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On properties and operations of complex neutrosophic soft groups

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Abstract

Complex neutrosophic soft groups represent a significant advancement in handling uncertainty by integrating the concepts of fuzzy logic, soft sets, and neutrosophic logic. These groups generalize complex fuzzy soft groups and introduce an additional dimension through neutrosophic membership functions, namely truth, indeterminacy, and falsity. This creates a richer framework for dealing with uncertainty and ambiguity, making it well-suited for managing complex data structures in real-world applications. We explore some important definitions and theoretical frameworks surrounding complex neutrosophic soft groups, highlighting the unique aspect of neutrosophic membership functions. Additionally, we present an overview of neutrosophic soft groups, exploring some of their key operations and fundamental properties. We then examine the basics of homogeneous complex neutrosophic soft sets and their roles in establishing complex neutrosophic soft groups.

Keywords: Neutrosophic logic; Complex neutrosophic soft set; Complex neutrosophic soft group; Homogenous neutrosophic soft group

1 Introduction

Uncertainty and imprecision are inherent aspects of various real-world problems, particularly in fields such as data analysis, artificial intelligence, and decision-making. Conventional mathematical frameworks often struggle to address these complexities effectively [19], leading to the development of advanced models, including fuzzy and neutrosophic logic. Neutrosophic logic has a wide range of applications [8, 18] in various fields, including information retrieval, engineering and control systems, decision support systems, and sustainability management.

The neutrosophic set (NS), proposed by Smarandache [17] in 1998, generalizes the concept of fuzzy sets. The NS framework serves as an advanced method for dealing with ambiguity by incorporating truth, indeterminacy, and falsity membership functions, allowing for a more flexible approach to managing uncertain data in real-world scenarios. The complex fuzzy set (CFS) that takes a different approach by extending the membership functions into the complex plane's unit circle was introduced by Ramot et al. [14]. In 2010, Nadia [12] merged the principles of CFS with those of soft sets, resulting in the complex fuzzy soft sets, which offer a unique perspective on data analysis.

Maji [10] explored the principles of NS and incorporated them into soft sets to create the neutrosophic soft set (NSS). The author defined certain operations and applications, particularly in the context of decision-making for NSS. In 2016, Ali and Smarandache [2] developed a hybrid model, namely a complex neutrosophic set (CNS) that combined CFS with NS. The CNS framework can address multiple aspects of uncertainty, including incompleteness, indeterminacy, and inconsistency, while also incorporating periodicity in a single set. Later, Bera and Mahapatra [4] incorporated the concept of groups into NSS and introduced the notion of neutrosophic soft groups. They provided detailed examples to illustrate their approach and defined the Cartesian product for neutrosophic soft groups, contributing to the development of algebraic structures within this framework.

Broumi et al. [6] studied both CNS and soft set models, leading to the development of the complex neutrosophic soft set (CNSS) model. They tested the CNSS model in a decision-making context, demonstrating its effectiveness in capturing indeterminacy when dealing with uncertainty. The CNSS model represents a better generalization compared to fuzzy soft sets, NS, and CFS. In 2017, Alsarahead and Ahmad [3] employed Rosenfeld's technique [15] and extended it to the complex plane's unit circle by applying Ramot's approach, leading to the notion of the complex fuzzy soft group (CFSG). This development integrates fuzzy logic concepts with soft set theory in a complex plane context.

In this paper, we explore the unique properties and operations associated with complex neutrosophic soft groups (CNSGs) that extend the concept of CFSG by incorporating phase terms to represent three distinct neutrosophic membership functions, namely truth, indeterminacy, and falsity, as opposed to the single membership function used in fuzzy logic. This extension allows CNFGs to encapsulate a broader range of uncertainty and ambiguity, providing a more comprehensive framework for managing complex data structures. We also explore the fundamental characteristics of neutrosophic soft groups and discuss the homogeneous complex neutrosophic soft sets as well as their contributions to the formation of complex neutrosophic soft groups.

2 Preliminaries

This section introduces definitions of various elements within the neutrosophic theory and outlines the gradual development of concepts related to NS, NSS, CNS, and CNSS. Additionally, some definitions concerning the properties and operations of neutrosophic theory are also presented.

Definition 2.1 ([2]) Let S be a CNS in the universe $\mathbb{U}$, described by three member terms $T_S(\mathbb{u})$, $I_S(\mathbb{u})$, and $F_S(\mathbb{u})$. respectively. For any $\mathbb{u} \in \mathbb{U}$, the three-member terms fall within the complex plane unit circle as shown below:

$$T_S(\mathbb{u}) = p_S(\mathbb{u}) \cdot e^{j\mu_S(\mathbb{u})},$$
$$I_S(\mathbb{u}) = q_S(\mathbb{u}) \cdot e^{j\nu_S(\mathbb{u})},$$
$$F_S(\mathbb{u}) = r_S(\mathbb{u}) \cdot e^{j\omega_S(\mathbb{u})},$$

where $p_S(\mathbb{u}), q_S(\mathbb{u}), r_S(\mathbb{u}) \in [0,1]$ such that $0 \le p_S(\mathbb{u}) + q_S(\mathbb{u}) + r_S(\mathbb{u}) \le 3$.

Definition 2.2 ([1]) Let (f, A) be a soft set over a group $\mathbb{X}$, then (f, A) denotes a soft group over $\mathbb{X}$ if and only if $f(a) < \mathbb{X}$ for each $a \in A$.

Definition 2.3 ([11]) Suppose that (f, A) is a fuzzy soft set over a group G. For all $a \in A$, $x, y \in G$, if

i. $f_a(xy) \geq \min\{f_a(x), f_a(y)\}$,
ii. $f_a(x^{-1}) \geq f_a(x)$,
iii. $\mu_{f_a}(xy) \geq \min\{\mu_{f_a}(x), \mu_{f_a}(y)\}$,
iv. $\mu_{f_a}(x^{-1}) \geq \mu_{f_a}(x)$ if $a \in A$, $x \in G$,

then (f, A) is a fuzzy soft group.

Definition 2.4 ([9]) Let G be an arbitrary group, $\gamma_G \in IFS(U)$ where $IFS(U)$ is an intuitionistic fuzzy set over the universe U. Then γ_G refers to an intuitionistic fuzzy soft group (IFSG) if $\gamma_G(x^{-1}) = \gamma_G(x)$ for all $x \in G$.

Definition 2.5 ([3]) Let G be a group and (f, A) be a homogenous complex fuzzy soft set on G. Then (f, A) is a CFSG over G if and only if the following hold:

i. $\mu_{f_a}(xy) \geq \min\{\mu_{f_a}(x), \mu_{f_a}(y)\}$, $\forall a \in A$, $x, y \in G$,
ii. $\mu_{f_a}(x^{-1}) \geq \mu_{f_a}(x)$ if $a \in A$, $x \in G$.

Definition 2.6 ([13]) Let $\{S = (x, \mu_S(x), \upsilon_S(x)) : x \in G\}$ be a complex intuitionistic fuzzy set on G. Then S is a complex intuitionistic fuzzy subgroup (CIF-subgroup) of G if for all $x, y \in G$ the following hold:

i. $\mu_S(xy) \geq \min\{\mu_S(x), \mu_S(y)\}$,
ii. $\mu_S(xy) \leq \max\{\nu_S(x), \nu_S(y)\}$,
iii. $\mu_S(x^{-1}) \geq \mu_S(x)$,
iv. $\nu_S(x^{-1}) \leq \nu_S(x)$.

