Journal of Inequalities and Applications
a SpringerOpen Journal

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Common best proximity point theorems in Hausdorff topological spaces

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Abstract

In the present paper, we have obtained common best proximity point theorems of nonself maps in Hausdorff topological space. Further, our results extend the results due to Gerald F. Jungck, thereby proving a generalized version of Kirk's theorem (J. London Math. 1(1):107–111, 1969).

Keywords: Common best proximity point; Hausdorff topological space

1 Introduction

Metric fixed point theory is an important tool in mathematics as it has numerous applications in the field of differential equations as well as integral equations. Let S, T be nonempty subsets of metric space (Ω, d) and Γ be a self mapping defined on Ω. Suppose $d(\zeta, \Gamma\zeta) = 0$ for some $\zeta \in \Omega$, then ζ is a fixed point of Γ. Sometimes it is not necessary that $d(\zeta, \Gamma\zeta) = 0$ will have a solution for $\Gamma : S \to T$. In that case, we will go for the $\min_{\zeta \in S} d(\zeta, \Gamma\zeta)$. In particular, if we have $\zeta \in S$ such that $d(\zeta, \Gamma\zeta) = D(S, T) = \inf\{d(\eta_1, \eta_2) : \eta_1 \in S, \eta_2 \in T\}$, then such ζ is referred to as a best proximity point of Γ. For more details on best proximity points, refer to [1, 2, 4, 6, 10, 12, 15–18, 20], and for more details on fixed points, refer to [5, 13].

When it comes to topological space, we cannot use the concept of metric space. Recently, Raj and Piramatchi [21] introduced the notion of $D_h(S, T)$ in the setting of topological space, equivalent to the concept of distance in metric spaces.

Definition 1 [21] Let S, T be nonempty subsets of a topological space Ω, and let $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $D_h(S, T) = \inf\{|h(\zeta, \eta)| : \zeta \in S, \eta \in T\}$. When Ω is a metric space, $D_h(S, T) = D(S, T) = \inf\{d(\eta_1, \eta_2) : \eta_1 \in S, \eta_2 \in T\}$ as $h = d$, the metric on Ω.

Using this notion, they introduced topological P-property and topologically r-contractive mapping, thereby proving the best proximity point theorem for cyclic mappings.

In [22], Dey et al. extended the Banach contraction principle to nonself maps in arbitrary topological space. With the help of a continuous function $g : \Omega \times \Omega \to \mathbb{R}$, they brought the idea of topologically Banach contraction for nonself maps and established the existence of best proximity point.

The above results are for a single nonself map. In the literature, there are not many results on common best proximity points for mappings in a topological space.

In [9], Jungck considered compatible maps of Hausdorff topological space and proved the existence of common fixed points by introducing the notion of proper orbits, thereby using it to obtain the fixed points.

The following theorem is one of his main results on common fixed points.

Theorem 1 [9] *Let Ω be a first countable Hausdorff topological space and Γ be a continuous self mapping on Ω. Suppose that Γ has proper orbits that are relatively compact, then Γ has a common fixed point with each $\Lambda \in E_\Gamma$. Furthermore, for all $\zeta \in \Omega$, there exists a subsequence of $\{\Lambda^n(\zeta)\}$ such that it converges to a fixed point of Γ.*

Inspired by works of Raj [21] and Dey [22], in our paper we consider Hausdorff topological space and prove common best proximity theorems for nonself maps. We extend the notions like orbits, proper orbits, and other related concepts to the nonself map in the framework of Hausdorff topological space, thereby proving the existence of best proximity points for that mapping. Moreover, we provide an example that validates our result. Further, we also obtain some results on common best proximity points for nonself mappings. When the respective space is a metric space and the map is a self map, our results imply the results of Jungck [9].

To know more about the common best proximity points, one can refer to [3, 7, 8, 19].

2 Preliminaries

Here we provide some definitions, notations, and concepts needed in the sequel.

Unless specified, we assume throughout S, T to be nonempty subsets of a topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ to be a continuous function.

