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HDG methods for the unilateral contact problem

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Abstract

This article presents the HDG approximation as a solution to the unilateral contact problem, leveraging the regularization method and an iterative procedure for resolution. In our study, u represents the potential (displacement of the elastic body) and $\boldsymbol{q}$ represents the flux (the force exerted on the body). Our analysis establishes that the utilization of polynomials of degree $k(k \geq 1)$ leads to achieving an optimal convergence rate of order $k+1$ in L^2-norm for both u and $\boldsymbol{q}$. Importantly, this optimal convergence is maintained irrespective of whether the domain is discretized through a structured or unstructured grid. The numerical results consistently align with the theoretical findings, underscoring the effectiveness and reliability of the proposed HDG approximation method for unilateral contact problems.

Keywords: HDG approximation; Unilateral contact problem; Regularization method; Convergence rate

1 Introduction

In this paper, we undertake an a priori error analysis of the hybridizable discontinuous Galerkin (HDG) method applied to the unilateral contact problem. To provide a specific application context, we consider a simplified model of a scalar two-dimensional unilateral contact problem with friction. This model adheres to the static Coulomb law and can be viewed as a simplified representation of the displacement field of an elastic body situated in a two-dimensional bounded domain with a smooth boundary. This elastic body is in unilateral frictional contact with a rigid foundation [1].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\operatorname{meas}(\Gamma_N) > 0$. Define $V = H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$, and consider an elliptic variational inequality of the second type:

$$\begin{aligned} &a(u, v-u) + j(v) - j(u) \geq (f, v-u), \quad \forall v \in V, \\ &a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx, \\ &j(v) = \int_{\Gamma_N} g|v|\,ds, \end{aligned} \tag{1}$$

where $g > 0$ is a constant representing the friction coefficient on the boundary. If $j(v) \equiv 0$, then problem (1) reduces to the classical second-order elliptic problem. The corresponding optimal functional problem is defined as

$$u = \arg\min_{v\in V} E(v), \quad E(v) = \frac{1}{2}a(v,v) + j(v) - (f,v). \tag{2}$$

Problem (1) can be also be described by the following equation [2]:

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ u = 0, & x \in \Gamma_D, \\ \left|\dfrac{\partial u}{\partial \boldsymbol{n}}\right| \le g & x \in \Gamma_N. \end{cases} \tag{3}$$

To overcome the challenges posed by the nondifferentiable functional term $j(\cdot)$, various numerical methods, such as finite and boundary element methods, have been employed to solve problem (1) [2–5]. In the past 30 years, the discontinuous Galerkin (DG) method has been widely used to solve various initial boundary value problems [6–10]. The discontinuous Galerkin (DG) method, extensively used in solving initial boundary value problems, has gained popularity in the last three decades. Wang et al. utilized DG methods to address problem (1) [11].

To address the challenges associated with the nondifferentiable functional term $j(\cdot)$, a regularization procedure has been proposed. Zhang [12] introduced a regularization technique, obtaining a nonlinear second-order elliptic problem with mixed boundary values. This problem is solved using the DG method and an iterative linearization procedure. If the solution u is sufficiently smooth, i.e., $u \in H^{1+s}(\Omega) \cap H^{1+\alpha}(\Gamma_N)$, $1/2 \le s \le k$, $s - 1/2 \le \alpha \le s$, [12] established the approximate convergence order with $h^s\|u\|_{H^{1+s}(\Omega)} + \frac{1}{\gamma}h^{1/2+\alpha}\|u\|_{H^{1+\alpha}(\Gamma_N)}$ in the energy norm. If $u \in H^{1+s}(\Omega)$, $s \ge 1/2$, then the approximate convergence order is $\left(1 + \frac{1}{\gamma}\right) h^{1+s}, 1/2 \le s \le k$ in L^2 norm.

In this work, we introduce the application of the hybridizable discontinuous Galerkin (HDG) method to a problem formulated with a gradient-potential approach, which is particularly suited for the unilateral contact problem under consideration. The approximate potential and flux, using polynomial spaces of degree k, were proven to converge with the optimal order of $k + 1$ in L^2 norm for any $k \ge 0$. After applying an element-by-element postprocessing scheme, the new potential approximation converges with order $k + 2$ for $k \ge 1$ and with order 1 for $k = 0$ [13–21]. This paper aims to present the approximation of problem (1) using the HDG method and the regularization method, providing an error analysis with supporting numerical experiments.

The organization of this paper is as follows: In Sect. 2, we collect some results on the regularization procedure. Section 3 describes the HDG scheme and proves the existence and uniqueness of the numerical solutions. In Sect. 4, we provide the error estimate. Section 5 offers numerical results to assess the convergence rates and accuracy. Section 6 concludes the paper and discusses potential future projects.

2 Regularization procedure

The key to regularizing problem (1) is to replace $j(\cdot)$ with a differentiable functional $j_\gamma(\cdot)$ that satisfies $\lim_{\gamma\to 0} j_\gamma(v) = j(v)$. Using the regularization method introduced by Zhang [12], we define the regularization functional as follows:

$$j_\gamma(v) = \int_{\Gamma_N} \psi(v)ds,$$

where the function $\psi(\cdot)$ is defined by

$$\psi(v) = \begin{cases} gv - \dfrac{\gamma}{2}g^2, & v \geq \gamma g, \\ \dfrac{1}{2\gamma}v^2, & |v| \leq \gamma g, \\ -gv - \dfrac{\gamma}{2}g^2, & v \leq -\gamma g. \end{cases}$$

Here, $\gamma > 0$ is a small parameter. It can be verified that the regularization functional satisfies the relationship

$$\left|j_\gamma(v) - j(v)\right| \leq \frac{1}{2}\gamma g^2 \operatorname{meas}(\Gamma_N).$$

Next, we consider the variational inequality problem

$$a(u, v-u) + j_\gamma(v) - j_\gamma(u) \geq \left(f, v-u\right), \quad \forall v \in V. \tag{4}$$

The properties of problem (4) are presented in the following theorem.

