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# Approximation properties of a modified Gauss–Weierstrass singular integral in a weighted space

Abhay Pratap Singh[1] and Uaday Singh[1*]

*Correspondence:
uadayfma@iitr.ac.in
[1]Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, Uttarakhand, India

**Abstract**

Singular integral operators play an important role in approximation theory and harmonic analysis. In this paper, we consider a weighted Lebesgue space $L^{p,w}$, define a modified Gauss–Weierstrass singular integral on it, and study direct and inverse approximation properties of the operator followed by a Korovkin-type approximation theorem for a function $f \in L^{p,w}$. We use the modulus of continuity of the functions to measure the rate of convergence.

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**Keywords:** Modulus of continuity; Gauss–Weierstrass singular operator; Weighted Lebesgue spaces; Weighted Hölder space

## 1 Introduction

The approximation of functions by singular integrals is a significant topic in the theory of partial differential equations and integro-differential equations. The Gauss–Weierstrass singular integral is closely connected to the initial value problem of the heat equation for an infinite rod [5, pp. 125–126]. Furthermore, this integral has practical applications in signal and image processing, serving as a Gaussian blur and low-pass filter [11]. In this study, we focus on the approximation properties of a modified Gauss–Weierstrass singular integral in weighted spaces.

Let $w$ be a positive weight function defined on $\mathbb{R}$ that satisfies the following conditions:

(1) $w(x)$ is an even function on $\mathbb{R}$ and nonincreasing for $x > 0$;

(2) $\mathcal{W}_\epsilon(s) = \int_0^\infty e^{-st^2} \frac{1}{w(t+\epsilon)} dt < \infty$, $s \geq 0$, $\epsilon \geq 0$;

(3) $\sup_{x\in\mathbb{R}} \left(\frac{w(x)}{w(x-h)}\right) \leq \frac{1}{w(h)}$, $h \in \mathbb{R}$.

For $1 \leq p < \infty$ and given a weight function $w$, we denote by $L^{p,w}$ the set of all real valued functions $f$ defined on $\mathbb{R}$ for which $|wf|^p$ is Lebesgue integrable on $\mathbb{R}$. For $p = \infty$ and given weight function $w$, we denote by $L^{\infty,w}$ the set of all real valued functions $f$ for which $wf$ is a uniformly continuous and bounded function on $\mathbb{R}$.

For $1 \leq p \leq \infty$, the norm of $f \in L^{p,w}$ is defined as

$$||f||_{p,w} = \begin{cases} (\int_{\mathbb{R}} |w(x)f(x)|^p dx)^{1/p}, & 1 \leq p < \infty; \\ \sup\limits_{x \in \mathbb{R}} w(x)|f(x)|, & p = \infty. \end{cases}$$

We define a modified Gauss–Weierstrass singular integral $W^*_{r,n}(f,x)$ of a function $f \in L^{p,w}$ as follows:

$$W^*_{r,n}(f,x) = \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(s_n(x)+t) e^{\frac{-t^2}{4r}} dt, x \in \mathbb{R}, r > 0, \tag{1}$$

where $r := r(n) \to 0$ as $n \to \infty$ and $s_n(x), n \in \mathbb{N}$ is a sequence of functions such that $s_n(x) \to x$ as $n \to \infty$, that is, for any $\epsilon > 0$, there exists a positive integer $n_0$ such that $|s_n(x) - x| < \epsilon$ for all $n \geq n_0(\epsilon)$ and $s'_n(x) = 1$.

Also the $k^{th}$ derivative of $W^*_{r,n}(f,x)$ is defined by

$$W^{(k)}_{r,n}(f,x) = \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f^{(k)}(s_n(x)+t) e^{\frac{-t^2}{4r}} dt.$$

If $s_n(x) = x$, then $W^*_{r,n}(f,x)$ reduces to the classical Gauss–Weierstrass singular integral

$$W_{r,n}(f,x) = \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(x+t) e^{\frac{-t^2}{4r}} dt, x \in \mathbb{R}, r > 0.$$

For $f \in L^{p,w}$, its modulus of continuity of order $j$ is defined by

$$\omega_j(f, L^{p,w}, t) = \sup_{0 \leq |k| \leq t} ||\Delta^j_k f(x)||_{p,w}, j \in \mathbb{Z}^+,$$

where $\Delta^j_k f(x) = \sum_{m=0}^{j} (-1)^{j-m} \binom{j}{m} f(x+mk)$.

We denote by $\Omega^2$ the set of all real valued functions $\mu$, where $\mu$ is continuous and nondecreasing, satisfies $\mu(0) = 0$ and $\frac{\mu(t)}{t^2}$ is nonincreasing for all positive values of $t$ [20, pp. 93–97]. For given $1 \leq p \leq \infty$ and $\mu \in \Omega^2$, we define the weighted Hölder space $H^{p,w,\mu}$ of order $j$ to be the set of all functions $f$ in $L^{p,w}$ for which

$$||f||_{p,w,\mu} = \sup_{h>0} \left( \frac{||\Delta^j_h f(.)||_{p,w}}{\mu(h)} \right) < \infty.$$

The norm in $H^{p,w,\mu}$ space is defined by

$$||f||^*_{p,w,\mu} = ||f||_{p,w} + ||f||_{p,w,\mu}. \tag{2}$$

We note that for $f \in H^{p,w,\mu}$,

$$\omega_j(f, L^{p,w}, t) \leq \mu(t) ||f||_{p,w,\mu}.$$

In particular, if $w(x) = e^{-2ax^2}$, $a > 0$, then $L^{p,w} = L^{p,a}$, $H^{p,w,\mu} = H^{p,a,\mu}$, and the corresponding norms are defined by $\|.\|_{p,w} = \|.\|_{p,2a}$ and $\|.\|_{p,w,\mu} = \|.\|_{p,2a,\mu}$ [21]. Further, for $s_n(x) = x - \frac{a}{2n}$, $a > 0$ and $r = \frac{1}{4n}$, $n \in \mathbb{N}$ our operator defined in (1) reduces to $(W^*_n f)(x)$ [21, p. 90].

We obtain the following important upper estimates for the norm of $W^*_{r,n}(f,x)$.

