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Quasinormed spaces generated by a quasimodular

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Abstract

In this paper, we introduce the notion of a quasimodular and we prove that the respective Minkowski functional of the unit quasimodular ball becomes a quasinorm. In this way, we refer to and complete the well-known theory related to the notions of a modular and a convex modular that lead to the F-norm and to the norm, respectively. We use the obtained results to consider the basic properties of quasinormed Calderón–Lozanovskiĭ spaces E_φ, where the lower Matuszewska–Orlicz index α_φ plays the key role. Our studies are conducted in a full possible generality.

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1 Introduction

The subject of quasinormed or F-normed spaces (which are a natural generalization of normed spaces) has been the object of research by many authors for a long time (see [1, 4, 10–16, 22, 25, 27]). Recall also that the so-called Δ-norm is a generalization of both a quasinorm and an F-norm (see [12]). From a different point of view, the theory of modular spaces has also been developed (see [21, 26]).

We will try to link these threads. Namely, we will introduce a new notion of the functional ρ that we will call a quasimodular. This is an essential generalization of the concept of a convex semimodular. Furthermore, this quasimodular ρ is defined in such a way that the respective Minkowski functional of the unit quasimodular ball gives a quasinorm. We also check if the basic properties from the classical theory of modular spaces are still true for quasimodular spaces.

Then, we apply our general concept to a special class of quasimodular spaces, that is, the quasinormed Calderón–Lozanovskiĭ spaces E_φ generated by a quasimodular ρ_φ^E. We focus mainly on the basic relations between a positive lower Matuszewska–Orlicz index α_φ^E and the fact that ρ_φ^E ($\|\cdot\|_\varphi$) is a quasimodular (a quasinorm), respectively. We study carefully these relations in both directions. We will finish the article by characterizing the quasinormed Orlicz–Lorentz (Orlicz) spaces as spaces generated by a suitable quasimodular. We admit nonconvex, nondecreasing, and degenerated Orlicz functions, which give the full generality of studies. Some geometric properties of these spaces have been studied in [14] and, in the particular case for $E = L^1$, in [16], but the authors consider them directly with a quasinorm (not as spaces generated by a "modular").

2 Preliminaries

Definition 2.1 Given a real vector space X the functional $x \mapsto \|x\|$ is called a *quasinorm* if the following three conditions are satisfied:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|$ for any $x \in X$ and $a \in \mathbb{R}$;
(iii) there exists $C = C_X \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

For $0 < p \leq 1$, the functional $x \mapsto \|x\|_1$ is called a *p-norm* if it satisfies the first two conditions of the quasinorm and the condition $\|x+y\|_1^p \leq \|x\|_1^p + \|y\|_1^p$ for any $x, y \in X$. Clearly, each p-norm is a quasinorm. By the *Aoki–Rolewicz theorem* (cf. [12, Theorem 1.3 on page 7], [25, page 86]), given a quasinorm $\| \cdot \|$, if $0 < p \leq 1$ is such that $C = 2^{1/p-1}$, then there exists a p-norm $\| \cdot \|_1$ which is equivalent to $\| \cdot \|$, that is

$$\|x\|_1 \leq \|x\| \leq 2C\|x\|_1 \tag{2.1}$$

for all $x \in X$. The quasinorm $\| \cdot \|$ induces a metric topology on X: in fact a metric can be defined by $d(x, y) = \|x - y\|_1^p$. We say that $X = (X, \| \cdot \|)$ is a *quasi-Banach space* if it is complete for this metric. Let us note that a lot of important information and results on quasi-Banach spaces can be found in [11], see also [12].

Recall that a quasi-Banach lattice E is called *order continuous* ($E \in$ (OC)) if for each sequence $x_n \downarrow 0$, that is $x_n \geq x_{n+1}$ and $\inf_n x_n = 0$, we have $\|x_n\|_E \to 0$ (see [17, 23, 28]).

3 Quasimodular spaces

In this section, we introduce the notions of a quasimodular and a quasimodular space.

Definition 3.1 Let X be a real linear space. We say that a function $\rho : X \to [0, \infty]$ is a quasimodular whenever for all $x, y \in X$ the following conditions are satisfied:

(i) $\rho(0) = 0$ and the condition $\rho(\lambda x) \leq 1$ for all $\lambda > 0$ implies that $x = 0$;
(ii) $\rho(-x) = \rho(x)$;
(iii) $\rho(\lambda x)$ is a nondecreasing function of λ, where $\lambda \geq 0$;
(iv) There is $M \geq 1$ such that

$$\rho(\alpha x + \beta y) \leq M\big[\rho(x) + \rho(y)\big]$$

provided $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$;

(v) There is a constant $p > 0$ such that for all $\varepsilon > 0$ and all $A > 0$ there exists $K = K(\varepsilon, A) \geq 1$ such that

$$\rho(ax) \leq Ka^p \rho(x) + \varepsilon$$

for any $0 < a \leq 1$ whenever $\rho(x) \leq A$.

Remark 3.2 (*i*) The above definition, in particular the condition (*v*), has been introduced in such a way as to cover the largest possible class of mappings ρ and to provide a quasinorm (see Theorem 3.4). As we will show in Theorem 4.10, condition (v) can be simplified in some particular cases (more precisely, some particular modulars satisfy condition (v) in a simpler or stronger form).

(ii) Obviously, a convex modular (more precisely a convex semimodular) defined in [26] is in particular a quasimodular. As we will show in Example 4.15 (ii) and (iii), the concepts of quasimodular and modular (more precisely a semimodular, which induces an F-norm) defined in [26] are incomparable.

If ρ is a quasimodular on X, then

$$X_\rho := \left\{x \in X\colon \lim_{\lambda\to 0} \rho(\lambda x) = 0\right\}$$

is called a quasimodular space. It is easy to show that X_ρ is a linear subspace of X. We also obtain the following:

Lemma 3.3 *For any quasimodular ρ, we have*

$$X_\rho = \left\{x \in X\colon \rho(\lambda x) < \infty \textit{ for some } \lambda > 0\right\}.$$

Proof Note that in order to prove this lemma, it is enough to show that if $\rho(\lambda_0 x) < \infty$ for some $\lambda_0 > 0$, then $\lim_{\lambda\to 0}\rho(\lambda x) = 0$. Let $x \in X$ and $\rho(\lambda_0 x) < \infty$ for some $\lambda_0 > 0$. We prove that for any $\varepsilon \in (0, \rho(\lambda_0 x))$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(\lambda x) < \varepsilon$, whenever $\lambda < \delta$. Let $\varepsilon \in (0, \rho(\lambda_0 x))$ be fixed and take $K = K(\frac{\varepsilon}{2}, \rho(\lambda_0 x))$ from condition (v) of Definition 3.1. Then, for $\lambda < \delta$, where $\delta = \lambda_0(\frac{\varepsilon}{2K\rho(\lambda_0 x)})^{1/p}$, we obtain

$$\rho(\lambda x) = \rho\left(\frac{\lambda}{\lambda_0}\lambda_0 x\right) \le K\left(\frac{\lambda}{\lambda_0}\right)^p \rho(\lambda_0 x) + \varepsilon/2 < \varepsilon. \qquad \square$$

Theorem 3.4 *Let ρ be a quasimodular on X. Then, the functional*

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho(x/\lambda) \le 1\right\}$$

is a quasinorm on X_ρ.

