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The logarithmic Sobolev inequality on the Heisenberg group and applications to the uncertainty inequality and heat equation

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Abstract

We consider the logarithmic Sobolev inequality on the Heisenberg group. One can derive the logarithmic Sobolev inequality from the Sobolev inequality, and we consider an application to the uncertainty inequality on the Heisenberg group. Moreover, one can also obtain a dissipative estimate of a solution of the heat equation on the Heisenberg group from the logarithmic Sobolev inequality.

Keywords: Logarithmic Sobolev inequality; Heisenberg group; Heat equation; Shannon inequality; Uncertainty inequality; Sobolev inequality

1 Introduction

We consider Sobolev's and a related inequality on the Heisenberg group $\mathbb{H}^n$. Sobolev's inequality is an inequality that holds in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ on n-dimensional Euclidean space $\mathbb{R}^n$, and is an essential tool in the study of nonlinear partial differential equations. The logarithmic Sobolev inequality is one of a variety of Sobolev's inequality, an inequality that implies that the functional appearance in the Boltzmann–Gibbs–Shannon entropy is bounded by a Sobolev norm. In this paper, we derive the logarithmic Sobolev inequality on the Heisenberg group.

We first consider the n-dimensional Euclidean space case. The sharp form of Sobolev's inequality is given in the following form: for any $f \in H^1(\mathbb{R}^n)$,

$$\|f\|^2_{L^{2^*}(\mathbb{R}^n)} \le S_n \|\nabla f\|^2_{L^2(\mathbb{R}^n)}, \tag{1.1}$$

where $n \ge 3$, $2^* = 2n/(n-2)$,

$$\nabla = {}^t\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right),$$

and $H^1(\mathbb{R}^n) \equiv \{f \in L^2(\mathbb{R}^n);\ |\nabla f| \in L^2(\mathbb{R}^n)\}$. The constant

$$S_n = \frac{1}{n(n-2)} 2^{2(1-\frac{1}{n})} \pi^{-(\frac{1}{n}+1)} \Gamma\left(\frac{n+1}{2}\right)^{\frac{2}{n}}$$

is the best possible, and this constant is attained by the Aubin–Talenti function:

$$f(x) = (1 + |x|^2)^{-\frac{n-2}{2}} \tag{1.2}$$

up to conformal automorphism (see Aubin [1] and Talenti [18]). To be precise, Talenti [18] identified the sharp constant of Sobolev's inequality in general cases $W^{1,p}(\mathbb{R}^n) \subset L^{np/(n-p)}(\mathbb{R}^n)$ by finding that (1.2) attains the best possible constant. Lieb [13] proved the sharp version of the Hardy–Littlewood–Sobolev inequality and showed the sharp constant with the extremal function of Sobolev's inequality (1.1) (see also [14]).

On the other hand, the following logarithmic Sobolev inequality was obtained by Stam [17] and Gross [10]: For any $f \in H^1(\mathbb{R}^n) \setminus \{0\}$,

$$\int_{\mathbb{R}^n} |f(x)|^2 \log \frac{|f(x)|^2}{\|f\|^2_{L^2(\mathbb{R}^n)}} \, dx \le \frac{n}{2} \|f\|^2_{L^2(\mathbb{R}^n)} \log \left(\frac{2}{n\pi e \|f\|^2_{L^2(\mathbb{R}^n)}} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx \right), \tag{1.3}$$

where the constant appearing on the right-hand side is the best possible. Moreover, this constant is attained by the Gaussian function

$$f(x) = e^{-|x|^2}$$

up to conformal automorphism. The optimal constant and extremal function of the inequality (1.3) were given by Weissler [22] and Carlen [3], respectively. In Lieb and Loss [14], we realize that the inequality (1.3) is equivalent to the L^p-L^q dissipative estimate of a solution of the heat equation

$$\begin{cases} \partial_t u = \Delta u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\partial_t = \partial/\partial t$ and $\Delta u = \nabla \cdot \nabla u$ (see also Davies [5], Carlen and Loss [4]). In fact, the extremal function of the inequality (1.3), that is, the Gaussian function coincides with the heat kernel by a conformal automorphism.

Concerning Sobolev's inequality on the other manifold, we introduce the Heisenberg group:

Definition (Heisenberg group) The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is defined by

$$(z,s) \cdot (z',s') = (z + z', t + t' + 2\mathrm{Im}(z \cdot \bar{z'}))$$

for $(z,s), (z',s') \in \mathbb{C}^n \times \mathbb{R}$, where

$$z \cdot \bar{z'} = \sum_{j=1}^{n} z_j \bar{z'_j}$$

when $z = (z_1, z_2, \ldots, z_n)$, $z' = (z'_1, z'_2, \ldots, z'_n)$.