For any $\epsilon > 0$ and for any fixed $n \geq n_0(\epsilon) \in \mathbb{N}$, using the properties of $w$ and $f \in L^{\infty,w}$, we have

$$\begin{aligned}
||W^*_{r,n}(f,x)||_{\infty,w} &= \sup_{x\in\mathbb{R}} \left( w(x)|W^*_{r,n}(f,x)| \right) \\
&= \sup_{x\in\mathbb{R}} \left( w(x) \left| \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(s_n(x)+t) e^{\frac{-t^2}{4r}} dt \right| \right) \\
&\leq \frac{1}{\sqrt{4\pi r}} \sup_{x\in\mathbb{R}} \left( \int_{\mathbb{R}} \left| \frac{w(x)}{w(s_n(x)+t)} w(s_n(x)+t) f(s_n(x)+t) \right| e^{\frac{-t^2}{4r}} dt \right) \\
&\leq \frac{1}{\sqrt{\pi r}} \left( \int_0^\infty \frac{1}{w(\epsilon+t)} e^{\frac{-t^2}{4r}} dt \right) ||f||_{\infty,w} \\
&= \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) ||f||_{\infty,w}.
\end{aligned}$$

For $f \in L^{p,w}$, $1 \leq p < \infty$, using the generalized Minkowski inequality and the properties of $w$, we have

$$\begin{aligned}
||W^*_{r,n}(f,x)||_{p,w} &= \left( \int_{\mathbb{R}} |w(x) W^*_{r,n}(f,x)|^p dx \right)^{\frac{1}{p}} \\
&= \left( \int_{\mathbb{R}} \left| w(x) \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(s_n(x)+t) e^{\frac{-t^2}{4r}} dt \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \frac{1}{\sqrt{4\pi r}} \left( \int_{\mathbb{R}} e^{\frac{-t^2}{4r}} \right. \\
&\quad \times \left. \left( \int_{\mathbb{R}} \left| \frac{w(x)}{w(s_n(x)+t)} w(s_n(x)+t) f(s_n(x)+t) \right|^p dx \right)^{\frac{1}{p}} dt \right).
\end{aligned}$$

Since the derivative of $s_n(x)$ is uniformly bounded, the substitution $s_n(x) + t = u$ yields

$$\begin{aligned}
||W^*_{r,n}(f,x)||_{p,w} &\leq \frac{1}{\sqrt{4\pi r}} \left( \int_{\mathbb{R}} e^{\frac{-t^2}{4r}} \frac{1}{w(\epsilon+t)} \left( \int_{\mathbb{R}} \left| w(u) f(u) \right|^p du \right)^{\frac{1}{p}} dt \right) \\
&\leq \frac{1}{\sqrt{\pi r}} \left( \int_0^\infty \frac{1}{w(\epsilon+t)} e^{\frac{-t^2}{4r}} dt \right) ||f||_{p,w} \\
&= \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) ||f||_{p,w}.
\end{aligned}$$

Thus, we have

$$||W^*_{r,n}(f,x)||_{p,w} \leq \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) ||f||_{p,w}, n \geq n_0(\epsilon) \in \mathbb{N}, 1 \leq p \leq \infty.$$

Using similar calculations, for $f \in L^{p,w}$, $1 \leq p \leq \infty$, with $f^{(k)} \in L^{p,w}$, we can prove that

$$||W^{(k)}_{r,n}(f,x)||_{p,w} \leq \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) ||f^{(k)}||_{p,w}, n \geq n_0(\epsilon) \in \mathbb{N}, 1 \leq p \leq \infty. \tag{3}$$

Now, for $f \in H^{p,w,\mu}$, $1 \le p < \infty$, we have

$$\begin{aligned}
||W^*_{r,n}(f,x)||_{p,w,\mu} &= \sup_{h>0} \frac{||\Delta^j_h(W^*_{r,n}(f,x))||_{p,w}}{\mu(h)} \\
&= \sup_{h>0} \frac{\| W^*_{r,n}\left(\Delta^j_h f(x)\right)\|_{p,w}}{\mu(h)} \\
&\le \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon\left(\frac{1}{4r}\right) \sup_{h>0} \frac{||\Delta^j_h(f(x))||_{p,w}}{\mu(h)} \\
&= \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon\left(\frac{1}{4r}\right) ||f||_{p,w,\mu}.
\end{aligned} \tag{4}$$

Combining (4) and (2), we get

$$||W^*_{r,n}(f,x)||^*_{p,w,\mu} \le \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon\left(\frac{1}{4r}\right) ||f||^*_{p,w,\mu}, n \ge n_0(\epsilon) \in \mathbb{N}, 1 \le p \le \infty.$$

Hence, for any $\epsilon > 0$, the integral $W^*_{r,n}(f,x), n \ge n_0(\epsilon) \in \mathbb{N}$ is a well-defined bounded linear operator on the spaces $L^{p,w}$ and $H^{p,w,\mu}$ whenever $\mathcal{W}_\epsilon\left(\frac{1}{4r}\right)$ exists. An analogous property of integral (1) for $w(x) = e^{-2ax^2}$ and $s_n(x) = x - \frac{a}{2n}$, $a > 0$, $n \in \mathbb{N}$ can be seen in [21, p. 91, Lemma 2.3].

These operators are very useful in approximation theory, and their approximation properties have been studied by many researchers [6–8, 14, 17]. This study has further been extended to different weighted spaces [1, 4, 21, 22]. More precisely, Yilmaz [22] investigated the problem for the Gauss–Weierstrass singular integral within the space $L^{p,w}, 1 \le p \le \infty$, where $w(x) = e^{-a|x|^2}, a > 0$. The same author [21] also studied the approximation properties of the operator defined in (1) with $w(x) = e^{-a|x|^2}$ and $s_n(x) = x - \frac{a}{2n}, a > 0, n \in \mathbb{N}$. Bogalska et al. [4] studied the approximation properties of the Gauss–Weierstrass singular integral for functions of two variables in the exponentially weighted space, and Agratini et al. [1] studied the problem in the polynomial weighted space and calculated the rate of approximation using the weighted modulus of continuity. They also proved that the operators defined in (1) give the faster convergence results as compared to classical Gauss–Weierstrass operators with $s_n(x) = x - \frac{a}{2n}, a > 0, n \in \mathbb{N}$ [1, pp. 1195–1197]. Some important linear operators based on the Kantorovich-type modification can be found in [2] and [12]. In these articles, the authors explored direct theorems related to generalized Baskakov operators for the class of functions having a derivative coinciding almost everywhere with a function of bounded variation. Grewal et al. [10] investigated the approximation properties of a general family of positive linear operators defined on $[0, \infty]$ in weighted spaces by utilizing the weighted modulus of continuity. Some researchers [13, 15, 16] introduced a new type of sequence of linear positive operators for approximating Lebesgue integrable functions. Shvai et al. [18] studied approximation properties of the Gauss–Weierstrass singular operators in the neighborhood of a point of $\mathbb{R}$. Bardaro et al. [3] defined a class of operators that fixes exponential functions. They proved the Korovkin-type approximation theorem and a Voronovskaja-type formula for such operators.