Definition 2 Consider a mapping $\Gamma : S \to T$, $\Gamma(S) \subseteq T_0$, where $T_0 = \{\eta \in T \mid D_h(S,T) = |h(\zeta,\eta)| \text{ for some } \zeta \in S\}$. Let $\zeta_0 \in S$. Since $\Gamma(S) \subseteq T_0$, there exists $\zeta_1 \in S$ such that $|h(\zeta_1, \Gamma\zeta_0)| = D_h(S,T)$. Similarly, choose $\zeta_2 \in S$ such that $|h(\zeta_2, \Gamma\zeta_1)| = D_h(S,T)$. Proceeding like this, we obtain a collection of elements $\{\zeta_n\}$ in S. Define $O_h(\zeta_0, \Gamma) = \{\zeta_n \mid D_h(S,T) = |h(\zeta_n, \Gamma\zeta_{n-1})|, n \in \mathbb{N}\}$. $O_h(\zeta_0, \Gamma)$ is defined as the proximal orbit of a nonself map Γ starting at $\zeta_0 \in S$ with respect to the nonnegative function h.

Here, unlike self maps, ζ_i's need not be unique while constructing the proximal orbits. Refer to the example given below.

Example 1 Consider $\Omega = \mathbb{R}$ with the usual topology, and the continuous function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $h(\zeta,\eta) = \min\{\zeta,\eta\}$. Let $S = \{5,9,7,1\}$ and $T = \{2,3,4,8\}$. Here $D_h(S,T) = 1$. Now, define $\Gamma : S \to T$ as follows:

$$\Gamma(5) = 2,\ \Gamma(9) = 8,\ \Gamma(7) = 3,\ \Gamma(1) = 4.$$

The proximal orbits of Γ at each element in S are given by

$$O_h(5,\Gamma) = O_h(1,\Gamma) = O_h(7,\Gamma) = \{1\} \text{ and } O_h(9,\Gamma) = \{5,1\}, \{7,1\}, \{1\}.$$

Here, the proximal orbits of Γ at $5, 7$, and 1 are unique. However, the proximal orbit of Γ at $9 \in S$ is not unique. Because, while constructing, we can choose our ζ_1 to be any element from $\{5, 7, 1\}$ and ζ_2 will be always $\{1\}$. So, here we will be having three different proximal orbits.

Suppose that we have $h : \Omega \times \Omega \to \mathbb{R}$ and $|h(\zeta_1, \eta)| = |h(\zeta_2, \eta)| \implies \zeta_1 = \zeta_2$ for $\zeta_1, \zeta_2, \eta \in \Omega$. Then, while constructing orbits, ζ_i's will be unique.

Definition 3 The proximal orbit of Γ at ζ_0 with respect to h $(O_h(\zeta_0, \Gamma))$ is proper iff either $O_h(\zeta_0, \Gamma) = \{\zeta_0\}$ or $\exists m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $cl(O_h(\zeta_n, \Gamma)) \subset cl(O_h(\zeta_0, \Gamma))$, where $\zeta_n \in cl(O_h(\zeta_0, \Gamma))$. If $O_h(\zeta, \Gamma)$ is proper for all $\zeta \in S_1 \subset S$, then the mapping Γ is said to have proper proximal orbits on S_1. If $S_1 = S$, then we say that Γ has proper proximal orbits.

Definition 4 Let $S_1 \neq \emptyset$ be a subset of S. S_1 is said to be a proximally-Γ-invariant subset of S if for all $\zeta \in S_1$ there exists $\eta \in S_1$ such that $|h(\eta, \Gamma\zeta)| = D_h(S, T)$.

The following example illustrates the definitions provided so far.

Example 2 Consider $\Omega = \mathbb{R}$ with the usual topology, and the continuous function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $h(\zeta, \eta) = \zeta^2 - \eta^2$. Let $S, T \subset \mathbb{R}$ as $S = \mathbb{N} \cup \{0\}$ and $T = \mathbb{Z}^-$. Here $D_h(S, T) = 0$. Now, define $\Gamma : S \to T, \Gamma(\zeta) = -(\zeta + 1)$. The proximal orbit starting at $\zeta = 0$ is given by $O_h(0, \Gamma) = \{1, 2, 3, \ldots\}$. It can be verified that the proximal orbits are proper. Also, here $O_h(\zeta_0, \Gamma)$ is a proximally-Γ-invariant subset of S.

Definition 5 Let $\Gamma, \Lambda : S \to T$ be two nonself maps. An element $\zeta \in S$ is a common best proximity point of the given pair (Γ, Λ) if $|h(\zeta, \Gamma\zeta)| = |h(\zeta, \Lambda\zeta)| = D_h(S, T)$.

