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Asymptotic stability and bifurcations of a perturbed McMillan map

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Abstract

This paper presents various bifurcations of the McMillan map under perturbations of its coefficients, such as period-doubling, pitchfork, and hysteresis bifurcation. The associated existence regions are located. Using the quasi-Lyapunov function method, the existence of asymptotically stable fixed point is also demonstrated.

Mathematics Subject Classification: 39A11; 39A20

Keywords: McMillan map; Period-doubling bifurcation; Pitchfork bifurcation; Hysteresis bifurcation; Asymptotic stability

1 Introduction

McMillan first introduced a symplectic map to describe the motion of a particle through a periodically repeated focusing system containing lumped nonlinear impulses [1]. McMillan map is not only a model of accelerator lattice, but is at the core of general symplectic dynamics of the plane. Zolkin et al. considered the McMillan sextupole and octupole integrable mappings and provided complete description of all stable trajectories including parametrization of invariant curves [2]; the second one is sometimes referred to as canonical McMillan map. Both of them are the natural extensions of the optical function formalism used in accelerator physics. Gubser et al. obtained new analytic solutions describing motions of closed segmented strings in AdS_3 in terms of elliptic functions, which exactly solve instances of the McMillan map [3]. Danilov et al. deduced a generalization of the McMillan map to N-body nonlinear integrable system, which can be realized in particle accelerators [4]. The McMillan map exhibits plentiful dynamical behaviors such as bifurcations, asymptotic stability, and various kinds of chaotic attractors depending on the choice of its coefficients [5]. The standard form of the McMillan map is a rational integrable mapping of the form

$$H(x,y) = (y, -x + \frac{\alpha + \beta y}{1 - y^2}),$$

where α and β are constants. The map possesses the following biquadratic integral:

$$x^2y^2 - (x^2 + y^2) + \beta xy + \alpha(x + y) = \mathcal{K}.$$

Here, $\mathcal{K}$ is a parameter that indicates each invariant curve in a two-dimensional phase plane.

The McMillan map is also known as the autonomous discrete Painlevé II equation. Discrete versions of the Painlevé equations occur frequently in problems in mathematical physics. For example, the discrete Painlevé I equation has been obtained in the theory of orthogonal polynomials [6]. The Bäcklund transformation of the continuous Painlevé IV equation led to corresponding discrete Painlevé IV equation [7]. Similarity reduction [8] of the modified KdV equation led to the discrete Painlevé II equation given by

$$x_{n+2} = -x_n + \frac{a + \zeta_{n+1}x_{n+1}}{1 - x_{n+1}^2}. \tag{1}$$

This map also has been found from unitary matrix models of quantum gravity [9]. When ζ_{n+1} is a constant and not a function of variable n, (1) is referred to as the autonomous discrete Painlevé II equation. It admits a two-parameter family of finite-order meromorphic solutions, which is an indicator of integrability in difference equations [10]. In this paper, we consider the following second order difference equation

$$x_{n+2} = \gamma x_n + \frac{\alpha + \beta x_{n+1}}{1 - x_{n+1}^2}, \tag{2}$$

where $-1 < \gamma < 1$ or $-3 < \gamma < -1$, $\alpha \geq 0$, and β is a real number. Recurrence (2) is precisely the McMillan map for $\gamma = -1$. Recurrence (2) can be expressed equivalently by the planar mapping

$$F(x, y) = (y, \gamma x + \frac{\alpha + \beta y}{1 - y^2}). \tag{3}$$

Here, we investigate the asymptotic stability of equilibrium of the perturbed McMillan map (2) by the quasi-Lyapunov function method (see [11]). The bifurcation and asymptotic stability of equilibrium of difference equations are considered in numerous papers (see [11–14] and the references cited therein). For example, Merino proved the global attractivity of a difference equation via Lyapunov function method (see [14]).

2 Lyapunov function, equilibria, and 2-cycle

Many researchers [2–4] studied various properties of solutions of rational integrable maps using the parametrization of the invariant curves. Similarly, to investigate the asymptotic stability for the mapping F of (3), we introduce a function V as follows:

$$V(x, y) = x^2y^2 - x^2 - y^2 + \frac{2\beta}{1 - \gamma}xy + \frac{2\alpha}{1 - \gamma}(x + y), \tag{4}$$

where $-1 < \gamma < 1$ or $-3 < \gamma < -1$, $\alpha \geq 0$, and β is a real number. It is obvious that $V(x, y) = \mathcal{K}$ is the biquadratic integral of the mapping H for $\gamma = -1$. Now we present some propositions of the function $V(x, y)$ in a two-dimensional phase plane.