For $(z,s) \in \mathbb{H}^n$ and $\lambda > 0$, the dilation is defined by $\lambda(z,s) = (\lambda z, \lambda^2 s)$, and we denote $|(z,s)|$ by

$$|(z,s)| \equiv (|z|^4 + s^2)^{\frac{1}{4}}.$$

We denote the homogeneous dimension of $\mathbb{H}^n$ by $Q = 2n+2$. For $1 \leq p < \infty$, we define L^p space on $\mathbb{H}^n$ by

$$L^p(\mathbb{H}^n) \equiv \left\{ f : \mathbb{H}^n \to \mathbb{R};\ \|f\|_p = \left(\int_{\mathbb{H}^n} |f(z,s)|^p \, dz \, ds \right)^{\frac{1}{p}} < \infty \right\}.$$

For $j = 1, 2, \dots, n$, the vector field

$$S = \frac{\partial}{\partial s}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}$$

satisfies the Hörmander condition. Then, the operator

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is a hypoelliptic operator. When we set $\nabla_{\mathbb{H}} = {}^t(X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n)$, we define Sobolev's space $H^1(\mathbb{H}^n)$ by

$$H^1(\mathbb{H}^n) \equiv \left\{ f \in L^2(\mathbb{H}^n);\ \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} f(z,s)|^2 \, dz \, ds < \infty \right\},$$

where

$$|\nabla_{\mathbb{H}} f(z,s)| = \left[\sum_{j=1}^{n} \left(|X_j f(z,s)|^2 + |Y_j f(z,s)|^2 \right) \right]^{\frac{1}{2}}.$$

Sobolev's inequality on Lie groups was studied by Folland [7] and Fischer and Ruzhansky [6]. For the Heisenberg group, Jerison and Lee [12] identified the sharp constant in Sobolev's inequality, and Frank and Lieb [8] gave different proofs and derived the sharp Hardy–Littlewood–Sobolev inequality on the Heisenberg group.

Proposition 1.1 (Sobolev's inequality on the Heisenberg group [8, 12]) *For any* $f \in H^1(\mathbb{H}^n)$,

$$\|f\|_{\frac{2Q}{Q-2}}^2 \leq S_{\mathbb{H}} \|\nabla_{\mathbb{H}} f\|_2^2, \tag{1.4}$$

where the constant

$$S_{\mathbb{H}} = 2^{-\frac{2}{n+1}} \pi^{-1} n^{-2} (n!)^{\frac{1}{n+1}} = 2^{2(1-\frac{2}{Q})} \pi^{-1} (Q-2)^{-2} \Gamma\left(\frac{Q}{2}\right)^{\frac{2}{Q}}$$

is the best possible. Moreover, this constant is attained by

$$f(z,s) = cH(\delta(a\cdot(z,s))),$$

where

$$H(z,s) = \left((1+|z|^2)^2 + s^2\right)^{-\frac{Q-2}{4}}$$

for $c\in\mathbb{C}$, $\delta>0$, *and* $a\in\mathbb{H}^n$.

We derive the logarithmic Sobolev inequality on the Heisenberg group by using Sobolev's inequality (1.4).

Theorem 1.2 (The logarithmic Sobolev inequality on the Heisenberg group) *For any* $f\in H^1(\mathbb{H}^n)\setminus\{0\}$,

$$\int_{\mathbb{H}^n} |f(z,s)|^2 \log\frac{|f(z,s)|^2}{\|f\|_2^2}\,dz\,ds \le \frac{Q}{2}\|f\|_2^2\log\left(\frac{S_{\mathbb{H}}}{\|f\|_2^2}\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}} f(z,s)|^2\,dz\,ds\right), \tag{1.5}$$

where the constant $S_{\mathbb{H}}$ *is the same as the above.*

For the Heisenberg group, Inglis and Papageorgiou [11] and Papageorgiou [15] showed the L^q-type logarithmic Sobolev inequality ($1<q\le 2$). They considered the logarithmic Sobolev inequality for Gibbs measures on the infinite product of Heisenberg groups. Bonnefont, Chafaï, and Herry [2] showed a variant of the logarithmic Sobolev inequality on the Heisenberg group, and Gordina and Luo [9] derived the logarithmic Sobolev inequality on nonisotropic Heisenberg groups.

