Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximation properties by shifted knots type of α-Bernstein–Kantorovich–Stancu operators

Md. Nasiruzzaman[1*], Mohammad Dilshad[1], Bader Mufadhi Eid Albalawi[1] and Mohammad Rehan Ajmal[2]

*Correspondence:
mfarooq@ut.edu.sa;
nasir3489@gmail.com
[1]Department of Mathematics, Faculty of Science, University of Tabuk, P. O. Box 4279, Tabuk 71491, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Through the real polynomials of the shifted knots, the α-Bernstein–Kantorovich operators are studied in their Stancu form, and the approximation properties are obtained. We obtain some direct approximation theorem in terms of Lipschitz type maximum function and Peetre's K-functional, as well as Korovkin's theorem. Eventually, the modulus of continuity is used to compute the upper bound error estimation.

Mathematics Subject Classification: 41A25; 41A36; 33C45

Keywords: α-Bernstein operators; Korovkin's theorem; α-Bernstein–Kantorovich operators; Modulus of continuity; Rate of convergence

1 Introduction and preliminaries

In 1912, it was established that for all continuous function g on the interval $[0,1]$ the classical Bernstein polynomials $B_s(g;z)$ converge uniformly to g (see [9]). In 1968, for the real parameters α, β such that $0 \le \alpha \le \beta$, Stancu introduced another variant of Bernstein polynomial known as Bernstein–Stancu operators $B_{s,\alpha,\beta}(g;z)$ such that (see [28])

$$B_{s,\alpha,\beta}(g;z) = \sum_{i=0}^{s} \binom{s}{i} z^i (1-z)^{s-i} g\left(\frac{i+\alpha}{s+\beta}\right), \tag{1}$$

where $z \in [0,1]$, $z \in \mathbb{N}$, and Stancu in his investigation showed that $B_{s,\alpha,\beta}$ is uniformly convergent to the continuous function g in $[0,1]$.

In the year 2017, Chen et al. [12] proposed the generalization of Bernstein operators with shape parameter $\alpha \in [0,1]$. These operators are defined as follows:

$$B_{s,\alpha}(g;z) = \sum_{i=0}^{s} g_i \tilde{b}_{s,i}(\alpha;z), \quad z \in [0,1], \tag{2}$$

where $g \in C[0,1]$ and $g_i = g\left(\frac{i}{s}\right)$. For $s > 2$, the polynomial $\tilde{b}_{s,i}(\alpha;z)$ of degree s is defined by $\tilde{b}_{1,0}(\alpha;z) = 1 - z$, $\tilde{b}_{1,1}(\alpha;z) = z$, and

$$\begin{aligned}\tilde{b}_{s,i}(\alpha;z) &= \binom{s-2}{i}(1-\alpha)z^i(1-z)^{s-i-1} + \binom{s-2}{i-2}(1-\alpha)z^{i-1}(1-z)^{s-i} \\ &\quad + \binom{s}{i}\alpha z^i(1-z)^{s-i},\end{aligned}$$

and the binomial coefficient $\binom{s}{i}$ is given by

$$\binom{s}{i} = \begin{cases} \frac{s!}{i!(s-i)!} & \text{if } 0 \le i \le s, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Mohiuddine et al. [19] provided the Kantorovich variant of operators (2). Numerous authors have repeatedly presented various generalizations of the aforementioned operators and established approximation results. For instance, Schurer modification of operators (2) was covered in Özger et al. [23].

In a more recent work, the Bernstein–Stancu polynomials are presented using shifted knots with new parameters in the following way (see [13]):

$$S_{s,\alpha,\beta}(g;z) = \left(\frac{s+\beta_2}{s}\right)^s \sum_{i=0}^{s} \binom{s}{i} \left(z - \frac{\alpha_2}{s+\beta_2}\right)^i \left(\frac{s+\alpha_2}{s+\beta_2} - z\right)^{s-i} g\left(\frac{i+\alpha_1}{s+\beta_1}\right), \tag{4}$$

where $z \in [\frac{\alpha_2}{s+\beta_2}, \frac{s+\alpha_2}{s+\beta_2}]$ and α_i, β_i, $i = 1,2$, are positive real numbers provided $0 \le \alpha_2 \le \alpha_1 \le \beta_1 \le \beta_2$. Researchers in approximation processes have recently developed Bernstein type operators; for example, we choose [1, 21, 29]. The relevant results for this article have been studied in various functional spaces such as Phillips operators via q-Dunkl generalization [2], Approximation by α-Baskakov-Jain type operators [15], Durrmeyer-type generalization of μ-Bernstein operators [16], modified λ-Bernstein polynomial [7], Kantorovich q-Baskakov operators [6], Szász–Durrmeyer operators [5], Bézier bases with Schurer polynomials [22], Stancu-type λ-Schurer [4], λ-Bernstein operators [10, 11], q-Szász–Durrmeyer type operators [26], Stancu variant of generalized Baskakov operators [25], and modified Baskakov–Durrmeyer operators [24].

In this work, we highlight the following main notion. The fundamental definitions and characteristics of Bernstein operators are covered in the first part. In the second part, we employ the shifted knot parameters to construct a new family of generalized α-Bernstein operators, including the Stancu–Kantorovich variation. We demonstrate that these new operators are essential to the theory of uniform approximation. In the third, fourth, and fifth sections, we examine the new operators' shape-preserving features, rate of convergence, and several direct theorems.

Recently, the authors have developed the Stancu variation of Bernstein–Kantorovich operators based on the shape parameter α (fixed real) by [20] for the set of all continuous functions defined on $[0,1]$.

$$\mathcal{J}_{s,\alpha}^{s_1,t_1}(g;z) = (s+1+t_1) \sum_{i=0}^{s} R_{s,\alpha}^{*}(z) \int_{\frac{i+s_1}{s+1+t_1}}^{\frac{i+1+s_1}{s+1+t_1}} g(t)\,\mathrm{d}t, \tag{5}$$

where

$$R^*_{s,\alpha}(z) = \binom{s-2}{i}(1-\alpha)z^i(1-z)^{s-i-1} + \binom{s-2}{i-2}(1-\alpha)z^{i-1}(1-z)^{s-i} + \binom{s}{i}\alpha z^i(1-z)^{s-i},$$

and the parameters satisfy the condition $0 \leq s_1 \leq t_1$.

2 Construction of operators

We offer an extension of Stancu type α-Bernstein–Kantorovich operators in this section, exploring some approximation results through the use of shifted knots. Let $1 \leq p < \infty$, $\mathcal{I}_s = [\frac{\zeta_2}{\varkappa_2+1+s}, \frac{\zeta_2+1+s}{\varkappa_2+1+s}]$ and $s \geq 2$, $s \in \mathbb{N}$. Given $\mathcal{J}^*_{s,\alpha} : L_p[0,1] \to L_p(\mathcal{I}_s)$, we define shifted knots of α-Bernstein–Kantorovich–Stancu type polynomials (5) as follows: For every $g \in L_p[0,1]$ and any fixed real α, we write

$$\mathcal{J}^*_{s,\alpha}(g;z) = (\varkappa_1 + 1 + s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1}\sum_{i=0}^{s}\mathcal{Q}^*_{s,\alpha}(z)\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\,\mathrm{d}t, \tag{6}$$

where $\frac{\zeta_2}{\varkappa_2+1+s} \leq z \leq \frac{\zeta_2+1+s}{\varkappa_2+1+s}$, ζ_i, $\varkappa_i$ $(i = 1,2)$ are positive real numbers, indeed $0 \leq \zeta_2 \leq \zeta_1 \leq \varkappa_1 \leq \varkappa_2$. It is very clear to note that if $\zeta_2 = \varkappa_2 = 0$ then the α-Bernstein–Stancu variant of Kantorovich polynomial is given in [20]. And in case of $\zeta_1 = \zeta_2 = \varkappa_1 = \varkappa_2 = 0$ the classic α-Bernstein–Kantorovich polynomial is obtained in [19] and $\mathcal{Q}^*_{s,\alpha}(z)$ is given as follows:

$$\begin{aligned}\mathcal{Q}^*_{s,\alpha}(z) &= \binom{s-2}{i}(1-\alpha)\left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^i\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1} \\ &+ \binom{s-2}{i-2}(1-\alpha)\left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i-1}\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i} \\ &+ \binom{s}{i}\alpha\left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^i\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i}.\end{aligned}$$