3 Concepts of complex neutrosophic soft groups

We now explore the properties of CNSG by including comprehensive definitions and a series of theorems, each supported by relevant examples to highlight their significance and potential applications for this framework.

Definition 3.1 Let $(H, \mathbb{A})$ be a neutrosophic soft set over the universe $\mathbb{U}$ and $\mathbb{E}$ be a set of parameters, and $\mathbb{A} \subseteq \mathbb{E}$ such that

$$(H, \mathbb{A}) = \left\{\left\langle a, P_{H_a}(\mathbb{u}), Q_{H_a}(\mathbb{u}), R_{H_a}(\mathbb{u})\right\rangle : a \in \mathbb{A}, \mathbb{u} \in \mathbb{U}\right\},$$

where $P_{H_a}(\mathbb{u})$, $Q_{H_a}(\mathbb{u})$, and $R_{H_a}(\mathbb{u})$ describe the truth, indeterminacy, and falsity membership terms, respectively. Then the set

$$(H, \mathbb{A})_\pi = \left\{\left\langle a, P_{H_{\pi_a}}(\mathbb{u}), Q_{H_{\pi_a}}(\mathbb{u}), R_{H_{\pi_a}}(\mathbb{u})\right\rangle : a \in \mathbb{A}, \mathbb{u} \in \mathbb{U}\right\}$$

denotes a *π-neutrosophic soft set* when $P_{H_{\pi_a}}(\mathbb{u}) = 2\pi P_{H_a}(\mathbb{u})$, $Q_{H_{\pi_a}}(\mathbb{u}) = 2\pi Q_{H_a}(\mathbb{u})$, and $R_{H_{\pi_a}}(\mathbb{u}) = 2\pi R_{H_a}(\mathbb{u})$ for all $a \in \mathbb{A}$ and $\mathbb{u} \in \mathbb{U}$.

Definition 3.2 Let G be a group and

$$(H, \mathbb{A})_\pi = \left\{\left\langle a, P_{H_{\pi_a}}(\mathbb{u}), Q_{H_{\pi_a}}(\mathbb{u}), R_{H_{\pi_a}}(\mathbb{u})\right\rangle : a \in \mathbb{A}, \mathbb{u} \in \mathbb{U}\right\}$$

be a π-neutrosophic soft set of G. Then $(H,\mathbb{A})_\pi$ is called a π-neutrosophic soft group if for all $a \in \mathbb{A}$ and $\varkappa, y \in G$ the following hold:

1. $P_{H_{\pi_a}}(\varkappa y) \geq \min\left\{P_{H_{\pi_a}}(\varkappa), P_{H_{\pi_a}}(y)\right\}$,
2. $Q_{H_{\pi_a}}(\varkappa y) \leq \max\left\{Q_{H_{\pi_a}}(\varkappa), Q_{H_{\pi_a}}(y)\right\}$,
3. $R_{H_{\pi_a}}(\varkappa y) \leq \max\left\{R_{H_{\pi_a}}(\varkappa), R_{H_{\pi_a}}(y)\right\}$,
4. $P_{H_{\pi_a}}(\varkappa^{-1}) \geq P_{H_{\pi_a}}(\varkappa)$,
5. $Q_{H_{\pi_a}}\left(\varkappa^{-1}\right) \leq Q_{H_{\pi_a}}(\varkappa)$,
6. $R_{H_{\pi_a}}\left(\varkappa^{-1}\right) \leq R_{H_{\pi_a}}(\varkappa)$.

Proposition 3.3 *The π-neutrosophic soft set $(H,\mathbb{A})_\pi$ is a π-neutrosophic soft group if and only if $(H,\mathbb{A})$ is a neutrosophic soft group.*

Proof

($\Rightarrow$)

Let $(H,\mathbb{A})_\pi$ be a π-neutrosophic soft group. Since $(H,\mathbb{A})_\pi$ satisfies the six conditions as in Definition 3.2, $(H,\mathbb{A})$ is a neutrosophic soft group.

($\Leftarrow$)

Let $(H,\mathbb{A})$ be a neutrosophic soft group. Then $(H,\mathbb{A})$ is a neutrosophic subgroup. Since π-neutrosophic soft set $(H,\mathbb{A})_\pi$ is a π-neutrosophic soft group if and only if $(H,\mathbb{A})$ is a neutrosophic subgroup. Hence, $(H,\mathbb{A})_\pi$ is a π-neutrosophic soft group. □

Remark Let $(h,\mathbb{A})$ be a CNSS over the universe $\mathbb{U}$ such that, for all $a \in \mathbb{A}$ and $\varkappa \in \mathbb{U}$,

$$(h,\mathbb{A}) = \left\{\left\langle a, T_{h_a}(\varkappa), I_{h_a}(\varkappa), F_{h_a}(\varkappa)\right\rangle : a \in \mathbb{A}, \varkappa \in \mathbb{U}\right\},$$

where

$$T_{h_a}(\varkappa) = p_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)}, \varkappa \in \mathbb{U}$$
$$I_{h_a}(\varkappa) = q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)},$$
$$F_{h_a}(\varkappa) = r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)},$$

where $j = \sqrt{-1}$ exemplifying the three complex terms defined earlier. Then $(h,\mathbb{A})$ results in two real neutrosophic soft sets on $\mathbb{U}$ as shown below:

1. *The neutrosophic soft set*

$$(h,\mathbb{A}) = \left\{\left\langle a, p_{h_a}(\varkappa), q_{h_a}(\varkappa), r_{h_a}(\varkappa)\right\rangle : a \in \mathbb{A}, \varkappa \in \mathbb{U}\right\},$$

   where for all $a \in \mathbb{A}$, $\varkappa \in \mathbb{U}$, and $p_{h_a}(\varkappa)$, $q_{h_a}(\varkappa)$ and $r_{h_a}(\varkappa)$ represent the amplitude terms of the complex-valued membership functions $T_{h_a}(\varkappa)$, $I_{h_a}(\varkappa)$, and $F_{h_a}(\varkappa)$, respectively.

2. *The π-neutrosophic soft set*

$$(h,\mathbb{A}) = \left\{\left\langle a, \mu_{h_a}(\varkappa), \nu_{h_a}(\varkappa), \omega_{h_a}(\varkappa)\right\rangle : a \in \mathbb{A}, \varkappa \in \mathbb{U}\right\},$$

where for all $a \in \mathbb{A}$, $\varkappa \in \mathbb{U}$, and $\mu_{h_a}(\varkappa)$, $\nu_{h_a}(\varkappa)$ and $\omega_{h_a}(\varkappa)$ represent the phase terms of the complex-valued membership functions $T_{h_a}(\varkappa)$, $I_{h_a}(\varkappa)$, and $F_{h_a}(\varkappa)$, respectively.