Proof The condition $\|0\|_\rho = 0$ is obvious. Suppose $x \neq 0$. Then, by condition (i), there exists $\lambda > 0$ such that $\rho(\lambda x) > 1$ and, by condition (iii), $\|x\|_\rho \ge 1/\lambda$.

For any $x \in X_\rho$ and all $\alpha \in \mathbb{R}$, exactly the same way as in [26], we obtain

$$\|\alpha x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{\alpha x}{\lambda}\right) \le 1\right\} = |\alpha| \inf\left\{\lambda/|\alpha| > 0 : \rho\left(\frac{x}{\lambda/\alpha}\right) \le 1\right\} = |\alpha|\,\|x\|_\rho.$$

Finally, we prove the quasitriangle inequality. Let $0 < \varepsilon < \frac{1}{2M}$ be fixed and take $K = K(\varepsilon, 1)$ (where constants M and K arise from conditions (iv) and (v) of Definition 3.1). Defining

$$C = \left(\frac{K}{\frac{1}{2M} - \varepsilon}\right)^{1/p},$$

for all $x, y \in X_\rho$ and any $\delta > 0$, we obtain

$$\begin{aligned}
&\rho\left(\frac{x+y}{C(\|x\|_\rho + \|y\|_\rho + \delta)}\right) \\
&\quad = \rho\left(\frac{\|x\|_\rho + \delta/2}{\|x\|_\rho + \|y\|_\rho + \delta}\frac{x}{C(\|x\|_\rho + \delta/2)} + \frac{\|y\|_\rho + \delta/2}{\|x\|_\rho + \|y\|_\rho + \delta}\frac{y}{C(\|y\|_\rho + \delta/2)}\right) \\
&\quad \le M\left[\rho\left(\frac{x}{C(\|x\|_\rho + \delta/2)}\right) + \rho\left(\frac{y}{C(\|y\|_\rho + \delta/2)}\right)\right] \\
&\quad \le M\left[\frac{K}{C^p}\cdot\rho\left(\frac{x}{(\|x\|_\rho + \delta/2)}\right) + \varepsilon + \frac{K}{C^p}\cdot\rho\left(\frac{y}{(\|y\|_\rho + \delta/2)}\right) + \varepsilon\right] \\
&\quad \le 2M\left(\frac{K}{C^p} + \varepsilon\right) = 1,
\end{aligned}$$

whence

$$\|x+y\|_\rho \le C\big(\|x\|_\rho + \|y\|_\rho + \delta\big).$$

By the arbitrariness of δ we have $\|x+y\|_\rho \le C(\|x\|_\rho + \|y\|_\rho)$. □

By the definition of quasinorm, we obtain immediately the following:

Lemma 3.5 *Let ρ be a quasimodular on X. Then, for any $x \in X_\rho$ the following statements hold*:

(*i*) *If* $\rho(x) \le 1$, *then* $\|x\|_\rho \le 1$;

(*ii*) *If ρ is left continuous* ($\lim_{\lambda\to 1-}\rho(\lambda x) = \rho(x)$ *for all* $x \in X_\rho$), *then* $\rho(x) \le 1$ *whenever* $\|x\|_\rho \le 1$;

(*iii*) *If* $\|x\|_\rho < 1$, *then* $\rho(x) \le 1$;

(*iv*) *If ρ is right continuous* ($\lim_{\lambda\to 1+}\rho(\lambda x) = \rho(x)$ *for all* $x \in X_\rho$), *then* $\|x\|_\rho < 1$ *whenever* $\rho(x) < 1$.

Remark 3.6 As we will show in Example 4.15 (*iv*) the implication, if $\|x\|_\rho < 1$ then $\rho(x) < 1$, is not always true. Recall that this implication is true for any modular as well as for any convex modular (see [26]).

Lemma 3.7 *For each sequence* (x_n) *in* X_ρ *we have* $\lim_{n\to\infty}\|x_n\|_\rho = 0$ *if and only if* $\lim_{n\to\infty}\rho(\lambda x_n) = 0$ *for all* $\lambda > 0$.

Proof The implication, if $\lim_{n\to\infty}\rho(\lambda x_n) = 0$ for all $\lambda > 0$ then $\lim_{n\to\infty}\|x_n\|_\rho = 0$ is obvious. Let now $\|x_n\|_\rho \to 0$. Fix $\lambda > 0$ and $\varepsilon \in (0,1)$ and let $K = K(\varepsilon/2, 1)$ be the constant from condition (*v*) of Definition 3.1. Then, there exists $n_{\lambda,\varepsilon}$ such that $\|\lambda x_n\|_\rho \le (\varepsilon/4K)^{1/p}$ for all $n \ge n_{\lambda,\varepsilon}$. Hence, for each $a \in ((\varepsilon/4K)^{1/p}, (\varepsilon/2K)^{1/p})$ we obtain

$$\rho(\lambda x_n) = \rho\left(a\frac{\lambda x_n}{a}\right) \le Ka^p\rho\left(\frac{\lambda x_n}{a}\right) + \frac{\varepsilon}{2} \le \varepsilon.$$

By the arbitrariness of ε, for any $\lambda > 0$ we obtain $\lim_{n\to\infty}\rho(\lambda x_n) = 0$. □

4 Quasinormed Calderón–Lozanowskiĭ spaces

A triple (T,Σ,μ) stands for a positive, complete, and σ-finite measure space and $L^0 = L^0(T,\Sigma,\mu)$ denotes the space of all (equivalence classes of) Σ-measurable functions $x : T \to \mathbb{R}$. For every $x \in L^0$ we denote $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$. Moreover, for any $x, y \in L^0$, we write $x \leq y$, if $x(t) \leq y(t)$ almost everywhere with respect to the measure μ on the set T.

A quasinormed lattice [quasi-Banach lattice] $E = (E, \leq, \|\cdot\|_E)$ is called a *quasinormed ideal space* [*quasi-Banach ideal space* (or a *quasi-Köthe space*)] if it is a linear subspace of L^0 satisfying the following conditions:

(i) If $x \in L^0$, $y \in E$ and $|x| \leq |y|$ μ-a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$;

(ii) There exists $x \in E$ that is strictly positive on the whole T.

By E_+ we denote the positive cone of E, that is, $E_+ = \{x \in E : x \geq 0\}$. Let C_E be the constant from the quasitriangle inequality for E. In turn, by $E(w)$ we denote the weighted quasinormed ideal space, that is,

$$E(w) = \left\{x \in L^0 : xw \in E\right\}$$

with the quasinorm $\|x\|_{E(w)} = \|xw\|_E$, where $w : T \to (0,\infty)$ is a measurable weight function.

We say that a quasinormed ideal space E has the *Fatou property*, if for all $x \in L^0$ and any $(x_n)_{n=1}^\infty$ in E_+ such that $x_n \uparrow |x|$ μ-a.e and $\sup_{n\in N} \|x_n\|_E < \infty$, we obtain $x \in E$ and $\lim_n \|x_n\|_E = \|x\|_E$. It is well known that E has the Fatou property if and only if for each $x \in L^0$ and all $(x_n)_{n=1}^\infty$ in E such that $x_n \to x$ μ-a.e and $\liminf_{n\in N} \|x_n\|_E < \infty$, we have $x \in E$ and $\|x\|_E \leq \liminf_n \|x_n\|_E$ (cf. [2, Lemma 1.5 on page 4]).