Our motive to construct the Stancu type α-Bernstein–Kantorovich shifted knot operators gives a valuable research path to a researcher in the field of approximation theory. The researcher can use the shifted knot operators with a shape parameter $\alpha \in [0,1]$ in various Bernstein type operators and can explore them to get a better observation and more flexibility. For example, in our future work we commit to apply these operators to the Bernstein operators by Bézier bases function. For more related results and applications of Bernstein type and shifted knot type operators, we refer to [10, 11, 13, 29]. Moreover, the Bernstein and Bernstein–Stancu polynomials are used in many branches of mathematics and computer science (see [8, 18]).

The primary goal of this paper is to use the integral modulus of continuity and the modulus of continuity to study the approximation properties of operators (6). Ultimately, we derive the convergence of operators (6) using Peetre's K-functional and Lipschitz maximal functions. Let us do some simple manipulation here, so we take

$$J_1 = \sum_{i=0}^{s}\binom{s-2}{i}\left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^i\left(\frac{\zeta_2+1+s}{s+\varkappa_2} - z\right)^{s-i-1}\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\,\mathrm{d}t,$$

$$J_2 = \sum_{i=0}^{s} \binom{s-2}{i-2} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i-1} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i} \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t.$$

If it corresponds to term $i = s$ in J_1 and $i = 0$ in J_2, then $\binom{s-2}{s} = 0$ and $\binom{s-2}{-2} = 0$. Thus

$$J_1 = \sum_{i=0}^{s-1} \binom{s-2}{i} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1} \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t,$$

$$J_2 = \sum_{i=1}^{s} \binom{s-2}{i-2} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i-1} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i} \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t.$$

Clearly, replace i by $i + 1$ in J_2, then

$$J_2 = \sum_{i=0}^{s-1} \binom{s-2}{i-1} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1} \int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} g(t)\mathrm{d}t.$$

Therefore, it follows that

$$\begin{aligned} J_1 + J_2 &= \sum_{i=0}^{s-1} \left\{ \binom{s-2}{i} \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t + \binom{s-2}{i-1} \int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} g(t)\mathrm{d}t \right\} \\ &\quad \times \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1}, \end{aligned}$$

where $\binom{s-2}{i} = \left(1 - \frac{i}{s-1}\right)\binom{s-1}{i}$ and $\binom{s-2}{i-1} = \frac{i}{s-1}\binom{s-1}{i}$. Hence we have

$$J_1 + J_2 = \sum_{i=0}^{s-1} \binom{s-1}{i} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1} f_i, \tag{7}$$

where

$$f_i = \left(1 - \frac{i}{s-1}\right) \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t + \frac{i}{s-1} \int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} g(t)\mathrm{d}t. \tag{8}$$

In addition, we take any function $g \in L_p[0,1]$, then for all $z \in \mathcal{I}_s$, $s \geq 2$, $s \in \mathbb{N}$ operators (6) are also calculated as

$$\mathcal{M}^*_{s,\alpha}(g;z) = (\varkappa_1 + 1 + s) \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \left[(1-\alpha) \sum_{i=0}^{s-1} \mathcal{U}^*_{s,\alpha} f_i + \alpha \sum_{i=0}^{s} \mathcal{V}^*_{s,\alpha} g_i \right], \tag{9}$$

where f_i is defined by (8), α is any fixed real and

$$g_i = \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} g(t)\mathrm{d}t, \tag{10}$$

$$\mathcal{U}^*_{s,\alpha} = \binom{s-1}{i} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i-1},$$

$$\mathcal{V}^*_{s,\alpha} = \binom{s}{i} \left(z - \frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s} - z\right)^{s-i}.$$

Lemma 1 *Suppose that the test function $\chi_j(t) = t^\mu$ for $\mu = 0, 1, 2$, then for all $z \in \mathcal{I}_s$ operators (6) have the following identities*:

$$(1)\quad \mathcal{M}^*_{s,\alpha}(\chi_0(t); z) = \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2 (1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right);$$

$$\begin{aligned}(2)\quad \mathcal{M}^*_{s,\alpha}(\chi_1(t); z) &= \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2 \frac{1}{2(\varkappa_1 + 1 + s)}(1-\alpha) \\ &\quad \times \left[2s\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^3 + (1 + 2\zeta_1)\right] \\ &\quad + \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\frac{1}{2(\varkappa_1 + 1 + s)}\alpha \\ &\quad \times \left[s\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2 + (1 + 2\zeta_1)\right];\end{aligned}$$

$$\begin{aligned}(3)\quad \mathcal{M}^*_{s,\alpha}(\chi_2(t); z) &= \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2 \frac{1}{3(\varkappa_1 + 1 + s)^2}(1-\alpha) \\ &\quad \times \left[3s(s-2)\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2\right. \\ &\quad + \Big(3(1+2\zeta_1)(s-1) + 3(2+\zeta_1)\Big)\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right) \\ &\quad \left. + \left(1 + 3\zeta_1 + 3\zeta_1^2\right)\right] \\ &\quad + \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\frac{1}{3(\varkappa_1 + 1 + s)^2}\alpha \\ &\quad \times \left[3s(s-1)\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2\right. \\ &\quad \left. + 3(1+2\zeta_1)s\left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right) + \left(1 + 3\zeta_1 + 3\zeta_1^2\right)\right].\end{aligned}$$

Proof If $\mu = 0$, $\chi_0(t) = 1$, then we are able to see that from (9)

$$\begin{aligned}\mathcal{M}^*_{s,\alpha}(\chi_0(t); z) &= (\varkappa_1 + 1 + s)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^{s+1} \\ &\quad \times \left[(1-\alpha)\sum_{i=0}^{s-1}\mathcal{U}^*_{s,\alpha}\left\{\left(1 - \frac{i}{s-1}\right)\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} \mathrm{d}t + \frac{i}{s-1}\int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} \mathrm{d}t\right\}\right. \\ &\quad \left. + \alpha\sum_{i=0}^{s}\mathcal{V}^*_{s,\alpha}\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} \mathrm{d}t\right] \\ &= \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2 (1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right).\end{aligned}$$

For $\mu = 1$, $\chi_1(t) = t$, we get

$$\mathcal{M}^*_{s,\alpha}(\chi_1(t); z) = (\varkappa_1 + 1 + s)\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^{s+1}$$

$$\times \left[(1-\alpha) \sum_{i=0}^{s-1} \mathcal{U}_{s,\alpha}^{*} \left\{ \left(1-\frac{i}{s-1}\right) \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t\mathrm{d}t + \frac{i}{s-1} \int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} t\mathrm{d}t \right\} + \alpha \sum_{i=0}^{s} \mathcal{V}_{s,\alpha}^{*} \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t\mathrm{d}t \right].$$