Note that $\gamma \neq -1$, then a direct computation shows the following relation:

$$\begin{aligned}
&V(x,y) - V(F(x,y)) = V(x,y) - V(y,\gamma x + \tfrac{\alpha+\beta y}{1-y^2}) \\
&= y^2[x^2 - (\gamma x + \tfrac{\alpha+\beta y}{1-y^2})^2] - x^2 + (\gamma x + \tfrac{\alpha+\beta y}{1-y^2})^2 + \tfrac{2\beta}{1-\gamma} y[x - (\gamma x + \tfrac{\alpha+\beta y}{1-y^2})] \\
&\quad + \tfrac{2\alpha}{1-\gamma}[x - (\gamma x + \tfrac{\alpha+\beta y}{1-y^2})] \\
&= (x - \gamma x - \tfrac{\alpha+\beta y}{1-y^2})[\tfrac{1+\gamma}{1-\gamma}(\alpha + \beta y) - (1+\gamma)x(1-y^2)] \\
&= \tfrac{1+\gamma}{(1-\gamma)(y^2-1)}[(1-\gamma)x(1-y^2) - (\alpha+\beta y)]^2.
\end{aligned}$$

We denote by $\mathcal{T} = \{(x,y) | (1-\gamma)x(1-y^2) = \alpha + \beta y, (x^2-1)(y^2-1) \neq 0\}$, then the equalities $V(x,y) = V(F(x,y))$ and $F(x,y) = (y,x)$ hold for $(x,y) \in \mathcal{T}$. Now we consider the set $\mathcal{T}$. For each point $(x,y) \in \mathcal{T}$, we have

$$\begin{aligned}
&V(x,y) - V(F^2(x,y)) = V(x,y) - V(F(y,x)) = V(x,y) - V(x,\gamma y + \tfrac{\alpha+\beta x}{1-x^2}) \\
&= x^2[y^2 - (\gamma y + \tfrac{\alpha+\beta x}{1-x^2})^2] - y^2 + (\gamma y + \tfrac{\alpha+\beta x}{1-x^2})^2 + \tfrac{2\beta}{1-\gamma} x[y - (\gamma y + \tfrac{\alpha+\beta x}{1-x^2})] \\
&\quad + \tfrac{2\alpha}{1-\gamma}[y - (\gamma y + \tfrac{\alpha+\beta x}{1-x^2})] \\
&= \tfrac{1+\gamma}{(1-\gamma)(x^2-1)}[(1-\gamma)y(1-x^2) - (\alpha+\beta x)]^2.
\end{aligned}$$

To obtain the fixed points and 2-cycle of the mapping F of (3), we consider the following set of equations:

$$\begin{cases} (1-\gamma)x(1-y^2) = \alpha + \beta y, \\ (1-\gamma)y(1-x^2) = \alpha + \beta x. \end{cases} \tag{5}$$

For simplicity of presentation, we use G to denote the set of solutions of (5), i.e., $G = \{(x,y) | (1-\gamma)x(1-y^2) = \alpha + \beta y, (1-\gamma)y(1-x^2) = \alpha + \beta x\}$. It follows that inequality $V(F(x,y)) \neq V(x,y)$ or $V(F^2(x,y)) \neq V(x,y)$ holds for each $(x,y) \in \mathbb{R}^2 \setminus \{G \cup \mathcal{L}\}$, where $\mathcal{L}$ denotes the set $\{(x,y) | (x^2-1)(y^2-1) = 0\}$.

Proposition 1 *Let (x,y) be a point of the set $G \setminus \mathcal{L}$. Then this point (x,y) is a fixed point or 2-cycle of the mapping F of (3).*

Proof Choose $(x,y) \in G \setminus \mathcal{L}$ such that $x = y$. Then

$$F(x,y) = (y, \gamma x + \frac{\alpha + \beta y}{1-y^2}) = (y,x) = (x,y).$$

That is, this point (x,y) is a fixed point of the mapping F. Now we consider each $(x,y) \in G \setminus \mathcal{L}$ with $x \neq y$. It follows that $F(x,y) = (y,x) \neq (x,y)$ and

$$F^2(x,y) = F(y, \gamma x + \frac{\alpha+\beta y}{1-y^2}) = F(y,x) = (x, \gamma y + \frac{\alpha + \beta x}{1-x^2}) = (x,y).$$

It turns out that this point (x,y) is a 2-cycle of F. This completes the proof. □

Next we find the elements of the set $G \setminus \mathcal{L}$ and their existence regions (see Fig. 1).