Lemma 4.1 *Let E be a quasinormed ideal space.*

(i) *If* $\lim_{n\to\infty} \|x - x_n\|_E = 0$, *where* $x \in E$ *and* $(x_n)_{n=1}^\infty$ *is a sequence in E, then* $x_n \to x$ *locally in measure.*

(ii) *For any Cauchy sequence* $(x_n)_{n=1}^\infty$ *in E there exists* $x \in L^0$ *such that* $x_n \to x$ *locally in measure.*

Proof This lemma can be proved analogously as Theorem 1 on page 96 in [17]. Indeed, assuming in (2) on page 96 that $\|x_n - x\|_E < \frac{\varepsilon}{2^n C_E^n}$, we obtain $\|\chi_{B_n}\|_E < \frac{1}{2^n C_E^n}$ (see (5) on page 96) and, in consequence:

$$\|\chi_{D_n}\|_E \leq \|\chi_{C_{ms_m}}\|_E \leq \left\|\sum_{k=m+1}^{m+s_m} \chi_{B_k}\right\|_E \leq \sum_{k=m+1}^{m+s_m} C_E^{(k-m)} \|\chi_{B_k}\|_E < \sum_{k=m+1}^{m+s_m} \frac{C_E^{(k-m)}}{2^k C_E^k} < \frac{1}{2^m},$$

(see page 97, line 5). □

Lemma 4.2 [14, *Lemma* 2.1] *A quasinormed ideal space E with the Fatou property is complete.*

Proof We recall a short proof of this lemma for the sake of completeness. By the Aoki–Rolewicz theorem, it is enough to show that for any Cauchy sequence $(x_n)_{n=1}^\infty$ in E there exists $x \in E$ such that $\lim_{n\to\infty} \|x - x_n\|_E = 0$. If $(x_n)_{n=1}^\infty$ is a Cauchy sequence in E, then by Lemma 4.1(*ii*), $x_n \to x$ locally in measure for some $x \in L^0$. Without loss of generality (passing to subsequences and applying the double extract convergence theorem, if necessary), we can assume $x_n \to x$ μ-a.e. Hence, by the Fatou property, we obtain $x \in E$ and $\|x - x_n\|_E \le \liminf_{m\to\infty} \|x_m - x_n\|_E$ for each $n \in \mathbb{N}$, which completes the proof. □

The following basic fact, very well known for Banach ideal spaces (see [17, Lemma 2, p. 97]), is also true for quasi-Banach ideal spaces.

Lemma 4.3 *Let $(E, \|\cdot\|_E)$ be a quasi-Banach ideal space. If $\|x_n\|_E \to 0$, then there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$, an element $y \in E_+$ and a sequence $\varepsilon_k \downarrow 0$ such that $|x_{n_k}| \le \varepsilon_k \cdot y$ for each k.*

Proof If $\|x_n\|_E \to 0$, then we can find a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\|x_{n_k}\| < \frac{1}{C_E^k 2^k}$ for any $k \in \mathbb{N}$. Consequently,

$$\sum_{k=1}^{\infty} C_E^k \|k \cdot x_{n_k}\| \le \sum_{k=1}^{\infty} C_E^k \frac{k}{C_E^k 2^k} = \sum_{k=1}^{\infty} \frac{k}{2^k} < \infty.$$

By Theorem 1.1 from [25], $y := \sum_{k=1}^{\infty} |k \cdot x_{n_k}| \in E_+$. Thus, $|x_{n_k}| \le \frac{y}{k}$ for all k. Taking $\varepsilon_k = 1/k$ we complete the proof. □

From now on, we will assume that E is a quasi-Banach ideal space with the Fatou property. During our studies we will consider three natural classes of E:

(1) neither $L_\infty \subset E$ nor $E \subset L_\infty$;

(2) $L_\infty \subset E$;

(3) $E \subset L_\infty$.

Let $T = [0, \gamma)$, $0 < \gamma \le \infty$, and μ be the Lebesgue measure. Also, the space $E = L_p$, $0 < p < \infty$, belongs to the class (1) if $\gamma = \infty$ and the class (2) otherwise. Moreover, the space $L_1 \cap L_\infty$ belongs to the class (3) whenever $\gamma = \infty$. Let now $T = \mathbb{N}$ and $\mu = m$ be a counting measure. Then, the space l_p, $p \in (0, \infty)$, belongs to class (3), the weighted sequence space $l_1(w)$, $w = (w(n))_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty} w(n) < \infty$, belongs to class (2) and the Cesàro sequence space ces_p, $1 < p < \infty$ (see [19] for the respective definition), belongs to class (1).

Remark 4.4 Let $E \subset L_\infty$. Then, by the closed-graph theorem that is still true for quasi-Banach spaces (see [12, Theorem. 1.6]), the inclusion is continuous, whence there exists a constant $D_E > 0$ such that

$$\|x\|_{L_\infty} \le D_E \|x\|_E \tag{4.1}$$

for each $x \in E$. Defining

$$a_E = \inf\big\{\|\chi_A\|_E : \chi_A \in E, \mu(A) > 0\big\}, \tag{4.2}$$

by (4.1), we have $a_E \ge 1/D_E > 0$.

Definition 4.5 A function $\varphi : [0, \infty) \to [0, \infty]$ is called an *Orlicz function* if φ is nondecreasing, vanishing, and right continuous at 0, continuous on $(0, b_\varphi)$, where

$$b_\varphi = \sup\big\{u \ge 0 : \varphi(u) < \infty\big\}$$

and left continuous at b_φ. In the whole paper, excluding Remark 4.14 and Example 4.15 (*iii*), we will assume that $\lim_{u\to\infty}\varphi(u)=\infty$.

Let

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\}.$$

By φ^{-1} we denote the generalized inverse of the function φ defined by

$$\varphi^{-1}(v) = \inf\{u \geq 0 \colon \varphi(u) > v\} \quad \text{for } v \in [0,\infty) \text{ and } \varphi^{-1}(\infty) = \lim_{v\to\infty}\varphi^{-1}(v)$$

(see [20]).

The following lemma is an easy exercise (see [20, Lemma 3.1], for a little different convex case).

Lemma 4.6 *For any Orlicz function φ we have*:

(i) *Let $u \in [0, b_\varphi)$. Then, $\varphi^{-1}(\varphi(u)) > u$ if φ is constant on the interval $[u, u+\delta)$ for some $\delta > 0$ and $\varphi^{-1}(\varphi(u)) = u$ otherwise.*

(ii) *If $b_\varphi < \infty$, then $\varphi^{-1}(\varphi(b_\varphi)) = b_\varphi$ and $\varphi^{-1}(\varphi(u)) = b_\varphi < u$ for $b_\varphi < u < \infty$.*

(iii) *If either $b_\varphi = \infty$ or $b_\varphi < \infty$ with $\varphi(b_\varphi) = \infty$, then $\varphi(\varphi^{-1}(u)) = u$ for any $u \in [0,\infty)$.*

(iv) *If $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$, then $\varphi(\varphi^{-1}(u)) = u$ for $u \in [0, \varphi(b_\varphi)]$ and $\varphi(\varphi^{-1}(u)) = \varphi(b_\varphi) < u$ for $u > \varphi(b_\varphi)$.*

From (*i*) *and* (*ii*), *in particular, we obtain*:

(v) *$\varphi^{-1}(\varphi(u)) = u$ for $a_\varphi \leq u < b_\varphi$ if either $b_\varphi = \infty$ or $b_\varphi < \infty$ with $\varphi(b_\varphi) = \infty$ and φ is strictly increasing on $[a_\varphi, b_\varphi)$.*