Theorem 1 [12] *Let u_γ be the solution of problem* (4) *and u be the solution of problem* (1), *then*

(1) *u_γ and u have the following relationship*:

$$\|u - u_\gamma\|_{H^1(\Omega)} \leq \sqrt{\gamma}g\left(\operatorname{meas}(\Gamma_N)\right)^{1/2}. \tag{5}$$

(2) *u_γ satisfies the following variational equation*:

$$a(u,v) + \int_{\Gamma_N} \varphi(u)v ds = (f, v), \quad \forall v \in H_D^1(\Omega), \tag{6}$$

where $\varphi(t) \in H^1(-\infty, +\infty)$ is defined as

$$\varphi(v) = \psi'(v) = \begin{cases} g, & v \geq \gamma g, \\ \dfrac{v}{\gamma}, & |v| \leq \gamma g, \\ -g, & v \leq -\gamma g, \end{cases}$$

and the function $\varphi(\cdot)$ has the following properties:

$$|\varphi(u) - \varphi(v)| \leq \frac{1}{\gamma}|u - v|, \quad \forall u, v \in V, \tag{7}$$

$$(\varphi(u) - \varphi(v))(u - v) \geq 0, \quad \forall u, v \in V. \tag{8}$$

(3) *If* $u_\gamma \in H_D^1(\Omega)\cap H^{1+s}(\Omega)(s \geq 1/2)$, *then* u_γ *is the solution of the following second-order elliptic problems in the sense of variation*:

$$\begin{cases} -\Delta u_\gamma + u_\gamma = f, & x \in \Omega, \\ u_\gamma = 0, & x \in \Gamma_D, \\ \dfrac{\partial u_\gamma}{\partial \boldsymbol{n}} + \varphi(u_\gamma) = 0, & x \in \Gamma_N. \end{cases} \tag{9}$$

The analysis above suggests that, to solve problem (1), we can address problem (9).

Remark 1 Theorem 1 indicates that the solution of problem (9) converges to the solution of problem (1), as shown by results (5). Therefore, we focus on the approximation solution of (9) to obtain an approximation of (4). In the subsequent sections, our emphasis is on the HDG approximation of problem (9).

3 HDG scheme

Consider the following nonlinear second-order elliptic problem:

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ u = 0, & x \in \Gamma_D, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \varphi(u) = 0, & x \in \Gamma_N. \end{cases} \tag{10}$$

By introducing the auxiliary variable $\boldsymbol{q} = -\nabla u$, problem (10) can be rewritten as

$$\begin{cases} \boldsymbol{q} + \nabla u = 0, & x \in \Omega, \\ \nabla \cdot \boldsymbol{q} + u = f, & x \in \Omega, \\ u = 0, & x \in \Gamma_D, \\ -\boldsymbol{q} \cdot \boldsymbol{n} + \varphi(u) = 0, & x \in \Gamma_N. \end{cases} \tag{11}$$

In this section, we extend the hybridizable discontinuous Galerkin (HDG) method to the unilateral contact problem, closely following the approach and notation established in [13].

To convey our approach, we define a mesh of the domain Ω consisting of triangles, where each triangle is a shape-regular element. We denote the set of all triangle edges as Γ_h and distinguish between the interior edges Γ_h^i and the boundary edges Γ_h^∂. For clarity, an edge e is considered part of the interior if it is shared by two triangles in the mesh, whereas it is a boundary edge if it lies on the perimeter of the domain. We denote by $\mathcal{T}_h$ the collection of all triangular elements.

The numerical solution $(\boldsymbol{q}_h, u_h, \hat{\boldsymbol{u}})$ is approximated within the finite-dimensional spaces $V_h \times W_h \times M_h$, where these spaces consist of polynomial functions that are defined on each triangle for V_h and W_h, and on the edges of the mesh for M_h. It is important to note that while these functions are globally in $L^2(\Omega)$, they are locally conforming to H^1 and $H(\mathrm{div})$ spaces, respectively. Specifically, for each triangular element K, functions in V_h and W_h belong to $H^1(K)$ in addition to $L^2(K)$, and functions in M_h are at least in $H(\mathrm{div}, e)$

for each edge e. This local conformity is achieved through the polynomial spaces $\mathbb{P}_k(K)$ and $\mathbb{P}_k(e)$ defined on each element and edge, allowing for an appropriate approximation of the functions and their derivatives locally. Following the approach in [13], we ensure that these discrete spaces capture the local regularity characteristics of the problem.

$$\boldsymbol{V}_h = \left\{\boldsymbol{v} \in [L^2(\Omega)]^2 : \boldsymbol{v}|_K \in [\mathbb{P}_k(K)]^2, \forall K \in \mathcal{T}_h\right\},$$
$$W_h = \left\{w \in L^2(\Omega) : w|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\right\},$$
$$M_h = \left\{\mu \in L^2(\Gamma_h) : \mu|_K \in \mathbb{P}_k(e), \forall e \in \Gamma_h\right\},$$

where $\mathbb{P}_k(K)$ is the space of polynomials of total degree at most k.

In this paper, we use the following numerical flux, as in [22]:

$$\hat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \tau(u_h - \hat{u}_h)\boldsymbol{n}, \tag{12}$$

and if $e \in \Gamma_D$, then the value of $\hat{u}_h$ is set to be 0. Here, τ is a constant defined on each edge of the mesh to ensure the stability of the numerical scheme.