From a simple calculation we see that

$$\left(1-\frac{i}{s-1}\right) \int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t\mathrm{d}t + \frac{i}{s-1} \int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} t\mathrm{d}t = \frac{1}{2(\varkappa_1+1+s)^2} \left((1+2\zeta_1) + \frac{2si}{(s-1)} \right),$$

$$\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t\mathrm{d}t = \frac{1}{2(\varkappa_1+1+s)^2} \left(1+2\zeta_1+2i\right).$$

Therefore, we get

$$\begin{aligned}
\mathcal{M}_{s,\alpha}^{*}(\chi_1(t);z) &= (\varkappa_1+1+s) \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \frac{1}{2(\varkappa_1+1+s)^2}(1-\alpha) \\
&\times \left[\sum_{i=1}^{s-1} 2s \binom{s-2}{i-1} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-1-i} \right. \\
&\left. + \sum_{i=0}^{s-1} (1+2\zeta_1) \binom{s-1}{i} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-1-i} \right] \\
&+ \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \frac{1}{2(\varkappa_1+1+s)^2}\alpha \\
&\times \left[\sum_{i=1}^{s} 2s \binom{s-1}{i-1} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-i} \right. \\
&\left. + \sum_{i=0}^{s} (1+2\zeta_1) \binom{s}{i} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-i} \right];
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{s,\alpha}^{*}(\chi_1(t);z) &= \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \frac{1}{2(\varkappa_1+1+s)}(1-\alpha) \\
&\times \left[\sum_{i=0}^{s-2} 2s \binom{s-2}{i} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i+1} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-2-i} \right. \\
&\left. + (1+2\zeta_1) \left(\frac{s+1}{\varkappa_2+1+s}\right)^{s-1} \right] + \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \frac{1}{2(\varkappa_1+1+s)}\alpha \\
&\times \left[\sum_{i=0}^{s-1} 2s \binom{s-1}{i} \left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i+1} \left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-1-i} \right. \\
&\left. + (1+2\zeta_1) \left(\frac{s+1}{\varkappa_2+1+s}\right)^{s} \right];
\end{aligned}$$

$$\mathcal{M}_{s,\alpha}^{*}(\chi_1(t);z) = \left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1} \frac{1}{2(\varkappa_1+1+s)}(1-\alpha)$$

$$
\begin{aligned}
&\times \left[2s\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s-2}+(1+2\zeta_1)\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s-1}\right]\\
&+\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1}\frac{1}{2(\varkappa_1+1+s)}\alpha\\
&\times \left[s\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s-1}+(1+2\zeta_1)\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s}\right]\\
&=\left(\frac{\varkappa_2+1+s}{s+1}\right)^{2}\frac{1}{2(\varkappa_1+1+s)}(1-\alpha)\\
&\times \left[2s\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{3}+(1+2\zeta_1)\right]\\
&+\left(\frac{\varkappa_2+1+s}{s+1}\right)\frac{1}{2(\varkappa_1+1+s)}\alpha\\
&\times \left[s\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{2}+(1+2\zeta_1)\right].
\end{aligned}
$$

For $\mu = 2$, $\chi_2(t) = t^2$, we obtain

$$
\begin{aligned}
&\left(1-\frac{i}{s-1}\right)\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t^2\mathrm{d}t+\frac{i}{s-1}\int_{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}^{\frac{i+2+\zeta_1}{\varkappa_1+1+s}} t^2\mathrm{d}t\\
&=\frac{1}{3(\varkappa_1+1+s)^3}\left(\frac{3s}{(s-1)}i^2+\frac{3(1+2\zeta_1)(s-1)+3(2+\zeta_1)}{(s-1)}i+(1+3\zeta_1+3\zeta_1^2)\right),
\end{aligned}
$$

$$
\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}} t^2\mathrm{d}t=\frac{1}{3(\varkappa_1+1+s)^3}\left(1+3\zeta_1+3\zeta_1^2+(3+6\zeta_1)i+3i^2)\right).
$$

Therefore,

$$
\begin{aligned}
\mathcal{M}^*_{s,\alpha}(\chi_2(t);z) &= (\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1}\frac{1}{3(\varkappa_1+1+s)^3}(1-\alpha)\\
&\times \left[\sum_{i=0}^{s-3}3s(s-2)\binom{s-3}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i+2}\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-3-i}\right.\\
&+\sum_{i=0}^{s-2}\Big(3(1+2\zeta_1)(s-1)+3(2+\zeta_1)\Big)\\
&\times\binom{s-2}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i+1}\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-2-i}\\
&\left.+\sum_{i=0}^{s-1}\left(1+3\zeta_1+3\zeta_1^2\right)\binom{s-1}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i}\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-1-i}\right]\\
&+(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1}\frac{1}{3(\varkappa_1+1+s)^3}\alpha\\
&\times\left[\sum_{i=0}^{s-2}3s(s-1)\binom{s-2}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^{i+2}\left(\frac{s+\zeta_2}{\varkappa_2+1+s}-z\right)^{s-i-2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=0}^{s} \left(1 + 3\zeta_1 + 3\zeta_1^2\right) \binom{s}{i} \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^{i} \left(\frac{\zeta_2 + 1 + s}{\varkappa_2 + 1 + s} - z\right)^{s-i} \\
&+ \sum_{i=0}^{s-1} 3(1 + 2\zeta_1) s \binom{s-1}{i} \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^{i+1} \left(\frac{\zeta_2 + 1 + s}{\varkappa_2 + 1 + s} - z\right)^{s-i-1} \Bigg] \\
&= \left(\frac{\varkappa_2 + 1 + s}{s + 1}\right)^{s+1} \frac{1}{3(\varkappa_1 + 1 + s)^2} (1 - \alpha) \\
&\times \Bigg[3s(s-2) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2 \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s-3} \\
&+ \Big(3(1 + 2\zeta_1)(s-1) + 3(2 + \zeta_1)\Big) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right) \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s-2} \\
&+ \left(1 + 3\zeta_1 + 3\zeta_1^2\right) \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s-1} \Bigg] \\
&+ \left(\frac{\varkappa_2 + 1 + s}{s + 1}\right)^{s+1} \frac{1}{3(\varkappa_1 + 1 + s)^2} \alpha \\
&\times \Bigg[3s(s-1) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2 \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s-2} \\
&+ \left(1 + 3\zeta_1 + 3\zeta_1^2\right) \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s} \\
&+ 3(1 + 2\zeta_1) s \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right) \left(\frac{s+1}{\varkappa_2 + 1 + s}\right)^{s-1} \Bigg];
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}^*_{s,\alpha}(\chi_2(t); z) &= \left(\frac{\varkappa_2 + 1 + s}{s + 1}\right)^{2} \frac{1}{3(\varkappa_1 + 1 + s)^2} (1 - \alpha) \\
&\times \Bigg[3s(s-2) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2 \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^{2} \\
&+ \Big(3(1 + 2\zeta_1)(s-1) + 3(2 + \zeta_1)\Big) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right) \left(\frac{\varkappa_2 + 1 + s}{s+1}\right) \\
&+ \left(1 + 3\zeta_1 + 3\zeta_1^2\right) \Bigg] \\
&+ \left(\frac{\varkappa_2 + 1 + s}{s + 1}\right) \frac{1}{3(\varkappa_1 + 1 + s)^2} \alpha \\
&\times \Bigg[3s(s-1) \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right)^2 \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^{2} \\
&+ 3(1 + 2\zeta_1) s \left(z - \frac{\zeta_2}{\varkappa_2 + 1 + s}\right) \left(\frac{\varkappa_2 + 1 + s}{s+1}\right) + \left(1 + 3\zeta_1 + 3\zeta_1^2\right) \Bigg].
\end{aligned}
$$

This completes the proof of Lemma 1. □

Lemma 2 *Operators $\mathcal{M}^*_{s,\alpha}(.;.)$ have the central moments as follows*:

$$
\begin{aligned}
(1) \quad \mathcal{M}^*_{s,\alpha}(\chi_1(t) - z; z) &= \mathcal{M}^*_{s,\alpha}(\chi_1(t); z) - z\mathcal{M}^*_{s,\alpha}(\chi_0(t); z) \\
&= O\left(\frac{1}{s}\right)(z + 1);
\end{aligned}
$$

$$(2)\quad \mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z) = \mathcal{M}^*_{s,\alpha}(\chi_2(t);z) - 2z\mathcal{M}^*_{s,\alpha}(\chi_1(t);z) + z^2\mathcal{M}^*_{s,\alpha}(\chi_0(t);z) = O\left(\frac{1}{s^2}\right)(z+1)^2,$$

where $\mathcal{M}^*_{s,\alpha}(\chi_j(t);z)$ *for all* $j = 0,1,2$ *are defined by Lemma* 1.