(vi) *$\varphi^{-1}(\varphi(u)) = u$ for $a_\varphi \leq u \leq b_\varphi$ if $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$ and φ is strictly increasing on $[a_\varphi, b_\varphi]$.*

Finally, note that from (*iii*) *and* (*iv*) *and* (*i*) *and* (*ii*) *we obtain*

(vii) *$\varphi(\varphi^{-1}(u)) \leq u$ for all $u \in [0,\infty)$ and $u \leq \varphi^{-1}(\varphi(u))$ if $\varphi(u) < \infty$.*

Recall that for any Orlicz function φ the *lower Matuszewska–Orlicz index* α_φ for all arguments is defined by the formula

$$\alpha_\varphi^a = \sup\{p \in \mathbb{R} : \text{there exists } K \geq 1 \text{ such that } \varphi(au) \leq Ka^p\varphi(u) \\ \text{for any } u \geq 0 \text{ and } 0 < a \leq 1\}.$$

Analogously, the *lower Matuszewska–Orlicz indices* for large and for small arguments are defined as

$$\alpha_\varphi^\infty = \sup\{p \in \mathbb{R} : \text{there exist } K \geq 1 \text{ and } u_0 > 0 \text{ such that } \varphi(u_0) < \infty \text{ and} \\ \varphi(au) \leq Ka^p\varphi(u) \text{ for any } u \geq u_0 \text{ and } 0 < a \leq 1\}$$

and

$$\alpha_\varphi^0 = \sup\{p \in \mathbb{R} : \text{there exist } K \geq 1 \text{ and } u_0 > 0 \text{ such that } \varphi(au) \leq Ka^p\varphi(u)$$

$$\text{for any } 0 \le u \le u_0 \text{ and } 0 < a \le 1\Big\},$$

respectively.

Remark 4.7 (*i*) If $0 < a_\varphi < b_\varphi$, then $\alpha_\varphi^0 = \infty$ and we may extend the key inequality in the definition of α_φ^0 to any $u_0 > a_\varphi$ such that $\varphi(u_0) < \infty$. Indeed, let $p > 0$. For every $0 \le u \le a_\varphi$ we have $\varphi(au) = 0 = Ka^p\varphi(u)$ for each $K > 0$ and $0 < a \le 1$. Take any $u_0 > a_\varphi$ such that $\varphi(u_0) < \infty$. If $a_\varphi < u \le u_0$ and $0 < a < \frac{a_\varphi}{u_0}$, then we have

$$\varphi(au) \le \varphi(au_0) \le \varphi(a_\varphi) = 0 \le Ka^p\varphi(u)$$

for every $K > 0$. Moreover,

$$\sup_{a_\varphi < u \le u_0} \sup_{\frac{a_\varphi}{u_0} \le a \le 1} \frac{\varphi(au)}{a^p\varphi(u)} \le \sup_{\frac{a_\varphi}{u_0} \le a \le 1} \frac{1}{a^p} = \left(\frac{u_0}{a_\varphi}\right)^p.$$

Thus, for $K = (\frac{u_0}{a_\varphi})^p$ we have

$$\varphi(au) \le Ka^p\varphi(u)$$

for all $0 < a \le 1$ and $0 \le u \le u_0$.

(*ii*) If $a_\varphi = 0$ and $\alpha_\varphi^0 > 0$ then we may extend the key inequality in the definition of α_φ^0 to any u_1 such that $\varphi(u_1) < \infty$. Indeed, suppose there is $p > 0$, $u_0 > 0$ and $K > 0$ such that

$$\varphi(au) \le Ka^p\varphi(u) \tag{4.3}$$

for all $0 < a \le 1$ and $0 \le u \le u_0$. Take $u_1 > u_0$ satisfying $\varphi(u_1) < \infty$. Then,

$$\sup_{u_0 < u \le u_1} \sup_{\frac{u_0}{u_1} \le a \le 1} \frac{\varphi(au)}{a^p\varphi(u)} \le \sup_{\frac{u_0}{u_1} \le a \le 1} \frac{1}{a^p} = \left(\frac{u_1}{u_0}\right)^p.$$

Set $K_1 = (\frac{u_1}{u_0})^p$. Now, we claim that

$$K_2 := \sup_{u_0 < u \le u_1} \sup_{0 < a < \frac{u_0}{u_1}} \frac{\varphi(au)}{a^p\varphi(u)} < \infty.$$

Otherwise, for each $n \in \mathbb{N}$ we can find $u_0 < u_n \le u_1$ and $0 < a_n < \frac{u_0}{u_1}$ such that

$$\varphi(a_nu_n) > na_n^p\varphi(u_n).$$

Denote $b_n := \frac{a_nu_n}{u_0}$. Then, $b_n < 1$ and

$$\varphi(b_nu_0) = \varphi(a_nu_n) > na_n^p\varphi(u_n) = nb_n^p\left(\frac{u_0}{u_n}\right)^p\varphi(u_n) \ge nb_n^p\left(\frac{u_0}{u_1}\right)^p\varphi(u_0).$$

On the other hand, by inequality (4.3),

$$\varphi(b_nu_0) \le Kb_n^p\varphi(u_0),$$

which gives a contradiction and proves the claim. Finally, setting $K_3 = \max\{K, K_1, K_2\}$ we conclude that

$$\varphi(au) \le K_3 a^p \varphi(u)$$

for all $0 < a \le 1$ and $0 \le u \le u_1$.

(*iii*) If $\alpha_\varphi^\infty > 0$ then we may extend the key inequality in the definition of α_φ^∞ to any $u_1 > a_\varphi$. Indeed, suppose there is $p > 0$, $u_0 > 0$, and $K > 0$ such that $\varphi(u_0) < \infty$ and

$$\varphi(au) \le K a^p \varphi(u)$$

for all $0 < a \le 1$ and $u \ge u_0$. Take $u_1 < u_0$ satisfying $\varphi(u_1) > 0$. Then,

$$\sup_{u_1 \le u < u_0} \sup_{0<a\le 1} \frac{\varphi(au)}{a^p\varphi(u)} \le \sup_{0<a\le 1} \frac{\varphi(au_0)}{a^p\varphi(u_1)} \le \sup_{0<a\le 1} \frac{Ka^p\varphi(u_0)}{a^p\varphi(u_1)} = \frac{K\varphi(u_0)}{\varphi(u_1)}.$$

Taking $K_1 = \max\{K, \frac{K\varphi(u_0)}{\varphi(u_1)}\} = \frac{K\varphi(u_0)}{\varphi(u_1)}$ we obtain that

$$\varphi(au) \le K_1 a^p \varphi(u)$$

for all $0 < a \le 1$ and $u \ge u_1$.

(*iv*) From the above consideration, we conclude immediately that if $\alpha_\varphi^\infty > 0$ and $\alpha_\varphi^0 > 0$ then $\alpha_\varphi^a > 0$.

Example 4.8 Taking $\varphi_1(u) = \ln(1+u)$, for $u \ge 0$, we easily obtain

$$\lim_{u\to\infty} \frac{\varphi_1(au)}{\varphi_1(u)} = 1$$

for any $a \in (0,1)$, whence $\alpha_{\varphi_1}^\infty = 0$. Analogously, defining $\varphi_2(0) = 0$ and

$$\varphi_2(u) = \frac{1}{\ln(1+\frac{1}{u})} \quad \text{for } u > 0,$$

we obtain

$$\lim_{u\to 0^+} \frac{\varphi_2(au)}{\varphi_2(u)} = 1$$

for any $a \in (0,1)$ and, in consequence, $\alpha_{\varphi_2}^0 = 0$.