Definition 3.4 If $(h, \mathbb{A})$ and $(k, \mathbb{B})$ are two complex neutrosophic soft sets over the universe $\mathbb{U}$, which are characterized by the complex-valued membership functions

$$T_{h_a}(\varkappa) = p_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)}, I_{h_a}(\varkappa) = q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)}, F_{h_a}(\varkappa) = r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)},$$

$$T_{k_a}(\varkappa) = p_{k_a}(\varkappa)\, e^{j\mu_{k_a}(\varkappa)}, I_{k_a}(\varkappa) = q_{k_a}(\varkappa)\, e^{j\nu_{k_a}(\varkappa)}, \text{ and } F_{k_a}(\varkappa) = r_{k_a}(\varkappa)\, e^{j\omega_{k_a}(\varkappa)}.$$

Then,

i. The set $(h, \mathbb{A})$ is said to be homogenous-CNSS if for all $a \in \mathbb{A}$ and $\varkappa, y \in \mathbb{U}$, we have
   1. $p_{h_a}(\varkappa) \le p_{h_a}(y) \Leftrightarrow \mu_{h_a}(\varkappa) \le \mu_{h_a}(y)$,
   2. $q_{h_a}(\varkappa) \le q_{h_a}(y) \Leftrightarrow \nu_{h_a}(\varkappa) \le \nu_{h_a}(y)$,
   3. $r_{h_a}(\varkappa) \le r_{h_a}(y) \Leftrightarrow \omega_{h_a}(\varkappa) \le \omega_{h_a}(y)$.

ii. A complex neutrosophic soft set $(h, \mathbb{A})$ denotes a completely homogenous complex neutrosophic soft set if it is homogenous and if and only if for all $\varkappa \in \mathbb{U}$ and for all $a, b \in \mathbb{A}$, we have
   1. $p_{h_a}(\varkappa) \le p_{h_b}(\varkappa) \Leftrightarrow \mu_{h_a}(\varkappa) \le \mu_{h_b}(\varkappa)$,
   2. $q_{h_a}(\varkappa) \le q_{h_b}(\varkappa) \Leftrightarrow \nu_{h_a}(\varkappa) \le \nu_{h_b}(\varkappa)$,
   3. $r_{h_a}(\varkappa) \le r_{h_b}(\varkappa) \Leftrightarrow \omega_{h_a}(\varkappa) \le \omega_{h_b}(\varkappa)$.

iii. A complex neutrosophic soft set $(h, \mathbb{A})$ is referred to as homogenous with $(k, \mathbb{B})$ if and only if for all $a \in \mathbb{A} \cap \mathbb{B}$ and for all $\varkappa \in \mathbb{U}$, we have
   1. $p_{h_a}(\varkappa) \le p_{k_a}(\varkappa) \Leftrightarrow \mu_{h_a}(\varkappa) \le \mu_{k_a}(\varkappa)$,
   2. $q_{h_a}(\varkappa) \le q_{k_a}(\varkappa) \Leftrightarrow \nu_{h_a}(\varkappa) \le \nu_{k_a}(\varkappa)$,
   3. $r_{h_a}(\varkappa) \le r_{k_a}(\varkappa) \Leftrightarrow \omega_{h_a}(\varkappa) \le \omega_{k_a}(\varkappa)$.

Example 1 Let $\mathbb{U} = \{x_1, x_2, x_3\}$ be a universal set and $\mathbb{E}$ be a universal set of parameters and $\mathbb{A}, \mathbb{B} \subseteq \mathbb{E}$, where $\mathbb{A} = \{a_1, a_2\}$ and $\mathbb{B} = \{a_1, a_3, a_4, a_5\}$. Suppose that $(h, \mathbb{A})$ and $(k, \mathbb{B})$ are two CNSSs as follows:

$$(h, \mathbb{A}) = \left\{ \begin{array}{c} (a_1, \dfrac{x_1}{0.2e^{i\pi(0.3)}, 0.4e^{i\pi(0.5)}, 0.3e^{i\pi(0.6)}}, \dfrac{x_2}{0.1e^{i\pi(0.2)}, 0.5e^{i\pi(0.7)}, 0.4e^{i\pi(0.8)}}, \\ \dfrac{x_3}{0.3e^{i\pi(0.4)}, 0.4e^{i\pi(0.6)}, 0.9e^{i\pi(0.8)}}), \\ (a_2, \dfrac{x_1}{0.8e^{i\pi 0.5}, 0.6e^{i\pi(0.6)}, 0.7e^{i\pi(0.7)}}, \dfrac{x_2}{0.2e^{i\pi(0.3)}, 0.4e^{i\pi 0.2}, 0.1e^{i\pi(0.7)}}, \\ \dfrac{x_3}{0.8e^{i\pi(0.5)}, 0.5e^{i\pi(0.6)}, 0.3e^{i\pi(0.7)}}), \end{array} \right\}$$

$$(k,\mathbb{B}) = \left\{ \begin{array}{l} (a_1, \frac{x_1}{0.4e^{i\pi(0.5)}, 0.4e^{i\pi(0.6)}, 0.1e^{i\pi(0.5)}}, \frac{x_2}{0.3e^{i\pi(0.7)}, 0.6e^{i\pi(0.9)}, 0.5e^{i\pi(0.9)}}, \\ \frac{x_3}{0.9e^{i\pi(0.9)}, 0.5e^{i\pi(0.7)}, 0.9e^{i\pi(0.7)}}), \\ (a_3, \frac{x_1}{0.7e^{i\pi 0.3}, 0.1e^{i\pi(0.8)}, 0.3e^{i\pi(0.5)}}, \frac{x_2}{0.2e^{i\pi(0.1)}, 0.4e^{i\pi 0.2}, 0.7e^{i\pi(0.6)}}, \\ \frac{x_3}{0.1e^{i\pi(0.8)}, 0.5e^{i\pi(0.7)}, 0.7e^{i\pi(0.2)}}), \\ (a_4, \frac{x_1}{0.2e^{i\pi(0.3)}, 0.4e^{i\pi(0.5)}, 0.3e^{i\pi(0.6)}}, \frac{x_2}{0.1e^{i\pi(0.2)}, 0.5e^{i\pi(0.7)}, 0.4e^{i\pi(0.8)}}, \\ \frac{x_3}{0.3e^{i\pi(0.4)}, 0.4e^{i\pi(0.6)}, 0.9e^{i\pi(0.8)}}), \\ (a_5, \frac{x_1}{0.8e^{i\pi 0.5}, 0.6e^{i\pi(0.6)}, 0.7e^{i\pi(0.7)}}, \frac{x_2}{0.2e^{i\pi(0.3)}, 0.4e^{i\pi 0.2}, 0.1e^{i\pi(0.7)}}, \\ \frac{x_3}{0.8e^{i\pi(0.5)}, 0.5e^{i\pi(0.6)}, 0.3e^{i\pi(0.7)}}). \end{array} \right\}$$

By Definition 3.4 (i & ii), it can be shown from the stated conditions that $(h,\mathbb{A})$ is homogenous and completely homogenous. On the other hand, $(k,\mathbb{B})$ is not homogenous, and therefore, it is not completely homogenous. Since $a_1 \in \mathbb{B}$ and $x_1, x_2 \in \mathbb{U}$, then $0.3 \le 0.4$ but $0.7 \nleq 0.5$. By Definition 3.4 (iii), $(h,\mathbb{A})$ is homogenous with $(k,\mathbb{B})$.