Theorem 1 *Let* $z \in \mathcal{I}_s$, *then for the continuous function* g *on* $[0,1]$, *it follows that*

$$\lim_{s\to\infty} \|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)\|_{C(\mathcal{I}_s)} = 0.$$

Proof Taking into account the equality in Lemma 1, it is easy to write

$$\lim_{s\to\infty} \max_{z\in\mathcal{I}_s} |\mathcal{M}^*_{s,\alpha}(\chi_\mu(t);z) - z^\mu| = 0, \quad \mu = 0,1,2. \tag{11}$$

We are denoting

$$\mathcal{R}^*_{s,\alpha}(g;z) = \begin{cases} \mathcal{M}^*_{s,\alpha}(g;z) & \text{if } z \in \mathcal{I}_s, \\ g(z) & \text{if } z \in [0,1] \setminus \mathcal{I}_s. \end{cases} \tag{12}$$

It is easy to get

$$\|\mathcal{R}^*_{s,\alpha}(g;z) - g(z)\|_{C[0,1]} = \max_{z\in\mathcal{I}_s} |\mathcal{M}^*_{s,\alpha}(g;z) - g(z)|. \tag{13}$$

On the other hand, from (11) and (13) we obviously get

$$\lim_{s\to\infty} \|\mathcal{R}^*_{s,\alpha}(\chi_\mu(t);z) - z^\mu\|_{C[0,1]} = 0, \quad \mu = 0,1,2.$$

Applying the well-known Korovkin's theorem [17] to the sequence of operators $\mathcal{R}^*_{s,\alpha}$, from this fact it is easy to see for all $g \in C[0,1]$ on $[0,1]$

$$\lim_{s\to\infty} \|\mathcal{R}^*_{s,\alpha}(g;z) - g(z)\|_{C[0,1]} = 0.$$

Thus, (13) gives us

$$\lim_{s\to\infty} \max_{z\in\mathcal{I}_s} |\mathcal{M}^*_{s,\alpha}(g;z) - g(z)| = 0,$$

where $C[0,1]$ is the set of all continuous function g on $[0,1]$. This completes the proof. □

Theorem 2 *Let* $g \in L_p[0,1]$, $p \in [1,\infty)$, *then operators* (6) *satisfy*

$$\lim_{s\to\infty} \|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)\|_{L_p(\mathcal{I}_s)} = 0.$$

Proof Theorem 1 is taken into consideration for the proof along with the operators $\mathcal{R}^*_{s,\alpha}$ by (12). According to the Luzin theorem, there is a continuous function ϕ on $[0,1]$ such that $\|g-\phi\|_{L_p[0,1]} < \epsilon$ for given $\epsilon > 0$. Let $L_p[0,1] \to L_p[0,1]$ be the operator with (12) and $\|\mathcal{R}^*_{s,\alpha}\|$ be the operator norm. Then, for any $s \in \mathbb{N}$, it is sufficient to demonstrate that there exists a positive number M such that $\|\mathcal{R}^*_{s,\alpha}\| \leq M$ to support the conclusions made by Theorem 2. We employ Theorem 1 for this purpose, which states that for any $\epsilon > 0$, there exists a number $s_0 \in \mathbb{N}$, $s \geq s_0$ such that $\|\mathcal{R}^*_{s,\alpha}(\phi;z) - \phi(z)\|_{C[0,1]} < \epsilon$.

From these facts, we immediately consider

$$\begin{aligned}\|\mathcal{R}^*_{s,\alpha}(g;z) - g(z)\|_{L_p[0,1]} &\leq \|\mathcal{R}^*_{s,\alpha}(\phi;z) - \phi(z)\|_{C[0,1]} + \|g-\phi\|_{L_p[0,1]} \\ &\quad + \|\mathcal{R}^*_{s,\alpha}(g;z) - \mathcal{R}^*_{s,\alpha}(\phi;z)\|_{L_p[0,1]}.\end{aligned} \tag{14}$$

To get positive S, we apply Jensen's inequality to operators (6)

$$\begin{aligned}|\mathcal{M}^*_{s,\alpha}(g;z)|^p &\leq \left\{(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s+1}\sum_{i=0}^{s}\mathcal{Q}^*_{s,\alpha}(z)\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|\mathrm{d}t\right\}^p \\ &\leq \sum_{i=0}^{s}\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s}\mathcal{Q}^*_{s,\alpha}(z)\left\{(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|\mathrm{d}t\right\}^p \\ &\leq \sum_{i=0}^{s}\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s}\mathcal{Q}^*_{s,\alpha}(z)(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^p\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|^p\mathrm{d}t \\ &\leq \left[(1-\alpha)\sum_{i=0}^{s-1}\binom{s-1}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^i\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-1-i}\right. \\ &\quad \left. + \alpha\sum_{i=0}^{s}\binom{s}{i}\left(z-\frac{\zeta_2}{\varkappa_2+1+s}\right)^i\left(\frac{\zeta_2+1+s}{\varkappa_2+1+s}-z\right)^{s-i}\right] \\ &\quad \times\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s}(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^p\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|^p\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|\mathcal{M}^*_{s,\alpha}(g;z)|^p\mathrm{d}z &\leq \left[\sum_{i=0}^{s-1}\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s}\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s+1}\frac{1}{s+1}(1-\alpha)\right. \\ &\quad \left.+\sum_{i=0}^{s}\left(\frac{\varkappa_2+1+s}{s+1}\right)^{s}\left(\frac{s+1}{\varkappa_2+1+s}\right)^{s+1}\frac{1}{(s+1)}\alpha\right] \\ &\quad \times(\varkappa_1+1+s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^p\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|^p\mathrm{d}t \\ &\leq \left(\frac{\varkappa_1+1+s}{s+1}\right)\left(\frac{\varkappa_2+1+s}{s+1}\right)^{p-1}\sum_{i=0}^{s}\int_{\frac{i+\zeta_1}{\varkappa_1+1+s}}^{\frac{i+\zeta_1+1}{\varkappa_1+1+s}}|g(t)|^p\mathrm{d}t \\ &\leq \left(\frac{\varkappa_2+1+s}{s+1}\right)^p\|g(t)\|^p_{L_p[0,1]}.\end{aligned}$$

In the next, from operators (12), we have the inequality $\int_{[0,1]\setminus\mathcal{I}_s}|g(z)|^p\mathrm{d}z \leq \|g\|^p_{L_p[0,1]}$. Thus we find that

$$\int_0^1|\mathcal{R}^*_{s,\alpha}(g;z)|^p\mathrm{d}z \leq \left[1+\left(\frac{\varkappa_2+1+s}{s+1}\right)^p\right]\|g\|^p_{L_p[0,1]}. \tag{15}$$