Definition 6 Let $\Gamma : S \to T$ be a continuous nonself map. Define E_Γ as the collection of all continuous mappings $\Lambda : S \to T$ such that:

(i) $\Lambda(S) \subseteq T_0$,
(ii) $M = \{\zeta \in S | \Gamma\zeta = \Lambda\zeta\} \neq \emptyset$, and
(iii) $|h(\eta, \Gamma\zeta)| = |h(\beta, \Lambda\zeta)| = D_h(S, T) \implies \Gamma\beta = \Lambda\eta, \ \forall \zeta \in M, \forall \eta, \beta \in S.$

Example 3 Consider $\Omega = \mathbb{R}^2$ with the usual topology and the continuous function $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as $h((\zeta, \eta), (\lambda, \beta)) = |\zeta - \lambda| + |\eta - \beta|$. Let $S = \{0\} \times [0, 1]$ and $T = \{1\} \times [0, 1]$. Here $D_h(S, T) = 1$. Now, define functions $\Gamma_1, \Gamma_2, \Gamma_3 : S \to T$ as $\Gamma_1(0, \zeta) = (1, 2\zeta - \zeta^2)$, $\Gamma_2(0, \zeta) = (1, \zeta^2)$, and $\Gamma_3(0, \zeta) = (1, \zeta^3)$. Here, $\Gamma_2(S) \subseteq T_0$ and $M = \{(0, 0), (0, 1)\} \neq \emptyset$. It can be verified that Γ_1 and Γ_2 do not satisfy the third condition. On the other hand, $\Gamma_3(S) \subseteq T_0$ and $M = \{(0, 0), (0, 1)\} \neq \emptyset$; and all the conditions are satisfied. Consider $\Gamma_2 : S \to T$. Here $E_{\Gamma_2} \neq \emptyset$ as $\{\Gamma_4 : S \to T$ such that $\Gamma_4(0, \zeta) = (1, \zeta^m)$, $m \in \mathbb{N}$, and $m > 2\} \subset E_{\Gamma_2}$. Also, $\Gamma_2(S) \subseteq T_0$ and $M = \{(0, 0), (0, 1)\} \neq \emptyset$. Clearly, the third condition holds for functions Γ_2 and Γ_4.

Definition 7 A point $\zeta \in S$ is called a recurrent point iff ζ is a limit point of $O_h(\zeta, \Gamma)$. Moreover, $\zeta \in S$ is a nontrivial periodic proximal point iff $\zeta = \zeta_n$ for some $\zeta_n \in O_h(\zeta, \Gamma), n \in \mathbb{N}$. But $\zeta \neq \zeta_1$, where $|h(\zeta_1, \Gamma\zeta)| = D_h(S, T)$.

Definition 8 A continuous nonself map $\Gamma : S \to T$ is compact iff T has a compact set T_1 and $\Gamma(S) \subset T_1$.

If Ω is a metric space, then for some $\zeta_0 \in \Omega$ we get $O(\zeta_0, \Gamma) = \{\zeta_n \mid d(\zeta_n, \Gamma\zeta_{n-1}) = D(S,T)\}$, where d is the respective metric and $n \in \mathbb{N}$. Let $S, T \neq \emptyset$ be subsets of a metric space (Ω, d).

Definition 9 A nonself mapping $\Gamma : S \to T$ is said to have proximal diminishing orbital diameters (proximal *d.o.d.*) if for all $\zeta_0 \in S$, $\delta(O(\zeta_0, \Gamma)) < \infty$ and whenever $\delta(O(\zeta_0, \Gamma)) > 0$, $\exists\ m = m_{\zeta_0} \in \mathbb{N}$ such that $\delta(O(\zeta_0, \Gamma)) > \delta(O(\zeta_m, \Gamma))$ for all $m \geq m_{\zeta_0}$, where $\zeta_m \in O(\zeta_0, \Gamma) = \{\zeta_l \mid d(\zeta_l, \Gamma\zeta_{l-1}) = D(S,T)\}$.

Definition 10 A nonself mapping $\Gamma : S \to T$ is said to have proximal orbits with proximal diminishing closure diameters (proximal d.c.d.) iff for all $\zeta_0 \in S$, $\delta(cl(O(\zeta_0, \Gamma))) < \infty$ and whenever $\delta(cl(O(\zeta_0, \Gamma))) > 0$, $\exists\ m = m_{\zeta_0} \in \mathbb{N}$ such that $\delta(cl(O(\zeta_0, \Gamma))) > \delta(cl(O(\zeta_m, \Gamma)))$ for all $m \geq m_{\zeta_0}$, where $\zeta_m \in O(\zeta_0, \Gamma) = \{\zeta_l \mid d(\zeta_l, \Gamma\zeta_{l-1}) = D(S,T)\}$.