In the HDG method, the numerical flux of $\hat{\boldsymbol{q}}_h$ on the boundaries satisfies the property of conservation, i.e.,

$$\left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu\right\rangle_{\partial\mathcal{T}_h} = \left\langle \varphi\left(\hat{u}_h\right), \mu\right\rangle_{\Gamma_N}, \ \ \forall \mu \in M_h, \tag{13}$$

and the expression on each element K is

$$\left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu\right\rangle_{\partial K} = \left\langle \varphi\left(\hat{u}_h\right), \mu\right\rangle_{\partial K\cap\Gamma_N}.$$

In light of the aforementioned notations and definitions, the HDG scheme corresponding to (11) is formulated as follows: Find a solution $\left(\boldsymbol{q}_h, u_h, \hat{u}_h\right) \in \boldsymbol{V}_h \times W_h \times M_h$ that satisfies the following system of equations:

$$(\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \left\langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{14}$$
$$(u_h, w)_{\mathcal{T}_h} - (\boldsymbol{q}_h, \nabla w)_{\mathcal{T}_h} + \left\langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \tag{15}$$
$$\left\langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\right\rangle_{\partial\mathcal{T}_h} = \left\langle \varphi(\hat{u}_h), \mu\right\rangle_{\Gamma_N}, \tag{16}$$
$$\left\langle \hat{u}_h, \mu\right\rangle_{\Gamma_D} = 0, \tag{17}$$

where $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$.

For the sake of simplicity, we assume that the value of τ on each edge $e \in \Gamma_h$ remains constant. This assumption streamlines the formulation while maintaining computational manageability.

Theorem 2 *There is a unique solution of HDG formulation* (14)–(17).

Proof To establish the existence of a solution, we employ the iterative method. The goal is to derive the existence by introducing the following iterative system, where $\hat{u}_h^n$ represents a known prior value of $\hat{u}$ on Γ_N:

$$\begin{cases} (\boldsymbol{q}_h^{n+1}, \boldsymbol{v})_{\mathcal{T}_h} - (u_h^{n+1}, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle \hat{u}_h^{n+1}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \\ (u_h^{n+1}, w)_{\mathcal{T}_h} - (\boldsymbol{q}_h^{n+1}, \nabla w)_{\mathcal{T}_h} + \left\langle \hat{\boldsymbol{q}}_h^{n+1} \cdot \boldsymbol{n}, w \right\rangle_{\partial \mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \\ \left\langle \hat{\boldsymbol{q}}_h^{n+1} \cdot \boldsymbol{n}, \mu \right\rangle_{\partial \mathcal{T}_h} = \langle \varphi(\hat{u}_h^n), \mu \rangle_{\Gamma_N}, \\ \langle \hat{u}_h^{n+1}, \mu \rangle_{\Gamma_D} = 0, \end{cases} \tag{18}$$

for all $(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Notably, (18) is the HDG formulation corresponding to the problem

$$\begin{cases} \boldsymbol{q} + \nabla u = 0, & x \in \Omega, \\ \nabla \cdot \boldsymbol{q} + u = f, & x \in \Omega, \\ u = 0, & x \in \Gamma_D, \\ \boldsymbol{q} \cdot \boldsymbol{n} = \varphi(\hat{u}_h^n), & x \in \Gamma_N. \end{cases}$$

Referring to [13] Theorem 3.5, under the assumption that the solution is sufficiently smooth and the mesh is sufficiently refined, it is established that problem (18) possesses a unique solution $(\boldsymbol{q}_h^{n+1}, u_h^{n+1}, \hat{u}_h^{n+1})$ with the following estimate:

$$\begin{aligned} \|u_h^{n+1}\|_{L^2(\mathcal{T}_h)} + \|\boldsymbol{q}_h^{n+1}\|_{L^2(\mathcal{T}_h)} + \|\hat{u}_h^{n+1}\|_{L^2(\partial \mathcal{T}_h)} &\le C(\|f\|_{L^2(\Omega)} + \|\varphi(\hat{u}_h^n)\|_{L^2(\Gamma_N)}) \\ &\le C(\|f\|_{L^2(\Omega)} + g \operatorname{meas}(\Gamma_N)), \end{aligned}$$

which implies that $(\boldsymbol{q}_h^{n+1}, u_h^{n+1}, \hat{u}_h^{n+1})$ is uniformly bounded in $\boldsymbol{V}_h \times W_h \times M_h$. Moreover, there exists a subsequence of $\{(\boldsymbol{q}_h^n, u_h^n, \hat{u}_h^n)\}$ such that $(\boldsymbol{q}_h^n, u_h^n, \hat{u}_h^n)$ converges to $(\boldsymbol{q}_h, u_h, \hat{u}_h)$. Leveraging the uniformly continuous property of the function $\varphi(\cdot)$, it is concluded that $(\boldsymbol{q}_h, u_h, \hat{u}_h)$ is a solution to (14)–(17). Next, we prove the uniqueness.

Moving on to the proof of uniqueness. Taking $f = 0$, let $\boldsymbol{v} = \boldsymbol{q}_h$ in equation (14), $w = u_h$ in equation (15), and $\mu = \hat{u}_h$ in equations (16)–(17), we obtain

$$\left(\boldsymbol{q}_h, \boldsymbol{q}_h\right)_{T_h} - \left(u_h, \nabla \cdot \boldsymbol{q}_h\right)_{T_h} + \left\langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \right\rangle_{\partial T_h} = 0, \tag{19}$$

$$\left(u_h, u_h\right)_{T_h} - \left(\boldsymbol{q}_h, \nabla u_h\right)_{T_h} + \left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, u_h \right\rangle_{\partial T_h} = 0, \tag{20}$$

$$\left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \hat{u}_h \right\rangle_{\partial T_h} = \left\langle \varphi\left(\hat{u}_h\right), \hat{u}_h \right\rangle_{\Gamma_N}, \tag{21}$$

$$\left\langle \hat{u}_h, \hat{u}_h \right\rangle_{\Gamma_D} = 0. \tag{22}$$

Utilizing the property of φ given in equation (8), we know that

$$0 = \left\langle \varphi\left(\hat{u}_h\right), \hat{u}_h \right\rangle_{\Gamma_N} \ge 0,$$

which implies $\hat{u}_h|_{\Gamma_N} = 0$, and therefore $\hat{u}_h|_{\partial\Omega} = 0$.