In this paper, we study the approximation properties of the integral defined in (1) in the weighted Lebesgue spaces $L^{p,w}$ and the weighted Hölder space $H^{p,w,\mu}$ with a certain weight function. We also show that many of the theorems in the literature dealing with approximation of functions by the Gauss–Weierstrass singular integrals are the special cases of our results. We also prove the Korovkin-type results for functions belonging to the space $L^{p,w}$. Finally, we prove the inverse approximation theorem for the space $L^{p,w}$.

## 2 Direct approximation theorems

In this section, we study the direct approximation theorems for the functions belonging to classes $L^{p,w}$ and $H^{p,w,\mu}$, $1 \le p \le \infty$, $\mu \in \Omega$. We obtain the upper estimates for the deviation $W^*_{r,n}(f,x) - f(x)$ under respective norm in terms of $\omega_j(f, L^{p,w}, r), j = 1, 2$. We denote by $r_0$ the maximum values of $r$ for which $\mathcal{W}_0\left(\frac{1}{4r}\right)$ exists.

In [19, p. 1554], the authors derived the following lemma for any function $f \in L^{p,w}$, $1 \le p \le \infty$.

**Lemma 2.1** *Let $f \in L^{p,w}$, $1 \le p \le \infty$. Then*

$$\omega_2(f, L^{p,w}, \lambda t) \le (1+\lambda)^2 \frac{1}{w(\lambda t)} \omega_2(f, L^{p,w}, t) \text{ for any } \lambda, t > 0. \tag{5}$$

Now, we state main theorems of this section.

**Theorem 2.2** *Let $f \in L^{p,w}$, $1 \le p \le \infty$. Then, for any $\epsilon > 0$, there exists $n \ge n_0(\epsilon) \in \mathbb{N}$ such that*

$$\begin{aligned} ||W^*_{r,n}(f,x) - f(x)||_{p,w} &\le \omega_2(f, L^{p,w}, \epsilon) \\ &\quad + \frac{\omega_2(f, L^{p,w}, r)}{(\sqrt{4\pi}) r^{\frac{5}{2}}} \left( r^2 \mathcal{W}_0\left(\frac{1}{4r}\right) - (1+2r)\mathcal{W}_0'\left(\frac{1}{4r}\right) + \frac{2r}{w(1)} \right), \end{aligned}$$

*where $r \in (0, r_0]$.*

**Theorem 2.3** *Let $f \in L^{p,w}$, $1 \le p \le \infty$. Then, for any $\epsilon > 0$, there exists $n \ge n_0(\epsilon) \in \mathbb{N}$ such that*

$$||W^*_{r,n}(f,x) - f(x)||_{p,w} \le \omega\left(f, L^{p,\omega}, \epsilon\right) + \omega\left(f, L^{p,\omega}, r\right) \frac{1}{\sqrt{4\pi r}} \mathcal{W}_0\left(\frac{1}{4r}\right),$$

*where $r \in (0, r_0]$.*

*Proof of Theorem 2.2* Using (1), we can write

$$\begin{aligned} &W^*_{r,n}(f,x) - f(x) \\ &= \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(s_n(x)+t) e^{\frac{-t^2}{4r}} dt - \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} f(x) e^{\frac{-t^2}{4r}} dt \\ &= \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt \\ &= \frac{1}{\sqrt{4\pi r}} \int_{-\infty}^{0} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt + \frac{1}{\sqrt{4\pi r}} \int_{0}^{\infty} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt \\ &= \frac{1}{\sqrt{4\pi r}} \int_{0}^{\infty} (f(s_n(x)-t) - f(x)) e^{\frac{-t^2}{4r}} dt + \frac{1}{\sqrt{4\pi r}} \int_{0}^{\infty} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt \end{aligned} \tag{6}$$

$$= \frac{1}{\sqrt{4\pi r}} \int_0^\infty \left(f(s_n(x)+t) - 2f(x) + f(s_n(x)-t)\right) e^{\frac{-t^2}{4r}}\, dt.$$

Using Lemma 2.1 and the generalized Minkowski inequality, we have

$$\begin{aligned}
&\| W^*_{r,n}(f,x) - f(x)\|_{p,w} \\
&= \left\| \frac{1}{\sqrt{4\pi r}} \int_0^\infty \left(f(s_n(x)+t) - 2f(x) + f(s_n(x)-t)\right) e^{\frac{-t^2}{4r}}\, dt \right\|_{p,w} \\
&= \left( \int_{\mathbb{R}} \left| w(x) \frac{1}{\sqrt{4\pi r}} \int_0^\infty (f(s_n(x)+t) - 2f(x) + f(s_n(x)-t)) e^{\frac{-t^2}{4r}}\, dt \right|^p dx \right)^{\frac{1}{p}} \\
&= \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \left( \int_{\mathbb{R}} w(x) |f(s_n(x)+t) - 2f(x) + f(s_n(x)-t)|^p dx \right)^{\frac{1}{p}} dt \\
&= \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \|f(s_n(x)+t) - 2f(x) + f(s_n(x)-t)\|_{p,w} dt \\
&\le \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \omega_2\left(f, L^{p,w}, s_n(x) - x + t\right) dt \\
&\le \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \omega_2(f, L^{p,w}, |s_n(x) - x + t|) dt \\
&\le \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \omega_2(f, L^{p,w}, |s_n(x) - x|) dt + \frac{1}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \omega_2(f, L^{p,w}, |t|) dt \\
&\le \omega_2(f, L^{p,w}, \epsilon) + \frac{\omega_2(f, L^{p,w}, r)}{\sqrt{4\pi r}} \int_0^\infty e^{\frac{-t^2}{4r}} \left(1 + \frac{t}{r}\right)^2 \frac{1}{w(t)} dt \text{ for } n \ge n_0(\epsilon) \in \mathbb{N} \\
&\le \omega_2(f, L^{p,w}, \epsilon) \\
&\quad + \frac{\omega_2(f, L^{p,w}, r)}{\sqrt{4\pi r}} \left( \int_0^\infty \frac{1}{w(t)} e^{\frac{-t^2}{4r}} dt + \frac{1}{r^2} \int_0^\infty \frac{t^2}{w(t)} e^{\frac{-t^2}{4r}} dt + \frac{2}{r} \int_0^\infty \frac{t}{w(t)} e^{\frac{-t^2}{4r}} dt \right) \\
&\le \omega_2(f, L^{p,w}, \epsilon) \\
&\quad + \frac{\omega_2(f, L^{p,w}, r)}{\sqrt{4\pi r}} \left( \mathcal{W}_0\left(\frac{1}{4r}\right) - \frac{1}{r^2} \mathcal{W}_0'\left(\frac{1}{4r}\right) + \frac{2}{r} \left( \frac{1}{w(1)} - \mathcal{W}_0'\left(\frac{1}{4r}\right) \right) \right) \\
&\le \omega_2(f, L^{p,w}, \epsilon) + \frac{\omega_2(f, L^{p,w}, r)}{(\sqrt{4\pi}) r^{\frac{5}{2}}} \left( r^2 \mathcal{W}_0\left(\frac{1}{4r}\right) - (1 + 2r) \mathcal{W}_0'\left(\frac{1}{4r}\right) + \frac{2r}{w(1)} \right).
\end{aligned}$$