One of the applications of the logarithmic Sobolev inequality (1.5) is to derive the Heisenberg uncertainty inequality on the Heisenberg group. The Heisenberg uncertainty implies that

$$\frac{n}{2}\|f\|_{L^2(\mathbb{R}^n)}^2 \le \left(\int_{\mathbb{R}^n}|x|^2|f(x)|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^n}|\nabla f(x)|^2\,dx\right)^{\frac12}. \tag{1.6}$$

For the Heisenberg group $\mathbb{H}^n$, Thangavelu [19], Sitaram, Sundari, and Thangavelu [16], and Xiao and He [24] considered the Heisenberg uncertainty inequality (see also [20, 21]). In order to prove the Heisenberg uncertainty inequality on the Heisenberg group, we derive Shannon's inequality. Define the weighted Lebesgue space $L_1^2(\mathbb{H}^n)$ by

$$L_1^2(\mathbb{H}^n) \equiv \{f\in L^2(\mathbb{H}^n);\ |(\cdot,\cdot)|f\in L^2(\mathbb{H}^n)\}.$$

Then, the following inequality holds:

Theorem 1.3 (The Shannon inequality) *For any* $f\in L_1^2(\mathbb{H}^n)\setminus\{0\}$,

$$-\int_{\mathbb{H}^n}|f(z,s)|^2\log\frac{|f(z,s)|^2}{\|f\|_2^2}\,dz\,ds \le \frac{Q}{2}\|f\|_2^2\log\left(\frac{C_n}{\|f\|_2^2}\int_{\mathbb{H}^n}|(z,s)|^2|f(z,s)|^2\,dz\,ds\right), \tag{1.7}$$

where the constant

$$C_n = \frac{\pi e}{n+1} n^{\frac{1}{n+1}} \pi^{-\frac{1}{2n+2}} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right)^{\frac{1}{n+1}}$$

is the best possible. Moreover, this constant is attained by

$$f(x) = e^{-|(z,s)|^2}$$

up to conformal automorphism.

Combining the inequalities (1.3) and (1.7), we obtain the Heisenberg uncertainty inequality on the Heisenberg group and the explicit constant:

Corollary 1.4 *For any* $f \in L^2_1(\mathbb{H}^n) \cap H^1(\mathbb{H}^n)$,

$$D_n \|f\|_2^2 \le \left(\int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz \, ds \right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} f(z,s)|^2 \, dz \, ds \right)^{\frac{1}{2}}, \tag{1.8}$$

where the constant D_n *is given by*

$$D_n = S_{\mathbb{H}}^{-\frac{1}{2}} C_n^{-\frac{1}{2}}.$$

This paper is constructed in the following sections. In Sect. 2, we give the proofs of Theorems 1.2 and 1.3, and Corollary 1.4. We also consider the behavior of the constant appearing in the inequalities as $n \to \infty$. For other applications, we show the estimate of solutions to the heat equation on the Heisenberg group in Sect. 3. In the Appendix, we add some calculations.

In what follows, $\| \cdot \|_p$ denotes the $L^p(\mathbb{H}^n)$-norm for $1 \le p < \infty$. We denote the gamma function by $\Gamma(\cdot)$.

2 Proof of the results and remarks on the constant

Proof of Theorem 1.2 If we set

$$d\mu = \frac{|f(z,s)|^2}{\|f\|_2^2} \, dz \, ds,$$

then we obtain $\int_{\mathbb{H}^n} d\mu = 1$. Furthermore, since the logarithmic function $\log x$ is concave, by Jensen's inequality, we have

$$\begin{aligned}
\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2} \, dz \, ds &= \frac{Q-2}{2} \|f\|_2^2 \int_{\mathbb{H}^n} \log \frac{|f(z,s)|^{\frac{4}{Q-2}}}{\|f\|_2^{\frac{4}{Q-2}}} \, d\mu \\
&\le \frac{Q-2}{2} \|f\|_2^2 \log \int_{\mathbb{H}^n} \frac{|f(z,s)|^{\frac{4}{Q-2}}}{\|f\|_2^{\frac{4}{Q-2}}} \, d\mu \\
&= \frac{Q-2}{2} \|f\|_2^2 \log \int_{\mathbb{H}^n} \frac{|f(z,s)|^{\frac{2Q}{Q-2}}}{\|f\|_2^{\frac{2Q}{Q-2}}} \, dz \, ds.
\end{aligned}$$