Theorem 2 *Suppose that $-3 < \gamma < -1$ or $-1 < \gamma < 1$, $\alpha \geq 0$, and $\beta \in \mathbb{R}$. Then the following statements are true.*

(i) *If* $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3 < (\frac{\alpha}{1-\gamma})^2$, $\beta \neq \alpha$, *or* $(\alpha,\beta) = (0, 1-\gamma)$, *then there exist a* 2*-cycle and an equilibrium point of the mapping* F, *where the fixed point lies on* $\Gamma : x - y = 0$.
(ii) *If* $\frac{1}{4}(1-\gamma) < \beta = \alpha$, *then the set* $G \setminus \mathcal{L}$ *is empty.*
(iii) *If* $0 < \frac{4}{27}(1-\frac{\beta}{1-\gamma})^3 = (\frac{\alpha}{1-\gamma})^2$, $\alpha \neq \frac{1}{4}(1-\gamma)$, *and* $\alpha \neq 2(1-\gamma)$, *then there exist a* 2*-cycle and two equilibrium points of the mapping* F, *where the fixed points lie on* Γ.
(iv) *If* $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $4(\frac{\beta}{1-\gamma}+1) > -(\frac{\alpha}{1-\gamma})^2$, *and* $\beta \neq \pm\alpha$, *then there exist a* 2*-cycle and three equilibrium points of the mapping* F, *where the fixed points lie on* Γ.
(v) *If* $(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}) \in \{(0,0), (\frac{1}{4}, \frac{1}{4}), (2,-2)\}$, *then the mapping* F *has a unique fixed point, which lies on* Γ.
(vi) *If* $0 < -\beta = \alpha$, $\alpha \neq 2(1-\gamma)$ *or* $0 < \beta = \alpha < \frac{1}{4}(1-\gamma)$, *then there exist two equilibrium points of the mapping* F, *which lie on* Γ.
(vii) *If* $4(\frac{\beta}{1-\gamma}+1) \le -(\frac{\alpha}{1-\gamma})^2$ *and* $(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}) \neq (2,-2)$, *then there exist three fixed points of the mapping* F, *which lie on* Γ.

Proof Let us first consider the solutions of system (5). Using the first equation in (5), we get $\alpha = (1-\gamma)x(1-y^2) - \beta y$. Substituting it into the second equation of (5) gives the following equality:

$$(1-\gamma)(y - x^2 y - x + xy^2) = \beta(x-y).$$

It follows that

$$(x-y)[(1-\gamma)(1+xy) + \beta] = 0.$$

Consequently, the solutions of (5) satisfy at least one of the equalities $x = y$ and $xy = -1 - \frac{\beta}{1-\gamma}$.

Now we consider the case $xy = -1 - \frac{\beta}{1-\gamma}$. Using this equality to eliminate β from the first equation in (5) gives $x + y = \frac{\alpha}{1-\gamma}$. If $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) > 0$, then there are two distinct real roots λ_1 and λ_2 of the quadratic equation $X^2 - \frac{\alpha}{1-\gamma}X - (1+\frac{\beta}{1-\gamma}) = 0$. Therefore (λ_1, λ_2) and (λ_2, λ_1) are the solutions of the set of equations (5). Note that the point (λ_1, λ_2) is precisely an element of the set $\mathcal{L}$ for $\beta = \pm\alpha$. Proposition 1 tells us that there exists a 2-cycle $\{(\lambda_1,\lambda_2), (\lambda_2,\lambda_1)\}$ of the mapping F for $\beta \neq \pm\alpha$ and $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) > 0$. If $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) = 0$, then there exists a repeated real root λ for the equation $X^2 - \frac{\alpha}{1-\gamma}X - (1+\frac{\beta}{1-\gamma}) = 0$. Using Proposition 1, we obtain that (λ,λ) is a fixed point of the mapping F for $4(\frac{\beta}{1-\gamma}+1) = (\frac{\alpha}{1-\gamma})^2$ and $(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}) \neq (2,-2)$. If $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) < 0$, then there is no real solution (x,y) of system (5) that satisfies $xy = -1 - \frac{\beta}{1-\gamma}$.

