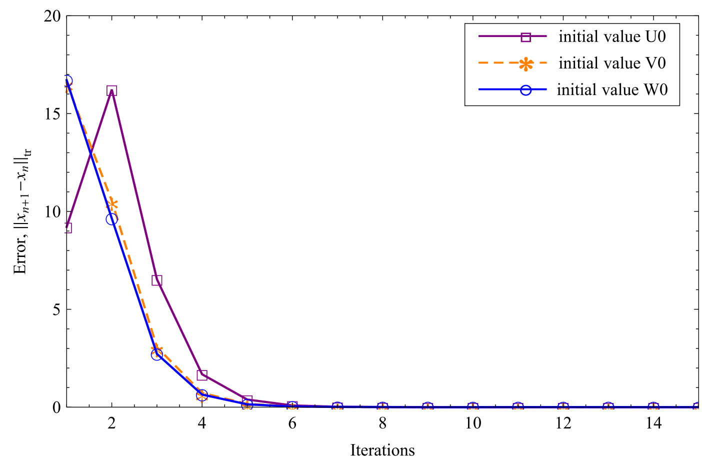

Figure 3 Convergence behavior

Acknowledgements
The authors would like D. Santina and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Author contributions
Conceptualization, K.N.V.V.V.P, V.M., Z.D.M, D.S. and N.M.; formal analysis, K.N. V.V. V.P, V.M., Z.D.M., D.S. and N.M.; writing—original draft preparation, K.N.V.V.V.P, V.M., Z.D.M, D.S. and N.M.; writing—review and editing, K.N.V.V.V.P, V.M., Z.D.M, D.S. and N.M.. All authors have read and agreed to the published version of the manuscript.

Funding
This research receives no external funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University), Bilaspur, C.G., 495009, India. [2]Faculty of Electrical Engineering, University of Banja Luka, Patre 5, Banja Luka, 78000, Bosnia and Herzegovina. [3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia.

Received: 26 April 2024 Accepted: 10 July 2024 Published online: 25 July 2024

References

1. Kirk, W.A.: Contraction mappings and extensions. In: Handbook of Metric Fixed Point Theory, pp. 1–34. Kluwer, Dordrecht (2001)
2. Shatanawi, W., Shatnawi, T.A.M.: New fixed point results in controlled metric type spaces based on new contractive conditions. AIMS Math. **8**(4), 9314–9330 (2023)
3. Rezazgui, A.Z., Tallafha, A.A., Shatanawi, W.: Common fixed point results via $A\nu-\alpha$-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space. AIMS Math. **8**(3), 7225–7241 (2023)
4. Joshi, M., Tomar, A., Abdeljawad, T.: On fixed points, their geometry and application to satellite web coupling problem in S-metric spaces. AIMS Math. **8**(2), 4407–4441 (2021). https://doi.org/10.3934/math.2023220
5. Kannan, R.: Some results on fixed points. Bull. Calcutta Math. Soc. **60**, 71–76 (1968)
6. Reich, S.: Fixed point of contractive functions. Boll. Unione Mat. Ital. (9) **5**, 26–42 (1972)
7. Ćirić, L.: Fixed point theorems for multi-valued contractions in complete metric spaces. J. Math. Anal. Appl. **348**(1), 499–507 (2008)
8. Reich, S.: Some remarks concerning contraction mappings. Can. Math. Bull. **14**(1), 121–124 (1971)
9. Rus, I.A.: Generalized Contractions and Applications. Cluj University Press, Cluj-Napoca (2001)
10. Karapınar, E.: Revisiting the Kannan type contractions via interpolation. Adv. Theory Nonlinear Anal. Appl. **2**(2), 85–87 (2018)
11. Karapınar, E., Agarwal, R., Aydi, H.: Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces. Mathematics **6**(11), 256 (2018)