By Sobolev's inequality on the Heisenberg group (1.4)

$$\int_{\mathbb{H}^n} |f(z,s)|^{\frac{2Q}{Q-2}}\, dz\, ds \le S_{\mathbb{H}}^{\frac{Q}{Q-2}} \|\nabla_{\mathbb{H}} f\|_2^{\frac{2Q}{Q-2}},$$

we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2} dzds &\le \frac{Q-2}{2}\|f\|_2^2 \log \frac{S_{\mathbb{H}}^{\frac{Q}{Q-2}} \|\nabla_{\mathbb{H}} f\|_2^{\frac{2Q}{Q-2}}}{\|f\|_2^{\frac{2Q}{Q-2}}} \\ &\le \frac{Q}{2}\|f\|_2^2 \log \frac{S_{\mathbb{H}}\|\nabla_{\mathbb{H}} f\|_2^2}{\|f\|_2^2},\end{aligned}$$

which implies the desired inequality (1.5). □

Remark In Euclidean space $\mathbb{R}^n$, by the Stirling approximation for the Gamma function

$$\Gamma(z) \approx (2\pi)^{\frac{1}{2}} e^{-z} z^{z-\frac{1}{2}} \quad \text{for } z \gg 1,$$

the sharp constant S_n in (1.1) is, for n large enough:

$$\begin{aligned}\frac{n\pi e}{2} S_n &= \frac{1}{n-2} 2^{1-\frac{2}{n}} \pi^{-\frac{1}{n}} e \Gamma\left(\frac{n+1}{2}\right)^{\frac{2}{n}} \\ &\approx \frac{1}{n-2} 2^{1-\frac{2}{n}} \pi^{-\frac{1}{n}} e \left(\frac{4\pi}{n+1}\right)^{\frac{1}{n}} \left(\frac{n+1}{2e}\right)^{1+\frac{1}{n}} \quad \text{for } n \gg 1 \\ &= \frac{n+1}{n-2}(2e)^{-\frac{1}{n}} \approx 1 \quad \text{for } n \gg 1,\end{aligned}$$

which implies that the sharp constant S_n in (1.1) is approximated by the one in the logarithmic Sobolev inequality $2/(n\pi e)$ for n large enough.

On the other hand, in the Heisenberg group $\mathbb{H}^n$, the constant $S_{\mathbb{H}}$ in (1.4) is approximated by $2/(Q\pi e)$ as the following:

$$\begin{aligned}\frac{Q\pi e}{2} S_{\mathbb{H}} &= \frac{Q}{(Q-2)^2} 2^{1-\frac{4}{Q}} e \Gamma\left(\frac{Q}{2}\right)^{\frac{2}{Q}} \\ &\approx \frac{Q}{(Q-2)^2} 2^{1-\frac{4}{Q}} e \left(\frac{4\pi}{Q}\right)^{\frac{1}{Q}} \frac{Q}{2e} \quad \text{for } Q \gg 1 \\ &= \frac{Q^2}{(Q-2)^2}\left(\frac{\pi}{4Q}\right)^{\frac{1}{Q}} \approx 1 \quad \text{for } Q \gg 1\end{aligned}$$

when n is large enough. By the above observation, we conjugate that the sharp constant in (1.5) coincides with $2/(Q\pi e)$ or approximates it for n large enough.

As a corollary, one can obtain the parametric logarithmic Sobolev inequality on the Heisenberg group. The following inequality is useful for estimating solutions to the heat equation on the Heisenberg group:

Corollary 2.1 (The parametric logarithmic Sobolev inequality on the Heisenberg group) *For any $f \in H^1(\mathbb{H}^n) \setminus \{0\}$ and $a > 0$,*

$$\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2}\, dz\, ds \le a\|\nabla_{\mathbb{H}} f\|_2^2 + \frac{n}{2}\|f\|_2^2 \log \frac{nS_{\mathbb{H}}}{2ae}. \tag{2.1}$$

Proof of Corollary 2.1 For any $a > 0$,

$$\begin{aligned}\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2}\, dz\, ds &\le \frac{Q}{2}\|f\|_2^2 \log \frac{S_{\mathbb{H}}\|\nabla_{\mathbb{H}} f\|_2^2}{\|f\|_2^2} \\ &= \frac{Q}{2}\|f\|_2^2 \log \frac{S_{\mathbb{H}}}{a} + \frac{n}{2}\|f\|_2^2 \log \frac{a\|\nabla_{\mathbb{H}} f\|_2^2}{\|f\|_2^2}.\end{aligned}$$