Adding equation (19) to equation (20), we get

$$0 = \left\|\boldsymbol{q}_h\right\|_{T_h}^2 + \|u_h\|_{T_h}^2 - \left(u_h, \nabla \cdot \boldsymbol{q}_h\right)_{T_h} + \left\langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n} \right\rangle_{\partial T_h} - \left(\boldsymbol{q}_h, \nabla u_h\right)_{T_h} + \left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, u_h \right\rangle_{\partial T_h}.$$

By integration by parts, we can express this as

$$
\begin{aligned}
0 &= \left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 - \left\langle u_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\right\rangle_{\partial T_h} + \left\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, u_h\right\rangle_{\partial T_h} \\
&= \left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 + \left\langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \left(\hat{\boldsymbol{q}}_h - \boldsymbol{q}_h\right) \cdot \boldsymbol{n}, u_h\right\rangle_{\partial T_h}.
\end{aligned}
$$

Given the uniformity of $\hat{u}_h$, $\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}$ on the element boundaries, we have

$$
\left\langle \hat{u}_h, \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = \langle \varphi(\hat{u}_h), \hat{u}_h\rangle_{\Gamma_N},
$$

which leads to

$$
\begin{aligned}
0 &= \left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 + \left\langle \hat{u}_h, \boldsymbol{q}_h \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \left(\hat{\boldsymbol{q}}_h - \boldsymbol{q}_h\right) \cdot \boldsymbol{n}, u_h\right\rangle_{\partial\mathcal{T}_h} - \left\langle \hat{u}_h, \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\right\rangle_{\partial T_h} \\
&\quad + \langle \varphi(\hat{u}_h), \hat{u}_h\rangle_{\Gamma_N} \\
&= \left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 + \left\langle \hat{u}_h, \left(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h\right) \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \left(\hat{\boldsymbol{q}}_h - \boldsymbol{q}_h\right) \cdot \boldsymbol{n}, u_h\right\rangle_{\partial T_h} + \langle \varphi(\hat{u}_h), \hat{u}_h\rangle_{\Gamma_N} \\
&= \left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 + \left\langle \left(\hat{\boldsymbol{q}}_h - \boldsymbol{q}_h\right) \cdot \boldsymbol{n}, u_h - \hat{u}_h\right\rangle_{\partial T_h} + \langle \varphi(\hat{u}_h), \hat{u}_h\rangle_{\Gamma_N}.
\end{aligned}
$$

Substituting the definition of $\hat{\boldsymbol{q}}_h$, from equation (12) we get

$$
\left\|\boldsymbol{q}_h\right\|_{\mathcal{T}_h}^2 + \|u_h\|_{\mathcal{T}_h}^2 + \tau \left\|u_h - \hat{u}_h\right\|_{\partial\mathcal{T}_h}^2 + \langle \varphi(\hat{u}_h), \hat{u}_h\rangle_{\Gamma_N} = 0,
$$

which implies

$$
\boldsymbol{q}_h = 0, u_h = 0, \hat{u}_h|_{\partial\mathcal{T}_h} = u_h|_{\partial\mathcal{T}_h} = 0. \qquad \square
$$

4 Error estimate

Before presenting the estimates for $\|u - u_h\|_{L^2(\Omega)}$ and $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}$, we begin by revisiting the HDG projection (Π_V, Π_W) [22] defined as follows:

$$
(\Pi_V \boldsymbol{q}, \boldsymbol{v})_K = (\boldsymbol{q}, \boldsymbol{v})_K, \quad \forall \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^2, \tag{23}
$$

$$
(\Pi_W u, w)_K = (u, w)_K, \quad \forall w \in \mathbb{P}_{k-1}(K), \tag{24}
$$

$$
\langle \Pi_V \boldsymbol{q} \cdot \boldsymbol{n} + \tau \Pi_W u, \mu\rangle_e = \langle \boldsymbol{q} \cdot \boldsymbol{n} + \tau u, \mu\rangle_e, \quad \forall \mu \in \mathbb{P}_k(e). \tag{25}
$$

For $k \geq 1$, $\tau|_{\partial K} \geq$, and $\tau_K^{\max} > 0$, projection is defined by (23)–(25). Additionally, there exists a constant C independent of K and τ such that the following estimates hold:

$$
\|\Pi_V \boldsymbol{q} - \boldsymbol{q}\|_K \leq C h_K^{\ell_q+1} |\boldsymbol{q}|_{H^{\ell_q+1}(K)} + C h_K^{\ell_u+1} \tau_K^* |u|_{H^{\ell_u+1}(K)}, \tag{26}
$$

$$
\|\Pi_W u - u\|_K \leq C h_K^{\ell_u+1} |u|_{H^{\ell_u+1}(K)} + C \frac{h_K^{\ell_q+1}}{\tau_K^{\max}} |\nabla \cdot \boldsymbol{q}|_{H^{\ell_q}(K)}. \tag{27}
$$

Here, $0 \leq \ell_q$, $\ell_u \leq k$, $\tau_K^* = \max \tau|_{\partial K \setminus e^*}$, $\tau|_{e^*} = \tau_K^{\max}$. Also, $|\cdot|_{H^k(K)}$ denotes the semi-norm, which is defined as $|v|_{H^k(K)} = \left(\sum_{|\alpha|=k} \|D^\alpha v\|_{L^2(K)}^2\right)^{1/2}$.