Next we study the case $x = y$. Substituting it into the first equation in (5) gives the cubic equation

$$z^3 + (\frac{\beta}{1-\gamma} - 1)z + \frac{\alpha}{1-\gamma} = 0. \tag{6}$$

For simplicity of presentation, we define $g(z) = z^3 + (\frac{\beta}{1-\gamma} - 1)z + \frac{\alpha}{1-\gamma}$ and compute its derivative

$$g'(z) = 3z^2 + (\frac{\beta}{1-\gamma} - 1).$$

If $\frac{\beta}{1-\gamma}-1\geq 0$, then $g'(z)\geq 0$. Therefore, we obtain that the cubic equation (6) has a unique real root z^* for $\frac{\beta}{1-\gamma}-1\geq 0$. Notice that the equation $g(z)=0$ has a solution -1 for $\beta=\alpha$. Therefore, the mapping F has a unique fixed point for $\beta\neq\alpha$ and $\frac{\beta}{1-\gamma}-1\geq 0$, whereas the set $G\setminus\mathcal{L}$ is empty for $\frac{1-\gamma}{4}<\beta=\alpha$. If $\frac{\beta}{1-\gamma}-1<0$, then the equation $g'(z)=0$ has two distinct real roots $\tilde{z}_1=\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}$ and $\tilde{z}_2=-\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}$. It is easy to show that the equation $g(z)=0$ has a unique real root for $g(\tilde{z}_1)>0$ and $\frac{\beta}{1-\gamma}-1<0$. That is, there exists a unique fixed point (l_1,l_1) of the mapping F for $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3<(\frac{\alpha}{1-\gamma})^2$, $\beta\neq\alpha$ and $\frac{\beta}{1-\gamma}-1<0$. Moreover, we deduce that $l_1<-2\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}$. However, the set $G\setminus\mathcal{L}$ is empty for $\frac{1}{4}(1-\gamma)<\beta=\alpha<1-\gamma$. If $g(\tilde{z}_1)=0$ and $\frac{\beta}{1-\gamma}-1<0$, then the equation $g(z)=0$ has two distinct real roots $l_{21}=\tilde{z}_1$ and $l_{22}=-2\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}$. Therefore, the mapping F has two fixed points for $0<\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3=(\frac{\alpha}{1-\gamma})^2$, $\alpha\neq\frac{1}{4}(1-\gamma)$ and $\alpha\neq 2(1-\gamma)$. Finally, we consider the subcase $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3>(\frac{\alpha}{1-\gamma})^2$. This means that $\frac{\beta}{1-\gamma}-1<0$, $g(\tilde{z}_1)<0$, and $g(\tilde{z}_2)>0$. It turns out that the equation $g(z)=0$ has three distinct real roots l_{31}, l_{32}, and l_{33} such that

$$\begin{aligned}-2\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}<l_{33}<-\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}<l_{32}<\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}\\ <l_{31}<2\sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}.\end{aligned}\tag{7}$$

Note that the equation $g(z)=0$ has a solution 1 for $\beta=-\alpha$. Therefore, the mapping F has three fixed points for $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3>(\frac{\alpha}{1-\gamma})^2$ and $\beta\neq\pm\alpha$. Nevertheless, for $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3>(\frac{\alpha}{1-\gamma})^2$ and $0<\beta=\alpha<\frac{1-\gamma}{4}$, the mapping F has two fixed points. Similarly, there exist two fixed points of F for $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3>(\frac{\alpha}{1-\gamma})^2$ and $0>\beta=-\alpha>-2(1-\gamma)$.

Since equation (6) has a triple root 0 when $\alpha=0$ and $\beta=1-\gamma$, we obtain that there exist a 2-cycle and an equilibrium point of the mapping F for $(\alpha,\beta)=(0,1-\gamma)$. It is easy to show that the mapping F has a unique fixed point for $(\frac{\alpha}{1-\gamma},\frac{\beta}{1-\gamma})\in\{(0,0),(\frac{1}{4},\frac{1}{4}),(2,-2)\}$. The above results are summarized in Theorem 2. □

Bifurcations of fixed points and 2-cycle are summarized in Fig. 1. The seven associated existence regions are also marked on the parameter plane. Theorem 2 shows that a pitchfork bifurcation occurs when $\frac{\beta}{1-\gamma}$ passes through the point 1 along the line $\frac{\alpha}{1-\gamma}=0$. When $\frac{\beta}{1-\gamma}$ increasingly crosses L_2, a 2-cycle of the mapping F appears. That is, L_2 is a period-doubling bifurcation curve. A complete hysteresis bifurcation arises as the parameter $\frac{\beta}{1-\gamma}$ is increased along the curve $\frac{1}{2}(\frac{\beta}{1-\gamma})^2=\frac{\alpha}{1-\gamma}-\frac{1}{4}$ in the range from -2 to $\frac{1}{2}$. For parameters $\gamma=-1$, $\alpha=0$, and $-2\leq\beta\leq 2$, the corresponding constant level sets $V(x,y)=\mathcal{K}$ of the mapping F are provided in Fig. 5 by Zolkin et al. in [2].