Here, we use the essential inequality:

$$\alpha\beta \le e^{\alpha} + \beta(\log\beta - 1)$$

for $\alpha, \beta > 0$. By this inequality, we have

$$\log \frac{a\|\nabla_{\mathbb{H}} f\|_2^2}{\|f\|_2^2} \le a\frac{\|\nabla_{\mathbb{H}} f\|_2^2}{\|f\|_2^2} - 1.$$

Thus, we obtain

$$\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2}\, dz\, ds \le \frac{Q}{2}\|f\|_2^2 \log \frac{S_{\mathbb{H}}}{a} + \frac{n}{2} a\|\nabla_{\mathbb{H}} f\|_2^2 - 1.$$

Replacing $na/2 \to a$, we conclude that the inequality (2.1) holds. □

Proof of Theorem 1.3 Let ϕ be given by

$$\phi(z,s) = k_{n,b} e^{-|(z,s)|^2},$$

where the constant $k_{n,b}$ is defined below. By the polar transformation:

$$\int_{\mathbb{H}^n} e^{-2|(z,s)|^2}\, dz\, ds = \left|\Sigma^{n-1}\right| \int_0^{\infty} e^{-2r^2} r^{2n+1}\, dr,$$

where

$$\left|\Sigma^{n-1}\right| = \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}.$$

Changing variables with $2r^2 = s$, since $r = \sqrt{s/2}$, $dr = 2^{-3/2}s^{-1/2}\, ds$:

$$\int_0^{\infty} e^{-2r^2} r^{2n+1}\, dr = 2^{-n-2}\int_0^{\infty} e^{-s}s^n\, ds = 2^{-n-2}\Gamma(n+1).$$

Thus, we have

$$\int_{\mathbb{H}^n} e^{-2|(z,s)|^2}\, dz\, ds = 2^{-n-1} n\pi^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)},$$

that is,

$$k_n = 2^{\frac{n+1}{2}} n^{-\frac{1}{2}} \pi^{-\frac{2n+1}{4}} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right)^{\frac{1}{2}}.$$

For any $f \in L^2_1(\mathbb{H}^n)$ with $\|f\|_2 = 1$, we consider the relative entropy of f and ϕ. By the Jensen inequality, we have

$$\int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|\phi(z,s)|^2}{|f(z,s)|^2} \, dz\, ds \leq \log \int_{\mathbb{H}^n} |\phi(z,s)|^2 \, dz\, ds = 0.$$

Thus, we obtain

$$-\int_{\mathbb{H}^n} |f(z,s)|^2 \log |f(z,s)|^2 \, dz\, ds \leq 2 \int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz\, ds - 2 \log k_n. \tag{2.2}$$

For $\lambda > 0$, set f_λ to be the scaling of f by

$$f_\lambda(z,s) \equiv \lambda^{n+1} f(\lambda z, \lambda^2 s),$$

which preserves L^2-norm. Substituting f_λ into inequality (2.2), the left-hand side of (2.2) is written by

$$-\int_{\mathbb{H}^n} |f_\lambda(z,s)|^2 \log |f_\lambda(z,s)|^2 \, dz\, ds = -\int_{\mathbb{H}^n} |f(z,s)|^2 \log |f(z,s)|^2 \, dz\, ds - (2n+2) \log \lambda,$$

and the right-hand side is expressed by

$$2 \int_{\mathbb{H}^n} |(z,s)|^2 |f_\lambda(z,s)|^2 \, dz\, ds - 2 \log k_n = 2\lambda^{-2} \int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz\, ds - 2 \log k_n.$$

Combining both sides, we have

$$-\int_{\mathbb{H}^n} |f(z,s)|^2 \log |f(z,s)|^2 \, dz\, ds \leq 2\lambda^{-2} \int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz\, ds + (2n+2) \log \frac{\lambda}{k_n^{\frac{1}{n+1}}}$$

for any $\lambda > 0$. Optimizing the right-hand side in this inequality with

$$\lambda_0 = \left(\frac{2}{n+1} \right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz\, ds \right)^{\frac{1}{2}},$$

we obtain the desired inequality (1.7). □

Remark We remark that the constant C_n in the inequality (1.7) is approximated by $(2\pi e)/Q$ as the following:

$$\frac{Q}{2\pi e} C_n = n^{\frac{1}{n+1}} \pi^{-\frac{1}{2n+2}} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right)^{\frac{1}{n+1}}$$