Definition 3.5 Let $(h,\mathbb{A})$ be a homogenous complex neutrosophic soft set on a group G. Then $(h,\mathbb{A})$ denotes a complex neutrosophic soft group on G if and only if, for all $a \in \mathbb{A}$ and $\varkappa, \mathrm{y} \in G$, we have

1. $T_{h_a}(\varkappa\mathrm{y}) \ge \min\left\{T_{h_a}(\varkappa), T_{h_a}(\mathrm{y})\right\}$,
2. $I_{h_a}(\varkappa\mathrm{y}) \le \max\left\{I_{h_a}(\varkappa), I_{h_a}(\mathrm{y})\right\}$,
3. $F_{h_a}(\varkappa\mathrm{y}) \le \max\left\{F_{h_a}(\varkappa), F_{h_a}(\mathrm{y})\right\}$,
4. $T_{h_a}(\varkappa^{-1}) \ge T_{h_a}(\varkappa)$,
5. $I_{h(a)}\left(\varkappa^{-1}\right) \le I_{h_a}(\varkappa)$,
6. $F_{h_a}\left(\varkappa^{-1}\right) \le F_{h_a}(\varkappa)$.

Definition 3.6 Suppose that $(h,\mathbb{A})$ and $(k,\mathbb{B})$ are two CNSSs. Then $(h,\mathbb{A})$ is a complex neutrosophic soft subgroup of $(k,\mathbb{B})$ if it satisfies the following:

1. $(h,\mathbb{A}) \subseteq (k,\mathbb{B})$ where $\subseteq$ exemplifies a CNS-subset.
2. $(h,\mathbb{A})$ and $(k,\mathbb{B})$ are both CNS-groups.

The following theorem demonstrates the relation between complex neutrosophic soft groups and neutrosophic soft groups.

Theorem 3.7 *Let $(h,\mathbb{A})$ be a homogenous complex neutrosophic soft set on a group G. Then $(h,\mathbb{A})$ is a CNS-group of G if and only if*

i. *The neutrosophic soft set $(h,\mathbb{A})$ is a neutrosophic soft group.*
ii. *The π-neutrosophic soft set $(h,\mathbb{A})$ is a π-neutrosophic soft group (π-NS-group).*

Proof To prove this theorem, the previously defined six conditions need to be satisfied.

$(\Rightarrow)$

Let $(h,\mathbb{A})$ be a homogeneous complex neutrosophic soft group and $\varkappa, \mathrm{y} \in G$. Then, for all $a \in \mathbb{A}$, we have

$$P_{h_a}(\varkappa\mathrm{y})\, e^{j\mu_{h_a}(\varkappa\mathrm{y})} = T_{h_a}(\varkappa\mathrm{y})$$

$$\begin{aligned}
&\geq \min\left\{T_{h_a}(x), T_{h_a}(y)\right\}\\
&= \min\left\{P_{h_a}(x)\, e^{j\mu_{h_a}(x)}, P_{h_a}(y)\, e^{j\mu_{h_a}(y)}\right\}\\
&= \min\left\{P_{h_a}(x), P_{h_a}(y)\right\}\cdot e^{j\min\left\{\mu_{h_a}(x),\mu_{h_a}(y)\right\}}.
\end{aligned}$$

Since $(h,\mathbb{A})$ is homogeneous,we have

$$P_{h_a}(xy) \geq \min\left\{P_{h_a}(x), P_{h_a}(y)\right\} \text{ and } \mu_{h_a}(xy) \geq \min\left\{\mu_{h_a}(x), \mu_{h_a}(y)\right\}.$$

Similarly,

$$\begin{aligned}
q_{h_a}(xy)\, e^{j\nu_{h_a}(xy)} &= I_{h_a}(xy)\\
&\leq \max\left\{I_{h_a}(x), I_{h_a}(y)\right\},\\
&= \max\left\{q_{h_a}(x)\, e^{j\nu_{h_a}(x)}, q_{h_a}(y)\cdot e^{j\nu_{h_a}(y)}\right\}\\
&= \max\left\{q_{h_a}(x), q_{h_a}(y)\right\}\cdot e^{j\max\left\{\nu_{h_a}(x),\nu_{h_a}(y)\right\}}.
\end{aligned}$$

Since $(h,\mathbb{A})$ is homogeneous,

$$q_{h_a}(xy) \leq \max\left\{q_{h_a}(x), q_{h_a}(y)\right\} \text{ and } \nu_{h_a}(xy) \leq \max\left\{\nu_{h_a}(x), \nu_{h_a}(y)\right\}.$$

Using a similar approach, we get

$$\begin{aligned}
&r_{h_a}(xy) \leq \max\left\{r_{h_a}(x), r_{h_a}(y)\right\},\\
&\omega_{h_a}(xy) \leq \max\left\{\omega_{h_a}(x), \omega_{h_a}(y)\right\}.
\end{aligned}$$

Now, by applying the fourth condition of Definition 2.5, we have

$$\begin{aligned}
P_{h_a}(x^{-1})e^{j\mu_{h_a}(x^{-1})} &= T_{h_a}(x^{-1})\\
&\geq T_{h_a}(x)\\
&= P_{h_a}(x)\, e^{j\mu_{h_a}(x)},
\end{aligned}$$

which implies that

$$P_{h_a}(x^{-1}) \geq P_{h_a}(x) \text{ and } \mu_{h_a}(x^{-1}) \geq \mu_{h_a}(x),$$

where $(h,\mathbb{A})$ is homogeneous.

Similarly,

$$\begin{aligned}
q_{h_a}(x^{-1})e^{j\nu_{h_a}(x^{-1})} &= I_{h_a}(x^{-1})\\
&\leq I_{h_a}(x)\\
&= q_{h_a}(x)\cdot e^{j\nu_{h_a}(x)}.
\end{aligned}$$

Since $(h, \mathbb{A})$ is homogeneous, then

$$q_{h_a}(\varkappa^{-1}) \leq q_{h_a}(\varkappa),$$
$$\nu_{h_a}(\varkappa^{-1}) \leq \nu_{h_a}(\varkappa).$$

Similarly, we can get

$$r_{h_a}(\varkappa^{-1}) \leq r_{h_a}(\varkappa) \text{ and } \omega_{h_a}(\varkappa^{-1}) \leq \omega_{h_a}(\varkappa).$$

Therefore

1. $P_{h_a}(\varkappa y) \geq \min\left\{P_{h_a}(\varkappa), P_{h_a}(y)\right\}$,
2. $q_{h_a}(\varkappa y) \leq \max\left\{q_{h_a}(\varkappa), q_{h_a}(y)\right\}$,
3. $r_{h_a}(\varkappa y) \leq \max\left\{r_{h_a}(\varkappa), r_{h_a}(y)\right\}$,
4. $P_{h_a}\left(\varkappa^{-1}\right) \geq P_{h_a}(\varkappa)$,
5. $q_{h_a}\left(\varkappa^{-1}\right) \leq q_{h_a}(\varkappa)$,
6. $r_{h_a}\left(\varkappa^{-1}\right) \leq r_{h_a}(\varkappa)$,

which implies that $(h, \mathbb{A})$ is a neutrosophic soft group. This proves the first item and

1. $\mu_{h_a}(\varkappa y) \geq \min\left\{\mu_{h_a}(\varkappa), \mu_{h(b)}(y)\right\}$,
2. $\nu_{h_a}(\varkappa y) \leq \max\left\{\nu_{h(b)}(\varkappa), \nu_{h(b)}(y)\right\}$,
3. $\omega_{h_a}(\varkappa y) \leq \max\left\{\omega_{h_a}(\varkappa), \omega_{h_a}(y)\right\}$,
4. $\mu_{h_a}\left(\varkappa^{-1}\right) \geq \mu_{h_a}(\varkappa)$,
5. $\nu_{h_a}(\varkappa^{-1}) \leq \nu_{h_a}(\varkappa)$,
6. $\omega_{h_a}\left(\varkappa^{-1}\right) \leq \omega_{h_a}(\varkappa)$.