3 Asymptotic stability of the perturbed McMillan map

In this section we employ the Lyapunov function $V(x,y)$ in (4) to study the asymptotic stability of the solutions of equation (2). We first find the local extreme values of $V(x,y)$. Notice that $V(x,y)$ has continuous higher partial derivatives in $\mathbb{R}^2$. Taking partial derivatives and setting them equal to 0 gives

$$\begin{cases}\frac{\partial V}{\partial x}=2xy^2-2x+\frac{2\beta}{1-\gamma}y+\frac{2\alpha}{1-\gamma}=0,\\ \frac{\partial V}{\partial y}=2x^2y-2y+\frac{2\beta}{1-\gamma}x+\frac{2\alpha}{1-\gamma}=0.\end{cases}\tag{8}$$

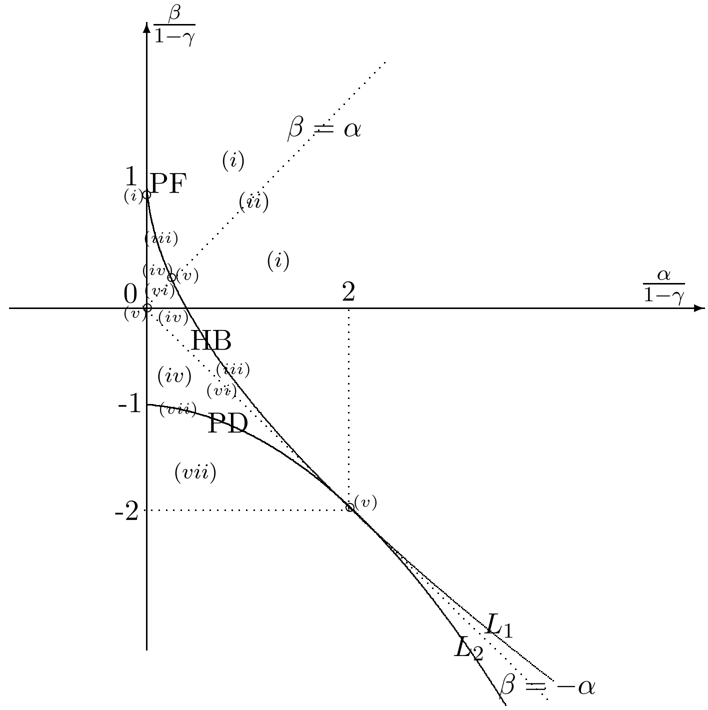

Figure 1 Bifurcation diagram for the perturbed McMillan map. Two critical curves L_1: $(\frac{\alpha}{1-\gamma})^2 = \frac{4}{27}(1-\frac{\beta}{1-\gamma})^3$ and L_2: $4(\frac{\beta}{1-\gamma}+1) = -(\frac{\alpha}{1-\gamma})^2$. Label HB corresponds to hysteresis bifurcation. Pitchfork and period-doubling bifurcations are marked with PF and PD

It is no difficult to see that this system of equations is equivalent to (5). That is, the fixed points and periodic solution of the mapping F are the stationary points of $V(x,y)$. If $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, then there exist three different solutions (l_{31}, l_{31}), (l_{32}, l_{32}), and (l_{33}, l_{33}) for system (5) such that relations (7) and $g(\tilde{z}_1) < 0$ hold. Note that we have $g(\frac{1}{2}\frac{\alpha}{1-\gamma}) = \frac{1}{2}\frac{\alpha}{1-\gamma}[(\frac{1}{2}\frac{\alpha}{1-\gamma})^2 + \frac{\beta}{1-\gamma} + 1] \geq 0$ for $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) > 0$. Therefore, for $\frac{4}{27}(1-\frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma}+1) > 0$ and $\frac{\beta}{1-\gamma} > -2$, we have $\frac{1}{2}\frac{\alpha}{1-\gamma} \leq l_{32} < \tilde{z}_1 = \sqrt{\frac{1}{3}(1-\frac{\beta}{1-\gamma})}$. That is, in this case the relations

