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Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces

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Abstract

Let $\mathbb{S}^{n-1}$ denote unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure. Let $\Phi \in L^s(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero such that $\int_{\mathbb{S}^{n-1}} \Phi(y')d\sigma(y') = 0$, where $y' = y/|y|$ for any $y \neq 0$. The commutators of variable Marcinkiewicz fractional integral operator is defined as

$$[b,\mu_\Phi]^m_\beta(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\leq s} \frac{\Phi(x-y)[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}} f(y)dy \right|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}}.$$

In this paper, we obtain the boundedness of the commutators of the variable Marcinkiewicz fractional integral operator on grand variable Herz spaces $\dot{K}^{\alpha(\cdot),q),\theta}_{p(\cdot)}(\mathbb{R}^n)$.

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1 Introduction

It is well known that Beurling [8] and Herz [16] introduced new spaces to characterize certain properties of functions. These spaces are known as Herz spaces, and numerous studies involving them can be found in the literature. One of the main reasons is that the Hardy space theory associated with Herz spaces is very rich. These new Hardy spaces serve as a localized version of the ordinary Hardy spaces and can sometimes be better substitutes when considering, for instance, the boundedness of non-translation invariant singular integral operators.

Nowadays, there is a vast boom of research related to both the study of the Herz spaces themselves and the operator theory in these spaces. This is caused by the influence of some possible applications in modeling with nonstandard local growth (in differential equations, fluid mechanics, elasticity theory; see, for example, [1–4, 7, 23, 25, 26]). There was a vast boom of research in the so-called variable exponent spaces. We refer, for example, to the papers [9, 11–14], see also references therein.

In [20], authors considered variable potential operators $I^{\beta(x)}$ to prove a Sobolev-type theorem for the potential operator from the Lebesgue space $L^{p(\cdot)}$ into the weighted Lebesgue space $L^{q(\cdot)}_w$ in $\mathbb{R}^n$, under the conditions that $p(x)$ is satisfying the logarithmic condition locally and at infinity. It was not supposed that $p(x)$ is constant at infinity but also assumed that $p(x)$ took its minimal value at infinity.

Many studies related to Herz spaces and their variations can be found, including variable Herz spaces, continual Herz spaces, grand variable Herz spaces, grand weighted spaces, and grand weighted Herz–Morrey spaces. For details, see [5, 6, 15, 17, 18, 21, 22, 27, 29–36] and references therein.

Motivated by the above results, in this paper, we prove the boundedness of the commutators of variable Marcinkiewicz fractional integral operator on grand variable Herz spaces $\dot{K}^{\alpha(\cdot),q),\theta}_{p(\cdot)}(\mathbb{R}^n)$.

2 Preliminaries

If H is a measurable set in $\mathbb{R}^n$ and $q(\cdot)\colon H \to [1,\infty)$ is a measurable function, we suppose that

$$1 \le q_-(H) \le q(h) \le q_+(H) < \infty, \tag{2.1}$$

where $q_- := \operatorname{ess\,inf}_{h\in H} q(h)$, $\quad q_+ := \operatorname{ess\,sup}_{h\in H} q(h)$.

(a) Lebesgue space with variable exponent $L^{q(\cdot)}(H)$ is defined as

$$L^{q(\cdot)}(H) = \left\{ f \textit{ measurable}: \int_H \left(\frac{|f(y)|}{\gamma}\right)^{q(y)} dy < \infty, \textit{where } \gamma \textit{ is a constant} \right\}.$$

Norm in $L^{q(\cdot)}(H)$ is defined as

$$\|f\|_{L^{q(\cdot)}(H)} = \inf\left\{ \gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma}\right)^{q(y)} dy \le 1 \right\}.$$

(b) The space $L^{q(\cdot)}_{\mathrm{loc}}(H)$ can be defined as

$$L^{q(\cdot)}_{\mathrm{loc}}(H) := \left\{ f : f \in L^{q(\cdot)}(K) \textit{ for all compact subsets } K \subset H \right\}.$$

Let us recall the well-known log-Hölder continuity condition (or the Dini–Lipschitz condition) for $q : H \mapsto (0,\infty)$: there is a positive constant C such that for all $x, y \in H$ with $|x-y| \le \frac{1}{2}$,

$$|q(x) - q(y)| \le \frac{C}{-\ln|x-y|}. \tag{2.2}$$

Further, we say that $q(\cdot)$ satisfies the decay condition if there exists $q_\infty := q(\infty) = \lim_{|x|\to\infty} q(x)$, and there is a positive constant $C_\infty > 0$ such that

$$|q(h) - q_\infty| \le \frac{C_\infty}{\ln(e+|h|)}. \tag{2.3}$$

We will also need the log-Hölder continuity condition at 0 for $q(\cdot)$: there are constants $C_0 > 0$ such that for all $|h| \leq \frac{1}{2}$,

$$|q(h) - q(0)| \leq \frac{C_0}{\ln |h|}. \tag{2.4}$$

The best possible constant C in (2.2) (resp. C_∞ in (2.3)) is called log-Hölder continuity or log-Dini–Lipschitz constant (resp. decay constant) for the exponent $q(\cdot)$.