Hence the proof is completed. □

*Proof of Theorem* 2.3 Using (6) and the generalized Minkowski inequality, we have

$$\begin{aligned}
&\| W^*_{r,n}(f,x) - f(x)\|_{p,w} \\
&= \left\| \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt \right\|_{p,w} \\
&= \left( \int_{\mathbb{R}} w(x) \left| \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} (f(s_n(x)+t) - f(x)) e^{\frac{-t^2}{4r}} dt \right|^p dx \right)^{\frac{1}{p}} \\
&\le \frac{1}{\sqrt{4\pi r}} \left( \int_{\mathbb{R}} w(x) \left( \int_{\mathbb{R}} |f(s_n(x)+t) - f(x) + f(x+t) - f(x+t)|^p e^{\frac{-t^2}{4r}} dt \right) dx \right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\left(\int_{\mathbb{R}} w(x)|f(s_n(x)+t)-f(x+t)|^p dx\right)^{\frac{1}{p}} dt\right)\\
&\quad+\frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\left(\int_{\mathbb{R}} w(x)|f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}} dt\right)\\
&\leq \frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\|f(s_n(x)+t)-f(x+t)\|_{p,\omega} dt\right)\\
&\quad+\frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\|f(x+t)-f(x)\|_{p,\omega} dt\right)\\
&\leq \frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\omega\left(f,L^{p,\omega},\epsilon\right) dt\right)+\frac{1}{\sqrt{4\pi r}}\left(\int_{\mathbb{R}} e^{\frac{-t^2}{4r}}\omega\left(f,L^{p,\omega},t\right) dt\right).\\
&\leq \omega\left(f,L^{p,\omega},\epsilon\right)+\omega\left(f,L^{p,\omega},r\right)\frac{1}{\sqrt{4\pi r}}\int_{\mathbb{R}}\left(1+\frac{t}{r}\right)\frac{1}{w(t)}e^{\frac{-t^2}{4r}} dt \text{ for } n\geq n_0(\epsilon)\in\mathbb{N}\\
&\leq \omega\left(f,L^{p,\omega},\epsilon\right)+\omega\left(f,L^{p,\omega},r\right)\frac{1}{\sqrt{4\pi r}}\int_{\mathbb{R}}\frac{1}{w(t)}e^{\frac{-t^2}{4r}} dt\\
&\quad+\omega\left(f,L^{p,\omega},r\right)\frac{1}{r\sqrt{4\pi r}}\int_{\mathbb{R}}\frac{t}{w(t)}e^{\frac{-t^2}{4r}} dt\\
&\leq \omega\left(f,L^{p,\omega},\epsilon\right)+\omega\left(f,L^{p,\omega},r\right)\frac{1}{\sqrt{4\pi r}}\mathcal{W}_0\left(\frac{1}{4r}\right).
\end{aligned}
$$

Hence the proof is completed. □

*Remark* 1 Theorem 2.2 provides the rate of approximation for the function $f \in L^{p,w}$ if its second modulus of continuity $\omega_2(f, L^{p,w}, t)$ is known. Similarly, Theorem 2.3 determines the rate of approximation for the function $f \in L^{p,w}$ when its first modulus of continuity $\omega(f, L^{p,w}, t)$ is given.

The following corollaries can be derived from Theorem 2.2 and Theorem 2.3.

**Corollary 2.4** *Let $f \in L^{p,w}$ with $f^{(k)} \in L^{p,w}$, $1 \leq p \leq \infty$. Then, for any $\epsilon > 0$, there exists $n \geq n_0(\epsilon) \in \mathbb{N}$ such that*

$$
\begin{aligned}
&||W_{r,n}^{(k)}(f,x)-f^{(k)}(x)||_{p,w}\\
&\leq \omega_2(f,L^{p,w},\epsilon)+\frac{\omega_2(f,L^{p,w},r)}{\sqrt{4\pi}r^{\frac{5}{2}}}\left(r^2\mathcal{W}_0\left(\frac{1}{4r}\right)-(1+2r)\mathcal{W}_0'\left(\frac{1}{4r}\right)+\frac{2r}{w(1)}\right)
\end{aligned}
$$

*and*

$$
||W_{r,n}^{(k)}(f,x)-f^{(k)}(x)||_{p,w} \leq \omega\left(f,L^{p,\omega},\epsilon\right)+\omega\left(f,L^{p,\omega},r\right)\frac{1}{\sqrt{4\pi r}}\mathcal{W}_0\left(\frac{1}{4r}\right),
$$

*where $r \in (0, r_0]$.*

*Remark* 2 Theorem 3.1 and Theorem 3.2 of [21, p. 92–94] are the particular cases of Theorem 2.2 and Theorem 2.3, respectively, for $w(x) = e^{-2ax^2}$ and $s_n(x) = x - \frac{a}{2n}$, $a > 0$, $n \in \mathbb{N}$.

**Corollary 2.5** *Let* $f \in H^{p,w,\mu}$, $1 \leq p \leq \infty$, *and* $\mu \in \Omega^2$. *Then, for any* $\epsilon > 0$, *there exists* $n \geq n_0(\epsilon) \in \mathbb{N}$ *such that*

$$||W^*_{r,n}(f,x) - f(x)||_{p,w}$$
$$\leq \mu(\epsilon)||f||^*_{p,w,\mu} + \frac{\mu(r)||f||^*_{p,w,\mu}}{\sqrt{4\pi}r^{\frac{5}{2}}}\left(r^2\mathcal{W}_0\left(\frac{1}{4r}\right) - (1+2r)\mathcal{W}_0'\left(\frac{1}{4r}\right) + \frac{2r}{w(1)}\right)$$

and

$$||W^*_{r,n}(f,x) - f(x)||_{p,w} \leq \mu(\epsilon)||f||^*_{p,w,\mu} + \mu(r)||f||^*_{p,w,\mu}\frac{1}{\sqrt{4\pi r}}\mathcal{W}_0\left(\frac{1}{4r}\right)$$

for every $r \in (0, r_0]$.

Now, we shall give an analogue of Theorem 2.3 in the weighted Hölder space specified by moduli of continuity. For $\eta, \mu \in \Omega^2$, let us assume that $\phi(t) = \frac{\mu(t)}{\eta(t)}$, $t > 0$, is an increasing function so that $H^{p,w,\mu} \subset H^{p,w,\eta}$. With this notation, we state our next theorem as follows.