For any pair E and φ we define the *lower Matuszewska–Orlicz index* α_φ^E, by the formula

$$\alpha_\varphi^E := \begin{cases} \alpha_\varphi^a, & \text{when neither } L_\infty \subset E \text{ nor } E \subset L_\infty, \\ \alpha_\varphi^\infty, & \text{when } L_\infty \subset E, \\ \alpha_\varphi^0, & \text{when } E \subset L_\infty. \end{cases}$$

Given a quasi-Banach ideal space E and an Orlicz function φ, we define on L^0 a functional ρ_φ^E, by

$$\rho_\varphi^E(x) := \begin{cases} \|\varphi(|x|)\|_E & \text{if } \varphi(|x|) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

It is well known that if φ is a convex Orlicz function and E is a Banach ideal space then ρ_φ^E is a convex modular. The result below concerns the more general case (the index α_φ^E plays a role of substitution of convexity of φ).

Theorem 4.9 *Let E be a quasi-Banach ideal space and φ be an Orlicz function. If $\alpha_\varphi^E > 0$, then ρ_φ^E is quasimodular (see Definition 3.1).*

Proof Obviously, $\rho_\varphi^E(0) = 0$ and $\rho_\varphi^E(-x) = \rho_\varphi^E(x)$ for any $x \in L^0$. Let $x \neq 0$. Then, there exist $A \in \Sigma$ with $\mu(A) > 0$ and $n \in \mathbb{N}$ such that $\frac{1}{n}\chi_A \leq |x|$. If $\varphi(|x|) \notin E$, then $\rho_\varphi^E(x) = \infty > 1$, while, if $\varphi(|x|) \in E$, by $\varphi(\frac{1}{n})\chi_A \leq \varphi(|x|)$, we obtain $\chi_A \in E$. Since $\lim_{u\to\infty}\varphi(u) = \infty$, we can find $\lambda_A > 0$ such that $\varphi(\lambda_A/n) > 1/\|\chi_A\|_E$ and, in consequence, $\rho_\varphi^E(\lambda_A x) = \|\varphi(\lambda_A|x|)\|_E \geq \|\varphi(\lambda_A/n)\chi_A\|_E > 1$.

For every $x \in L^0$ and all $0 \leq \lambda_1 \leq \lambda_2$ we have $\varphi(\lambda_1|x(t)|) \leq \varphi(\lambda_2|x(t)|)$ for μ-a.e. $t \in T$, whence $\rho_\varphi^E(\lambda_1 x) \leq \rho_\varphi^E(\lambda_2 x)$.

Let now $x, y \in L^0$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Then,

$$\varphi\bigl(\bigl|\alpha x(t) + \beta y(t)\bigr|\bigr) \leq \varphi\bigl(\max\bigl(\bigl|x(t)\bigr|, \bigl|y(t)\bigr|\bigr)\bigr) \leq \varphi\bigl(\bigl|x(t)\bigr|\bigr) + \varphi\bigl(\bigl|y(t)\bigr|\bigr)$$

for μ-a.e. $t \in T$. Hence, if $\varphi(|x|) \in E$ and $\varphi(|y|) \in E$, we obtain

$$\begin{aligned} \rho_\varphi^E(\alpha x + \beta y) &= \bigl\|\varphi\bigl(|\alpha x + \beta y|\bigr)\bigr\|_E \leq \bigl\|\varphi\bigl(|x|\bigr) + \varphi\bigl(|y|\bigr)\bigr\|_E \\ &\leq C_E\bigl(\bigl\|\varphi\bigl(|x|\bigr)\bigr\|_E + \bigl\|\varphi\bigl(|y|\bigr)\bigr\|_E\bigr) = C_E\bigl(\rho_\varphi^E(x) + \rho_\varphi^E(y)\bigr). \end{aligned}$$

Obviously, the inequality $\rho_\varphi^E(\alpha x + \beta y) \leq C_E(\rho_\varphi^E(x) + \rho_\varphi^E(y))$ holds true, when $\varphi(|x|) \notin E$ or $\varphi(|y|) \notin E$.

Finally, we will prove that ρ_φ^E satisfies condition (ν). Without loss of generality, we can suppose that $0 < \rho_\varphi^E(x) < \infty$ (then, in particular, we have $a_\varphi < b_\varphi$). We will consider three cases.

If $L_\infty \subset E$, then for any $\varepsilon > 0$ there exists $u_1 \in (a_\varphi, b_\varphi)$ such that $C_E \cdot \varphi(u_1)\|\chi_T\|_E < \varepsilon$. Since $\alpha_\varphi^E = \alpha_\varphi^\infty > 0$, by Remark 4.7, we obtain that there exist numbers $p > 0$ and $K = K(\varepsilon) \geq 1$ such that $\varphi(au) \leq Ka^p\varphi(u)$ for all $u \geq u_1$ and $0 < a \leq 1$. Defining $B = \{t \in T : |x(t)| \geq u_1\}$, for any $a \in (0, 1]$ we obtain

$$\begin{aligned} \rho_\varphi^E(ax) &= \bigl\|\varphi\bigl(a|x|\bigr)\bigr\|_E \leq C_E\bigl(\bigl\|\varphi\bigl(a|x|\bigr)\chi_B\bigr\|_E + \bigl\|\varphi\bigl(a|x|\bigr)\chi_{T\setminus B}\bigr\|_E\bigr) \\ &\leq C_E Ka^p\bigl\|\varphi\bigl(|x|\bigr)\bigr\|_E + C_E\varphi(u_1)\|\chi_T\|_E \leq C_E Ka^p\rho_\varphi^E(x) + \varepsilon. \end{aligned} \tag{4.4}$$

In the case when neither $L_\infty \subset E$ nor $E \subset L_\infty$, analogously as above, we obtain

$$\rho_\varphi^E(ax) = \bigl\|\varphi\bigl(a|x|\bigr)\bigr\|_E \leq Ka^p\bigl\|\varphi\bigl(|x|\bigr)\bigr\|_E = Ka^p\rho_\varphi^E(x). \tag{4.5}$$

Let now $E \subset L_\infty$, take $A > 0$ and assume that $\rho_\varphi^E(x) = \|\varphi(|x|)\|_E \le A$. By the closed-graph theorem, there exists a constant $D_E > 0$ such that $\|\varphi(|x|)\|_{L^\infty} \le D_E\|\varphi(|x|)\|_E \le AD_E$. Thus, $\varphi(|x(t)|) \le \min(AD_E, \varphi(b_\varphi))$ for μ-a.e. $t \in T$, whence, by Lemma 4.6, $|x(t)| \le u_2$ for the same t, where $u_2 = \varphi^{-1}(\min(AD_E, \varphi(b_\varphi)))$. Simultaneously, by $\alpha_\varphi^E = \alpha_\varphi^0 > 0$ and Remark 4.7, we obtain that there exist $p > 0$ and $K = K(A) \ge 1$ such that $\varphi(au) \le Ka^p\varphi(u)$ for any $u \le u_2$ and $0 < a \le 1$. Therefore, $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x)$. □

Now, we will show that, depending on the embeddings between E and L_∞, the condition $\alpha_\varphi^E > 0$ is even equivalent to a certain condition that is close to the point (v) of Definition 3.1.