Lemma 1 *Let $S, T \neq \emptyset$ be subsets of a semimetric space Ω, and $\Gamma : S \to T$. Suppose that Γ has proximal orbits with proximal d.c.d., then Γ has proper proximal orbits.*

Proof Suppose $O(\zeta_0, \Gamma) \neq \{\zeta_0\}$ for some $\zeta_0 \in S$. Then $\delta(cl(O(\zeta_0, \Gamma))) > 0$. Given Γ has proximal d.c.d., we can find $m_{\zeta_0} \in \mathbb{N}$ such that $\delta(cl(O(\zeta_0, \Gamma))) > \delta(cl(O(\zeta_m, \Gamma)))$, $\forall m \geq m_{\zeta_0}$, where $\zeta_m \in O(\zeta_0, \Gamma)$. Also, by construction of orbits, $O(\zeta_m, \Gamma) \subset O(\zeta_0, \Gamma)$, $\forall\, m \geq m_{\zeta_0} \in \mathbb{N}$, provided $\zeta_m \in O(\zeta_0, \Gamma)$. Therefore, for all $m \geq m_{\zeta_0} \in \mathbb{N}$, we get $cl(O(\zeta_m, \Gamma)) \subset cl(O(\zeta_0, \Gamma))$, $\zeta_m \in O(\zeta_0, \Gamma)$. This implies that the proximal orbits of Γ are proper. □

When Ω is a metric space, proximal d.c.d. become proximal *d.o.d*. Hence we have the following lemma, which can be proved similarly.

Lemma 2 *Let S, T be nonempty subsets of a metric space Ω, and $\Gamma : S \to T$. Suppose that Γ has proximal orbits with proximal d.o.d., then Γ has proper proximal orbits.*

3 Main results

Theorem 2 *Let $S, T \neq \emptyset$ be subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let S_0 be nonempty and $\Gamma : S \to T$ be continuous. If Γ has proper proximal orbits on S that are relatively compact, then any nonempty, closed, and proximally-Γ-invariant subset of S contains a best proximity point of Γ. In particular, the closure of each proximal orbit $O_h(\zeta, \Gamma)$ has a best proximity point of Γ.*

Proof Let $M \neq \emptyset$ be a proximally-Γ-invariant and closed subset of S. Let $\zeta_0 \in M$. By the proximally-Γ-invariant property of M, there exists $\zeta_1 \in M$ such that $|h(\zeta_1, \Gamma\zeta_0)| = D_h(S,T)$. Proceeding like this, we have $O_h(\zeta_0, \Gamma) = \{\zeta_n \mid |h(\zeta_n, \Gamma\zeta_{n-1})| = D_h(S,T)\} \subset M$. Since M is closed, $cl(O_h(\zeta_0, \Gamma)) \subset M$.

By the construction of proximal orbits, we have

$$O_h(\zeta_1, \Gamma) \subset O_h(\zeta_0, \Gamma) \subset M \implies cl(O_h(\zeta_1, \Gamma)) \subset cl(O_h(\zeta_0, \Gamma)) \subset M. \tag{1}$$

By the relative compactness of proximal orbits, $cl(O_h(\zeta_0, \Gamma))$ is compact. Now, we claim that $cl(O_h(\zeta_0, \Gamma))$ is proximally-Γ-invariant.

Let $cl(O_h(\zeta_0, \Gamma)) = U_0$ and $\Gamma(U_0) = C_0$.

Define $W_0 = \{\eta \in C_0 \mid |h(\zeta, \eta)| = D_h(S,T)$ for some $\zeta \in U_0\}$. It is enough to prove that $\Gamma(U_0) \subseteq W_0$.

Clearly,

$$\Gamma(O_h(\zeta_0,\Gamma)) \subset W_0 \subset C_0. \tag{2}$$

Given Γ is continuous,

$$\Gamma(U_0) \subset cl(\Gamma(O_h(\zeta_0,\Gamma))). \tag{3}$$

Thus we have

$$\Gamma(O_h(\zeta_0,\Gamma)) \subset W_0 \implies cl(\Gamma(O_h(\zeta_0,\Gamma))) \subset cl(W_0). \tag{4}$$

From Equations (2), (3), and (4), we have

$$\Gamma(O_h(\zeta_0,\Gamma)) \subset W_0 \subset C_0 = \Gamma(U_0) \subset cl(\Gamma(O_h(\zeta_0,\Gamma))) \subset cl(W_0).$$

$$W_0 \subset C_0 = \Gamma(U_0) \subset cl(W_0). \tag{5}$$

Since Ω is a Hausdorff topological space and Γ is continuous, we get C_0 is a closed and, therefore, compact subset of Ω. Hence, for a net $(\zeta_\alpha)_{\alpha\in I} \in W_0$, where I is the directed set such that $(\zeta_\alpha)_{\alpha\in I}$ converges to ζ for some $\zeta \in T$, then $\zeta \in C_0$.