**Theorem 2.6** *Let* $f \in H^{p,w,\mu}$, $1 \leq p \leq \infty$. *Then, for any* $\epsilon > 0$, *there exist* $n \geq n_0(\epsilon) \in \mathbb{N}$ *and function* $N_\epsilon(r)$ *such that*

$$||W^*_{r,n}(f,x) - f(x)||^*_{p,w,\eta} \leq N_\epsilon(r)\phi(r)||f||^*_{p,w,\mu} \tag{7}$$

*for every* $r \in (0, r_0]$.

For the proof of Theorem 2.6, we need the following lemma.

**Lemma 2.7** *Let* $f \in L^{p,w}$, $1 \leq p \leq \infty$. *Then*

$$||f(x+h)||_{p,w} \leq \frac{1}{w(h)}||f||_{p,w}, \quad h \in \mathbb{R}.$$

*Proof* For $1 \leq p < \infty$, we have

$$\begin{aligned} ||f(x+h)||_{p,w} &= \left(\int_{\mathbb{R}} |w(x)f(x+h)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}} |w(u-h)f(u)|^p du\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}} \left|\frac{w(u-h)}{w(u)}\right| |w(u)f(u)|^p du\right)^{\frac{1}{p}} \\ &\leq \frac{1}{w(h)}||f||_{p,w}, \end{aligned}$$

in view of the properties of $w$.

Similarly, for $p = \infty$, we have

$$\begin{aligned} ||f(x+h)||_{\infty,w} &= \sup_{x\in\mathbb{R}} w(x)\left|f(x+h)\right| \\ &= \sup_{u\in\mathbb{R}} w(u-h)\left|f(u)\right| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{u\in\mathbb{R}} \left| \frac{w(u-h)}{w(u)} \right| w(u) \left| f(u) \right| \\
&\leq \frac{1}{w(h)} ||f||_{\infty,w}.
\end{aligned}
$$

Thus the proof is completed. □

*Proof of Theorem* 2.6 Let $r$ be a fixed point in $(0, r_0] \cap (0, 1]$. Then, by using the definition of weighted Hölder spaces and (2), we can write

$$
\begin{aligned}
||W^*_{r,n}(f,x) - f(x)||^*_{p,w,\eta} \leq{}& ||W^*_{r,n}(f,x) - f(x)||_{p,w} \\
&+ \sup_{h\in(0,r]} \frac{||\Delta^2_h \left( W^*_{r,n}(f,x) - f(x) \right) ||_{p,w}}{\eta(h)} \\
&+ \sup_{h\in(r,1]} \frac{||\Delta^2_h \left( W^*_{r,n}(f,x) - f(x) \right) ||_{p,w}}{\eta(h)}.
\end{aligned} \tag{8}
$$

Now, we estimate the right-hand side of (8) by using Corollary 2.5 and Lemma 2.7. We have

$$
||W^*_{r,n}(f,x) - f(x)||_{p,w} \leq \left( 1 + \frac{1}{\sqrt{4\pi r}} \mathcal{W}_0 \left( \frac{1}{4r} \right) \right) \eta(r)\phi(r)||f||^*_{p,w,\mu}, \tag{9}
$$

$$
\sup_{h\in(0,r]} \frac{||\Delta^2_h \left( W^*_{r,n}(f,x) - f(x) \right) ||_{p,w}}{\eta(h)} \leq \left( 1 + \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) \right) \phi(r)||f||^*_{p,w,\mu}, \tag{10}
$$

and

$$
\begin{aligned}
&\sup_{h\in(r,1]} \frac{||\Delta^2_h \left( W^*_{r,n}(f,x) - f(x) \right) ||_{p,w}}{\eta(h)} \\
&= \sup_{h\in(r,1]} \frac{\| W^*_{r,n}(f,x+h) - f(x+h)\|_{p,w} + \| W^*_{r,n}(f,x-h) - f(x-h)\|_{p,w}}{\eta(h)} \\
&\quad - \frac{2\| W^*_{r,n}(f,x) - f(x)\|_{p,w}}{\eta(h)} \\
&\leq \frac{1}{w(1)} \sup_{h\in(r,1]} \frac{\| W^*_{r,n}(f,x) - f(x)\|_{p,w}}{\eta(h)} \\
&\leq \frac{1}{w(1)} \left( 1 + \frac{1}{\sqrt{4\pi r}} \mathcal{W}_0 \left( \frac{1}{4r} \right) \right) \phi(r)||f||^*_{p,w,\mu}.
\end{aligned} \tag{11}
$$

Collecting all the estimates from (9) to (11) and using (8), we obtain (7), where $N_\epsilon(r) = \max\left\{ \left( 1 + \frac{1}{\sqrt{4\pi r}} \mathcal{W}_0 \left( \frac{1}{4r} \right) \right) \eta(r), \left( 1 + \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) \right), \frac{1}{w(1)} \left( 1 + \frac{1}{\sqrt{4\pi r}} \mathcal{W}_0 \left( \frac{1}{4r} \right) \right) \right\}$. □

## 3 Numerical example

In this section, we give an example to demonstrate the convergence of $W^*_{r,n}(f,x)$.

Let $f(x) = e^{x^2}$ and $w(x) = e^{-ax^2}$, we can observe that $f(x) \in L^{1,w}$ for every $a > 1$. Now, if we fix $r = \frac{1}{4n}$ and $s_n(x) = x + \frac{a}{2n}$, $n \in \mathbb{N}$, then

$$
W^*_n(e^{x^2}, x) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{(x+\frac{a}{2n}+t)^2} e^{-nt^2}\, dt = \sqrt{\frac{n}{n-1}} e^{(x+\frac{a}{2n})^2 \frac{n}{n-1}}. \tag{12}
$$

**Table 1** Approximation error for $a = 2$

| $a$ | $n$ | Error |
|---|---|---|
| 2 | $10^2$ | 0.01836 |
| | $10^3$ | 0.0017787 |
| | $10^4$ | 0.0001773 |
| | $10^5$ | $1.7722 \times 10^{-5}$ |
| | $10^6$ | $1.7658 \times 10^{-6}$ |
| | $10^7$ | $1.3455 \times 10^{-7}$ |
| | $10^8$ | $1.3455 \times 10^{-8}$ |

**Table 2** Approximation error for $a = 3$

| $a$ | $n$ | Error |
|---|---|---|
| 3 | $10^2$ | 0.0099397 |
| | $10^3$ | 0.00094528 |
| | $10^4$ | $9.4052 \times 10^{-5}$ |
| | $10^5$ | $9.4074 \times 10^{-6}$ |
| | $10^6$ | $9.1193 \times 10^{-7}$ |
| | $10^7$ | $9.8958 \times 10^{-8}$ |
| | $10^8$ | $9.8958 \times 10^{-9}$ |