$$\approx \left(\frac{e^{-\frac{n}{2}} \left(\frac{n}{2}\right)^{\frac{n}{2}-\frac{1}{2}}}{e^{-\frac{n+1}{2}} \left(\frac{n+1}{2}\right)^{\frac{n}{2}}} \right)^{\frac{1}{n+1}} \quad \text{for } Q \gg 1$$

$$= \left(e^{\frac{1}{2}} \left(\frac{n}{2}\right)^{-\frac{1}{2}} \left(\frac{n}{n+1}\right)^{\frac{n}{2}} \right)^{\frac{1}{n+1}} \approx 1 \quad \text{for } Q \gg 1$$

when n is large enough. The constant $(2\pi e)/Q$ corresponds to the optimal constant of the Shannon inequality on the n-dimensional Euclidean space.

Proof of Corollary 1.4 Combining inequalities (1.3) and (1.7), we have

$$\begin{aligned}
&-\frac{Q}{2}\|f\|_2^2 \log\left(\frac{C_n}{\|f\|_2^2} \int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz \, ds \right) \\
&\leq \int_{\mathbb{H}^n} |f(z,s)|^2 \log \frac{|f(z,s)|^2}{\|f\|_2^2} \, dz \, ds \\
&\leq \frac{Q}{2}\|f\|_2^2 \log\left(\frac{S_{\mathbb{H}}}{\|f\|_2^2} \int_{\mathbb{H}^2} |\nabla_{\mathbb{H}} f(z,s)|^2 \, dz \, ds \right),
\end{aligned}$$

and thus, we obtain

$$\frac{Q}{2}\|f\|_2^2 \log\left(\frac{S_{\mathbb{H}} C_n}{\|f\|_2^4} \int_{\mathbb{H}^2} |\nabla_{\mathbb{H}} f(z,s)|^2 \, dz \, ds \int_{\mathbb{H}^n} |(z,s)|^2 |f(z,s)|^2 \, dz \, ds \right) \geq 0.$$

Thus, we obtain the desired inequality (1.8). □

Remark By the above computations of the constants appearing in inequalities (1.5) and (1.7), we see that

$$\lim_{n\to\infty} \frac{2}{Q} D_n = 1.$$

We note that the constant $Q/2$ corresponds to the optimal constant of the Heisenberg uncertainty inequality (1.6) on the n-dimensional Euclidean space.

3 Application to the heat equation on the Heisenberg group

For other applications, one can derive the dissipative estimate of solutions to the heat equation on the Heisenberg group. We consider the Cauchy problem of the heat equation on the Heisenberg group:

$$\begin{cases} \partial_t u = \Delta_{\mathbb{H}} u, & t > 0, \ (z,s) \in \mathbb{H}^n, \\ u(0,z,s) = u_0(z,s), & (z,s) \in \mathbb{H}^n, \end{cases} \tag{3.1}$$

where $u = u(t,z,s)$ is the unknown function, and $u_0 \in L^q(\mathbb{H}^n)$ $(q \geq 1)$ is the given initial data. One can derive a L^p-L^q dissipative estimate for a solution of (3.1) via the parametric logarithmic Sobolev inequality (2.1) on the Heisenberg group.

Theorem 3.1 *Let u be a solution of* (3.1) *with $u_0 \in L^q(\mathbb{H}^n)$, $1 \le q < \infty$. Then, for $t > 0$ and $q < p \le \infty$,*

$$\|u(t)\|_p \le D t^{-\frac{Q}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_q, \tag{3.2}$$

where

$$D = D(n,p,q) = \left(\frac{C_q}{C_p}\right)^{\frac{n}{2}} \left(\frac{2nS_{\mathbb{H}}}{e^3(\frac{1}{q}-\frac{1}{p})}\right)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}, \quad C_p = p^{\frac{1}{p}}\left(1-\frac{1}{p}\right)^{1-\frac{1}{p}}$$

and the constant $S_{\mathbb{H}}$ is the same as the above.