Therefore we can write expression (15)

$$\|\mathcal{R}^*_{s,\alpha}(g;z)\|_{L_p[0,1]} \leq (2+\varkappa_2)\|g\|_{L_p[0,1]} \leq M\|g\|_{L_p[0,1]}.$$

From the above fact it follows that, for all $s \in \mathbb{N}$, there exists a positive M such that $\|\mathcal{R}^*_{s,\alpha}\| \leq M$. Therefore, from (14) we deduce that

$$\begin{aligned}\|\mathcal{R}^*_{s,\alpha}(g;z)-g(z)\|_{L_p[0,1]} &\leq \|\mathcal{R}^*_{s,\alpha}(\phi;z)-\phi(z)\|_{L_p[0,1]} \\ &\quad + \|\mathcal{R}^*_{s,\alpha}\|\|g-\phi\|_{L_p[0,1]} + \|g-\phi\|_{L_p[0,1]} \\ &\leq 2\epsilon + \epsilon M.\end{aligned}$$

Similarly, taking into account the above inequality, we also see

$$\begin{aligned}\|\mathcal{R}^*_{s,\alpha}(g;z)-g(z)\|_{L_p[0,1]} &= \left(\int_0^1 |\mathcal{R}^*_{s,\alpha}(g;z)-g(z)|^p \mathrm{d}z\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathcal{I}_s} |\mathcal{M}^*_{s,\alpha}(g;z)-g(z)|^p \mathrm{d}z\right)^{\frac{1}{p}} \\ &= \|\mathcal{M}^*_{s,\alpha}(g;z)-g(z)\|_{L_p(\mathcal{I}_s)} \\ &\leq 2\epsilon + \epsilon M.\end{aligned}$$

These explanations give us $\lim_{s\to\infty} \|\mathcal{M}^*_{s,\alpha}(g;z)-g(z)\|_{L_p(\mathcal{I}_s)} = 0$, this completes the proof of Theorem 2. □

3 Rate of convergence of $\mathcal{M}^*_{s,\alpha}$

Finding the degree of approximation for a series of operators $\mathcal{M}^*_{s,\alpha}$ is our goal in this section. Recalling the modulus of continuity of function g (see [3]), we employ $g \in C[0,1]$, the set of all continuous functions on $[0,1]$, and a positive value δ^*, thus

$$\omega^*(g;\delta^*) = \sup\{|g(z_1)-g(z_2)| : z_1, z_2 \in [0,1], |z_1 - z_2| \leq \delta^*\}. \tag{16}$$

As a result, we express the general estimate theorem developed by Shisha and Mond (see [27]) in terms of the modulus of continuity as follows to ascertain the order of convergence.

Theorem 3 [27] *Let $[c,d] \subset [a,b]$ and $\{L_s\}_{s\geq 1}$ be the sequence of positive linear operators from $C[a,b]$ to $C[c,d]$.*

(1) *If $g \in C[a,b]$ and $z \in [c,d]$, then we have*

$$\begin{aligned}|L_s(g;z)-g(z)| &\leq |g(z)||L_s(\chi_0(t);z)-1| \\ &\quad + \left\{L_s(\chi_0(t);z) + \frac{1}{\delta^*}\sqrt{L_s((t-z)^2;z)}\sqrt{L_s(\chi_0(t);z)}\right\}\omega^*(g;\delta^*).\end{aligned}$$

(2) *If $g' \in C[a,b]$ and $z \in [c,d]$, then we have*

$$\begin{aligned}|L_s(g;z)-g(z)| &\leq |g(z)||L_s(\chi_0(t);z)-1| + |g'(z)||L_s(t-z;z)| \\ &\quad + L_s((t-z)^2;z)\left\{\sqrt{L_s(\chi_0(t);z)} + \frac{1}{\delta^*}\sqrt{L_s((t-z)^2;z)}\right\}\omega^*(g';\delta^*).\end{aligned}$$

Theorem 4 *Let the function g be continuous on $[0,1]$, then for each $z \in \mathcal{I}_s$ operators (6) satisfy the inequality*

$$|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)| \leq \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)|g(z)| + 2\left\{\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2(1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\right\}\omega^*\left(g; \sqrt{\delta^*_{s,\alpha}(z)}\right),$$

where $\delta^*_{s,\alpha}(z) = \frac{1}{\left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)}\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)$.

Proof If we utilize Lemma 2 and take into account (1) of Theorem 3, we can write

$$\begin{aligned}
&|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)| \\
&\quad\leq |g(z)||\mathcal{M}^*_{s,\alpha}(\chi_0(t);z) - 1| \\
&\qquad + \left\{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z) + \frac{1}{\delta^*}\sqrt{\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)}\sqrt{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z)}\right\}\omega^*(g;\delta^*) \\
&\quad= \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)|g(z)| \\
&\qquad + \omega^*(g;\delta^*)\left\{\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2(1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\right. \\
&\qquad \left. + \frac{1}{\delta^*}\sqrt{\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)}\sqrt{\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2(1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)}\right\}.
\end{aligned}$$

If we choose $\delta^* = \sqrt{\delta^*_{s,\alpha}(z)} = \frac{1}{\sqrt{\left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)}}\sqrt{\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)}$, then we get the desired result. □

Theorem 5 *Let $z \in \mathcal{I}_s$, then for every $g \in C[0,1]$, it follows that*

$$|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)| \leq \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)\max_{z\in\mathcal{I}_s}|g(z)| + 2\left\{\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)^2(1-\alpha) + \alpha\left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\right\}\omega^*\left(g; \sqrt{\tilde{\delta}}\right),$$

where $\tilde{\delta} = \max_{z\in\mathcal{I}_s}\delta^*_{s,\alpha}(z)$.

Proof We easily obtain the desired result due to the monotonicity of the modulus of continuity. □

Remark 1 The order of the local approximation is estimated by Theorem 4, and the global order is estimated by Theorem 5 when we examine the approximation for each point $z \in \mathcal{I}_s$.

Theorem 6 *Let $g' \in C[0,1]$, then for each $z \in \mathcal{I}_s$ we have*

$$|\mathcal{M}^*_{s,\alpha}(g;z) - g(z)| \leq \left(\frac{\varkappa_2 + 1 + s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)|g(z)| + \left|O\left(\frac{1}{s}\right)(1+z)\right||g'(z)|$$

$$+ 2O\left(\frac{1}{s}\right)\left\{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z)\right\}^{\frac{1}{2}}(z+1)\omega^*\left(g';\sqrt{\delta^*_{s,\alpha}(z)}\right),$$

where $\mathcal{M}^*_{s,\alpha}(\chi_0(t);z) = \left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)$ *and* $\delta^*_{s,\alpha}(z)$ *is defined in Theorem 4.*