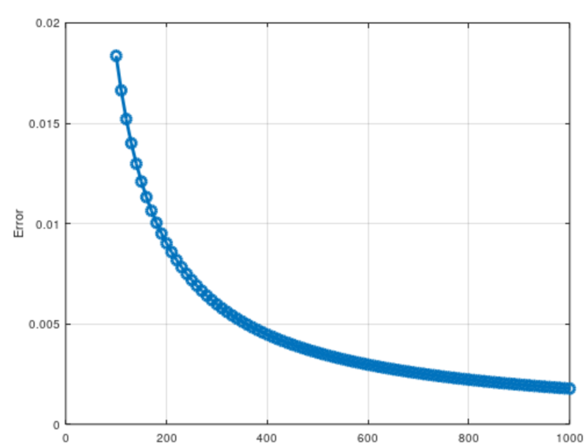


**Figure 1** Error with respect to $n$ for $a = 3$

Now

$$\text{Error} = \| W_n^*(e^{x^2}, x) - e^{x^2} \|_{1,e^{-ax^2}} = \int_{\mathbb{R}} e^{-ax^2} \left( \sqrt{\frac{n}{n-1}} e^{\left(x+\frac{a}{2n}\right)^2 \frac{n}{n-1}} - e^{x^2} \right) dx.$$

In the above example, we used Matlab 7.0 for computations and figure creation. For large values of $n$, Table 1 displays the approximation error by the operator $W_n^*(f,x)$ for $a = 2$ with the corresponding graphical representation shown in Fig. 1. Similarly, for $a = 3$, Table 2 presents the approximation error by the operator $W_n^*(f,x)$, and Fig. 2 provides the graphical depiction of this approximation error. The figures and tables indicate that as the value of $a$ increases from 2 to 3, the approximation error between the operator $W_n^*(f,x)$ and the function $f$ decreases with increasing $n$.

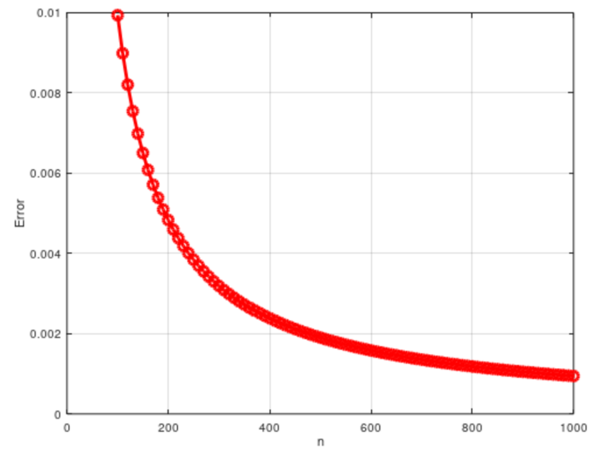


**Figure 2** Error with respect to $n$ for $a = 3$

## 4 Korovkin-type approximation theorem

Agratini et al. [1] showed that $(W_n^* f)(x), n \geq 1$ is a sequence of linear operators on $L^{p,w}, 1 \leq p \leq \infty$. For $e_j = t^j, t \in \mathbb{R}, j \geq 0, e_0 = 1$ works as a fixed point for the operator $W_{r,n}^*(f,x), n \geq 1$. If one chooses $w(x) = e^{2ax}$, $a > 0$, and $s_n(x) = x - \frac{a}{2n}$, $n \geq 1$, then $W_{r,n}^*$ fixes not only $e_0$ but also $w(x)$ [1, p. 1198, Lemma 6]. The authors [1, p. 1191, Lemma 2] also proved that $W_{r,n}^*(e_0,x) = 1$, $W_{r,n}^*(e_2,x) = s_n(x)$, and $W_{r,n}^*(e_2,x) = s_n^2(x) + \frac{1}{2n}$, $n \in \mathbb{N}$.

Here, we study the Korovkin-type results for $W_{r,n}^*(f,x)$ in the space $L^{p,w}, 1 \leq p \leq \infty$, where the weight function $w \in L^p(\mathbb{R}), 1 \leq p \leq \infty$. More precisely, we prove the following.

**Theorem 4.1** *Let $f \in L^{p,w}$, $1 \leq p \leq \infty$, such that $w(x)$ is an even function on $\mathbb{R}$ and $\int_0^\infty (w(x))^p dx < \infty$. Then, for any $\epsilon > 0$, there exists $n \geq n_0(\epsilon) \in \mathbb{N}$ such that*

$$(1) \quad \| W_{r,n}^*(e_0) - e_0 \|_{p,w} = 0; \tag{13}$$

$$(2) \quad \| W_{r,n}^*(e_1) - e_1 \|_{p,w} \leq 2\epsilon \left( \int_0^\infty (w(x))^p dx \right)^{\frac{1}{p}}; \tag{14}$$

$$(3) \quad \| W_{r,n}^*(e_2) - e_2 \|_{p,w} \leq 2(\epsilon + 2r) \left( \int_0^\infty (w(x))^p dx \right)^{\frac{1}{p}}; \tag{15}$$

*and*

$$\lim_{n\to\infty} \| W_{r,n}^*(f,x) - f(x) \|_{p,w} = 0. \tag{16}$$

*Proof* To prove (16), we have to show that

$$\lim_{n\to\infty} \| W_{r,n}^*(e_j) - e_j \|_{p,w} = 0 \text{ for every } j = 0, 1, 2. \tag{17}$$

Since $W^*_{r,n}(e_0) = 1$, (17) is already satisfied for $j = 0$. Using the triangle inequality, we have

$$\begin{aligned} W^*_{r,n}(e_1)(x) - e_1(x) &= \frac{1}{\sqrt{4\pi r}} \int_R (s_n(x) + t) e^{\frac{-t^2}{4r}} dt - \frac{x}{\sqrt{4\pi r}} \int_{\mathbb{R}} e^{\frac{-t^2}{4r}} dt \\ &= \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} (s_n(x) - x + t) e^{\frac{-t^2}{4r}} dt \\ &\leq \frac{\epsilon}{\sqrt{4\pi r}} \int_{\mathbb{R}} e^{\frac{-t^2}{4r}} dt + \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} t e^{\frac{-t^2}{4r}} dt \text{ for } n \geq n_0(\epsilon) \in \mathbb{N} \\ &= \epsilon. \end{aligned}$$