Proposition 3 [22] *Let $w \in W_h$, $(\Phi, \Psi) \in \boldsymbol{V} \times W$, then we have*

$$(w, \nabla \cdot \Phi)_K = (w, \nabla \cdot \Pi_V \Phi)_K + \langle w, \tau(\Pi_W \Psi - \Psi)\rangle_{\partial K} \tag{28}$$

or

$$(w, \nabla \cdot \Phi)_K = -(\nabla w, \Pi_V \Phi)_K + \langle w, \Pi_V \Phi \cdot \boldsymbol{n}\rangle_{\partial K} + \langle w, \tau(\Pi_W \Psi - \Psi)\rangle_{\partial K}. \tag{29}$$

Given that τ is a constant on each edge, the L^2-projection $P_M : W \to M_h$ possesses the following property:

$$\langle \tau(P_M u - u), \mu\rangle_e = 0, \ \ \forall \mu \in M_h. \tag{30}$$

This property highlights that the L^2-projection of u onto M_h with respect to τ is orthogonal to u in the inner product defined on each element.

In the subsequent analysis, the derivation of error estimates relies on the error equations and the approximation characteristics of the HDG projection. To facilitate this discussion, let us first introduce the error equations.

Proposition 4 *Let $\varepsilon_h^u = \Pi_W u - u_h$, $\varepsilon_h^q = \Pi_V \boldsymbol{q} - \boldsymbol{q}_h$, $\varepsilon_h^{\hat{u}} = P_M u - \hat{u}_h$. The error equations are expressed as follows*:

$$\left(\varepsilon_h^q, \boldsymbol{v}\right)_{\mathcal{T}_h} - \left(\varepsilon_h^u, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^{\hat{u}}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_h} = (\Pi_V \boldsymbol{q} - \boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{31}$$

$$\begin{aligned}\left(\varepsilon_h^u, w\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q, \nabla w\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^q \cdot \boldsymbol{n}, w\right\rangle_{\partial \mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), w\right\rangle_{\partial \mathcal{T}_h} \\ = (\Pi_W u - u, w)_{\mathcal{T}_h},\end{aligned} \tag{32}$$

$$\left\langle \varepsilon_h^q \cdot \boldsymbol{n} + \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \mu\right\rangle_{\partial \mathcal{T}_h} = \left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \mu\right\rangle_{\Gamma_N} \tag{33}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}_h \times W_h$.

These equations characterize the errors in the approximations of u, $\boldsymbol{q}$, and $\hat{u}$ by their respective HDG projections.

Proof The exact solution $\boldsymbol{q}$ and u satisfy the following coupled equations:

$$(\boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h} - (u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle u, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h} = 0, \tag{34}$$

$$(u, w)_{\mathcal{T}_h} - (\boldsymbol{q}, \nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{q} \cdot \boldsymbol{n}, w\rangle_{\partial \mathcal{T}_h} = \left(f, w\right)_{\mathcal{T}_h}. \tag{35}$$

Analyzing equation (34), we obtain

$$\begin{aligned}&(\boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h} - (u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle u, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h} \\ &\quad = (\Pi_V \boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h} - (\Pi_V \boldsymbol{q} - \boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_h} - (\Pi_W u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle P_M u, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h},\end{aligned}$$

utilizing the properties of the HDG projection. Similarly, for equation (35), we have

$$(u,w)_{\mathcal{T}_h} - (\boldsymbol{q},\nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{q}\cdot\boldsymbol{n}, w\rangle_{\partial\mathcal{T}_h} = (\Pi_W u, w)_{\mathcal{T}_h} - (\Pi_W u - u, w)_{\mathcal{T}_h} - (\Pi_V \boldsymbol{q}, \nabla w)_{\mathcal{T}_h} + \langle \Pi_V\boldsymbol{q}\cdot\boldsymbol{n} + \tau\Pi_W u, w\rangle_{\partial\mathcal{T}_h} - \langle \tau u, w\rangle_{\partial\mathcal{T}_h}.$$

By collecting the above calculations, we deduce

$$(\Pi_V\boldsymbol{q},\boldsymbol{v})_{\mathcal{T}_h} - (\Pi_W u, \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle P_M u, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = (\Pi_V\boldsymbol{q}-\boldsymbol{q},\boldsymbol{v})_{\mathcal{T}_h}, \tag{36}$$

$$\begin{aligned}&(\Pi_W u, w)_{\mathcal{T}_h} - (\Pi_V\boldsymbol{q},\nabla w)_{\mathcal{T}_h} + \langle \Pi_V\boldsymbol{q}\cdot\boldsymbol{n} + \tau\Pi_W u, w\rangle_{\partial\mathcal{T}_h} - \langle \tau P_M u, w\rangle_{\partial\mathcal{T}_h}\\ &\quad = \left(f,w\right)_{\mathcal{T}_h} + (\Pi_W u - u, w)_{\mathcal{T}_h}.\end{aligned}\tag{37}$$

By subtracting (14) from (36), we obtain (31). By subtracting (15) from (37), we have

$$\begin{aligned}(\Pi_W u - u, w)_{\mathcal{T}_h} &= \left(\varepsilon_h^u, w\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q,\nabla w\right)_{\mathcal{T}_h} + \langle \Pi_V\boldsymbol{q}\cdot\boldsymbol{n} + \tau\Pi_W u, w\rangle_{\partial\mathcal{T}_h}\\ &\quad - \langle \tau P_M u, w\rangle_{\partial\mathcal{T}_h} - \left\langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h}\\ &= \left(\varepsilon_h^u, w\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q,\nabla w\right)_{\mathcal{T}_h} + \langle \Pi_V\boldsymbol{q}\cdot\boldsymbol{n} + \tau\Pi_W u - \tau P_M u, w\rangle_{\partial\mathcal{T}_h}\\ &\quad - \left\langle \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau\left(u_h - \hat{u}_h\right), w\right\rangle_{\partial\mathcal{T}_h}\\ &= \left(\varepsilon_h^u, w\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q,\nabla w\right)_{\mathcal{T}_h} + \left\langle\varepsilon_h^q\cdot\boldsymbol{n}, w\right\rangle_{\partial\mathcal{T}_h} + \left\langle\tau\left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), w\right\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Finally, we prove equation (33). Using definitions (23)–(25) of (Π_V, Π_W) and the property (30) of P_M, we get