Theorem 4.10 (i) *Assume that neither $L_\infty \subset E$ nor $E \subset L_\infty$. Then, $\alpha_\varphi^a > 0$ if and only if there exist constants $p > 0$ and $K \ge 1$ such that for all $x \in L^0$ and $0 < a \le 1$ we have $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x)$.*

(ii) *Let $L_\infty \subset E$. Then, $\alpha_\varphi^\infty > 0$ if and only if there is a constant $p > 0$ such that for all $v_0 > 0$ there exists $K = K(v_0) \ge 1$ such that for each $x \in L^0$ satisfying $|x(t)| \ge v_0$ for μ-a.e. $t \in T$ and any $0 < a \le 1$ we have $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x)$.*

(iii) *Let $E \subset L_\infty$. Then, $\alpha_\varphi^0 > 0$ if and only if there is a constant $p > 0$ such that for every $A > 0$ there exists $K = K(A) \ge 1$ such that for any $x \in L^0$ satisfying $\rho_\varphi^E(x) \le A$ and every $0 < a \le 1$ we have $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x)$.*

Proof The necessity of statements (i) and (iii) follows from the proof of Theorem 4.9. Now, we will show the sufficiency of (iii). Let $D \in \Sigma$ be such that $\mu(D) > 0$ and $\chi_D \in E$. Take $u_0 > 0$ satisfying $0 < \varphi(u_0) < \infty$ and define $A := \varphi(u_0)\|\chi_D\|_E$. Then, for any $u \le u_0$ and any $a \in (0, 1]$ we obtain

$$\begin{aligned}\varphi(au)\|\chi_D\|_E &= \big\|\varphi(au)\chi_D\big\|_E = \rho_\varphi^E(au\chi_D)\\ &\le K(A)a^p\rho_\varphi^E(u\chi_D) = K(A)a^p\big\|\varphi(u)\chi_D\big\|_E = K(A)a^p\varphi(u)\|\chi_D\|_E.\end{aligned}$$

Hence, by the arbitrariness of u and a, we obtain $\alpha_\varphi^0 > 0$. Analogously, we can prove the sufficiency of the condition $\alpha_\varphi^a > 0$ in (i).

Now, we will prove statement (ii), that is, let $L_\infty \subset E$. First, let us note that, by the proof of Theorem 4.9, we obtain the implication: if $\alpha_\varphi^\infty > 0$, then there is a constant $p > 0$ such that for each $\varepsilon > 0$ there exists $K = K(\varepsilon) \ge 1$ such that $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x) + \varepsilon$ for any $x \in L^0$ and $0 < a \le 1$.

Let $\alpha_\varphi^\infty > 0$ and take $p \in (0, \alpha_\varphi^\infty)$, $v_0 > 0$ and $x \in L^0$ such that $\rho_\varphi^E(x) < \infty$ and $|x(t)| \ge v_0$ for μ-a.e. $t \in T$. By Remark 4.7 (especially (iii) and (iv)), there exists $K = K(v_0)$ such that

$$\varphi\big(a\big|x(t)\big|\big) \le Ka^p\varphi\big(\big|x(t)\big|\big)$$

for all $a \in (0, 1]$ and μ-a.e. $t \in T$, whence $\rho_\varphi^E(ax) \le Ka^p\rho_\varphi^E(x)$.

Finally, we will prove the opposite implication. Take $u_0 > 0$ satisfying $0 < \varphi(u_0) < \infty$ and define $v_0 = u_0$. Then, for any $u \ge u_0$ and any $a \in (0, 1]$ we have

$$\varphi(au)\|\chi_T\|_E = \rho_\varphi^E(au\chi_T) \le K(v_0)a^p\rho_\varphi^E(u\chi_T) = K(v_0)a^p\varphi(u)\|\chi_T\|_E$$

and, in consequence, $\alpha_\varphi^\infty > 0$. □

Definition 4.11 Let a quasi-Banach ideal space E and an Orlicz function φ be such that $\alpha_\varphi^E > 0$. Then, the Calderón–Lozanovskiĭ space E_φ is defined by

$$E_\varphi = \left\{x \in L^0 : \lim_{\lambda\to 0} \rho_\varphi^E(\lambda x) = 0\right\}.$$

By Theorem 4.9 and Lemma 3.3, E_φ is a quasimodular space and

$$E_\varphi = \left\{x \in L^0 : \rho_\varphi^E(\lambda x) < \infty \text{ for some } \lambda > 0\right\}.$$

Moreover, by Theorem 3.4, the functional

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : \rho_\varphi^E(x/\lambda) \le 1\right\},$$

is a quasinorm, called a *Luxemburg–Nakano quasinorm*. It is easy to show that $E_\varphi = (E_\varphi, \le, \|\cdot\|_\varphi)$ is a quasinormed ideal space.

Remark 4.12 It is known that there is a connection between the Banach ideal space E_φ (where φ is a convex Orlicz function and E is a Banach ideal space) and the normed Calderón–Lozanovskii interpolation construction $\psi(E, L_\infty)$ (where ψ is a homogeneous, concave function on $\mathbb{R}_+^2$) – see [24, Example 2, p. 178]. However, there is a similar relation if φ is a nonconvex Orlicz function, E is a quasi-Banach ideal space, and ψ is positively homogeneous, nondecreasing with respect to each variable. On the other hand, such an approach leads to quasi-Banach lattices $\psi(E_0, E_1)$ that have application in interpolation theory (see [27]).

We also obtain the following:

Lemma 4.13 *For any quasi-Banach ideal space E and any Orlicz function φ the following assertions hold*:

(i) *For each $x \in E_\varphi$ the function $f_x(\alpha) := \rho_\varphi^E(\alpha x)$, for $\alpha > 0$, is nondecreasing and left continuous.*
(ii) *For any $x \in E_\varphi$ we have $\|x\|_\varphi \le 1$ if and only if $\rho_\varphi^E(x) \le 1$.*
(iii) *The quasinormed ideal space E_φ has the Fatou property and, in consequence, E_φ is complete.*

Proof The assertion (i) follows immediately from the properties of φ and E (recall that E has the Fatou property). Next, by (i) and Lemma 3.5, we obtain (ii). Proceeding analogously as in [6, Theorem 12] (see also [14, Lemma 2.2(ii)]), we obtain that E_φ has the Fatou property. Hence, by Lemma 4.2, E_φ is complete. □

Remark 4.14 Now, we will show the naturalness of the assumption $\lim_{u\to\infty}\varphi(u) = \infty$ in the definition of Orlicz function (see Definition 4.5). Obviously, if $\alpha_\varphi^a > 0$ or $\alpha_\varphi^\infty > 0$, then $\lim_{u\to\infty}\varphi(u) = \infty$. Simultaneously, for $\varphi(u) = \min(u^2, 1)$, we have $\alpha_\varphi^0 > 0$ and $\lim_{u\to\infty}\varphi(u) = 1$. Let $E \subset L_\infty$, $\lim_{u\to\infty}\varphi(u) < \infty$ and $\alpha_\varphi^E = \alpha_\varphi^0 > 0$. Then, condition (i) of Definition 3.1 holds whenever $\lim_{u\to\infty}\varphi(u) > 1/a_E$, where a_E is defined by formula (4.2), and, in consequence, ρ_φ^E is quasimodular. Moreover, defining the new Orlicz function ψ, by $\psi(u) = \varphi(u)$ for $u \in [0, \varphi^{-1}(1/a_E)]$ and $\psi(u) = u - (\varphi^{-1}(1/a_E) - 1/a_E)$ for $u > \varphi^{-1}(1/a_E)$, we obtain $\lim_{u\to\infty}\psi(u) = \infty$ and $(E_\varphi, \|\cdot\|_\varphi) \equiv (E_\psi, \|\cdot\|_\psi)$.