This implies that $(h, \mathbb{A})$ is a π-NS-group. This proves the second item.

$(\Leftarrow)$

If $(h, \mathbb{A})$ is a neutrosophic soft group and $(h, \mathbb{A})$ is a π-neutrosophic soft group, then for all $a \in \mathbb{A}$,

1. $P_{h_a}(\varkappa y) \geq \min\left\{P_{h(b)}(\varkappa), P_{h(b)}(y)\right\}$,
2. $q_{h_a}(\varkappa y) \leq \max\left\{q_{h_a}(\varkappa), q_{h_a}(y)\right\}$,
3. $r_{h_a}(\varkappa y) \leq \max\left\{r_{h_a}(\varkappa), r_{h_a}(y)\right\}$,
4. $P_{h_a}(\varkappa^{-1}) \geq P_{h_a}(\varkappa)$,
5. $q_{h_a}(\varkappa^{-1}) \leq q_{h_a}(\varkappa)$,
6. $r_{h_a}(\varkappa^{-1}) \leq r_{h_a}(\varkappa)$,

and

1. $\mu_{h_a}(\varkappa y) \geq \min\left\{\mu_{h_a}(\varkappa), \mu_{h_a}(y)\right\}$,
2. $\nu_{h_a}(\varkappa y) \leq \max\left\{\nu_{h_a}(\varkappa), \nu_{h_a}(y)\right\}$,
3. $\omega_{h_a}(\varkappa y) \leq \max\left\{\omega_{h_a}(\varkappa), \omega_{h_a}(y)\right\}$,
4. $\mu_{h_a}(\varkappa^{-1}) \geq \mu_{h_a}(\varkappa)$,
5. $\nu_{h_a}(\varkappa^{-1}) \leq \nu_{h_a}(\varkappa)$,
6. $\omega_{h_a}\left(\varkappa^{-1}\right) \leq \omega_{h_a}(\varkappa)$.

Now,

$$\begin{aligned} T_{h_a}(\varkappa y) &= P_{h_a}(\varkappa y)\, e^{j\mu_{h_a}(\varkappa y)} \\ &\geq \min\left\{P_{h_a}(\varkappa), P_{h_a}(y)\right\} \cdot e^{j\min\left\{\mu_{h_a}(\varkappa), \mu_{h_a}(y)\right\}} \end{aligned}$$

$$= \min\left\{P_{h_a}(x)\, e^{j\mu_{h_a}(x)}, P_{h(b)}(y)\, e^{j\mu_{h_a}(y)}\right\}.$$

Since $(h, \mathbb{A})$ is homogeneous,

$$T_{h_a}(xy) \geq \min\left\{T_{h_a}(x), T_{h_a}(y)\right\}.$$

In a similar manner,

$$\begin{aligned} I_{h_a}(xy) &= q_{h_a}(xy)\, e^{j\nu_{h_a}(xy)} \\ &\leq \max\left\{I_{h_a}(x), I_{h_a}(y)\right\} \cdot e^{j\max\left\{\nu_{h_a}(x),\nu_{h_a}(y)\right\}} \\ &= \max\left\{q_{h_a}(x)\, e^{j\nu_{h_a}(x)}, q_{h_a}(y)\, e^{j\nu_{h_a}(y)}\right\}. \end{aligned}$$

Since $(h, \mathbb{A})$ is homogeneous,

$$I_{h_a}(xy) \leq \max\left\{I_{h_a}(x), I_{h_a}(y)\right\}.$$

In the same manner, we prove the inequality

$$\begin{aligned} F_{h_a}(xy) &= r_{h_a}(xy)\, e^{j\omega_{h_a}(xy)} \\ &\leq \max\left\{F_{h_a}(x), F_{h_a}(y)\right\} \cdot e^{j\max\left\{\omega_{h_a}(x),\omega_{h_a}(y)\right\}} \\ &= \max\left\{r_{h_a}(x)\, e^{j\omega_{h_a}(x)}, r_{h_a}(y)\, e^{j\omega_{h_a}(y)}\right\}. \end{aligned}$$

Since $(h, \mathbb{A})$ is homogeneous, we have

$$F_{h_a}(xy) \leq \max\left\{F_{h_a}(x), F_{h_a}(y)\right\}.$$

On the other hand,

$$\begin{aligned} T_{h_a}(x^{-1}) &= P_{h_a}(x^{-1}) e^{j\mu_{h_a}}(x^{-1}) \\ &\geq P_{h_a}(x)\, e^{j\mu_{h_a}(x)} \qquad \text{(homogeneity)} \\ &= T_{h_a}(x). \end{aligned}$$

Also,

$$\begin{aligned} I_{h_a}(x^{-1}) &= q_{h_a}(x^{-1}) e^{j\nu_{h_a}}(x^{-1}) \\ &\leq q_{h_a}(x)\, e^{j\nu_{h_a}(x)} \qquad \text{(homogeneity)} \\ &= I_{h_a}(x). \end{aligned}$$

Similarly, it can easily be proven that

$$\begin{aligned} &F_{h_a}(x^{-1}) \leq F_{h_a}(x) \\ &F_{h_a}(x^{-1}) = r_{h_a}(x^{-1}) e^{j\omega_{h_a}(x^{-1})} \end{aligned}$$

$$\le r_{h_a}(x)\, e^{j\omega_{h_a}(x)} \qquad \text{(homogeneity)}$$
$$= F_{h_a}(x).$$

Therefore, $(h, \mathbb{A})$ is a homogenous CNS-group. □

Theorem 3.8 *Let $(h, \mathbb{A})$ be a homogenous complex neutrosophic soft set on a group G. Then $(h, \mathbb{A})$ is a homogenous complex neutrosophic soft group if and only if, for all $a \in \mathbb{A}$, $x, y \in G$,*

1. $T_{h_a}\left(xy^{-1}\right) \ge \min\left\{T_{h_a}(x), T_{h_a}(y)\right\}$,
2. $I_{h_a}\left(xy^{-1}\right) \le \max\left\{I_{h_a}(x), I_{h_a}(y)\right\}$,
3. $F_{h_a}\left(xy^{-1}\right) \le \max\left\{F_{h_a}(x), F_{h_a}(y)\right\}$.

Proof

($\Rightarrow$)

Let $(h, \mathbb{A})$ be a homogenous CNSG, and $x, y \in G$. Then, for all $a \in \mathbb{A}$, we prove the two parts related to the complex truth as well as falsity membership functions. We begin with the truth membership proof as follows:

Since $(h, \mathbb{A})$ is a homogenous CNSG, we obtain

$$T_{h_a}(xy) \ge \min\left\{T_{h_a}(x), T_{h_a}(y)\right\}.$$

Since $T_{h_a}\left(x^{-1}\right) \ge T_{h_a}(x)$, we have

$$T_{h_a}\left(xy^{-1}\right) \ge \min\left\{T_{h_a}(x), T_{h_a}\left(y^{-1}\right)\right\}$$
$$\ge \min\left\{T_{h_a}(x), T_{h_a}(y)\right\}.$$

Next, the falsity of membership can be proven as follows:

We have $F_{h_a}(xy) \le \max\left\{F_{h_a}(x), F_{h_a}(y)\right\}$.