Proof We consider (2) of Theorem 3 and Lemma 1; as a result, we can write quickly

$$\begin{aligned}
|\mathcal{M}^*_{s,\alpha}(g;z)-g(z)| &\le \left(\frac{\varkappa_2+1+s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)|g(z)| + \left|O\left(\frac{1}{s}\right)(1+z)\right||g'(z)| \\
&+ \sqrt{\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)}\sqrt{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z)} \\
&\times \left\{1+\frac{1}{\delta^*}\frac{\sqrt{\mathcal{M}^*_{s,\alpha}((\chi_1(t)-z)^2;z)}}{\sqrt{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z)}}\right\}\omega^*\left(g';\delta^*\right) \\
&\le \left(\frac{\varkappa_2+1+s}{s+1}\right)\left(\frac{\varkappa_2}{s+1}\right)(1-\alpha)|g(z)| + \left|O\left(\frac{1}{s}\right)(1+z)\right||g'(z)| \\
&+ O\left(\frac{1}{s}\right)\left\{\mathcal{M}^*_{s,\alpha}(\chi_0(t);z)\right\}^{\frac{1}{2}}(z+1)2\omega^*\left(g';\sqrt{\delta^*_{s,\alpha}(z)}\right),
\end{aligned}$$

which completes the proof asserted by Theorem 6. □

4 Some direct approximation theorem to $\mathcal{M}^*_{s,\alpha}$

Here, too, we examine the approximation in L_p spaces thanks to some preliminary data. The L_p spaces on $\mathcal{I}_s$ are taken for this purpose for every $p \in [1,\infty)$ as $L_p(\mathcal{I}_s)$. Lastly, we derive the corresponding estimate in the $L_p(\mathcal{I}_s)$ spaces. In fact, the integral modification in terms of modulus of continuity for every $g \in L_p(\Theta_\varkappa)$ is provided by

$$\hat{\omega}_{1,p}(g,t) = \sup_{z\in[0,1]}\sup_{0<\varkappa\le t}\|g(z+\varkappa)-g(z)\|_{L_p(\Theta_\varkappa)} \quad (1\le p<\infty), \tag{17}$$

where $\|.\|_{L_p(\Theta_\varkappa)}$ denotes the L_p norm over $\Theta_\varkappa = [0,1-\varkappa]$. To measure the smoothness, we use Peetre's K-functional, and for this purpose we take $\mathcal{Z}_{1,p}(\Theta_\varkappa) = \{g,g' \in L_p(\Theta_\varkappa) :$ g is absolutely continuous$\}$. Let $1 \le p < \infty$, then for any $g \in L_p(\Theta_\varkappa)$ Peetre's K-functional is defined as follows:

$$\mathcal{K}_p(g,t) = \inf_{f\in\mathcal{Z}_{1,p}(\Theta_\varkappa)}\left(\|g-f\|_{L_p(\Theta_\varkappa)} + t\|f'\|_{L_p(\Theta_\varkappa)}\right). \tag{18}$$

Moreover, for the positive constants c_1 and c_2, the equivalence of relations between integral modulus of continuity and Peetre's K-functional is given by the inequality (see [14])

$$c_1\hat{\omega}_{1,p}(g,t) \le \mathcal{K}_p(g,t) \le c_2\hat{\omega}_{1,p}(g,t). \tag{19}$$

Taking into account operators (6) and Lemma 1, let us denote $\mathcal{M}_i = [\frac{i+\zeta_1}{\varkappa_1+1+s}, \frac{i+\zeta_1+1}{\varkappa_1+1+s}]$

$$\mathcal{T}^*_{s,\alpha}(.;z) = \frac{1}{\left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)}\mathcal{M}^*_{s,\alpha}(.;z) \tag{20}$$

and

$$\mathcal{P}^*_{s,i}(z) = \frac{(\varkappa_1 + 1 + s)\left(\frac{\varkappa_2+1+s}{s+1}\right)^s}{\left(\frac{\varkappa_2+1+s}{s+1}\right)(1-\alpha)+\alpha}\,\mathcal{Q}^*_{s,\alpha}(z). \tag{21}$$

Theorem 7 *Let $g \in \mathcal{Z}_{1,p}[0,1]$, $p > 1$, then operators $\mathcal{T}^*_{s,\alpha}$ by (20) satisfy the inequality*

$$||\mathcal{T}^*_{s,\alpha}(g;z) - g(z)||_{L_p(\mathcal{I}_s)} \le 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\max_{z\in\mathcal{I}_s}\left(\mathcal{T}^*_{s,\alpha}(\chi_2(t);z)\right)^{\frac{1}{2}}||g'||_{L_p[0,1]},$$

where $\mathcal{T}^*_{s,\alpha}(\chi_2(t);z)$ is defined by (20).

Proof For any fixed $z \in \mathcal{I}_s$, we easily get that

$$\begin{aligned}
\left|\mathcal{T}^*_{s,\alpha}(g;z) - g(z)\right| &= \left|\sum_{i=0}^{s}\mathcal{P}^*_{s,i}(z)\int_{\mathcal{M}_i}\big(g(t)-g(z)\big)\,\mathrm{d}t\right| \\
&\le \sum_{i=0}^{s}\mathcal{P}^*_{s,i}(z)\int_{\mathcal{M}_i}\int_z^t\left|g'(\lambda)\right|\mathrm{d}\lambda\,\mathrm{d}t \\
&\le \Phi_{g'}(z)\sum_{i=0}^{s}\mathcal{P}^*_{s,i}(z)\int_{\mathcal{M}_i}|t-z|\,\mathrm{d}t,
\end{aligned}$$

where $\Phi_{g'}(z) = \sup_{t\in[0,1]}\frac{1}{t-z}\int_z^t\left|g'(\lambda)\right|\mathrm{d}\lambda$ for $t \ne z$ is the Hardy–Littlewood majorant of g'. Applying the well-known Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\left|\mathcal{T}^*_{s,\alpha}(g;z) - g(z)\right| &\le \Phi_{g'}(z)\left(\sum_{i=0}^{s}\mathcal{P}^*_{s,i}(z)\right)^{\frac{1}{2}}\left(\sum_{i=0}^{s}\mathcal{P}^*_{s,i}(z)\int_{\mathcal{M}_i}(t-z)^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \Phi_{g'}(z)\max_{z\in\mathcal{I}_s}\left(\mathcal{T}^*_{s,\alpha}((\chi_2(t);z)\right)^{\frac{1}{2}},
\end{aligned}$$

where from the Hardy–Littlewood theorem [30] we get

$$\int_0^1\Phi_{g'}(z)\mathrm{d}z \le 2\left(\frac{p}{p-1}\right)^p\int_0^1\left|g'(z)\right|^p\mathrm{d}z \quad (1<p<\infty).$$

Thus, for any $1 < p < \infty$, we get that

$$||\mathcal{T}^*_{s,\alpha}(g;z) - g(z)||_{L_p(\mathcal{I}_s)} \le 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\max_{z\in\mathcal{I}_s}\left(\mathcal{T}^*_{s,\alpha}((t-z)^2;z)\right)^{\frac{1}{2}}||g'||_{L_p[0,1]}. \qquad \square$$

Theorem 8 *For any $g \in L_p[0,1]$ with the number $1 < p < \infty$, operators (20) verify the inequality*

$$||\mathcal{T}^*_{s,\alpha}(g;z) - g(z)||_{L_p(\mathcal{I}_s)} \le \mathcal{C}\hat{\omega}_{1,p}\left(g,\zeta_{s,\alpha}(z)\right),$$

where $\mathcal{C} = 2c_2 + 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)$ is a positive constant and $\zeta_{s,\alpha}(z) = \max_{z\in\mathcal{I}_s}\left(\mathcal{T}^*_{s,\alpha}(\chi_2(t);z)\right)^{\frac{1}{2}}$.

Proof Let us denote

$$\left|\left|\mathcal{T}^*_{s,\alpha}(\varphi;z)-\varphi(z)\right|\right|_{L_p[0,1]} \le \begin{cases} 2\,||\varphi||_{L_p[0,1]} & \text{if } \varphi\in L_p[0,1], \\ 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\zeta_{s,\alpha}(z)\,||\varphi||_{L_p[0,1]} & \text{if } \varphi\in \mathcal{Z}_{1,p}[0,1], \end{cases} \tag{22}$$

where $\zeta_{s,\alpha}(z)=\max_{z\in\mathcal{I}_s}\left(\mathcal{T}^*_{s,\alpha}\left(\chi_2(t);z\right)\right)^{\frac{1}{2}}$.