We use these notations in this article:

(i) $B(x,r)$ is the ball of radius r and center at the point x.

(ii) $B_k := B(0,2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.

(iii) $R_{t,\tau} := B_\tau \setminus B_t = \{x : t < |x| < \tau\}$ is a spherical layer.

(iv) $R_k := B_k \setminus B_{k-1}$.

(v) $\chi_k := \chi_{R_k}$, $\chi_{t,\tau} := \chi_{R_{t,\tau}}$.

(vi) Let $f \in L^1_{\text{loc}}(H)$ be a locally integrable function, then the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(x)|dx, \quad (x \in H),$$

where

$$B(x,r) := \{y \in H : |x-y| < r\}.$$

(vii) The set $\mathcal{P}(H)$ consists of all measurable functions $q(\cdot)$ satisfying $q_- > 1$ and $q_+ < \infty$.

(viii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ consists of all functions $q \in \mathcal{P}(H)$ satisfying (2.1) and (2.2).

(ix) $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,\infty}(H)$ are the subsets of $\mathcal{P}(H)$ and values of these subsets lies in $[1,\infty)$, which satisfy condition (2.3) and conditions (2.3) and (2.4), respectively.

(x) In what follows,
we denote $\chi_l = \chi_{R_l}$, $R_l = B_l \setminus B_{l-1}$, $B_l = B(0,2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$ for all $l \in \mathbb{Z}$.

(xi) By $p'(x) = p(x)/(p(x)-1)$, we denote the conjugate exponent of $p(\cdot)$.

(xii) C is a constant,
its value varies from line to line and is independent of main parameters involved.

Lemma 2.1 [27] *Let $D > 1$ and $q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{c_0} r^{\frac{n}{q(0)}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \leq c_0 r^{\frac{n}{q(0)}}, \text{ for } 0 < r \leq 1 \tag{2.5}$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{q_\infty}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}} \leq c_\infty r^{\frac{n}{q_\infty}}, \text{ for } r \geq 1, \tag{2.6}$$

respectively, where $c_0 \geq 1$ and $c_\infty \geq 1$ depend on D but do not depend on r.

The Hölder inequality in the variable exponent case has the following form:

$$\int_{\Omega} \left| f(x)g(x) \right| dx \leq \kappa \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $\kappa = \frac{1}{p_-} + \frac{1}{(p')_-}$, and Ω is a measurable subset of $\mathbb{R}^n$.

The next statement is the generalized Hölder inequality for variable exponent Lebesgue spaces (see [10]):

Lemma 2.2 *Let H be a measurable subset of $\mathbb{R}^n$ and $p(\cdot)$ be an exponent such that $1 \leq p_-(H) \leq p_+(H) < \infty$ and $q_-, r_- > 1$. Then*

$$\|fg\|_{L^{r(\cdot)}(H)} \leq 2^{1/r_-} \|f\|_{L^{p(\cdot)}(H)} \|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f \in L^{p(\cdot)}(H)$, $g \in L^{q(\cdot)}(H)$ and $\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$.

Definition 2.3 (BMO space) A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|B|} \int_B |u(y) - u_B| dy$ is bounded. Mathematically,

$$\|u\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |u(y) - u_B| dy < \infty.$$

Lemma 2.4 ([19]) *Let $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{N}$, $i, j \in \mathbb{Z}$ with $i < j$. Then, we have*

$$\begin{aligned} C^{-1} \|b\|^m_{BMO(\mathbb{R}^n)} \leq \sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}} \|(b - b_B)^m \chi_B\|_{L^{q(\cdot)}} \\ \leq C \|b\|^m_{BMO(\mathbb{R}^n)}, \\ \|(b - b_{B_i})^m \chi_{B_j}\|_{L^{q(\cdot)}} \leq C(j - i)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q(\cdot)}}. \end{aligned}$$

Definition 2.5 Let $0 < p \leq \infty$, $q \in \mathcal{P}(\mathbb{R}^n)$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{K^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{L^{q(\cdot)}} + \left(\sum_{k \geq 1} \left\| 2^{\alpha(\cdot)} f \chi_k \right\|^p_{L^{q(\cdot)}} \right)^{1/p} < \infty.$$

The homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{R}^n)} := \left(\sum_{k \in \mathbb{Z}} \left\| 2^{\alpha(\cdot)} f \chi_k \right\|^p_{L^{q(\cdot)}} \right)^{1/p} < \infty.$$

Next we will define grand variable Herz spaces.