Now, we will show (among others) that the notion of modular and quasimodular are incomparable.

Example 4.15 Assume $T = [0,\infty)$ and μ is the Lebesgue measure.

(*i*) If $E = L_1$ and $\varphi(u) = u^p$ for $u \geq 0$, $p > 0$, then ρ_φ^E is a quasimodular (a convex modular for $p > 1$) as well as a modular (see [26]). We have

$$\|x\|_\varphi = \|x\|_p = \left(\int_0^\infty |x(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Simultaneously, the F-norm is given by

$$\big|\|x\|\big|_\varphi := \inf\{\lambda > 0 \colon \rho_\varphi^E(x/\lambda) \leq \lambda\} = \left(\int_0^\infty |x(t)|^p\,dt\right)^{\frac{1}{1+p}}$$

(see [26]). Note also that $|\|x_n\||_\varphi \to 0$ if and only if $\|x\|_\varphi \to 0$, but there do not exist constants $A, B > 0$ such that $A\|x\|_\varphi \leq |\|x\||_\varphi \leq B\|x\|_\varphi$ for all $x \in L_\varphi$.

(*ii*) Let $E = L_{(1/4)}$ and $\varphi(u) = u^2$ for $u \geq 0$. Obviously, ρ_φ^E is a quasimodular. Simultaneously, for $x = \chi_{[0,1)}$ and $y = \chi_{[1,2)}$ we obtain

$$\rho_\varphi^E\left(\frac{1}{2}x + \frac{1}{2}y\right) = 4 > 2 = \rho_\varphi^E(x) + \rho_\varphi^E(y).$$

Thus, ρ_φ^E is not a modular.

(*iii*) If $E = L_1$ and $\varphi(u) = \arctan(u)$ for $u \geq 0$ (see Definition 4.5), then ρ_φ^E is a modular (see [26]). Now, we will show that ρ_φ^E is not a quasimodular, more precisely, ρ_φ^E does not satisfy condition (v) of Definition 3.1. Let $p > 0$ and take $A = \pi/2$ and $\varepsilon = \pi/8$. Then, for $x_n = n\chi_{[0,1)}$ and $a_n = 1/n$, we obtain $\rho_\varphi^E(x_n) \leq \pi/2$, $\rho_\varphi^E(a_n x_n) = \pi/4$, and $\lim_{n\to\infty}(a_n)^p = 0$. In consequence, for any $K \geq 1$ there exists $n \in \mathbb{N}$ such that $\rho_\varphi^E(a_n x_n) > K a_n^p \rho_\varphi^E(x_n) + \pi/8$.

(*iv*) Let $E = L_1$, $\varphi(u) = u$ for $u \in [0,1]$ and $\varphi(u) = \max(1, u-1)$ for $u > 1$. For $x = \chi_{[0,1)}$ we have $\rho_\varphi^E(x) = \rho_\varphi^E(2x) = 1$. Simultaneously, $\rho_\varphi^E(x/\lambda) > 1$ for $\lambda < 1/2$, so $\|x\|_\varphi = \frac{1}{2}$ (see Remark 3.6).

Recall the notion of uniform monotonicity (see, for example, [3, 9, 22]) that plays an important role in the theory of Banach lattices. A quasi-Banach lattice $(E, \|\cdot\|_E)$ is said to be *uniformly monotone* provided for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in E_+$ with $\|x\|_E = 1$ we have $\|x + y\|_E \geq 1 + \delta$ whenever $\|y\|_E \geq \varepsilon$.

Lemma 4.16 *If E is uniformly monotone, then for each $\varepsilon_1 > 0$ and $A > 0$ there exists $\delta_1 = \delta_1(\varepsilon_1, A) = \delta(\frac{\varepsilon_1}{A}) > 0$ (here the function $\delta(\cdot)$ comes from the definition of the uniform monotonicity) such that for all $x, y \in E_+$ we have $\|x + y\|_E \geq \|x\|_E(1 + \delta_1)$ whenever $\|y\|_E \geq \varepsilon_1$ and $\|x\|_E \leq A$.*

Proof The proof can be found in [9] (the same for the quasinorm). However, we will need the precise form of the function $\delta_1(\varepsilon_1, A)$, so we present the proof for the reader's convenience.

Let $\varepsilon_1 > 0$ and $A > 0$. Take $x, y \in E_+$ such that $\|y\|_E \geq \varepsilon_1$ and $\|x\|_E \leq A$. Denote

$$\widetilde{x} = \frac{x}{\|x\|_E} \quad \text{and} \quad \widetilde{y} = \frac{y}{\|x\|_E}.$$

Then, $\|\widetilde{x}\|_E = 1$ and $\|\widetilde{y}\|_E \geq \frac{\varepsilon_1}{A}$. By the uniform monotonicity of E, there exists $\delta_1 = \delta_1(\varepsilon_1, A) = \delta(\frac{\varepsilon_1}{A}) > 0$ such that $\|\widetilde{x} + \widetilde{y}\|_E \geq 1 + \delta_1$, whence

$$\|x + y\|_E \geq \|x\|_E(1 + \delta_1). \qquad \square$$

Recall that the assumption $\alpha_\varphi^E > 0$ is important to show that ρ_φ^E is a quasimodular (see Theorem 4.9 and also Theorem 4.10) and, consequently, that $\|\cdot\|_\varphi$ is a quasinorm. However, if we know nothing about the index α_φ^E then we may still define a functional

$$\rho_\varphi^E(x) := \begin{cases} \|\varphi(|x|)\|_E & \text{if } \varphi(|x|) \in E, \\ \infty & \text{otherwise} \end{cases}$$

and the set

$$E_\varphi = \left\{x \in L^0 : \lim_{\lambda \to 0} \rho_\varphi^E(\lambda x) = 0\right\}.$$

Then, it is easy to see that E_φ is a linear space and we may consider the functional

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : \rho_\varphi^E(x/\lambda) \leq 1\right\}, \quad \text{for } x \in E_\varphi,$$

which satisfies the conditions (*i*) and (*ii*) of the quasinorm definition. We are going to show that, under some natural assumptions, the condition $\alpha_\varphi^E > 0$ can be even necessary, that is to say, if the functional $\|\cdot\|_\varphi$ is a quasinorm, then $\alpha_\varphi^E > 0$. Since the result below is only some illustration of how natural is the assumption that $\alpha_\varphi^E > 0$, we will limit ourselves only to one case of the ideal space E.