12. Wangwe, L., Kumar, S.: Fixed point results for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space with an applications. J. Anal. **31**(1), 387–404 (2023)
13. Gautam, P., Kaur, C.: Fixed points of interpolative Matkowski type contraction and its application in solving non-linear matrix equations. Rend. Circ. Mat. Palermo (2) **72**(3), 2085–2102 (2023)
14. Taş, N.: Interpolative contractions and discontinuity at fixed point. Appl. Gen. Topol. **24**(1), 145–156 (2023)
15. Muhammad, R., Shagari, M.S., Azam, A.: On interpolative fuzzy contractions with applications. Filomat **37**(1), 207–219 (2023)
16. Gordji, M.E., Ramezani, M., De La Sen, M., Cho, Y.J.: On orthogonal sets and Banach fixed point theorem. Fixed Point Theory **18**(2), 569–578 (2017)
17. Nazam, M., Javed, K., Arshad, M.: The (ψ, ϕ)-orthogonal interpolative contractions and an application to fractional differential equations. Filomat **37**(4), 1167–1185 (2023)
18. Samet, B., Turinici, M.: Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications. Commun. Math. Anal. **13**(2), 82–97 (2012)
19. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435–1443 (2004)
20. Nieto, J.J., Rodríguez-López, R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**(3), 223–239 (2005)
21. Roldán, A., Karapinar, E.: Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ, φ)-contractivity conditions. Fixed Point Theory Appl. **2013**, 158 (2013)
22. Roldán-López-de-Hierro, A.F., Shahzad, N.: Some fixed/coincidence point theorems under (ψ, φ)-contractivity conditions without an underlying metric structure. Fixed Point Theory Appl. **2014**, 218 (2014)
23. Ben-El-Mechaiekh, H.: The ran-reurings fixed point theorem without partial order: a simple proof. J. Fixed Point Theory Appl. **16**, 373–383 (2014)
24. Berzig, M., Karapinar, E.: Fixed point results for $(\alpha\psi, \beta\varphi)$-contractive mappings for a generalized altering distance. Fixed Point Theory Appl. **2013**, 205 (2013)
25. Berzig, M.: Coincidence and common fixed point results on metric spaces endowed with an arbitrary binary relation and applications. J. Fixed Point Theory Appl. **12**(1–2), 221–238 (2012)
26. Ghods, S., Gordji, M.E., Ghods, M., Hadian, M.: Comment on "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces" [Lakshmikantham and Ciric, Nonlinear Anal. TMA 70 (2009) 4341-4349]. J. Comput. Anal. Appl. **14**, 958–966 (2012)
27. Alam, A., Imdad, M.: Relation-theoretic contraction principle. J. Fixed Point Theory Appl. **31**, 693–702 (2015)
28. Zada, M.B., Sarwar, M.: Common fixed point theorems for rational F_R-contractive pairs of mappings with applications. J. Inequal. Appl. **2019**, 11 (2019)
29. Prasad, K.N.V.V.V., Mishra, V., Mitrovic, Z.D., Aloqaily, A., Mlaiki, N.: Fixed point results for generalized almost contractions and application to a nonlinear matrix equation. AIMS Math. **9**(5), 12287–12304 (2024)
30. Shil, S., Nashine, H.K.: Unique positive definite solution of non-linear matrix equation on relational metric spaces. Fixed Point Theory **24**, 367–382 (2023)
31. Antal, S., Khantwal, D., Negi, S., Gairola, U.C.: Fixed points theorems for (ϕ, ψ, p)-weakly contractive mappings via w-distance in relational metric spaces with applications. Filomat **37**, 7319–7328 (2023)
32. Khantwal, D., Aneja, S., Prasad, G., Joshi, B.C., Gairola, U.C.: Multivalued relation-theoretic graph contraction principle with applications. Int. J. Nonlinear Anal. Appl. **13**(2), 2961–2971 (2022)
33. Din, F.U., Alshaikey, S., Ishtiaq, U., Din, M., Sessa, S.: Single and multi-valued ordered-theoretic Perov fixed-point results for θ-contraction with application to nonlinear system of matrix equations. Mathematics **12**(9), 1302 (2024)
34. Choudhury, B.S., Chakraborty, P.: Fixed point problem of a multi-valued Kannan–Geraghty type contraction via w-distance. J. Anal. **31**(1), 439–458 (2023)
35. Alam, A., Imdad, M.: Relation-theoretic metrical coincidence theorems. Filomat **17**(14), 4421–4439 (2017)
36. Alam, A., Imdad, M.: Nonlinear contractions in metric spaces under locally T-transitive binary relations. Fixed Point Theory **19**(1), 13–24 (2018)
37. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α–ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl. **75**(4), 2154–2165 (2012)
38. Ran, A.C., Reurings, M.C.: On the nonlinear matrix equation $X + A^*f(X)A = Q$: solutions and perturbation theory. Linear Algebra Appl. **346**(1–3), 15–26 (2002)

Publisher's Note