Definition 2.6 Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $1 \leq p < \infty$, $\theta > 0$ and $q : \mathbb{R}^n \to [1, \infty)$, $\theta > 0$. Then the grand variable Herz space $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)} = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} = \sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}}^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}$$
$$= \sup_{\psi>0}\psi^{\frac{\theta}{p(1+\psi)}}\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),p(1+\psi)}(\mathbb{R}^n)}.$$

The next proposition is the generalization of variable exponents Herz spaces in [6]. We omit the proof of proposition 2.6 since it is essentially similar to the proof given in [6], and with slight modification, we can obtain the following result in grand variable Herz spaces.

Proposition 2.7 *Let α, p, q be as defined in definition 2.5, then*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} = \sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}}^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}$$
$$\approx \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\psi)}\|f\chi_k\|_{L^{q(\cdot)}}^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}$$
$$+ \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty p(1+\psi)}\|f\chi_k\|_{L^{q(\cdot)}}^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}.$$

The Riesz-type potential operator of variable order $\beta(x)$ is defined by

$$I^{\beta(x)}f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\beta(x)}}dy, \quad 0 < \beta(x) < n. \tag{2.7}$$

Note that the $\beta(x)$ is the order of the Riesz potential operator, which is variable.

We are assuming that the order of Riesz potential operator $\beta(x)$ is not continuous rather, we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

$$\beta_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n}\beta(x) > 0, \tag{2.8}$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x)\beta(x) < n, \tag{2.9}$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(\infty)\beta(x) < n. \tag{2.10}$$

The following proposition is one of the main requirements for proving our main results. The given proposition was proved in [20] and is commonly known as a Sobolev theorem for the Riesz potential operator in Lebesgue spaces.

Proposition 2.8 *Suppose that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}_{0,\infty}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, and assume that*

$$1 < p(\infty) \le p(x) \le p_+ < \infty.$$

Let $\beta(x)$ satisfy the above conditions (2.8), (2.9), *and* (2.10). *Then, we have the following weighted Sobolev-type estimate for the operator $I^{\beta(z)}$*,

$$\|(1+|x|)^{-\lambda(z)} I^{\beta(z)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n}$$

is the Sobolev exponent.

$$\lambda(z) = C_\infty \beta(x)\left(1-\frac{\beta(x)}{n}\right) \le C_\infty \frac{n}{4},$$

where C_∞ denotes the Dini–Lipschitz constant from inequality (2.3) *in which $a(\cdot)$ is replaced by $p(\cdot)$.*

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ ($n \ge 2$) with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Phi(y')\, d\Phi(y') = 0, \tag{2.11}$$

where $y' = y/|y|$, and y is not zero. The Marcinkiewicz integral was introduced by Stein [28] in connection with Littlewood–Paley g-function on $\mathbb{R}^n$ as:

$$\mu_\Phi(f)(x) = \left(\int_0^\infty |R_{\Phi,s}(f)(x)|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}},$$

where

$$R_{\Phi,s}(f)(x) = \int_{|x-y|\le s} \frac{\Phi(x-y)}{|x-y|^{n-\beta(x)-1}} f(y)\,dy.$$

Let b be a locally integrable function on $\mathbb{R}^n$. The commutators on the variable Marcinkiewicz fractional integral operator are defined by

$$[b,\mu_\Phi]^m_\beta(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le s} \frac{\Phi(x-y)[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}} f(y)\,dy\right|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}}.$$

Lemma 2.9 ([24]) *If $a > 0$, $s \in [1,\infty]$, $0 < d \le s$ and $-m + (m-1)ds < u < \infty$, then*

$$\left(\int_{|y|\le a|x|} |y|^u |\Phi(x-y)|^d\, dy\right)^{1/d} \le |x|^{(u+m)/d} \|\Phi\|_{L^s(\mathbb{S}^{m-1})}.$$

3 The commutators of variable Marcinkiewicz fractional integral operator on grand variable Herz spaces

Theorem 3.1 *Let $0 < \nu \leq 1$, $\alpha(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q^- \leq q^+ < \infty$, $1 \leq p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be homogeneous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q'^-$. Let α be such that:*

(i) $-\frac{n}{q_1(0)} - \nu - \frac{n}{s} < \alpha(0) < \frac{n}{q_1'(0)} - \nu - \frac{n}{s}$

(ii) $-\frac{n}{q_{1\infty}} - \nu - \frac{n}{s} < \alpha_\infty < \frac{n}{q_{1\infty}'} - \nu - \frac{n}{s}$.