$$\begin{aligned}\left\langle\varepsilon_h^q\cdot\boldsymbol{n} + \tau\left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right),\mu\right\rangle_{\partial\mathcal{T}_h} &= \left\langle\left(\Pi_V\boldsymbol{q} - \boldsymbol{q}_h\right)\cdot\boldsymbol{n} + \tau\left(\Pi_W u - u_h - P_M u + \hat{u}_h\right),\mu\right\rangle_{\partial\mathcal{T}_h}\\ &= \left\langle\left(\boldsymbol{q} - \boldsymbol{q}_h\right)\cdot\boldsymbol{n} + \tau\left(u - u_h - P_M u + \hat{u}_h\right),\mu\right\rangle_{\partial\mathcal{T}_h}\\ &= \left\langle\left(\boldsymbol{q} - \boldsymbol{q}_h\right)\cdot\boldsymbol{n} + \tau\left(u - u_h - u + \hat{u}_h\right),\mu\right\rangle_{\partial\mathcal{T}_h},\end{aligned}$$

so we have

$$\begin{aligned}\left\langle\varepsilon_h^q\cdot\boldsymbol{n} + \tau\left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right),\mu\right\rangle_{\partial\mathcal{T}_h} &= \langle\boldsymbol{q}\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h} - \left\langle\boldsymbol{q}_h\cdot\boldsymbol{n} + \tau\left(u_h - \hat{u}_h\right),\mu\right\rangle_{\partial\mathcal{T}_h}\\ &= \langle\boldsymbol{q}\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h} - \left\langle\hat{\boldsymbol{q}}_h\cdot\boldsymbol{n},\mu\right\rangle_{\partial\mathcal{T}_h}\\ &= \left\langle\varphi\left(u\right) - \varphi\left(\hat{u}_h\right),\mu\right\rangle_{\Gamma_N}.\end{aligned}$$

□

Proposition 5 *Let $(\boldsymbol{q}, u)$ and $(\boldsymbol{q}_h, u_h, \hat{u}_h)$ be the solutions of (11) and (14)–(17) respectively, then we have*

$$\begin{aligned}&\left(\varepsilon_h^q,\varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\varepsilon_h^u,\varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle\tau\left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right),\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right\rangle_{\partial\mathcal{T}_h}\\ &\quad = \left(\Pi_V\boldsymbol{q} - \boldsymbol{q},\varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\Pi_W u - u,\varepsilon_h^u\right)_{\mathcal{T}_h} - \left\langle\varphi\left(u\right) - \varphi\left(\hat{u}_h\right),\varepsilon_h^{\hat{u}}\right\rangle_{\Gamma_N}.\end{aligned}\tag{38}$$

Proof By taking $\boldsymbol{v} = \varepsilon_h^q$ in (31), $w = \varepsilon_h^u$ in (32), and $\mu = \varepsilon_h^{\hat{u}}$ in (33), we obtain the following expressions:

$$\left(\varepsilon_h^q, \varepsilon_h^q\right)_{\mathcal{T}_h} - \left(\varepsilon_h^u, \nabla \cdot \varepsilon_h^q\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^{\hat{u}}, \varepsilon_h^q \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} = \left(\Pi_V \boldsymbol{q} - \boldsymbol{q}, \varepsilon_h^q\right)_{\mathcal{T}_h},$$

$$\left(\varepsilon_h^u, \varepsilon_h^u\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q, \nabla \varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^q \cdot \boldsymbol{n}, \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} = \left(\Pi_W u - u, \varepsilon_h^u\right)_{\mathcal{T}_h},$$

$$\left\langle \varepsilon_h^q \cdot \boldsymbol{n} + \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^{\hat{u}} \right\rangle_{\partial \mathcal{T}_h} = \left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \varepsilon_h^{\hat{u}} \right\rangle_{\Gamma_N}.$$

By summing the first two equations and subtracting the third equation, we obtain

$$\begin{aligned}
&\left(\Pi_V \boldsymbol{q} - \boldsymbol{q}, \varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\Pi_V u - u, \varepsilon_h^u\right)_{\mathcal{T}_h} - \left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \varepsilon_h^{\hat{u}} \right\rangle_{\Gamma_N} \\
&\quad = \left(\varepsilon_h^q, \varepsilon_h^q\right)_{\mathcal{T}_h} - \left(\varepsilon_h^u, \nabla \cdot \varepsilon_h^q\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^{\hat{u}}, \varepsilon_h^q \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} \\
&\qquad + \left(\varepsilon_h^u, \varepsilon_h^u\right)_{\mathcal{T}_h} - \left(\varepsilon_h^q, \nabla \varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^q \cdot \boldsymbol{n}, \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} \\
&\qquad - \left\langle \varepsilon_h^q \cdot \boldsymbol{n} + \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^{\hat{u}} \right\rangle_{\partial \mathcal{T}_h}.
\end{aligned}$$

Further simplifying through integration by parts, we have

$$\begin{aligned}
&\left(\Pi_V \boldsymbol{q} - \boldsymbol{q}, \varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\Pi_V u - u, \varepsilon_h^u\right)_{\mathcal{T}_h} - \left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \varepsilon_h^{\hat{u}} \right\rangle_{\Gamma_N} \\
&\quad = \left(\varepsilon_h^q, \varepsilon_h^q\right)_{\mathcal{T}_h} - \left\langle \varepsilon_h^u, \varepsilon_h^q \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} + \left\langle \varepsilon_h^{\hat{u}}, \varepsilon_h^q \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} \\
&\qquad + \left(\varepsilon_h^u, \varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^q \cdot \boldsymbol{n}, \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} \\
&\qquad - \left\langle \varepsilon_h^q \cdot \boldsymbol{n} + \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^{\hat{u}} \right\rangle_{\partial \mathcal{T}_h}.
\end{aligned}$$