Since $F_{h_a}(x^{-1}) \le F_{h_a}(x)$,

$$F_{h_a}\left(xy^{-1}\right) \le \max\left\{F_{h_a}(x), F_{h_a}\left(y^{-1}\right)\right\}$$
$$\le \max\left\{F_{h_a}(x), F_{h_a}(y)\right\}.$$

The indeterminacy membership can be proven as follows:

$$I_{h_a}(xy) \le \max\left\{I_{h_a}(x), I_{h_a}(y)\right\}.$$

Since $I_{h_a}(x^{-1}) \le I_{h_a}(x)$,

$$I_{h_a}\left(xy^{-1}\right) \le \max\left\{I_{h_a}(x), I_{h_a}\left(y^{-1}\right)\right\}$$
$$\le \max\left\{I_{h_a}(x), I_{h_a}(y)\right\}.$$

($\Leftarrow$)

If e is a unit of a group G, then

$$T_{h_a}(x^{-1}) = T_{h_a}\left(e \cdot x^{-1}\right) \ge \min\left\{T_{h_a}(e), T_{h_a}(x)\right\}$$

$$
\begin{aligned}
&= \min\left\{T_{h_a}\left(\varkappa\cdot\varkappa^{-1}\right), T_{h_a}(\varkappa)\right\}\\
&\geq \min\left\{T_{h_a}(\varkappa), T_{h_a}(\varkappa), T_{h_a}(\varkappa)\right\}\\
&= T_{h_a}(\varkappa).\\
T_{h_a}(\varkappa^{-1}) &\geq T_{h_a}(\varkappa).
\end{aligned}
$$

Now, similarly we prove that $F_{h_a}(\varkappa^{-1}) \leq F_{h_a}(\varkappa)$ as follows:

$$
\begin{aligned}
F_{h_a}(\varkappa^{-1}) = F_{h_a}\left(e\cdot\varkappa^{-1}\right) &\leq \max\left\{F_{h_a}(e), F_{h_a}(\varkappa)\right\}\\
&= \max\left\{F_{h_a}\left(\varkappa\cdot\varkappa^{-1}\right), F_{h_a}(\varkappa)\right\}\\
&\leq \max\left\{F_{h_a}(\varkappa), F_{h_a}(\varkappa), F_{h_a}(\varkappa)\right\}\\
&= F_{h_a}(\varkappa).\\
F_{h_a}\left(\varkappa^{-1}\right) &\leq F_{h_a}(\varkappa).
\end{aligned}
$$

The indeterminacy membership can be proven in the same manner as it has been done with the falsity membership function, as follows:

$$
\begin{aligned}
I_{h_a}(\varkappa^{-1}) = I_{h_a}\left(e\cdot\varkappa^{-1}\right) &\leq \max\left\{I_{h_a}(e), I_{h_a}(\varkappa)\right\}\\
&= \max\left\{I_{h_a}\left(\varkappa\cdot\varkappa^{-1}\right), I_{h_a}(\varkappa)\right\}\\
&\leq \max\left\{I_{h_a}(\varkappa), I_{h_a}(\varkappa), I_{h_a}(\varkappa)\right\}\\
&= I_{h_a}(\varkappa).\\
I_{h_a}(\varkappa^{-1}) &\leq I_{h_a}(\varkappa).
\end{aligned}
$$

Now, we proceed to the next conditions of the complex neutrosophic soft group. We first start by proving the truth membership function as follows:

$$
\begin{aligned}
T_{h_a}(\varkappa y) &\geq T_{h_a}\left(\varkappa\cdot(y^{-1})^{-1}\right)\\
&\geq \min\left\{T_{h_a}(\varkappa), T_{h_a}(y^{-1})\right\}\\
&\geq \min\left\{T_{h_a}(\varkappa), T_{h_a}(y)\right\}.
\end{aligned}
$$

Now, for the falsity membership, we have

$$
\begin{aligned}
F_{h_a}(\varkappa y) &= F_{h_a}\left(\varkappa\cdot(y^{-1})^{-1}\right)\\
F_{h_a}(\varkappa y) &\leq \max\left\{F_{h_a}(\varkappa), F_{h_a}\left(y^{-1}\right)\right\}\\
&\leq \max\left\{F_{h_a}(\varkappa), F_{h_a}(y)\right\}.
\end{aligned}
$$

The indeterminacy membership part can be proven in the same manner as it has been done with the falsity membership part.

$$
I_{h_a}(\varkappa y) = I_{h_a}\left(\varkappa\cdot(y^{-1})^{-1}\right)
$$

$$\begin{aligned} I_{h_a}(\varkappa y) &\le \max\left\{I_{h_a}(\varkappa), I_{h_a}\left(y^{-1}\right)\right\} \\ &\le \max\left\{I_{h_a}(\varkappa), I_{h_a}(y)\right\}. \end{aligned}$$

□

Theorem 3.9 *Let* $(h, \mathbb{A})$ *be a homogenous complex neutrosophic soft group on a group* G. *If* e *is the unit element of* G, *then for all* $a \in \mathbb{A}$, $\varkappa \in G$, *the following six conditions hold*:

1. $T_{h_a}(e) \ge T_{h_a}(\varkappa)$,
2. $F_{h_a}(e) \le F_{h_a}(\varkappa)$,
3. $I_{h_a}(e) \le I_{h_a}(\varkappa)$,
4. $T_{h_a}(\varkappa^{-1}) \ge T_{h_a}(\varkappa)$,
5. $F_{h_a}\left(\varkappa^{-1}\right) \le F_{h_a}(\varkappa)$,
6. $I_{h_a}(\varkappa^{-1}) \le I_{h_a}(\varkappa)$.

Proof Conditions (1–3):

Let $e \in G$ and $\varkappa \in G$. By Definition 3.5,

$$\begin{aligned} T_{h_a}(e) &= T_{h_a}\left(\varkappa \cdot (\varkappa^{-1})\right) \\ &\ge \min\left\{T_{h_a}(\varkappa), T_{h_a}(\varkappa^{-1})\right\} \\ &= \min\left\{P_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)}, P_{h_a}(\varkappa^{-1}) e^{j\mu_{h_a}(\varkappa^{-1})}\right\} \\ &\ge \min\left\{P_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)}, P_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)}\right\} \\ &= P_{h_a}(\varkappa)\, e^{j\mu_{h_a}(\varkappa)} \\ &= T_{h_a}(\varkappa). \end{aligned}$$

Therefore, $T_{h_a}(e) \ge T_{h_a}(\varkappa)$.

For falsity membership, we have

$$\begin{aligned} F_{h_a}(e) &= F_{h_a}\left(\varkappa \cdot (\varkappa^{-1})\right) \\ &\le \max\left\{F_{h_a}(\varkappa), F_{h_a}(\varkappa^{-1})\right\} \\ &= \max\left\{q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)}, q_{h_a}(\varkappa^{-1}) e^{j\nu_{h_a}(\varkappa^{-1})}\right\} \\ &\le \max\left\{q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)}, q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)}\right\} \\ &= q_{h_a}(\varkappa)\, e^{j\nu_{h_a}(\varkappa)} \\ &= F_{h_a}(\varkappa). \end{aligned}$$

Hence, $F_{h_a}(e) \le F_{h_a}(\varkappa)$.