Then

$$\|(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g)\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof Let $g \in \dot{K}^{\alpha(\cdot),p),\theta}_{q_2(\cdot)}$, and $g(x) = \sum_{l=-\infty}^{\infty} g(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} g_l(x)$, we have,

$$\begin{aligned}
&\|(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g)\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g)\|^{p(1+\psi)}_{L^{q_2(\cdot)}}\right)^{\frac{1}{p(1+\psi)}} \\
&\leq \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|^{p(1+\psi)}_{L^{q_2(\cdot)}}\right)\right)^{\frac{1}{p(1+\psi)}} \\
&\leq \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\
&+ \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{l=k+1}^{\infty} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\
&=: E_1 + E_2.
\end{aligned}$$

For E_1, splitting E_1 using Minkowski's inequality, we have

$$\begin{aligned}
E_1 &\leq \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\
&+ \sup_{\psi>0}\left(\psi^\theta \sum_{k=0}^{\infty}\left(\sum_{l=-\infty}^{k} \|2^{k\alpha(\cdot)}\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\
&=: E_{11} + E_{12}.
\end{aligned}$$

For estimating E_{11}, we use the facts that, for each $k \in \mathbb{Z}$ and $l \leq k$ and a.e. $x \in R_k$, $y \in R_l$, we know that $|x-y| \approx |x| \approx 2^k$,

$$\left|[b,\mu_\Phi]^m_\beta(g_l)(x)\right| \leq \left(\int_o^{|x|}\left|\int_{|x-y|\leq t} \frac{\Phi(x-y)[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}} g_l(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Phi(x-y)[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}}g_l(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$=: I_{11}+I_{12}.$$

By the virtue of the mean value theorem we obtain,

$$\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|\le\frac{|y|}{|x-y|^3}. \tag{3.1}$$

For I_{11}, using Minkowski's inequality, generalized Hölder's inequality, and inequality (3.1), we have

$$\begin{aligned}
I_{11}&\le\int_{\mathbb{R}^n}\frac{|\Phi(x-y)|[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}}|g_l(y)|\left(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le\int_{\mathbb{R}^n}\frac{|\Phi(x-y)|[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}}|g_l(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}dy\\
&\le\int_{\mathbb{R}^n}\frac{|\Phi(x-y)||[b(x)-b(y)]^m}{|x-y|^{n-1-\beta(x)}}|g_l(y)|\left|\frac{|y|}{|x-y|^3}\right|^{1/2}dy\\
&\le\frac{2^{l/2}}{|x|^{n+\frac{1}{2}}.|x|^{-\beta(x)}}\int_{R_l}|\Phi(x-y)|[b(x)-b(y)]^m|g(y)|dy\\
&\le 2^{(l-k)/2}2^{-kn}|x|^{\beta(x)}\|g_l\|_{L^{q_1(\cdot)}}\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}.\\
&\le 2^{(l-k)/2}2^{-kn}|x|^{\beta(x)}\left\{|b(x)-b_{B_l}|^m\int_{R_l}|\Phi(x-y)||g_l(y)|dy\right.\\
&\qquad\left.+\int_{R_l}|b(y)-b_{B_l}|^m|\Phi(x-y)||g_l(y)|dy\right\}\\
&\le 2^{(l-k)/2}2^{-kn}|x|^{\beta(x)}\|g_l(y)\|_{L^{q_1(\cdot)}}\Big(|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\\
&\qquad+\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\Big).
\end{aligned}$$

Similarly, for I_{12}, we have

$$\begin{aligned}
I_{12}&\le\int_{\mathbb{R}^n}\frac{|\Phi(z-1-y)|}{|x-y|^{n-1-\beta(x)}}|g_l(y)|\left(\int_{|x|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le\int_{\mathbb{R}^n}\frac{|\Phi(x-y)|}{|x-y|^{n-1-\beta(x)}}|g_l(y)|dy\\
&\le|x|^{-n}|x|^{\beta(x)}\int_{R_l}|\Phi(x-y)||g(y)|dy
\end{aligned}$$

$$\leq 2^{-kn}|x|^{\beta(x)}\|g_l(y)\|_{L^{q_1(\cdot)}}\left\{|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\right.$$
$$\left.+\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\right\}.$$