Proof of Theorem 3.1 The proof of Theorem 3.1 is based on the argument in Lieb and Loss [14]. We take a function $r : [0,t) \to \mathbb{R}$ such that

$$r(0) = q, \quad r(t) = p, \quad r'(\tau) > 0 \quad \text{for } 0 \le \tau < t. \tag{3.3}$$

This function has been given later. For a moment, suppose that $u(\tau) \in L^{r(\tau)}(\mathbb{H}^n)$, then we have formally,

$$\begin{aligned}\frac{d}{d\tau}\|u(\tau)\|_{r(\tau)} &= \frac{r'(\tau)}{r(\tau)^2}\int_{\mathbb{H}^n} |u(\tau)|^{r(\tau)} \log \frac{|u(\tau)|^{r(\tau)}}{\|u(\tau)\|_{r(\tau)}^{r(\tau)}}\, dz\, ds \\ &\quad + \int_{\mathbb{H}^n} |u(\tau)|^{r(\tau)-2} u(\tau)\partial_\tau u(\tau)\, dz\, ds.\end{aligned}$$

From the second term on the right-hand side, using (3.1) and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} |u(\tau)|^{r(\tau)-2}u(\tau)\partial_\tau u(\tau)\, dz\, ds &= \int_{\mathbb{H}^n} |u(\tau)|^{r(\tau)-2}u(\tau)\Delta_{\mathbb{H}} u(\tau)\, dz\, ds \\ &= -\frac{4(r(\tau)-1)}{r(\tau)^2}\int_{\mathbb{H}^n} \left|\nabla_{\mathbb{H}} u(\tau)^{\frac{r(\tau)}{2}}\right|^2 dz\, ds.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{d}{d\tau}\|u(\tau)\|_{r(\tau)} \\ &\quad = \frac{r'(\tau)}{r(\tau)^2}\left(\int_{\mathbb{H}^n} |u(\tau)|^{r(\tau)} \log \frac{|u(\tau)|^{r(\tau)}}{\|u(\tau)\|_{r(\tau)}^{r(\tau)}}\, dz\, ds - \frac{4(r(\tau)-1)}{r'(\tau)}\int_{\mathbb{H}^n} \left|\nabla_{\mathbb{H}} u(\tau)^{\frac{r(\tau)}{2}}\right|^2 dz\, ds\right).\end{aligned}$$

By (2.1) with

$$a = \frac{4(r(\tau)-1)}{r'(\tau)},$$

we have

$$\frac{d}{d\tau}\|u(\tau)\|_{r(\tau)} \le -\frac{nr'(\tau)}{2r(\tau)^2}\|u(\tau)\|_{r(\tau)} \log\left(\frac{2nS_{\mathbb{H}}}{e}\frac{r(\tau)-1}{r'(\tau)}\right). \tag{3.4}$$

Here, we set

$$r(\tau) = \frac{pqt}{pt - (p-q)\tau},$$

then r satisfies the condition (3.3). Putting r to (3.4), we obtain

$$\frac{\frac{d}{d\tau}\|u(\tau)\|_{r(\tau)}}{\|u(\tau)\|_{r(\tau)}} \le -\frac{n}{2t}\left(\frac{1}{q}-\frac{1}{p}\right)\log\left[\frac{2nS_{\mathbb{H}}}{e}\frac{\{pt-(p-q)\tau\}\{pqt-pt+(p-q)\tau\}}{pq(p-q)t}\right].$$

Integrating both sides from 0 to t, we obtain

$$\log\frac{\|u(t)\|_p}{\|u_0\|_q} \le -\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)\log\frac{2nS_{\mathbb{H}}t}{e^3(\frac{1}{q}-\frac{1}{p})} - \frac{n}{2}\log\frac{C_p}{C_q},$$

which implies the desired inequality (3.2):

$$\|u(t)\|_p \le Dt^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_q.$$

□

Remark The explicit form of the heat kernel of the heat equation (3.1) is known (see [21]), and Xiao and He [23] proved the Hardy–Littlewood–Sobolev inequality on the Heisenberg group by using the estimate of the heat kernel. One can also derive a L^p-L^q dissipative estimate for a solution of the heat equation (3.1) by using this estimate and Young's inequality. Furthermore, we conjugate that the heat kernel of the heat equation (3.1) is the extremal function of (1.5).