Similarly, the condition for indeterminacy membership function can be proven in the same manner.

$$\begin{aligned} I_{h_a}(e) &= I_{h_a}\left(\varkappa \cdot (\varkappa^{-1})\right) \\ &\le \max\left\{I_{h_a}(\varkappa), I_{h_a}(\varkappa^{-1})\right\} \\ &= \max\left\{r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)}, r_{h_a}(\varkappa^{-1}) e^{j\omega_{h_a}(\varkappa^{-1})}\right\} \end{aligned}$$

$$\leq \max\left\{r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)}, r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)}\right\}$$
$$= r_{h_a}(\varkappa)\, e^{j\omega_{h_a}(\varkappa)}$$
$$= I_{h_a}(\varkappa).$$

Therefore, $I_{h_a}(e) \leq I_{h_a}(\varkappa)$.

Conditions (4–6):

Let $\varkappa \in G$. Since $(h, \mathbb{A})$ is a homogenous complex neutrosophic soft group, we have

$$T_{h_a}\left(\varkappa^{-1}\right) \geq T_{h_a}(\varkappa).$$

Now, since $T_{h_a}(\varkappa) = T_{h_a}(\varkappa^{-1})^{-1} \geq T_{h_a}(\varkappa^{-1})$, we have

$$T_{h_a}(\varkappa) \geq T_{h_a}\left(\varkappa^{-1}\right).$$

Then,

$$T_{h_a}(\varkappa) = T_{h_a}(\varkappa^{-1}).$$

Looking into the falsity membership

$$F_{h_a}(\varkappa^{-1}) \leq F_{h_a}(\varkappa),$$

we need to show that $F_{h_a}(\varkappa) \leq F_{h_a}\left(\varkappa^{-1}\right)$.

Since $F_{h_a}(\varkappa) = F_{h_a}(\varkappa^{-1})^{-1} \leq F_{h_a}(\varkappa^{-1})$, we have

$$F_{h_a}(\varkappa) \leq F_{h_a}\left(\varkappa^{-1}\right).$$

Hence,

$$F_{h_a}(\varkappa) = F_{h_a}(\varkappa^{-1}).$$

For Condition 6, it can be proven in the same way as in Condition 5. Given

$$I_{h_a}(\varkappa^{-1}) \leq I_{h_a}(\varkappa),$$

we need to show that $I_{h_a}(\varkappa) \leq I_{h_a}(\varkappa)^{-1}$.

Since $I_{h_a}(\varkappa) = I_{h_a}(\varkappa^{-1})^{-1} \leq I_{h_a}(\varkappa^{-1})$, we have

$$I_{h_a}(\varkappa) \leq I_{h_a}\left(\varkappa^{-1}\right)$$

Therefore,

$$I_{h_a}(\varkappa) = I_{h_a}(\varkappa^{-1}).$$

□

Theorem 3.10 *Let G be a group, $(h,\mathbb{A})$ and $(k,\mathbb{B})$ be two homogenous-NSSs on G. If $(h,\mathbb{A})$ and $(k,\mathbb{B})$ are two homogenous neutrosophic soft groups on G, then $(h,\mathbb{A})\cap(k,\mathbb{B})$ is a homogenous neutrosophic soft group.*

Proof Let $\mathbb{x},\mathbb{y}\in G$ and $\varepsilon\in\mathbb{A}\cap\mathbb{B}$. By Theorem 3.8, it is sufficient to show that

$$\begin{aligned}
&T_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\geq\min\left\{T_{h\cap k_\varepsilon}(\mathbb{x}),T_{h\cap k_\varepsilon}(\mathbb{y})\right\},\\
&F_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\leq\max\left\{F_{h\cap k_\varepsilon}(\mathbb{x}),F_{h\cap k_\varepsilon}(\mathbb{y})\right\},\\
&I_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\leq\max\left\{I_{h\cap k_\varepsilon}(\mathbb{x}),I_{h\cap k_\varepsilon}(\mathbb{y})\right\}.
\end{aligned}$$

By looking at the first condition, we have

$$\begin{aligned}
T_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)&=P_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\mu_{h\cap k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})}\\
&=\min\left\{P_{h_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right),P_{k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\right\}\cdot e^{j\min\left\{\mu_{h_\varepsilon}(\mathbb{x}\mathbb{y}^{-1}),\mu_{k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})\right\}}\\
&=\min\left\{\begin{array}{l}P_{h_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\mu_{h_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})},\\ P_{k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\mu_{k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})}\end{array}\right\}\\
&\geq\min\left\{\begin{array}{l}\min\left\{P_{h_\varepsilon}(\mathbb{x})\cdot e^{j\mu_{h_\varepsilon}(\mathbb{x})},P_{h_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{h_\varepsilon}(\mathbb{y})}\right\},\\ \min\left\{P_{k_\varepsilon}(\mathbb{x})\cdot e^{j\mu_{k_\varepsilon}(\mathbb{x})},P_{k_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{k_\varepsilon}(\mathbb{y})}\right\}\end{array}\right\}\\
&=\min\left\{\begin{array}{l}\min\left\{P_{h_\varepsilon}(\mathbb{x})\cdot e^{j\mu_{h_\varepsilon}(\mathbb{x})},P_{k_\varepsilon}(\mathbb{x})\cdot e^{j\mu_{k_\varepsilon}(\mathbb{x})}\right\},\\ \min\left\{P_{h_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{h_\varepsilon}(\mathbb{y})},P_{k_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{k_\varepsilon}(\mathbb{y})}\right\}\end{array}\right\}\\
&=\min\left\{\begin{array}{l}\min\left\{P_{h_\varepsilon}(\mathbb{x}),P_{k_\varepsilon}(\mathbb{x})\right\}\cdot e^{j\min\left\{\mu_{h_\varepsilon}(\mathbb{x}),\mu_{k_\varepsilon}(\mathbb{x})\right\}},\\ \min\left\{P_{h_\varepsilon}(\mathbb{y}),P_{k_\varepsilon}(\mathbb{y})\right\}\cdot e^{j\min\left\{\mu_{h_\varepsilon}(\mathbb{y}),\mu_{k_\varepsilon}(\mathbb{y})\right\}}\end{array}\right\}\\
&=\min\left\{\begin{array}{l}P_{h\cap k_\varepsilon}(\mathbb{x})\cdot e^{j\mu_{h\cap k_\varepsilon}(\mathbb{x})},\\ P_{h\cap k_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{h\cap k_\varepsilon}(\mathbb{y})}\end{array}\right\}\\
&=\min\left\{T_{h\cap k_\varepsilon}(\mathbb{x}),T_{h\cap k_\varepsilon}(\mathbb{y})\right\}.
\end{aligned}$$