Claim: $\zeta \in W_0$.

Since $(\zeta_\alpha)_{\alpha\in I} \in W_0$, by the definition of W_0, there exists $(\eta_\alpha)_{\alpha\in I} \in cl(O_h(\zeta_0,\Gamma))$ such that $|h(\eta_\alpha,\zeta_\alpha)| = D_h(S,T)$. Since U_0 is compact, there exists a subnet $(\eta_\beta)_{\beta\in J}$ of $(\eta_\alpha)_{\alpha\in I}$ that converges to some $\eta \in U_0$. Choose $(\zeta_\beta)_{\beta\in J} \in W_0$ such that $|h(\eta_\beta,\zeta_\beta)| = D_h(S,T)$. Since h is continuous, as $\beta \to \infty$, we have $|h(\eta,\zeta)| = D_h(S,T)$ implies $\zeta \in W_0$.

Since W_0 is closed, Equation (5) can be written as

$$W_0 \subset C_0 = \Gamma(U_0) \subset W_0 \implies \Gamma(U_0) = W_0.$$

Hence, $U_0 = cl(O_h(\zeta_0,\Gamma))$ is proximally-Γ-invariant.

Now, consider Equation (1). Here, Zorn's lemma [14] asserts the existence of a minimal, nonempty, closed, and proximally-Γ-invariant compact subset M_1 of M.

Now, let $a_0 \in M_1$. Moreover, $cl(O_h(a_0,\Gamma)) \subset M_1$ as M_1 is closed. As above,

$$O_h(a_0,\Gamma) \subset cl(O_h(a_0,\Gamma)) \subset M_1.$$

Also, as above, we can verify that $cl(O_h(a_0,\Gamma))$ is a proximally-Γ-invariant and nonempty compact subset of M_1. By the minimality of M_1, $cl(O_h(a_0,\Gamma)) = M_1$. Consequently, $M_1 = cl(O_h(a_i,\Gamma))$, $\forall i = 0,1,2,3,\ldots,$ and $a_i \in O_h(a_0,\Gamma)$. Furthermore, $cl(O_h(a_i,\Gamma)) = O_h(a_0,\Gamma)$, $\forall\, i = 1,2,3,\ldots.$ Since Γ has proper proximal orbits, $O_h(a_0,\Gamma) = \{a_0\}$. That is, $|h(a_0,\Gamma a_0)| = D_h(S,T)$. Hence, $a_0 \in M_1 \subset P$ is the best proximity point of Γ in M.

Hence, for each $\zeta \in S$, $cl(O_h(\zeta,\Gamma))$ is a nonempty, proximally-Γ- invariant, and closed subset of S. And it is compact. Moreover, $cl(O_h(\zeta,\Gamma))$ has a best proximity point of Γ. □

Corollary 1 *Let S,T be nonempty subsets of a Hausdorff topological space Ω, and $h:\Omega\times\Omega\to\mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and $\Gamma: S \to T$ be continuous. Then Γ has* *a best proximity point iff there exists $\zeta \in S$ such that $cl(O_h(\zeta,\Gamma))$ is compact, and Γ has proper proximal orbits on $cl(O_h(\zeta,\Gamma))$.*

Proof Assume that Γ has a best proximity point, $\zeta \in S$. That is, $|h(\zeta,\Gamma\zeta)| = D_h(S,T)$, which implies $O_h(\zeta,\Gamma) = \{\zeta\} = cl(O_h(\zeta,\Gamma))$. Thus, $cl(O_h(\zeta,\Gamma))$ is compact. And Γ has proper proximal orbits. Hence the given condition is necessary.

Conversely, let $\zeta \in S$ such that $cl(O_h(\zeta,\Gamma))$ is compact and, in addition, assume that Γ has proper proximal orbits on $cl(O_h(\zeta,\Gamma))$.

Claim: $cl(O_h(\zeta,\Gamma))$ is proximally-Γ-invariant.

Let $cl(O_h(\zeta,\Gamma)) = U$ and $\Gamma(U) = C$. Define $W = \{\eta \in C \mid |h(\zeta,\eta)| = D_h(S,T)$ for some $\zeta \in U\}$. It is enough to prove that $\Gamma(U) \subseteq W$.