Appendix

For a measurable function $f(z,s)$, by the polar transformation, we have

$$\begin{aligned}\int_{\mathbb{H}^n} f(z,s)\,dz\,ds &= \int_{\mathbb{R}}\int_{\mathbb{S}^{2n-1}}\int_0^\infty f(\rho\sigma,s)\rho^{2n-1}\,d\rho\,d\sigma\,ds \\ &= \int_{\mathbb{S}^{2n-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\infty f(r\sigma\cos^{\frac{1}{2}}\theta, r^2\sin\theta)r^{2n+1}\cos^{n-1}\theta\,dr\,d\theta\,d\sigma.\end{aligned}$$

In particular, we see that

$$\left|\Sigma^{n-1}\right| \equiv \int_{\mathbb{S}^{2n-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-1}\theta\,d\theta\,d\sigma = |\mathbb{S}^{2n-1}|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-1}\theta\,d\theta.$$

Here, since $\cos\theta$ is even, we have

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-1}\theta\,d\theta &= 2\int_0^{\frac{\pi}{2}}\cos\theta(1-\sin^2\theta)^{\frac{n-2}{2}}\,d\theta = 2\int_0^1(1-t^2)^{\frac{n}{2}-1}\,dt \\ &= B\left(\frac{1}{2},\frac{n}{2}\right) = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.\end{aligned}$$

Thus, it holds that

$$\left|\Sigma^{n-1}\right| = \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)\,\Gamma\left(\frac{n+1}{2}\right)}.$$

Moreover, for a radial function $f(x) = f(r)$ $(r = |(z,s)|)$,

$$\int_{\mathbb{H}^n} f(z,s)\,dz\,ds = \left|\Sigma^{n-1}\right| \int_0^{\infty} f(r) r^{2n+1}\,dr.$$

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The authors declare no competing interests.

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References
1. Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev. J. Differ. Geom. **11**, 573–598 (1976)
2. Bonnefont, M., Chafaï, D., Herry, R.: On logarithmic Sobolev inequalities for the heat kernel on the Heisenberg group. Ann. Fac. Sci. Toulouse **29**, 335–356 (2020)
3. Carlen, E.A.: Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. **101**, 194–211 (1991)
4. Carlen, E.A., Loss, M.: Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation. Duke Math. J. **81**, 135–157 (1995)
5. Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge University Press, Cambridge (1989)
6. Fischer, V., Ruzhansky, M.: Quantization on Nilpotent Lie groups. Progress in Mathematics, vol. 314. Birkhäuser, Basel (2016)
7. Folland, G.B.: Subelliptic estimates and function spaces on nilpotent Lie groups. Ark. Mat. **13**, 161–207 (1975)
8. Frank, R.L., Lieb, E.H.: Sharp constants in several inequalities on the Heisenberg group. Ann. Math. **176**, 349–381 (2012)
9. Gordina, M., Luo, L.: Logarithmic Sobolev inequalities on non-isotropic Heisenberg groups. J. Funct. Anal. **283**, 109500, 33pp (2022)
10. Gross, D.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)
11. Inglis, J., Papageorgiou, I.: Logarithmic Sobolev inequalities for infinite dimensional Hörmander type generators on the Heisenberg group. Potential Anal. **31**, 79–102 (2009)
12. Jerison, D., Lee, J.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. Am. Math. Soc. **1**, 1–13 (1988)
13. Lieb, E.: Constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. **118**, 349–374 (1983)
14. Lieb, E., Loss, M.: Analysis, 2nd edn. Am. Math. Soc., Providence (2001)
15. Papageorgiou, I.: The logarithmic Sobolev inequality for Gibbs measures on infinite product of Heisenberg groups. Markov Process. Relat. Fields **20**, 705–749 (2014)
16. Sitaram, A., Sundari, M., Thangavelu, S.: Uncertainty principles on certain Lie groups. Proc. Indian Acad. Sci. Math. Sci. **105**, 135–151 (1995)
17. Stam, A.J.: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Inf. Control **2**, 102–112 (1959)
18. Talenti, G.: Best constant in Sobolev enequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)
19. Thangavelu, S.: Some restriction theorems for the Heisenberg group. Stud. Math. **99**, 11–21 (1991)
20. Thangavelu, S.: Harmonic Analysis on the Heisenberg Group. Progress in Mathematics, xiv+192pp. Birkhäuser Boston, Inc., Boston (1998)
21. Thangavelu, S.: An Introduction to the Uncertainty Principle. Progress in Mathematics, xiv+174pp. Birkhäuser Boston, Inc., Boston (2004)
22. Weissler, F.B.: Logarithmic Sobolev inequalities for the heat-diffusion semigroup. Trans. Am. Math. Soc. **237**, 255–269 (1978)
23. Xiao, J., He, J.: Riesz potential on the Heisenberg group. J. Inequal. Appl. **2011**, 498638, 13pp (2011)
24. Xiao, J., He, J.: Uncertainty inequalities for the Heisenberg group. Proc. Indian Acad. Sci. Math. Sci. **122**, 573–581 (2012)