Considering the second condition, we have

$$\begin{aligned}
F_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)&=q_{h\cap k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\nu_{h\cap k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})}\\
&=\max\left\{q_{h_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right),q_{k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\right\}\cdot e^{j\max\left\{\nu_{h_\varepsilon}(\mathbb{x}\mathbb{y}^{-1}),\nu_{k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})\right\}}\\
&=\max\left\{\begin{array}{l}q_{h_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\nu_{h_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})},\\ q_{k_\varepsilon}\left(\mathbb{x}\mathbb{y}^{-1}\right)\cdot e^{j\nu_{k_\varepsilon}(\mathbb{x}\mathbb{y}^{-1})}\end{array}\right\}\\
&\leq\max\left\{\begin{array}{l}\max\left\{q_{h_\varepsilon}(\mathbb{x})\cdot e^{j\nu_{h_\varepsilon}(\mathbb{x})},q_{h_\varepsilon}(\mathbb{y})\cdot e^{j\nu_{h_\varepsilon}(\mathbb{y})}\right\},\\ \max\left\{q_{k_\varepsilon}(\mathbb{x})\cdot e^{j\nu_{k_\varepsilon}(\mathbb{x})},q_{k_\varepsilon}(\mathbb{y})\cdot e^{j\nu_{k_\varepsilon}(\mathbb{y})}\right\}\end{array}\right\}\\
&=\max\left\{\begin{array}{l}\max\left\{q_{h_\varepsilon}(\mathbb{x})\,e^{j\nu_{h_\varepsilon}(\mathbb{x})},q_{k_\varepsilon}(\mathbb{x})\cdot e^{j\nu_{k_\varepsilon}(\mathbb{x})}\right\},\\ \max\left\{q_{h_\varepsilon}(\mathbb{y})\,e^{j\nu_{h_\varepsilon}(\mathbb{y})},q_{k_\varepsilon}(\mathbb{y})\cdot e^{j\nu_{k_\varepsilon}(\mathbb{y})}\right\}\end{array}\right\}\\
&=\max\left\{\begin{array}{l}\max\left\{q_{h_\varepsilon}(\mathbb{x}),q_{k(\varepsilon)}(\mathbb{x})\right\}\cdot e^{j\max\left\{\nu_{h_\varepsilon}(\mathbb{x}),\nu_{k_\varepsilon}(\mathbb{x})\right\}},\\ \max\left\{q_{h_\varepsilon}(\mathbb{y}),q_{k(\varepsilon)}(\mathbb{y})\right\}\cdot e^{j\max\left\{\nu_{h_\varepsilon}(\mathbb{y}),\nu_{k_\varepsilon}(\mathbb{y})\right\}}\end{array}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\left\{\begin{array}{l} q_{h\cap k_\varepsilon}(\varkappa)\cdot e^{j\nu_{h\cap k_\varepsilon}(\varkappa)}, \\ q_{h\cap k_\varepsilon}(\mathbb{y})\cdot e^{j\mu_{h\cap k_\varepsilon}(\mathbb{y})} \end{array}\right\} \\
&= \max\left\{F_{h\cap k_\varepsilon}(\varkappa), F_{h\cap k_\varepsilon}(\mathbb{y})\right\}.
\end{aligned}
$$

Looking into the final condition:

$$
\begin{aligned}
I_{h\cap k_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right) &= r_{h\cap k_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right)\cdot e^{j\nu_{h\cap k_\varepsilon}(\varkappa\mathbb{y}^{-1})} \\
&= \max\left\{r_{h_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right), r_{k_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right)\right\}\cdot e^{j\max\left\{\omega_{h_\varepsilon}(\varkappa\mathbb{y}^{-1}),\omega_{k_\varepsilon}(\varkappa\mathbb{y}^{-1})\right\}} \\
&= \max\left\{\begin{array}{l} r_{h_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right) e^{j\omega_{h_\varepsilon}(\varkappa\mathbb{y}^{-1})}, \\ r_{k_\varepsilon}\left(\varkappa\mathbb{y}^{-1}\right) e^{j\omega_{k_\varepsilon}(\varkappa\mathbb{y}^{-1})} \end{array}\right\} \\
&\le \max\left\{\begin{array}{l} \max\left\{r_{h_\varepsilon}(\varkappa)\, e^{j\omega_{h_\varepsilon}(\varkappa)}, r_{h_\varepsilon}(\mathbb{y})\, e^{j\omega_{h_\varepsilon}(\mathbb{y})}\right\} \\ \max\left\{r_{k_\varepsilon}(\varkappa)\, e^{j\omega_{k_\varepsilon}(\varkappa)}, r_{k_\varepsilon}(\mathbb{y})\, e^{j\omega_{k_\varepsilon}(\mathbb{y})}\right\} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \max\left\{r_{h_\varepsilon}(\varkappa)\, e^{j\omega_{h_\varepsilon}(\varkappa)}, r_{k_\varepsilon}(\varkappa)\, e^{j\omega_{k_\varepsilon}(\varkappa)}\right\} \\ \max\left\{r_{h_\varepsilon}(\mathbb{y})\, e^{j\omega_{h_\varepsilon}(\mathbb{y})}, r_{k_\varepsilon}(\mathbb{y})\, e^{j\omega_{k_\varepsilon}(\mathbb{y})}\right\} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \max\left\{r_{h_\varepsilon}(\varkappa), r_{k(\varepsilon)}(\varkappa)\right\}\cdot e^{j\max\left\{\omega_{h_\varepsilon}(\varkappa),\omega_{k_\varepsilon}(\varkappa)\right\}}, \\ \max\left\{r_{h_\varepsilon}(\mathbb{y}), r_{k(\varepsilon)}(\mathbb{y})\right\}\cdot e^{j\max\left\{\omega_{h_\varepsilon}(\mathbb{y}),\omega_{k_\varepsilon}(\mathbb{y})\right\}} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} r_{h\cap k_\varepsilon}(\varkappa)\, e^{j\nu_{h\cap k_\varepsilon}(\varkappa)}, \\ r_{h\cap k_\varepsilon}(\mathbb{y})\, e^{j\mu_{h\cap k_\varepsilon}(\mathbb{y})} \end{array}\right\} \\
&= \max\left\{I_{h\cap k_\varepsilon}(\varkappa), I_{h\cap k_\varepsilon}\mathbb{y}\right\}.
\end{aligned}
$$

Therefore, $(h,\mathbb{A})\cap(k,\mathbb{B})$ is a CNSG. □

4 Conclusions and future work

This research has established the complex neutrosophic soft group as a generalization of the complex fuzzy soft group. The π-neutrosophic soft group has also been defined to represent the phase term of the complex-valued membership functions. The notion of the complex neutrosophic soft group has been defined along with the study of some important operations and basic properties of these concepts. The definitions and frameworks for homogeneous complex neutrosophic soft sets and complete homogeneous complex neutrosophic soft sets, particularly as they relate to the CNSG, were introduced and discussed in detail. The main innovation of this study is the introduction of CNSG, with specific examples illustrating the critical role of the phase element in the context of neutrosophic soft group theory. This work not only enriches the field of soft set theory but also creates new pathways for future research and practical applications in dealing with uncertainty and complexity.

Since graphs [5, 21] are commonly used to model a variety of real-world problems and that learning-based approaches [20] have been proposed in addressing these problems, it is natural to ask if there exists a relationship between graphs and complex neutrosophic soft groups so that relevant approaches can be applied in dealing with uncertainty [7, 16] in a more effective manner. In addition, it might be worth exploring if neutrosophic logic could be further developed as a tool to produce more accurate results in decision-making scenarios.

Author contributions
F.R. and K.S.Y. wrote the main manuscript and N.M.A.N.L and K.S.Y. revised the manuscript. All authors reviewed the manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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