$$-(\frac{\beta}{1-\gamma}+1) < (\frac{1}{2}\frac{\alpha}{1-\gamma})^2 \leq l_{32}^2 < \frac{1}{3}(1-\frac{\beta}{1-\gamma})$$

hold. Here we use the second partials test to determine the extreme values of $V(x,y)$. The second order partial derivatives are

$$A = \frac{\partial^2 V}{\partial x^2} = 2y^2 - 2, \quad C = \frac{\partial^2 V}{\partial y^2} = 2x^2 - 2, \text{ and } B = \frac{\partial^2 V}{\partial x \partial y} = 4xy + \frac{2\beta}{1-\gamma}.$$

The determinant of the Hessian is

$$\begin{aligned} AC - B^2 &= 4\left[(x^2-1)(y^2-1) - (2xy + \frac{\beta}{1-\gamma})^2\right] \\ &= 4\left[1 - x^2 - y^2 - 3x^2y^2 - \frac{4\beta}{1-\gamma}xy - (\frac{\beta}{1-\gamma})^2\right]. \end{aligned}$$

Now we evaluate the determinant of the Hessian at the stationary point (l_{32}, l_{32}):

$$(AC - B^2)\big|_{(l_{32},l_{32})} = -4\big(3l_{32}^2 - 1 + \frac{\beta}{1-\gamma}\big)\big(1 + \frac{\beta}{1-\gamma} + l_{32}^2\big) > 0.$$

It is easy to see that the relations $l_{32}^2 < \tilde{z}_1^2 = \frac{1}{3}(1 - \frac{\beta}{1-\gamma}) < 1$ hold for $-2 < \frac{\beta}{1-\gamma} < 1$. Then we obtain $A\ |_{(l_{32},l_{32})}= \frac{\partial^2 V}{\partial x^2}\ |_{(l_{32},l_{32})}= 2l_{32}^2 - 2 < 0$. Hence the function V attains a local maximum value at (l_{32}, l_{32}) for $\frac{4}{27}(1 - \frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$, and $-2 < \frac{\beta}{1-\gamma} < 1$. Note that there exists a 2-cycle $\{(\lambda_1, \lambda_2), (\lambda_2, \lambda_1)\}$ of system (5) for $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$, where $\lambda_1 = \frac{1}{2}\frac{\alpha}{1-\gamma} - \sqrt{(\frac{1}{2}\frac{\alpha}{1-\gamma})^2 + 1 + \frac{\beta}{1-\gamma}}$ and $\lambda_2 = \frac{1}{2}\frac{\alpha}{1-\gamma} + \sqrt{(\frac{1}{2}\frac{\alpha}{1-\gamma})^2 + 1 + \frac{\beta}{1-\gamma}}$. It can easily be checked that the relations $V(\tilde{z}_1, \tilde{z}_1) \geq V(-\sqrt{\frac{1}{3}(1 - \frac{\beta}{1-\gamma})}, -\sqrt{\frac{1}{3}(1 - \frac{\beta}{1-\gamma})}\,)$ and $V(\frac{1}{2}\frac{\alpha}{1-\gamma}, \frac{1}{2}\frac{\alpha}{1-\gamma}) > V(\lambda_1, \lambda_2)$ hold for $\frac{\alpha}{1-\gamma} \geq 0$ and $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$. We define the set $D = \{(x, y) | V(x, y) > \mathcal{K}_{pm}\}$, where

$$\mathcal{K}_{pm} = \begin{cases} V(\tilde{z}_1, \tilde{z}_1), & \alpha = 0 \text{ and } 0 \leq \beta < 1 - \gamma, \\ \max\{V(\frac{1}{2}\frac{\alpha}{1-\gamma}, \frac{1}{2}\frac{\alpha}{1-\gamma}), V(\tilde{z}_1, \tilde{z}_1)\}, & \alpha > 0 \text{ and } \beta \geq -\alpha, \\ \max\{V(\lambda_1, \lambda_2), V(\tilde{z}_1, \tilde{z}_1)\}, & 0 \leq \alpha < -\beta. \end{cases}$$