We define $q_1(\cdot)$ by the relation $\frac{1}{q_1'(x)}=\frac{1}{q_1(x)}+\frac{1}{s}$. Using Lemma (2.9) and generalized Hölder's inequality, we have

$$\begin{aligned}\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}&\leq\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s}\|\chi_l(\cdot)\|_{L^{q_1(\cdot)}}\\&\leq 2^{-l\nu}\left(\int\limits_{2^{l-1}<|y|<2^l}|\Phi(x-y)|^s|y|^{s\nu}dy\right)^{1/s}\|\chi_l\|_{L^{q_1(\cdot)}}\\&\leq 2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}.\end{aligned}$$

Similarly, using Lemma (2.4), we have

$$\begin{aligned}&\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}}\\&\quad\leq\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s}\|(b(\cdot)-b_{B_l})^m\chi_l(\cdot)\|_{L^{q(\cdot)}}\\&\quad\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s}\\&\quad\leq C\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}.\end{aligned}$$

It is known [37] that

$$\begin{aligned}I^{\beta(\cdot)}((b(x)-b_{B_l})^m\chi_k)(x)&\geq I^{\beta(\cdot)}((b(x)-b_{B_l})^m\chi_k)(x).(\chi_k)(x)\\&=\int_{B_k}\frac{|b(x)-b_{B_l}|^m}{|x-y|^{n-\beta(x)}}dy.\chi_k(x)\\&\geq C|b(x)-b_{B_l}|^m|x|^{\beta(x)}.\chi_k(x)\\&\geq C|b(x)-b_{B_l}|^m|x|^{\beta(x)}.\chi_k(x).\end{aligned}$$

Consequently, by proposition (2.8), we have

$$\begin{aligned}&\|(b(x)-b_{B_l})^m|x|^{\beta(x)}\chi_k(x)(1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}}\\&\leq\|(1+|x|)^{-\lambda(x)}(I^{\beta(\cdot)}((b(x)-b_{B_l})^m\chi_k)(x))\|_{L^{q_2(\cdot)}}\\&\leq\|(b(x)-b_{B_l})^m\chi_k)(x)\|_{L^{q_1(\cdot)}}.\end{aligned}$$

As a result, we get

$$\begin{aligned}&\|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\\&\leq C2^{-kn}\|g_l\|_{L^{q_1(\cdot)}}\left\{\|(b(x)-b_{B_l})^m|x|^{\beta(x)}\chi_k(x)(1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\right.\\&\times\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}\end{aligned}$$

$$
\begin{aligned}
&+ \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \| |x|^{\beta(x)} \chi_k(x)(1+|x|)^{-\lambda(x)} \|_{L^{q_2(\cdot)}} \Big\} \\
&\leq C 2^{-kn} \|g_l\|_{L^{q_1(\cdot)}} \Big\{ (k-l)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q_1(\cdot)}} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \\
&+ \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \|\chi_k\|_{L^{q_1(\cdot)}} \Big\} \\
&\leq C 2^{-kn} \|g_l\|_{L^{q_1(\cdot)}} (k-l)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q(\cdot)}} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|\chi_l\|_{L^{q_1(\cdot)}} \\
&\leq C (k-l)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-kn} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\chi_k\|_{L^{q_1(\cdot)}} \|\chi_l\|_{L^{q_1(\cdot)}} \|g_l\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

Applying results to E_{11}, we can get

$$
\begin{aligned}
E_{11} &\leq C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg[\psi^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\psi)} \Bigg(\sum_{l=-\infty}^{k} 2^{(l-k)(n/q_1'(0)-\nu-\frac{n}{s})} (k-l)^m \\
&\quad \times \|g\chi_l\|_{L^{q_1(\cdot)}} \Bigg)^{p(1+\psi)} \Bigg]^{\frac{1}{p(1+\psi)}} \\
&\leq C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg[\psi^\theta \sum_{k=-\infty}^{-1} \Bigg(\sum_{l=-\infty}^{k} 2^{\alpha(0)l} \|g\chi_l\|_{L^{q_1(\cdot)}} 2^{b(l-k)} (k-l)^m \Bigg)^{p(1+\psi)} \Bigg]^{\frac{1}{p(1+\psi)}}.
\end{aligned}
$$