Similarly, we have

$$\begin{aligned} W^*_{r,n}(e_2)(x) - e_2(x) &= \frac{1}{\sqrt{4\pi r}} \int_R \left((s_n(x) + t)^2 - x^2\right) e^{\frac{-t^2}{4r}} dt \\ &\leq \epsilon + 2r \text{ for } n \geq n_0(\epsilon) \in \mathbb{N}, \end{aligned}$$

so that

$$\begin{aligned} \| W^*_{r,n}(e_1) - e_1 \|_{p,w} &= \left( \int_{\mathbb{R}} |w(x) \left( W^*_{r,n}(e_1)(x) - e_1(x) \right) |^p dx \right)^{\frac{1}{p}} \\ &= 2\epsilon \left( \int_0^\infty (w(x))^p dx \right)^{\frac{1}{p}} \text{ for } n \geq n_0(\epsilon) \in \mathbb{N} \end{aligned}$$

and

$$\begin{aligned} \| W^*_{r,n}(e_2) - e_2 \|_{p,w} &= \left( \int_{\mathbb{R}} |w(x) \left( W^*_{r,n}(e_2)(x) - e_2(x) \right) |^p dx \right)^{\frac{1}{p}} \\ &= 2(\epsilon + 2r) \left( \int_0^\infty (w(x))^p dx \right)^{\frac{1}{p}} \text{ for } n \geq n_0(\epsilon) \in \mathbb{N}. \end{aligned}$$

Using [9, p. 1047, Theorem 1] and (13) to (15), we get (16). □

**Corollary 4.2** *Let $f \in L^{p,a}$, $1 \leq p \leq \infty$. Then*

$$\lim_{n\to\infty} \| W^*_{r,n}(f, x) - f(x) \|_{p,2a} = 0.$$

This corollary is Theorem 3.4 of [21, p. 94] and can be obtained from Theorem 4.1 by taking $s_n(x) = x - \frac{a}{2n}$, $n \in \mathbb{N}$ and $w(x) = e^{-2ax^2}$.

## 5 Inverse approximation type theorem

Now, we shall prove the results that are related to inverse approximation by means of the modified Gauss–Weierstrass singular integral $W^*_{r,n}(f, x)$. We denote by $s_0$ the maximum value of $r$ for which $\mathcal{W}_\epsilon \left(\frac{1}{4r}\right)$ exists.

**Theorem 5.1** *Let $f \in L^{p,w}$, $1 \leq p \leq \infty$, and suppose $f^{(2)} \in L^{p,w}$. Then, for any $\epsilon > 0$, there exists $n \geq n_0(\epsilon) \in \mathbb{N}$ such that*

$$|| W^*_{r,n}(f, x) - f(x) ||_{p,w} \leq \mu(r) \tag{18}$$

*and*

$$||W_{r,n}^{(2)}(f,x) - f^{(2)}(x)||_{p,w} \leq \mu(r), \text{for every } r > 0,$$

*where $\mu$ is a given function belonging to $\Omega^2$. Then there exists a constant $c(\epsilon) > 0$ such that*

$$\omega_2(f, L^{p,w}, t) \leq ct^2 \int_t^1 \frac{\mu(x)}{x^4} dx$$

*for all $t \in \left(0, \frac{1}{2}\right) \cap (0, s_0]$.*

We need the following lemma for proving this result.

**Lemma 5.2** *Let $f \in L^{p,w}$, $1 \leq p \leq \infty$, and suppose $f^{(2)} \in L^{p,w}$. Then, for any $\epsilon > 0$, there exists $n \geq n_0(\epsilon) \in \mathbb{N}$ such that*

$$||\Delta_h^2 W_{r,n}^*(f,x)||_{p,w} \leq \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon\left(\frac{1}{4r}\right) h^2 \frac{1}{w(h)} ||f^{(2)}||_{p,w}$$

*for every $r \in (0, s_0]$ and $h \in \mathbb{R}$.*

*Proof* For $1 \leq p < \infty$, we have

$$\begin{aligned}
||\Delta_h^2 W_{r,n}^*(f,x)||_{p,w} &= \left(\int_{\mathbb{R}} \left| w(x) \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} W_{r,n}^{(2)}(f, s_n(x) + t_1 + t_2) dt_1 dt_2 \right|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}} \left| w(x) \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} W_{r,n}^{(2)}(f, x + \epsilon + t_1 + t_2) dt_1 dt_2 \right|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \left| \frac{w(u - t_1 - t_2)}{w(u)} w(u) W_{r,n}^{(2)}(f, u) \right|^p dt_1 dt_2 du\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the generalized Minkowski inequality and the properties of $w$, we get

$$\begin{aligned}
||\Delta_h^2 W_{r,n}^*(f,x)||_{p,w} &\leq \left(\int_{\mathbb{R}} \left| w(u) W_{r,n}^{(2)}(f,u) \right|^p du\right)^{\frac{1}{p}} \left(\int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{1}{w(t_1 + t_2)} dt_1 dt_2\right) \\
&= ||W_{r,n}^{(2)}(f,x)||_{p,w} \left(\int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \frac{1}{w(t_1 + t_2)} dt_1 dt_2\right) \\
&\leq \frac{1}{w(h)} h^2 ||W_{r,n}^{(2)}(f,x)||_{p,w} \\
&\leq \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon\left(\frac{1}{4r}\right) h^2 \frac{1}{w(h)} ||f^{(2)}||_{p,w}
\end{aligned}$$

in view of (3).

Similarly, for $p = \infty$, we have

$$\begin{aligned}
||\Delta_h^2 W^*_{r,n}(f,x)||_{\infty,w} &= \sup_{x\in\mathbb{R}} \left(w(x)|\Delta_h^2 W^*_{r,n}(f,x)|\right) \\
&= \sup_{x\in\mathbb{R}} \left( w(x) \left| \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} W^{(2)}_{r,n}(f, s_n(x) + t_1 + t_2) dt_1 dt_2 \right| \right) \\
&= \sup_{x\in\mathbb{R}} \left( w(x) \left| \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} W^{(2)}_{r,n}(f, x + \epsilon + t_1 + t_2) dt_1 dt_2 \right| \right) \\
&\le \left( \sup_{u\in\mathbb{R}} w(u) \left| W^{(2)}_{r,n}(f,u) \right| \right) \\
&\quad \times \left( \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \sup_{u\in\mathbb{R}} \left( \frac{w(u - t_1 - t_2)}{w(u)} \right) dt_1 dt_2 \right) \\
&\le ||W^{(2)}_{r,n}(f,x)||_{\infty,w} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \frac{1}{w(t_1 + t_2)} dt_1 dt_2 \\
&\le \frac{1}{w(h)} h^2 ||W^{(2)}_{r,n}(f,x)||_{\infty,w} \\
&\le \frac{1}{\sqrt{\pi r}} \mathcal{W}_\epsilon \left( \frac{1}{4r} \right) h^2 \frac{1}{w(h)} ||f^{(2)}||_{p,w}
\end{aligned}$$