For an arbitrary function $g\in\mathcal{Z}_{1,p}[0,1]$, the positive linear operators imply that

$$\begin{aligned} \left|\left|\mathcal{T}^*_{s,\alpha}(g;z)-g(z)\right|\right|_{L_p(\mathcal{I}_s)} &\le \left|\left|\mathcal{T}^*_{s,\alpha}(g-\phi;z)-(g-\phi)(z)\right|\right|_{L_p(\mathcal{I}_s)} \\ &\quad+\left|\left|\mathcal{T}^*_{s,\alpha}(\phi;z)-\phi(z)\right|\right|_{L_p(\mathcal{I}_s)} \\ &\le 2\left(\left|\left|g-\phi\right|\right|_{L_p[0,1]}+2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\zeta_{s,\alpha}(z)\left|\left|g'\right|\right|_{L_p[0,1]}\right) \\ &\le 2\mathcal{K}_p\left(g,2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\zeta_{s,\alpha}(z)\right) \\ &\le 2c_2\hat{\omega}_{1,p}\left(g,2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\zeta_{s,\alpha}(z)\right) \\ &\le 2c_2\left(1+2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\hat{\omega}_{1,p}\left(g,\zeta_{s,\alpha}(z)\right)\right). \end{aligned}$$

□

Theorem 9 *For an arbitrary function $g\in C^2[0,1]$, let the auxiliary operators $\mathcal{R}^*_{s,\alpha}$ be such that*

$$\mathcal{R}^*_{s,\alpha}(g;z)=\mathcal{T}^*_{s,\alpha}(g;z)+g(z)-g\left(\mathcal{T}^*_{s,\alpha}(\chi_1(t);z)\right). \tag{23}$$

Then, for every $\phi\in C^2[0,1]$, it follows that

$$\left|\mathcal{R}^*_{s,\alpha}(\phi;z)-\phi(z)\right|\le\left[\mathcal{T}^*_{s,\alpha}\left((\chi_1(t)-z)^2;z\right)+\left(\zeta^*_{s,\alpha}(z)-z\right)^2\right]\|\phi''\|,$$

where $\zeta^*_{s,\alpha}(z)=\frac{1}{\left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)}\mathcal{T}^*_{s,\alpha}\left(\chi_1(t);z\right)$ *and* $\mathcal{T}^*_{s,\alpha}\left(\chi_1(t);z\right)$ *is defined by* (20). *In addition,* $C^2[0,1]$ *is defined by*

$$C^2[0,1]=\{\phi:\phi\in C[0,1]\ \textit{such that}\ \phi',\phi''\in C[0,1]\}.$$

Proof For any $\phi\in C^2[0,1]$, we easily get $\mathcal{R}^*_{s,\alpha}(\chi_0(t);z)=1$ and

$$\mathcal{R}^*_{s,\alpha}(\chi_1(t);z)=\mathcal{T}^*_{s,\alpha}(\chi_1(t);z)+z-\mathcal{T}^*_{s,\alpha}(\chi_1(t);z)=z.$$

From the Taylor series, we see

$$\phi(\chi_1(t))=\phi(z)+(\chi_1(t)-z)\phi'(z)+\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\phi''(\xi)\mathrm{d}\xi.$$

By use of the linearity of $\mathcal{R}^*_{s,\alpha}$, we get

$$\begin{aligned}\mathcal{R}^*_{s,\alpha}(\phi;z)-\phi(z)&=\phi'(z)\mathcal{R}^*_{s,\alpha}(\chi_1(t)-z;z)+\mathcal{R}^*_{s,\alpha}\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\psi''(\xi)\mathrm{d}\xi;z\right)\\&=\mathcal{R}^*_{s,\alpha}\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\psi''(\xi)\mathrm{d}\xi;z\right)\\&=\mathcal{T}^*_{s,\alpha}\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\psi''(\varkappa)\mathrm{d}\xi;z\right)\\&-\int_z^{\zeta^*_{s,\alpha}(z)}\left(\zeta^*_{s,\alpha}(z)-\xi\right)\phi''(\xi)\mathrm{d}\xi,\end{aligned}$$

where $\zeta^*_{s,\alpha}(z)=\frac{1}{\left(\frac{\varkappa_2+1+s}{s+1}\right)^2(1-\alpha)+\alpha\left(\frac{\varkappa_2+1+s}{s+1}\right)}\mathcal{T}^*_{s,\alpha}(\chi_1(t);z)$. Thus

$$\begin{aligned}|\mathcal{R}^*_{s,\alpha}(\phi;z)-\phi(z)|&\le\left|\mathcal{T}^*_{s,\alpha}\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\phi''(\varkappa)\mathrm{d}\xi;z\right)\right|\\&+\left|\int_z^{\zeta^*_{s,\alpha}(z)}\left(\zeta^*_{s,\alpha}(z)-\xi\right)\phi''(\xi)\mathrm{d}\xi\right|.\end{aligned}$$

Since we know

$$\begin{aligned}\left|\mathcal{T}^*_{s,\alpha}\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\phi''(\varkappa)\mathrm{d}\xi;z\right)\right|&\le\mathcal{T}^*_{s,\alpha}\left|\left(\int_z^{\chi_1(t)}(\chi_1(t)-\xi)\phi''(\varkappa)\mathrm{d}\xi;z\right)\right|\\&\le\mathcal{T}^*_{s,\alpha}\int_z^{\chi_1(t)}\left|(\chi_1(t)-\xi)\phi''(\varkappa)\mathrm{d}\xi;z\right)\right|\\&\le\mathcal{T}^*_{s,\alpha}\left((\chi_1(t)-z)^2;z\right)\|\phi''\|\end{aligned}$$

and

$$\left|\int_z^{\zeta^*_{s,\alpha}(z)}\left(\zeta^*_{s,\alpha}(z)-\xi\right)\phi''(\xi)\mathrm{d}\xi\right|\le\left(\zeta^*_{s,\alpha}(z)-z\right)^2\|\phi''\|,$$

we get

$$|\mathcal{R}^*_{s,\alpha}(\phi;z)-\phi(z)|\le\left[\mathcal{T}^*_{s,\alpha}\left((\chi_1(t)-z)^2;z\right)+\left(\zeta^*_{s,\alpha}(z)-z\right)^2\right]\|\phi''\|.$$

This gives the complete proof of Theorem 9. □

5 Graphical analysis

In this section, we give a numerical example with illustrative graphics with the help of MATLAB.

Example 1 Let $g(y)=(y-\frac{1}{3})(y-\frac{3}{5})$, $\xi_1=4$, $\xi_2=3$, $\chi_1=1.5$, $\chi_2=4$, and $s\in\{20,30,110\}$. The convergence of the operator towards the function $g(y)$ is shown in Fig. 1.

From this example, we observe that approximation of function by the operators becomes better when we take larger values of s.