Let $\nu_1 = \frac{n}{q_1'(0)} - \nu - \frac{n}{s} - \alpha(0) > 0$, applying Hölder's inequality, $2^{-p(1+\psi)} < 2^{-p}$ and Fubini's theorem for series, we get

$$
\begin{aligned}
E_{11} \leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg[\psi^\theta \sum_{k=-\infty}^{-1} \Bigg(\sum_{l=-\infty}^{k} 2^{\alpha(0)p(1+\psi)l} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} 2^{\nu_1 p(1+\psi)(l-k)/2} \\
&\times \sum_{l=-\infty}^{k} 2^{\nu_1 (p(1+\psi))'(l-k)/2} (k-l)^{m(p(1+\psi))'/2} \Bigg)^{\frac{p(1+\psi)}{(p(1+\psi))'}} \Bigg]^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k} 2^{\alpha(0)p(1+\psi)l} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} 2^{\nu_1 p(1+\psi)(l-k)/2} \Bigg)^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)p(1+\psi)l} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \sum_{k=l}^{-1} 2^{\nu_1 p(1+\psi)} \Bigg)^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)p(1+\psi)l} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \sum_{k=l}^{-1} 2^{\nu_1 p(l-k)/2} \Bigg)^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)p(1+\psi)l} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \Bigg)^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{l=-\infty}^{\infty} \|2^{\alpha(\cdot)l} g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \Bigg)^{\frac{1}{p(1+\psi)}} \\
\leq& C\|b\|_{BMO)(\mathbb{R}^n)} \|g\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_1(\cdot)}}.
\end{aligned}
$$

Now, for E_{12}, using Minkowski's inequality, we have

$$\begin{aligned} E_{12} \leq & \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_{\infty}p(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_{\Phi}]_{\beta}^{m}(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ & + \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_{\infty}p(1+\psi)}\left(\sum_{l=0}^{k}\|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_{\Phi}]_{\beta}^{m}(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ & =: A_1 + A_2. \end{aligned}$$

The estimate for A_2 follows in a similar manner to E_{11} by replacing $q_1'(0)$ with $q_{1\infty}'$ and by the use $\nu_2 := \frac{n}{q_{1\infty}'} - \nu - \frac{n}{s} - \alpha_{\infty} > 0$. For A_1, we have

$$\begin{aligned} & \|[b,\mu_{\Phi}]_{\beta}^{m}(g\chi_l)\chi_k\|_{L^{q_2(\cdot)}} \\ & \quad \leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}^m 2^{-kn}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_l\|_{L^{q_1(\cdot)}}\|g_l\|_{L^{q_1(\cdot)}} \\ & \quad \leq C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}^m 2^{l(\frac{n}{q_1(0)}-\nu)}2^{k(\nu+\frac{n}{s}-\frac{n}{q_{1\infty}'})}\|g_l\|_{L^{q_1(\cdot)}}. \end{aligned}$$

Now using the fact that $-\frac{n}{q_{1\infty}'} + \nu + \frac{n}{s} + \alpha_{\infty} < 0$, we have

$$\begin{aligned} A_1 \leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_{\infty}p(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_{\Phi}]_{\beta}^{m}g_l\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ \leq & C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_{\infty}p(1+\psi)}\right. \\ & \left.\times\left(\sum_{l=-\infty}^{-1}(k-l)^m 2^{l(\frac{n}{q_1(0)}-\nu)}2^{k(\nu+\frac{n}{s}-\frac{n}{q_{1\infty}'})}\|g_l\|_{L^{q_1(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ \leq & C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{k(a_{\infty}+\nu+\frac{n}{s}-\frac{n}{q_{1\infty}'})p(1+\psi)}\right. \\ & \left.\times\left(\sum_{l=-\infty}^{-1}(k-l)^m 2^{l(\frac{n}{q_1(0)}-\nu)}\|g_l\|_{L^{q_1(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ \leq & C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=-\infty}^{-1}(k-l)^m 2^{l(\frac{n}{q_1(0)}-\nu)}\|g_l\|_{L^{q_1(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ \leq & C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=-\infty}^{-1}(k-l)^m 2^{l(\frac{n}{q_1(0)}-\nu-\alpha(0))}2^{l\alpha(0)}\|g_l\|_{L^{q_1(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}} \\ \leq & C\|b\|_{BMO(\mathbb{R}^n)}^m\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1}2^{\alpha(0)lp(1+\psi)}\|g_l\|_{L^{q_1(\cdot)}}^{p(1+\psi)}\right. \end{aligned}$$

$$\times \left(\sum_{k=0}^{\infty} \sum_{l=-\infty}^{-1} (k-l)^{m(p(1+\psi))'} 2^{l(\frac{n}{q_1(0)} - \nu - l\alpha(0)(p(1+\psi))')} \right)^{p(1+\psi)/(p(1+\psi))'} \Bigg)^{\frac{1}{p(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{k=0}^{\infty} \sum_{l=-\infty}^{-1} 2^{\alpha(0)lp(1+\psi)} \|g_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \right.$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)lp(1+\psi)} \|g_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \right)^{\frac{1}{p(1+\psi)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{l=-\infty}^{\infty} \|2^{\alpha(\cdot)l} g_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \right)^{\frac{1}{p(1+\psi)}}$$

$$\leq C(k-l)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-kn} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\chi_k\|_{L^{q_1(\cdot)}} \|\chi_l\|_{L^{q_1(\cdot)}} \|g_l\|_{L^{q_1(\cdot)}}.$$