It is easy to verify that there exists a fixed point $(l_{32}, l_{32}) \in D$ of the mapping F for $\frac{4}{27}(1 - \frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$ and $-2 < \frac{\beta}{1-\gamma} < 1$. In fact, D is a union of disjoint sets, we denote $\hat{D}$ as a simple connected subset of D such that $(l_{32}, l_{32}) \in \hat{D}$. According to the definition of set D, it follows that $\hat{D} \subset \{(x, y) | x^2 < 1, y^2 < 1\}$. Note that $V(l_{32}, l_{32})$ is a unique local maximum value in $\hat{D}$. Then $V(l_{32}, l_{32})$ is the maximum value in $\hat{D}$. Obviously, if $\gamma = -1$, then $V(F(x, y)) = V(x, y)$ for every point $(x, y) \in \hat{D}$. That is, in this case the level curve $V(x, y) = \mathcal{K}$ is an invariant of F. Therefore, every invariant curve $V(x, y) = \mathcal{K}$ of F is a topological circle surrounding the equilibrium (l_{32}, l_{32}) for $\gamma = -1$ and $\mathcal{K}_{pm} < \mathcal{K} < V(l_{32}, l_{32})$.

Theorem 3 *Suppose that* $\alpha \geq 0$, $\frac{4}{27}(1 - \frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$, *and* $-2 < \frac{\beta}{1-\gamma} < 1$. *Then the following statements are true*:

(i) *If* $-1 < \gamma < 1$, *then the equilibrium point* l_{32} *of equation* (2) *is stable and attracts every solution of equation* (2) *with the initial values* $(x_0, x_1) \in \hat{D}$.

(ii) *The equilibrium point* l_{32} *of equation* (2) *is unstable for* $-3 < \gamma < -1$.

Proof Let $\{x_n\}_{n=0}^{\infty}$ be the solution of equation (2) with proper initial values x_0 and x_1, then it is simple to show that $F(x_n, x_{n+1}) = (x_{n+1}, x_{n+2})$ for all $n \geq 0$. Therefore, it suffices to study the behavior of sequence $\{F^n(x_0, x_1)\}_{n=1}^{\infty}$ starting from each point $(x_0, x_1) \in \hat{D}$. Recall that there exists a fixed point (l_{32}, l_{32}) of the mapping F for $\frac{4}{27}(1 - \frac{\beta}{1-\gamma})^3 > (\frac{\alpha}{1-\gamma})^2$, $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$, and $-2 < \frac{\beta}{1-\gamma} < 1$. Meanwhile, $V(l_{32}, l_{32})$ is the strict maximum of V in $\hat{D}$.

We first consider the case $-1 < \gamma < 1$. It follows from $-1 < \gamma < 1$ that we deduce $\frac{1+\gamma}{1-\gamma} > 0$. Using the fact that $\hat{D} \subset \{(x, y) | x^2 < 1, y^2 < 1\}$, we obtain that the inequality $V(F^2(x, y)) > V(x, y)$ holds for each $(x, y) \in \hat{D} \setminus \{(l_{32}, l_{32})\}$. Then the sequence $\{V(F^{2n}(x_0, x_1))\}_{n=1}^{\infty}$ is increasing and bounded starting from each point $(x_0, x_1) \in \hat{D}$. It turns out that $\lim_{n\to\infty} V(F^{2n}(x_0, x_1)) = \tilde{k}$. We first show that $\tilde{k} = V(l_{32}, l_{32})$. If not, there would exist an initial point $(\tilde{x}_0, \tilde{x}_1) \in \hat{D}$ such that $\lim_{n\to\infty} V(F^{2n}(\tilde{x}_0, \tilde{x}_1)) = \tilde{k} < V(l_{32}, l_{32})$. Defining $E = \{(x, y) | V(F^2(x_0, x_1)) \leq V(x,$