Clearly,

$$\Gamma(O_h(\zeta,\Gamma)) \subset W \subset C. \tag{6}$$

Given Γ is continuous,

$$\Gamma(U) \subset cl(\Gamma(O_h(\zeta,\Gamma))). \tag{7}$$

We have

$$\Gamma(O_h(\zeta,\Gamma)) \subset W \implies cl(\Gamma(O_h(\zeta,\Gamma))) \subset cl(W). \tag{8}$$

From Equations (6), (7), and (8), we have

$$\begin{aligned} &\Gamma(O_h(\zeta,\Gamma)) \subset W \subset C = \Gamma(U) \subset cl(\Gamma(O_h(\zeta,\Gamma))) \subset cl(W). \\ &W \subset C = \Gamma(U) \subset cl(W). \end{aligned} \tag{9}$$

Since Ω is a Hausdorff topological space and Γ is continuous, we get C is a closed, compact subset of Ω. Hence, for a net $(\zeta_\alpha)_{\alpha\in I} \in W$, where I is the directed set such that $(\zeta_\alpha)_{\alpha\in I}$ converges to ζ for some $\zeta \in T$, then $\zeta \in C$.

Claim: $\zeta \in W$.

Since $(\zeta_\alpha)_{\alpha\in I} \in W$, by the definition of W, there exists $(\eta_\alpha)_{\alpha\in I} \in cl(O_h(\zeta,\Gamma))$ such that $|h(\eta_\alpha,\zeta_\alpha)| = D_h(S,T)$. Since U is compact, there exists a subnet $(\eta_\beta)_{\beta\in J}$ of $(\eta_\alpha)_{\alpha\in I}$ that converges to some $\eta \in U$. Choose $(\zeta_\beta)_{\beta\in J} \in W$ such that $|h(\eta_\beta,\zeta_\beta)| = D_h(S,T)$. Since h is continuous, as $\beta \to \infty$, we have $|h(\eta,\zeta)| = D_h(S,T)$ implies $\zeta \in W$.

Since W is closed, Equation (9) can be written as

$$W \subset C = \Gamma(U) \subset W \implies \Gamma(U) = W.$$

Hence, $U = cl(O_h(\zeta,\Gamma))$ is proximally-Γ-invariant. Therefore, by Theorem 2, the conclusion follows. Hence, the condition is sufficient. □

Corollary 2 *Let S,T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and $\Gamma : S \to T$ be continuous. Suppose that Γ has proper proximal orbits that are relatively compact, then Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$.*

Proof Let $\Lambda \in E_\Gamma$.

Claim: M is proximally-Γ-invariant.

Suppose not. That is, there exists $\zeta_0 \in M$ such that

$$|h(\eta, \Gamma\zeta_0)| > D_h(S, T), \forall \eta \in M. \tag{10}$$

Since $\Lambda(S) \subset T_0$, by the definition of T_0, $\exists \eta \in S$ such that $|h(\eta, \Lambda\zeta_0)| = D_h(S, T)$. Also, by the definition of M, $\Gamma\zeta_0 = \Lambda\zeta_0$. We have $|h(\eta, \Gamma\zeta_0)| = D_h(S, T)$ for some $\eta \in S$. Using the third condition in Definition 6, we get $\Gamma\eta = \Lambda\eta$, which implies $\eta \in M$. It contradicts Equation (10). Hence, M is proximally-Γ-invariant and proximally-Λ-invariant.

Clearly, M is closed. Now, here all the conditions of Theorem 2 hold. As a result, M has a best proximity point of Γ. Let us say $\zeta_0 \in M$. By the definition of M, $\zeta_0 \in M$ implies $|h(\zeta_0, \Gamma\zeta_0)| = D_h(S, T) = |h(\zeta_0, \Lambda\zeta_0)|$. Therefore, ζ_0 is a best proximity point of Λ. Hence ζ_0 is the common best proximity point of Γ and Λ. □

Combining Theorem 2 and Corollary 2, we obtain the result given below.

Theorem 3 *Let S, T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let S_0 be nonempty and $\Gamma : S \to T$ be continuous. Suppose that Γ has proper proximal orbits that are relatively compact, then Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$. Moreover, each nonempty, closed, and proximally-Γ-invariant subset of S has a best proximity point of Γ. In particular, the closure of every proximal orbit $O_h(\zeta, \Gamma)$ has a best proximity point of Γ.*

Proof The result follows from the proof of Theorem 2 and Corollary 2. □

If Ω is a compact Hausdorff topological space, and S is a closed subset of Ω, then S is compact.