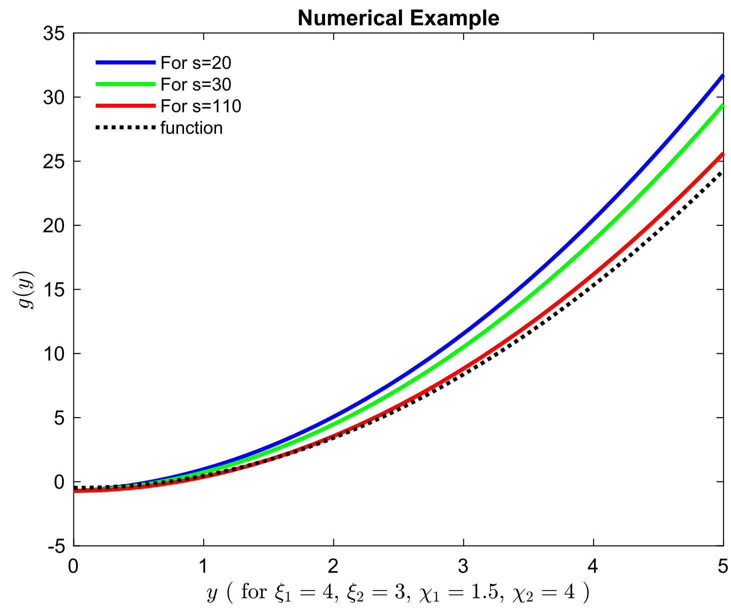

Figure 1 Convergence of the operator towards the function *g*(*y*)

6 Conclusions

It is very clear that for the choices of $\zeta_2 = \varkappa_2 = 0$, our new constructed operators reduced to α-Bernstein–Stancu variant of Kantorovich polynomial given by [20]. For the choices of $\zeta_1 = \zeta_2 = \varkappa_1 = \varkappa_2 = 0$, our new constructed operators reduced to the classical α-Bernstein–Kantorovich polynomial obtained in [19]. Therefore, based on our research, we can conclude that the shifted knot operators defined by equality (6), which we have named our new Bernstein–Kantorovich variation, are an improved version of the previous operators defined by [19, 20]. This article's major goal is to examine how operators that shifted knots of α-Bernstein–Kantorovich operators (6) in L_p spaces converge on $[0, 1]$.

Acknowledgment
The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for Funding this work through Research no. S-0157-1443.

Author contributions
Author 1: wrote the main manuscript text Authors 2,3,4: read and reviewed the manuscript

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Faculty of Science, University of Tabuk, P. O. Box 4279, Tabuk 71491, Saudi Arabia. [2]Physical Biochemistry Research Laboratory, Department of Biochemistry, Faculty of Science, University of Tabuk, P. O. Box 4279, Tabuk 71491, Saudi Arabia.

Received: 14 May 2024 Accepted: 17 June 2024 Published online: 27 June 2024

References

1. Alotaibi, A., Nasiruzzaman, M., Mohiuddine, S.A.: On the convergence of Bernstein-Kantorovich shifted knots operators by Schurer parameter. Complex Anal. Oper. Theory **18**, 4 (2024). https://doi.org/10.1007/s11785-023-01423-y
2. Alotaibi, A., Nasiruzzaman, M., Mursaleen, M.: Approximation by Phillips operators via q-Dunkl generalization based on a new parameter. J. King Saud Univ., Sci. **33**(4), Article id: 101413 (2021)
3. Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications. de Gruyter, Berlin (2011). https://doi.org/10.1515/9783110884586
4. Ansari, K.J., Özger, F., Özger, Z.Ö.: Numerical and theoretical approximation results for Schurer–Stancu operators with shape parameter λ. Comput. Appl. Math. **41**, Article id: 181 (2022)
5. Ayman-Mursaleen, M., Heshamuddin, M., Rao, N., Sinha, B.K., Yadav, A.K.: Hermite polynomials linking Szász-Durrmeyer operators. Comput. Appl. Math. **43**(4), Article id: 223 (2024)
6. Ayman-Mursaleen, M., Lamichhane, B.P., Kiliçman, A., Senu, N.: On q-statistical approximation of wavelets aided Kantorovich q-Baskakov operators. Filomat **38**(9), 3261–3274 (2024)
7. Ayman-Mursaleen, M., Nasiruzzaman, M., Rao, N., Dilshad, M., Nisar, K.S.: Approximation by the modified λ-Bernstein-polynomial in terms of basis function. AIMS Math. **9**(2), 4409–4426 (2024)
8. Babu, G.J., Canty, A.J., Chaubey, Y.P.: Application of Bernstein polynomials for smooth estimation of distribution and density function. J. Stat. Plan. Inference **105**(2), 377–392 (2002)
9. Bernstein, S.N.: Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités. Commun. Kharkov Math. Soc. **13**, 1–2 (1912)
10. Cai, Q.B.: The Bézier variant of Kantorovich type λ-Bernstein operators. J. Math. Anal. Appl. **2018**, Article id: 90 (2018)
11. Cai, Q.B., Lian, B.Y., Zhou, G.: Approximation properties of λ-Bernstein operators. J. Inequal. Appl. **2018**, Article id: 61 (2018)
12. Chen, X., Tan, J., Liu, Z., Xie, J.: Approximation of functions by a new family of generalized Bernstein operators. J. Math. Anal. Appl. **450**, 244–261 (2017)
13. Gadjiev, A.D., Ghorbanalizade, A.M.: Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables. Appl. Math. Comput. **216**, 890–901 (2010)
14. Johnen, H.: Inequalities connected with the moduli of smoothness. Mat. Vesn. **9**(24), 289–303 (1972)
15. Kajla, A., Mohiuddine, S.A., Alotaibi, A.: Approximation by α-Baskakov-jain type operators. Filomat **36**(5), 1733–1741 (2022)
16. Kajla, A., Mohiuddine, S.A., Alotaibi, A.: Durrmeyer-type generalization of μ-Bernstein operators. Filomat **36**(1), 349–360 (2022)
17. Korovkin, P.P.: Linear Operators and Approximation Theory. Hindustan Publishing Corporation, Delhi (1960)
18. Matsuoka, M.: An application of Bernstein polynomials in PAC model. In: Algorithmic Learning Theory. Lecture Notes in Comput. Sci., vol. 743, pp. 220–228. Springer, Berlin (1993)
19. Mohiuddine, S.A., Acar, T., Alotaibi, A.: Construction of a new family of Bernstein-Kantorovich operators. Math. Methods Appl. Sci. **40**(18), 7749–7759 (2017)
20. Mohiuddine, S.A., Özger, F.: Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**, Article id: 70 (2020)
21. Mursaleen, M., Ansari, K.J., Khan, A.: Approximation properties and error estimation of q-Bernstein shifted operators. Numer. Algorithms **84**, 207–227 (2020)
22. Özger, F.: On new Bézier bases with Schurer polynomials and corresponding results in approximation theory. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **69**(1), 376–393 (2020)
23. Özger, F., Srivastava, H.M., Mohiuddine, S.A.: Approximation of functions by a new class of generalized Bernstein–Schurer operators. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**, Article id: 173 (2020)
24. Rao, N., Nasiruzzaman, M., Heshamuddin, M., Shadab, M.: Approximation properties by modified Baskakov–Durrmeyer operators based on shape parameter-α. Iran. J. Sci. Technol., Trans. A, Sci. **45**, 1457–1465 (2021)
25. Rao, N., Wafi, A.: Stancu-variant of generalized Baskakov operators. Filomat **31**(9), 2625–2632 (2019)
26. Rao, N., Wafi, A., Acu, A.M.: q- Szász–Durrmeyer type operators based on Dunkl analogue. Complex Anal. Oper. Theory **13**, 915–934 (2019)
27. Shisha, O., Bond, B.: The degree of convergence of sequences of linear positive operators. Proc. Natl. Acad. Sci. USA **60**, 1196–1200 (1968)
28. Stancu, D.D.: Approximation of functions by a new class of linear polynomial operators. Rev. Roum. Math. Pures Appl. **13**, 1173–1194 (1968)
29. Sucu, S., Ibikli, E.: Approximation by means of Kantorovich-Stancu type operators. Numer. Funct. Anal. Optim. **34**(5), 557–575 (2013)
30. Zygmund, A.: Trigonometric Series I, II. Cambridge University Press, Cambridge (1959)

Publisher's Note