Now we estimate E_2, for each $k \in \mathbb{Z}$ and $l \geq k+1$ and a.e. $x \in R_k$, $y \in R_l$, we know that $|x-y| \approx |y| \approx 2^l$, we consider

$$\begin{aligned}\left|[b,\mu_\Phi]_\beta(g_l)(x)\right| \leq{}& \left(\int_o^{|y|} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)(b(x)-b(y))}{|x-y|^{n-1-\beta(x)}} g_l(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &+ \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)(b(x)-b(y))}{|x-y|^{n-1-\beta(x)}} g_l(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &=: I_{31} + I_{32}.\end{aligned}$$

Using similar arguments as used in I_{11}, we obtain

$$\begin{aligned}I_{31} \leq{}& 2^{(l-k)/2} 2^{-ln} |x|^{\beta(x)} \|g_l(y)\|_{L^{q_1(\cdot)}} \Big(|b(x)-b_{B_l}|^m \|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}} \\ &+ \|(b(\cdot)-b_{B_l})^m (\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}} \Big).\end{aligned}$$

Using the same arguments of I_{12}, we obtain

$$\begin{aligned}I_{32} \leq{}& 2^{-ln} |x|^{\beta(x)} \|g_l(y)\|_{L^{q_1(\cdot)}} \Big\{ |b(x)-b_{B_l}|^m \|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}} \\ &+ \|(b(\cdot)-b_{B_l})^m (\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q_1'(\cdot)}} \\ &\times \|\chi_k (1+|x|)^{-\lambda(x)} [b,\mu_\Phi]^m_\beta (g_l)\|_{L^{q_2(\cdot)}} \Big\} \\ \leq{}& C 2^{-ln} \|g_l\|_{L^{q_1(\cdot)}} \Big\{ \|(b(x)-b_{B_l})^m |x|^{\beta(x)} \chi_k(x) (1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}} \\ &\times 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \\ &+ \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \| |x|^{\beta(x)} \chi_k(x) (1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}} \Big\} \\ \leq{}& C 2^{-ln} \|g_l\|_{L^{q_1(\cdot)}} \Big\{ (k-l)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q_1(\cdot)}} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}}\end{aligned}$$

$$+ \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_l\|_{L^{q_1(\cdot)}} \|\chi_k\|_{L^{q_1(\cdot)}} \Big\}$$

$$\le C 2^{-ln} \|g_l\|_{L^{q_1(\cdot)}} (k-l)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q_1(\cdot)}} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|\chi_l\|_{L^{q_1(\cdot)}}$$

$$\le C(k-l)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|b\|^m_{BMO(\mathbb{R}^n)} 2^{-ln} 2^{-l\nu} 2^{k(\nu+\frac{n}{s})} \|\chi_k\|_{L^{q_1(\cdot)}} \|\chi_l\|_{L^{q_1(\cdot)}} \|g_l\|_{L^{q_1(\cdot)}}$$

$$\le C(k-l)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|b\|^m_{BMO(\mathbb{R}^n)} 2^{(k-l)(\nu+\frac{n}{s}+\frac{n}{q_{1\infty}})} \|g\chi_l\|_{L^{q_1(\cdot)}}.$$

Now splitting E_2, we have

$$E_2 \le \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{\infty} \left(\sum_{l=k+1}^{\infty} \|2^{k\alpha(\cdot)} \chi_k (1+|x|)^{-\lambda(x)} [b,\mu_\Phi]^m_\beta (g_l)\|_{L^{q_2(\cdot)}} \right)^{p(1+\psi)} \right)^{\frac{1}{p(1+\psi)}},$$

$$\le \sup_{\psi>0} \left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\psi)} \left(\sum_{l=k+1}^{\infty} \|\chi_k (1+|x|)^{-\lambda(x)} [b,\mu_\Phi]^m_\beta (g_l)\|_{L^{q_2(\cdot)}} \right)^{p(1+\psi)} \right)^{\frac{1}{p(1+\psi)}}$$