$y) \leq \tilde{k}, (x,y) \in \hat{D}\}$ and noting $\hat{D}$ as a simple connected set, we obtain that E is a compact closed and bounded set. Notice that the sequence $\{F^{2n}(\tilde{x}_0,\tilde{x}_1)\}_{n=1}^{\infty}$ is contained in E. It follows that there exists a subsequence $\{F^{2n_l}(\tilde{x}_0,\tilde{x}_1)\}_{l=1}^{\infty}$ such that $\lim\limits_{l\to\infty} F^{2n_l}(\tilde{x}_0,\tilde{x}_1) = (\tilde{x},\tilde{y})$ and $(\tilde{x},\tilde{y}) \in E$. It is clear that $(\tilde{x},\tilde{y}) \neq (l_{32},l_{32})$ as $\tilde{k} < V(l_{32},l_{32})$. Because V and F are continuous functions in $\hat{D}$, it follows that $V(\tilde{x},\tilde{y}) = \lim\limits_{l\to\infty} V(F^{2n_l}(\tilde{x}_0,\tilde{x}_1)) = \lim\limits_{n\to\infty} V(F^{2n}(\tilde{x}_0,\tilde{x}_1)) = \tilde{k}$. Next we consider another subsequence $\{F^2(F^{2n_l}(\tilde{x}_0,\tilde{x}_1))\}_{l=1}^{\infty}$ of $\{F^{2n}(\tilde{x}_0,\tilde{x}_1)\}_{n=1}^{\infty}$. Note that $\lim\limits_{l\to\infty} V(F^2(F^{2n_l}(\tilde{x}_0,\tilde{x}_1))) = \lim\limits_{n\to\infty} V(F^{2n}(\tilde{x}_0,\tilde{x}_1)) = \tilde{k}$ and $V(F^2(x,y)) > V(x,y)$ for $(x,y) \in \hat{D} \setminus \{(l_{32},l_{32})\}$, we obtain

$$\tilde{k} = \lim_{l\to\infty} V(F^2(F^{2n_l}(\tilde{x}_0,\tilde{x}_1))) = V(F^2 \lim_{l\to\infty}(F^{2n_l}(\tilde{x}_0,\tilde{x}_1))) = V(F^2(\tilde{x},\tilde{y})) > V(\tilde{x},\tilde{y}) = \tilde{k}.$$

This leads to a contradiction. Thus we conclude that the relation $\lim\limits_{n\to\infty} V(F^{2n}(x_0,x_1)) = V(l_{32},l_{32})$ holds for $(x_0,x_1) \in \hat{D}$. According to the fact that $V(l_{32},l_{32})$ is the strict maximum of V in $\hat{D}$, it follows that $\lim\limits_{n\to\infty} F^{2n}(x_0,x_1) = (l_{32},l_{32})$ for each $(x_0,x_1) \in \hat{D}$. Similarly, we obtain that the fixed point (l_{32},l_{32}) attracts every sequence $\{F^{2n+1}(x_0,x_1)\}_{n=1}^{\infty}$ with initial point $(x_0,x_1) \in \hat{D}$. Consequently, we infer that the equilibrium point l_{32} of equation (2) attracts every solution of equation (2) with the initial values $(x_0,x_1) \in \hat{D}$.

Finally, we consider the case $-3 < \gamma < -1$. It is apparent from $-3 < \gamma < -1$ that we obtain $\frac{1+\gamma}{1-\gamma} < 0$. From the fact that $\hat{D} \subset \{(x,y)|x^2 < 1, y^2 < 1\}$, it follows that the inequality $V(F^2(x,y)) < V(x,y)$ holds for each $(x,y) \in \hat{D} \setminus \{(l_{32},l_{32})\}$. Note that $V(l_{32},l_{32})$ is the strict maximum of V in $\hat{D}$. Thus we are led to the conclusion that the equilibrium point l_{32} of equation (2) is unstable for $-3 < \gamma < -1$. The proof is completed. □

Remark 4 Recall that there exists a 2-cycle $\{\lambda_1,\lambda_2\}$ of equation (2) for $(\frac{\alpha}{1-\gamma})^2 + 4(\frac{\beta}{1-\gamma} + 1) > 0$. In this case we compute the determinant of Hessian of $V(x,y)$ at (λ_1,λ_2):

$$\Big(\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \big(\frac{\partial^2 V}{\partial x \partial y}\big)^2\Big)\Big|_{(\lambda_1,\lambda_2)} = -4\Big[\big(\frac{\alpha}{1-\gamma}\big)^2 + 4\big(\frac{\beta}{1-\gamma} + 1\big)\Big] < 0.$$

It follows that $V(\lambda_1,\lambda_2)$ is not an extreme value of function $V(x,y)$. Similarly, we also deduce that $V(\lambda_2,\lambda_1)$ is not an extreme value of V. In fact, the 2-cycle $\{(\lambda_1,\lambda_2),(\lambda_2,\lambda_1)\}$ of the mapping F possesses both stable and unstable manifolds.

Author contributions
Lili Qian contributed to the conception of the study; Qiuying Lu contributed significantly to analysis and manuscript preparation; Guifeng Deng performed the data analyses and wrote the manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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