Rearranging and combining the five terms in the boundary integration, we get

$$\begin{aligned}
&\left(\Pi_V \boldsymbol{q} - \boldsymbol{q}, \varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\Pi_V u - u, \varepsilon_h^u\right)_{\mathcal{T}_h} - \left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \varepsilon_h^{\hat{u}} \right\rangle_{\Gamma_N} \\
&\quad = \left(\varepsilon_h^q, \varepsilon_h^q\right)_{\mathcal{T}_h} + \left\langle \varepsilon_h^{\hat{u}}, \varepsilon_h^q \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} + \left(\varepsilon_h^u, \varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^u \right\rangle_{\partial \mathcal{T}_h} \\
&\qquad - \left\langle \varepsilon_h^q \cdot \boldsymbol{n} + \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^{\hat{u}} \right\rangle_{\partial \mathcal{T}_h} \\
&\quad = \left(\varepsilon_h^q, \varepsilon_h^q\right)_{\mathcal{T}_h} + \left(\varepsilon_h^u, \varepsilon_h^u\right)_{\mathcal{T}_h} + \left\langle \tau \left(\varepsilon_h^u - \varepsilon_h^{\hat{u}}\right), \varepsilon_h^u - \varepsilon_h^{\hat{u}} \right\rangle_{\partial \mathcal{T}_h}
\end{aligned}$$

by using (33). This concludes the proof. □

Concerning the term $\left\langle \varphi(u) - \varphi\left(\hat{u}_h\right), \varepsilon_h^{\hat{u}} \right\rangle_{\Gamma_N}$, we can express it as follows:

$$\begin{aligned}
&\langle \varphi(u) - \varphi(\hat{u}), P_M u - \hat{u} \rangle_{\Gamma_N} \\
&\quad = \langle \varphi(u) - \varphi(\hat{u}), P_M u - u \rangle_{\Gamma_N} + \langle \varphi(u) - \varphi(\hat{u}), u - \hat{u} \rangle_{\Gamma_N} \\
&\quad \geq \langle \varphi(u) - \varphi(\hat{u}), P_M u - u \rangle_{\Gamma_N}
\end{aligned}$$

as derived from (8). By utilizing the definition of the function $\varphi(\cdot)$, we arrive at the inequality

$$\left|\langle\varphi(u)-\varphi(\hat{u}),P_M u-u\rangle_{\Gamma_N}\right|\le 2g\|P_M u-u\|_{\Gamma_h}.$$

This leads us to the following estimate:

$$\begin{aligned}(\varepsilon_h^q,\varepsilon_h^q)_{\mathcal{T}_h}+(\varepsilon_h^u,\varepsilon_h^u)_{\mathcal{T}_h}\le(\Pi_V\boldsymbol{q}-\boldsymbol{q},\varepsilon_h^q)_{\mathcal{T}_h}+(\Pi_W u-u,\varepsilon_h^u)_{\mathcal{T}_h}\\ +2g\|P_M u-u\|_{\Gamma_h}.\end{aligned}\tag{39}$$

By utilizing (39), the approximation properties of Π_V, Π_W, P_M, $\varphi(\cdot)$, and applying Cauchy's inequality, we can derive the following estimate.

Theorem 6 *Let* $(\boldsymbol{q},u)$ *and* $(\boldsymbol{q}_h,u_h,\hat{u}_h)$ *denote the solutions of* (11) *and* (14)–(17), *respectively. We can establish the following estimates*:

$$\begin{aligned}&\|\Pi_V\boldsymbol{q}-\boldsymbol{q}_h\|_{\mathcal{T}_h}+\|\Pi_W u-u_h\|_{\mathcal{T}_h}\\ &\quad\le C\left(\|\Pi_V\boldsymbol{q}-\boldsymbol{q}\|_{\mathcal{T}_h}+\|\Pi_W u-u_h\|_{\mathcal{T}_h}+\|P_M u-u\|_{\Gamma_N}\right).\end{aligned}\tag{40}$$

It is evident from (40) that when the solution is sufficiently smooth both in the region Ω and on the boundary Γ_N, the convergence order of the approximation is optimal.

5 Numerical examples

The HDG scheme for addressing unilateral contact problems is introduced using the regularization method, along with the provision of an a priori error estimate. Subsequently, to validate the theoretical findings, a numerical example is presented. This specific example is available in [12], Chap. 3, pages 91–92.

Example Consider the problem defined by equation (10) on the domain $\Omega=(0,1)^2$ with $g=1$, $\Gamma_N=\{(x,y)\in\partial\Omega: x=1\}$. The source term is represented as

$$f(x,y)=\left((2+\pi^2)\sin x-(1+\pi^2)\frac{\gamma\cos 1+\sin 1}{1+\gamma}x\right)\sin(\pi y),$$

where the γ is the regularization parameter from Sect. 2. The corresponding exact solution is given by

$$u(x,y)=\left(\sin x-\frac{\gamma\cos 1+\sin 1}{1+\gamma}x\right)\sin(\pi y).$$

In our numerical experiment, we employ piecewise linear polynomials and utilize an iterative method to solve the nonlinear equations. The iterative algorithm proceeds as follows: we start with an value $\hat{u}_h^0$ of u on the boundary Γ_N, and we iteratively solve problem (18) until the condition $\|\hat{u}_h^{n+1}-\hat{u}_h^n\|_{L^2(\Gamma_N)}<\varepsilon$ is satisfied, with $\varepsilon>0$ being sufficiently small. Here, we choose $\epsilon=10^{-2}$. The maximum number of iterations is set to 10, which was determined based on extensive experimentation. We found that this number of iterations is sufficient for the solution to converge to the desired accuracy in all our test cases. For our experiments, we also set $\hat{u}_h^0=0$, $\tau=1$.