$$+ \sup_{\psi>0} \left(\psi^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\psi)} \left(\sum_{l=k+1}^{\infty} \|\chi_k (1+|x|)^{-\lambda(x)} [b,\mu_\Phi]^m_\beta (g_l)\|_{L^{q_2(\cdot)}} \right)^{p(1+\psi)} \right)^{\frac{1}{p(1+\psi)}}$$

$$=: E_{21} + E_{22}.$$

For E_{22}, we have

$$E_{22} \le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\psi)}$$

$$\times \left(\sum_{l=k+1}^{\infty} 2^{(k-l)(\nu+\frac{n}{s}+\frac{n}{q_{1\infty}})} (k-l)^m \|g_l\|_{L^{q_1(\cdot)}} \right)^{p(1+\psi)} \Bigg)^{\frac{1}{p(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \Bigg(\psi^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\psi)} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}}$$

$$\times \left(\sum_{l=k+1}^{\infty} 2^{(k-l)(\nu+\frac{n}{s}+\frac{n}{q_{1\infty}})} (k-l)^m \right)^{p(1+\psi)} \Bigg)^{\frac{1}{p(1+\psi)}}.$$

Let $d = \frac{n}{q_{1\infty}} + \nu + \frac{n}{s} + \alpha_\infty > 0$. Then, we use Hölder's theorem for series and $2^{-p(1+\psi)} < 2^{-p}$ to obtain

$$E_{22} \le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \Bigg[\psi^\theta \sum_{k=0}^{\infty} \left(\sum_{l=k+1}^{\infty} 2^{l\alpha_\infty p(1+\psi)} \|g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} 2^{dp(1+\psi)(k-l)/2} \right)$$

$$\times \left(\sum_{l=k+1}^{\infty} (k-l)^{m(p(1+\psi))'/2} 2^{d(p(1+\psi))'(k-l)/2} \right)^{\frac{p(1+\psi)}{(p(1+\psi))'}} \Bigg]^{\frac{1}{p(1+\psi)}}$$

$$\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0} \left(\psi^\theta \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} 2^{l\alpha_\infty p(1+\psi)} \|g_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} 2^{dp(1+\psi)(k-l)/2} \right)^{\frac{1}{p(1+\psi)}}$$

$$
\begin{aligned}
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\psi)}\|g_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}} \sum_{k=0}^{l-1} 2^{dp(1+\psi)(k-l)/2}\right)^{\frac{1}{p(1+\psi)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\psi>0}\left(\psi^\theta \sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)} g\chi_l\|^{p(1+\psi)}_{L^{q_1(\cdot)}}\right)^{\frac{1}{p(1+\psi)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_1(\cdot)}}.
\end{aligned}
$$

Now, for E_{21}, using Minkowski's inequality, we have

$$
\begin{aligned}
E_{21} \le{}& \sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\psi)}\left(\sum_{l=k+1}^{-1} \|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}\\
&+\sup_{\psi>0}\left(\psi^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\psi)}\left(\sum_{l=0}^{\infty} \|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\right)^{p(1+\psi)}\right)^{\frac{1}{p(1+\psi)}}\\
=:{}&B_1+B_2.
\end{aligned}
$$

The estimate for B_1 follows in a similar manner to E_{22} with $q_{1\infty}$ replaced by $q_1(0)$ and using the fact that $\frac{n}{q_1(0)}+\nu+\frac{n}{s}+\alpha(0)>0$. For B_2, we have

$$
\begin{aligned}
&\|\chi_k(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g_l)\|_{L^{q_2(\cdot)}}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{\|(b(x)-b_{B_l})^m|x|^{\beta(x)}\chi_k(x)(1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}}\\
&\times 2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}\||x|^{\beta(x)}\chi_k(x)(1+|x|)^{-\lambda(x)}\|_{L^{q_2(\cdot)}}\Big\}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}\Big\{(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_l\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\Big\}\\
&\le C2^{-ln}\|g_l\|_{L^{q_1(\cdot)}}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}2^{-l\nu}2^{k(\nu+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_l\|_{L^{q_1(\cdot)}}\\
&\le C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-ln}2^{-l\nu}2^{k(\nu+\frac{n}{s})}2^{ln/q_{1\infty}}2^{kn/q_1(0)}\|g_l\|_{L^{q_1(\cdot)}}\\
&\le C(k-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l(\nu+\frac{n}{s}+\frac{n}{q_{1\infty}})}2^{k(\nu+\frac{n}{q_1(0)}+\frac{n}{s})}\|g_l\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

Using these estimates for B_2, we get the estimate for E_{21}.

Combining the estimates for E_1 and E_2 yields

$$
\|(1+|x|)^{-\lambda(x)}[b,\mu_\Phi]^m_\beta(g)\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},
$$

which ends the proof. □

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