Corollary 3 *Let S, T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and S be compact. Then any continuous nonself map $\Gamma : S \to T$ with proper proximal orbits has a best proximity point. Moreover, the closure of every proximal orbit contains a best proximity point of Γ. Also, Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$.*

Proof Given S is compact, and for any $\zeta \in S$, $cl(O_h(\zeta, \Gamma))$ is compact. That is, the proximal orbits are relatively compact. Now, we apply Theorem 3 to get the final result. □

Theorem 4 *Let S, T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and $\Gamma : S \to T$ be continuous. If Γ has neither recurrent points nor nontrivial periodic proximal points, then Γ has proper orbits.*

Proof Let $\zeta \in S$. We have to prove that $O_h(\zeta, \Gamma)$ is proper. If $|h(\zeta, \Gamma\zeta)| = D_h(S, T)$, then $O_h(\zeta, \Gamma) = \{\zeta\}$. Thus, the proof is over. So, let $|h(\zeta, \Gamma\zeta)| \neq D_h(S, T)$. Given Γ has no recurrent points, ζ is not a recurrent point of $O_h(\zeta, \Gamma)$. That is,

$$\exists \text{ a neighborhood } N(\zeta) \text{ of } \zeta \text{ such that } N(\zeta) \cap \left(\frac{O_h(\zeta, \Gamma)}{\{\zeta\}}\right) = \emptyset.$$

Also, ζ is not a nontrivial periodic proximal point. Therefore,

$$\frac{O_h(\zeta,\Gamma)}{\{\zeta\}} = O_h(\zeta_1,\Gamma), \text{ where } \zeta_1 \in S \text{ such that } |h(\zeta_1,\Gamma\zeta)| = D_h(S,T).$$

We get $N(\zeta) \cap O_h(\zeta_1,\Gamma) = \emptyset$. That is,

$$\zeta \notin O_h(\zeta_1,\Gamma) \implies cl(O_h(\zeta_1,\Gamma)) \subset cl(O_h(\zeta,\Gamma)).$$

Hence, $O_h(\zeta,\Gamma)$ is a proper proximal orbit. □

Now we are combining Corollary 3 and Theorem 4. Here, we remove the condition that the proximal orbits are proper.

Corollary 4 *Let S,T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let S_0 be nonempty and S be compact. Then any continuous nonself map $\Gamma : S \to T$ that has neither recurrent points nor nontrivial periodic proximal points has a best proximity point.*

Proof Here, Γ satisfies the hypothesis of Theorem 4. Hence, Γ has proper proximal orbits. Now, applying Corollary 3 yields that Γ has a best proximity point in S. □

Example 4 Consider $\mathbb{R}$ with the usual topology. Let $\Omega = \mathbb{R} - \{5\} \subset \mathbb{R}$ with the respective subspace topology and the continuous function $h : \Omega \times \Omega \to \mathbb{R}$ as $h(\zeta,\eta) = \zeta^2 - \eta^2$. Let $S = \{0,1,2,3,4\}$ and $T = \{-1,-2,-3,-4\}$. S_0 is nonempty. Here, $D_h(S,T) = 0$. Define $\Gamma : S \to T$ as follows:

$$\Gamma(0) = -4, \Gamma(4) = -3, \Gamma(3) = -2, \Gamma(2) = -1, \text{ and } \Gamma(1) = -1.$$

Subspace topologies of S and T are discrete. Hence, Γ is continuous, and now we can talk about the proximal orbits of Γ at each $\zeta \in S$.

$$O_h(0,\Gamma) = \{4,3,2,1\}, O_h(1,\Gamma) = \{1\}, O_h(2,\Gamma) = \{1\},$$
$$O_h(3,\Gamma) = \{2,1\}, O_h(4,\Gamma) = \{3,2,1\}.$$

Here, Γ has proper proximal orbits on $cl(O_h(0,\Gamma))$, and $cl(O_h(0,\Gamma))$ is compact. Further, Γ has neither recurrent points nor nontrivial periodic proximal points. For instance, let $\zeta = 1$ be a recurrent point. Then,

$$\forall \epsilon > 0,\ N(1,\epsilon) \cap \Big(\frac{O_h(1,\Gamma) = \{1\}}{\{1\}}\Big) \neq \emptyset.$$

However, it is an empty set. Therefore, $\zeta = 1$ is not a recurrent point of $O_h(1,\Gamma)$. Hence, Γ has a best proximity point on $cl(O_h(0,\Gamma))$ and $\zeta = 1$ is the best proximity point.