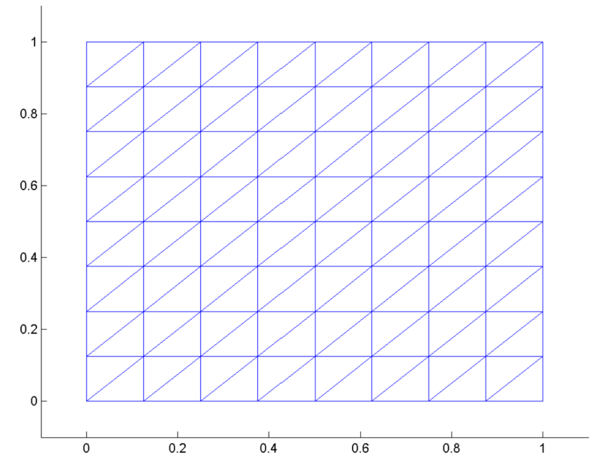

Figure 1 Structured grid ($h = 0.25$)

Table 1 L^2 error norm and convergence rates (structured grid)

	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\|u - u_h\|_{L^2(\Omega)}$	9.5131e-3	2.6936e-3	7.0421e-4	1.7914e-4	4.5117e-5
Convergence rate		1.8204	1.9355	1.9749	1.9893
$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}$	1.6841e-2	4.4903e-3	1.1464e-3	2.8890e-4	7.2466e-5
Convergence rate		1.9071	1.9697	1.9885	1.9952

Note: $\gamma = 0.01$

Table 2 L^2 error norm and convergence rates (structured grid)

	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\|u - u_h\|_{L^2(\Omega)}$	9.3833e-3	2.6589e-3	6.9533e-4	1.7689e-4	4.4554e-5
Convergence rate		1.8193	1.9351	1.9748	1.9892
$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}$	1.6829e-2	4.4908e-3	1.1472e-3	2.8925e-4	7.2577e-5
Convergence rate		1.9059	1.9689	1.9877	1.9947

Note: $\gamma = 0.001$

The results of the numerical experiment on a structured grid are presented in Fig. 1, with corresponding numerical values detailed in Table 1 and Table 2. Here, we select γ to be 0.01 and 0.001. Our observations reveal that the numerical convergence rates of $\|u - u_h\|$ and $\|\boldsymbol{q} - \boldsymbol{q}_h\|$ align with the theoretical predictions.

Next, we explore another scenario. Figure 2 illustrates the domain Ω divided by an unstructured grid, with corresponding numerical results detailed in Table 3 and Table 4. Similar to the structured grid case, γ is chosen to be 0.01 and 0.001. Once again, we observe that the numerical convergence rates of $\|u - u_h\|$ and $\|\boldsymbol{q} - \boldsymbol{q}_h\|$ are consistent with the theoretical predictions.

6 Conclusion

In summary, we have introduced the hybridizable discontinuous Galerkin (HDG) method as an effective solution to the unilateral contact problem. The inherent challenge of this problem lies in the presence of a nondifferentiable term in the bilinear form, introducing

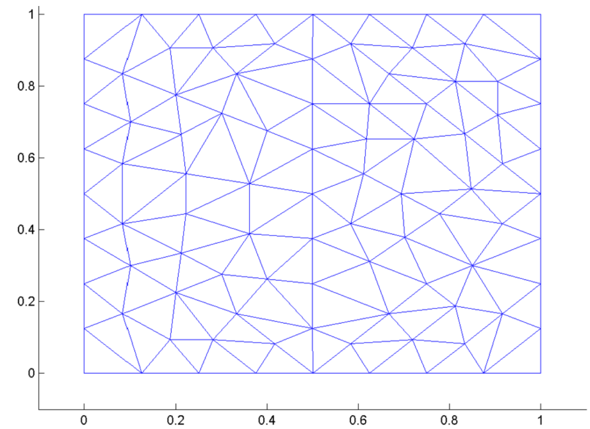

Figure 2 Unstructured grid ($h = 0.25$)

Table 3 L^2 error norm and convergence rates (unstructured grid)

	$h=\frac{1}{2}$	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$
$\Vert u-u_h\Vert_{L^2(\Omega)}$	9.5330e-3	2.5996e-3	6.2569e-4	1.5356e-4	3.9237e-5
Convergence rate		1.8746	2.0548	2.0266	1.9685
$\Vert \boldsymbol{q}-\boldsymbol{q}_h\Vert_{L^2(\Omega)}$	1.5506e-2	4.1182e-3	1.0384e-3	2.3494e-4	6.1449e-5
Convergence rate		1.9127	1.9877	2.1440	1.9348

Note: $\gamma = 0.01$

Table 4 L^2 error norm and convergence rates (unstructured grid)

	$h=\frac{1}{2}$	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$
$\Vert u-u_h\Vert_{L^2(\Omega)}$	9.3967e-3	2.5673e-3	6.1823e-4	1.5155e-4	3.8735e-5
Convergence rate		1.8719	2.0540	2.0283	1.9681
$\Vert \boldsymbol{q}-\boldsymbol{q}_h\Vert_{L^2(\Omega)}$	1.5477e-2	4.1163e-3	1.0359e-3	2.3479e-4	6.1397e-5
Convergence rate		1.9107	1.9905	2.1414	1.9351

Note: $\gamma = 0.001$

complexities that require a specialized approach. To address this, we employ the regularization method, successfully overcoming the associated difficulties.

Our HDG scheme for the nonlinear problem is presented, accompanied by a priori error estimates. Theoretical results are substantiated through numerical examples, demonstrating the validity of our approach. Notably, whether the domain is discretized using a structured grid or an unstructured grid, the numerical outcomes consistently align with our theoretical expectations, underscoring the practical efficacy of the HDG method.

An additional noteworthy feature of the HDG method is its capability for adaptive mesh calculation. This adaptive process, rooted in a posteriori error estimates, stands as a crucial advantage and warrants further exploration in future studies.

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Author contributions
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No datasets were generated or analysed during the current study.

Code availability
Available.

Declarations

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Not applicable.

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Yes.

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