Hence the proof is completed. □

*Proof of Theorem* 5.1 We can find two natural numbers $m$ and $n$ such that $0 < \frac{1}{2^n} < \frac{1}{2^m} \le r_0$. For all $x, h \in \mathbb{R}$ and $f \in L^{p,w}$, we can write

$$\begin{aligned}
\Delta_h^2 f(x) = \Delta_h^2 W^*_{2^{-m},n}(f,x) + \sum_{i=m}^{n-1} \Delta_h^2 \left( W^*_{2^{-i-1},n}(f,x) - W^*_{2^{-i},n}(f,x) \right) \\
+ \Delta_h^2 \left( f(x) - W^*_{2^{-n},n}(f,x) \right). \qquad (19)
\end{aligned}$$

Using Lemma 5.2, we get

$$||\Delta_h^2 W^*_{2^{-m},n}(f,x)||_{p,w} \le \sqrt{\frac{2^m}{\pi}} \mathcal{W}_\epsilon \left( \frac{1}{2^{-m+2}} \right) h^2 \frac{1}{w(h)} ||f^{(2)}||_{p,w}.$$

Now,

$$\begin{aligned}
\Delta_h^2 \left( W^*_{2^{-i-1},n}(f,x) - W^*_{2^{-i},n}(f,x) \right) &= \Delta_h^2 W^*_{2^{-i-1},n} \left( f - W^*_{2^{-i},n}(f), x \right) \\
&\quad + \Delta_h^2 W^*_{2^{-i},n} \left( W^*_{2^{-i-1},n}(f) - f, x \right)
\end{aligned}$$

so that

$$\begin{aligned}
\left|\left| \Delta_h^2 \left( W^*_{2^{-i-1},n}(f,x) - W^*_{2^{-i},n}(f,x) \right) \right|\right|_{p,w} &\le \left|\left| \Delta_h^2 W^*_{2^{-i-1},n} \left( f - W^*_{2^{-i},n}(f), x \right) \right|\right|_{p,w} \\
&\quad + ||\Delta_h^2 W^*_{2^{-i},n} \left( W^*_{2^{-i-1},n}(f) - f, x \right) ||_{p,w}.
\end{aligned}$$

Using Lemma 5.2, we get

$$
\begin{aligned}
&||\Delta_h^2\left(W^*_{2^{-i-1},n}(f,x)-W^*_{2^{-i},n}(f,x)\right)||_{p,w}\\
&\leq \sqrt{\frac{2^{i+1}}{\pi}}\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+1}}\right)h^2\frac{1}{w(h)}||f^{(2)}-W^{(2)}_{2^{-i},n}(f)||_{p,w}+\sqrt{\frac{2^i}{\pi}}h^2\frac{1}{w(h)}\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+2}}\right)\\
&\quad ||W^{(2)}_{2^{-i-1},n}(f)-f^{(2)}||_{p,w}\\
&\leq\left(\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+1}}\right)+\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+2}}\right)\right)\frac{1}{w(h)}h^2\sqrt{\frac{2^{i+1}}{\pi}}\mu(2^{-i}).
\end{aligned}
\tag{20}
$$

Using (18), we have

$$
\begin{aligned}
&||\Delta_h^2\left(f(x)-W^*_{2^{-n},n}(f,x)\right)||_{p,w}\\
&\leq \|f(x+h)-W^*_{2^{-n},n}(f,x+h)\|_{p,w}+\|f(x-h)-W^*_{2^{-n},n}(f,x-h)\|_{p,w}\\
&\quad -2\|f(x)-W^*_{2^{-n},n}(f,x)\|_{p,w}\\
&\leq \frac{1}{w(h)}\|f(x)-W^*_{2^{-n},n}(f,x)\|_{p,w}+\frac{1}{w(-h)}\|f(x)-W^*_{2^{-n},n}(f,x)\|_{p,w}+2\omega(2^{-n})\\
&\leq \frac{1}{w(h)}\mu(2^{-n})+\frac{1}{w(-h)}\mu(2^{-n})+2\mu(2^{-n})\leq 2\left(\frac{1}{w(h)}+1\right)\mu(2^{-n}).
\end{aligned}
\tag{21}
$$

Using (19) to (21), we get

$$
||\Delta_h^2(f(x))||_{p,w}\leq\frac{1}{w(h)}\left(\sqrt{\frac{2^m}{\pi}}Kh^2||f^{(2)}||_{p,w}+h^2K\sum_{i=m}^{n-1}\sqrt{\frac{2^{i+1}}{\pi}}\mu(2^{-i})+K'\mu(2^{-n})\right),
\tag{22}
$$

where $K(\epsilon)=\max\limits_{m\leq i\leq n-1}\left\{\mathcal{W}_\epsilon\left(\frac{1}{2^{-m+2}}\right),\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+1}}\right)+\mathcal{W}_\epsilon\left(\frac{1}{2^{-i+2}}\right)\right\}$ and $K'=1+w(0)$.

If $t\in\left(0,\frac{1}{2}\right)\cap(0,r_0]$, $|h|\leq t$, $m<n$, and $n$ is a natural number such that $2^{-n}\leq t<2^{-n+1}$, then from (22) we obtain

$$
\omega_2(f,L^{p,w},t)\leq d\left(t^2+t^2\sum_{i=m}^{n-1}\sqrt{\frac{2^{i+1}}{\pi}}\mu(2^{-i})+\mu(t)\right),
\tag{23}
$$

where $d(\epsilon)=\max\limits_{2^{-n}\leq t<2^{-n+1}}\frac{1}{w(t)}\left\{2^{3m+3}K||f^{(2)}||_{p,w},K(\epsilon),K'\right\}$.

We can write

$$
\sum_{i=m}^{n-1}2^{3i}\mu(2^{-i})\leq\frac{1}{ln2}\int_t^1\frac{\mu(x)}{x^4}dx.
\tag{24}
$$

It is easy to find constants $c_1$ and $c_2$ such that

$$
c_1t^2\leq\mu(t)\leq c_2t^2\int_t^1\frac{\mu(x)}{x^4}dx
\tag{25}
$$

for all $t\in\left(0,\frac{1}{2}\right)\cap(0,r_0]$ and $\mu\in\Omega^2$.

Collecting (23), (24), and (25), we obtain our result, where $c(\epsilon)=d\left(1+\frac{1}{ln2}+c_2\right)$. □

## 6 Conclusion

In the literature, numerous results have been obtained regarding the rate of approximation of functions in Hölder and Lebesgue spaces using the Gauss–Weierstrass singular integral. Specifically, this study focuses on the approximation of functions in weighted Lebesgue spaces and weighted Hölder spaces using the Gauss–Weierstrass singular integral. Another important category of theorems includes the Korovkin approximation type theorems and inverse approximation theorems. We establish the Korovkin-type theorem and the inverse approximation theorem for weighted Lebesgue spaces.

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## Declarations

**Competing interests**
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