Corollary 5 *Let S,T be nonempty subsets of a Hausdorff topological space Ω and $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and $\Gamma : S \to T_0$ be a homeomorphic nonself map with proper proximal orbits. Suppose that Γ is compact, then Γ has a best proximity point. Moreover, Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$.*

Proof Given Γ is compact, there exists a compact set $Q \subset T$ such that $\Gamma(S) \subset Q$, and given Γ is homeomorphic, we obtain $\Gamma^{-1}(Q)$ is compact. We have $\Gamma : S \to T_0$, and Γ has proper proximal orbits. Now, consider $\Gamma : \Gamma^{-1}(Q) \to T_0$. We get $\Gamma^{-1}(Q)$ is compact and $\Gamma^{-1}(Q)$ is proximally-Γ-invariant. Given Γ has proper proximal orbits, the restriction of the mapping Γ to $\Gamma^{-1}(Q)$ meets all the conditions in the hypothesis of Corollary 3. □

Corollary 6 *Let S, T be nonempty subsets of a first countable Hausdorff topological space Ω, and let $h : \Omega \times \Omega \to \mathbb{R}$ be a continuous function. Let $S_0 \neq \emptyset$ and $\Gamma : S \to T$ be continuous. Suppose that Γ has proper proximal orbits that are relatively compact, then Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$. Furthermore, for every $\zeta \in S$, there exists a subsequence of $\{O_h(\zeta, \Gamma)\}$ that converges to a best proximity point of Γ.*

Proof The proof is the same as that of Theorem 3. We need to prove the existence of a subsequence of $\{O_h(\zeta, \Gamma)\}$ that converges to a best proximity point of Γ for each $\zeta \in S$. Let $\zeta \in S$. Then $cl(O_h(\zeta, \Gamma))$ contains a best proximity point of Γ. Let it be η. We know that Ω is first countable. Hence, we can find a sequence in $O_h(\zeta, \Gamma)$ that converges to η. Since Γ has proximal orbits that are proper and relatively compact, any subsequence of $O_h(\zeta, \Gamma)$ converges to η. □

If Ω is a semimetric space, then Ω is both Hausdorff and first countable. The following corollary is for a semimetric space, and it is our extended version of Theorem 1.

Corollary 7 *Let $S, T \neq \emptyset$ be subsets of Ω where (Ω, d) is a semimetric space, and let $\Gamma : S \to T$ be continuous. Suppose that Γ has relatively compact proximal orbits that are either proper, or that it has proximal d.c.d., then Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$. Furthermore, for every $\zeta \in S$, some subsequence of $O_h(\zeta, \Gamma)$ converges to a best proximity point of Γ.*

Proof If Γ has proper orbits that are relatively compact, then the result follows from Corollary 6. On the other hand, if Γ has proximal d.c.d. that are relatively compact, then from Lemma 1 we know that proximal d.c.d. imply proper proximal orbits. Hence the result follows. □

If (Ω, d) is a metric space, and if we replace proximal d.c.d. with proximal *d.o.d.* in Corollary 7, it works since proximal *d.o.d.* yield proximal d.c.d. Hence, we obtain the following result.

Corollary 8 *Let $S, T \neq \emptyset$ be subsets of a compact metric space (Ω, d). Let $\Gamma : S \to T$ be a continuous mapping with proximal d.o.d. Thus, Γ shares a common best proximity point with each $\Lambda \in E_\Gamma$. Furthermore, for each $\zeta \in P$, some subsequence of $\{O(\zeta, \Gamma)\}$ has a limit that is a best proximity point of Γ.*

Consider the subsets of $\mathbb{R}^n$. We know that these subsets are compact iff they are bounded and closed. Hence, we obtain the following consequence of Corollary 7.

Corollary 9 *Let $S, T \neq \emptyset$ be subsets of $\mathbb{R}^n$ and Γ be a continuous nonself mapping $\Gamma : S \to T$ with proximal d.o.d., then Γ has a best proximity point. Furthermore, Γ shares a common* *best proximity point with each* $\Lambda \in E_{\Gamma}$*. Furthermore, each proximal orbit* $O(\zeta, \Gamma)$ *has a subsequence that converges to a best proximity point of* Γ*.*

Acknowledgements
The authors are grateful to the editor and the referees for their valuable comments and suggestions.

Author contributions
All authors contributed equally to this work.

Funding
Not applicable.

Data Availability
No datasets were generated or analysed during the current study.

Code availability
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Received: 7 March 2024 Accepted: 28 June 2024 Published online: 09 July 2024

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