Theorem 4.17 *Suppose φ is a finitely valued, strictly increasing Orlicz function. Let $(E, \|\cdot\|_E)$ be a uniformly monotone, p-normed ideal space over nonatomic measure space (T, Σ, μ) for some $0 < p \leq 1$. Assume that neither $L_\infty \subset E$ nor $E \subset L_\infty$. If $(E_\varphi, \|\cdot\|_\varphi)$ is a quasinormed space, then $\alpha_\varphi^a > 0$.*

Proof Denote by $C \geq 1$ the constant from the quasitriangle inequality for $(E_\varphi, \|\cdot\|_\varphi)$. Let $\delta_0 = \delta(1/2)$ be the constant from the definition of the uniform monotonicity of E. Fix $s > 0$. Recall also that if E is uniformly monotone, then E is order continuous (see [22, Proposition 2.4]). Thus, by the proof of Theorem 2.4 in [14], the function ν, defined by $\nu(A) = \|\chi_A\|_E$ for all $A \in \Sigma$, $\chi_A \in E$, is the submeasure in the sense of [5, Definition 1], whence by [5, Theorem 10], ν has the Darboux property. In consequence, we can find a set $A \in \Sigma$, $\chi_A \in E$ satisfying

$$\varphi(s) = \frac{1}{\|\chi_A\|_E(1 + \delta_0)}.$$

Take a set $B \in \Sigma$ of positive measure such that $\chi_B \in E$, $A \cap B = \emptyset$ and $\|\chi_B\|_E = \frac{1}{2}\|\chi_A\|_E$. Applying Lemma 4.16 we conclude that

$$\|\chi_{A\cup B}\|_E \geq \|\chi_A\|_E(1 + \delta_0). \tag{4.6}$$

It is well known that

$$\|\chi_A\|_\varphi = \frac{1}{\varphi^{-1}(\frac{1}{\|\chi_A\|_E})},$$

where φ^{-1} is the general right-inverse to φ. Indeed, $\|\varphi(\frac{\chi_A}{\lambda})\|_E \le 1$ if and only if $\frac{1}{\lambda} \le \varphi^{-1}(\frac{1}{\|\chi_A\|_E})$. In consequence,

$$C \ge \frac{\|\chi_A + \chi_B\|_\varphi}{\|\chi_A\|_\varphi + \|\chi_B\|_\varphi} \ge \frac{\|\chi_A + \chi_B\|_\varphi}{2\|\chi_A\|_\varphi} = \frac{\|\chi_{A\cup B}\|_\varphi}{2\|\chi_A\|_\varphi} = \frac{\varphi^{-1}(\frac{1}{\|\chi_A\|_E})}{2\varphi^{-1}(\frac{1}{\|\chi_{A\cup B}\|_E})}.$$

Moreover, also applying (4.6), we obtain

$$2C\varphi^{-1}\left(\frac{1}{\|\chi_A\|_E(1+\delta_0)}\right) \ge 2C\varphi^{-1}\left(\frac{1}{\|\chi_{A\cup B}\|_E}\right) \ge \varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)$$

and consequently we obtain

$$\varphi\left[2C\varphi^{-1}\left(\frac{1}{\|\chi_A\|_E(1+\delta_0)}\right)\right] \ge \varphi\left[\varphi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)\right] = \frac{1}{\|\chi_A\|_E}.$$

Taking $C_1 = 2C$, we have

$$(1+\delta_0)\varphi(s) \le \varphi\left[2C\varphi^{-1}\big(\varphi(s)\big)\right] = \varphi[2Cs] = \varphi[C_1 s]$$

for each $s > 0$. For every $a \ge 1$ there is $m \in \mathbb{N}$ such that $C_1^{m-1} \le a < C_1^m$. Fix $p > 0$ satisfying $p = \frac{\ln(1+\delta_0)}{\ln C_1}$. Then, $(1+\delta_0)^{m-1} = (C_1^{m-1})^p$ and

$$\varphi(as) \ge \varphi\big(C_1^{m-1}s\big) \ge (1+\delta_0)^{m-1}\varphi(s) = \big(C_1^{m-1}\big)^p\varphi(s) \ge \left(\frac{a}{C_1}\right)^p \varphi(s) = a^p C_1^{-p}\varphi(s)$$

for each $s > 0$. Setting $u := as$ and $b := 1/a$ we conclude that

$$\varphi(bu) \le b^p C_1^p \varphi(u)$$

for any $u > 0$ and each $b \in (0,1]$. This means that $\alpha_\varphi^a > 0$. □

5 The quasinormed Orlicz–Lorentz spaces and Orlicz spaces

Take $I = [0,1]$ or $I = [0,\infty)$ with the Lebesgue measure μ. Let $\omega : I \to \mathbb{R}_+$ be a measurable function with $\int_0^t \omega(s)\,ds < \infty$ for each $t \in I$. We assume that there is a constant $C > 0$ such that $\int_0^{2t} \omega(s)\,ds \le C\int_0^t \omega(s)\,ds$ for each $t \in \frac{1}{2}I$, which implies that the space

$$\Lambda_{1,\omega} = \left\{f \in L^0 : \|f\|_\omega = \int_I f^*(s)\omega(s)\,ds < \infty\right\},$$

where f^* is the nonincreasing rearrangement of f – see [2, 23], is the quasinormed ideal space with the Fatou property (see [13]) and it is called the Lorentz funtion space $\Lambda_{1,\omega}$. The Lorentz sequence space $\lambda_{1,\omega}$ over the counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$ we define analogously (see [15]).

Note that,

(1) neither $L_\infty \subset \Lambda_{1,\omega}$ nor $\Lambda_{1,\omega} \subset L_\infty$, whenever $I = [0,\infty)$ and $\int_0^\infty \omega(s)\,ds = \infty$;

(2) $L_\infty \subset \Lambda_{1,\omega}$,whenever $I = [0,1]$ or ($I = [0,\infty)$ and $\int_0^\infty \omega(s)\,ds < \infty$);

(3) $\lambda_{1,\omega} \subset l_\infty$ (furthermore, $\lambda_{1,\omega} = l_\infty$ provided $\sum_{i=1}^\infty \omega(i) < \infty$).

If E is a Lorentz function (sequence) space $\Lambda_{1,w}$ ($\lambda_{1,w}$), then Calderón–Lozanovskiĭ space E_φ is the corresponding Orlicz–Lorentz function (sequence) space $\Lambda_{\varphi,w}$ ($\lambda_{\varphi,w}$). If $E = L_1$ ($E = l_1$), then the space E_φ becomes the Orlicz function (sequence) space L_φ (l_φ) (cf. [16]). On the other hand, if $\varphi(u) = u^p$, $1 \le p < \infty$ $[0 < p < 1]$ then E_φ is the p-convexification [concavification] $E^{(p)}$ of E with the quasinorm $\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}$. If $\varphi(u) = 0$ for $0 \le u \le 1$ and $\varphi(u) = \infty$ for $u > 1$, then $E_\varphi = L_\infty$ ($E_\varphi = l_\infty$) with equality of the norms.

It is well known that Orlicz–Lorentz spaces $\Lambda_{\varphi,\omega}$ (in particular, the Lorentz spaces $\Lambda_{p,\omega}$ or the Orlicz spaces L_φ) have been studied directly by many authors (see, for example, [13–16] and the references therein).

Applying Theorems 3.4 and 4.9 with $E = \Lambda_{1,w}$ or $E = \lambda_{1,w}$ we conclude immediately:

Corollary 5.1 (*i*) *Let* $E = \Lambda_{1,w}(I,\Sigma,\mu)$ *be such that* $\mu(I) < \infty$ *or* ($\mu(I) = \infty$ *and* $\int_0^\infty \omega(s)\,ds < \infty$). *If* $\alpha_\varphi^\infty > 0$, *then the functional* $\rho_\varphi^{\Lambda_{1,w}}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_{\varphi,w}$ *is a quasinorm* (*called a Luxemburg–Nakano quasinorm*) *in* $\Lambda_{\varphi,w}$.

(*ii*) *Let* $E = \Lambda_{1,w}(I,\Sigma,\mu)$ *be such that* $\mu(I) = \infty$ *and* $\int_0^\infty \omega(s)\,ds = \infty$. *If* $\alpha_\varphi^a > 0$, *then the functional* $\rho_\varphi^{\Lambda_{1,w}}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_{\varphi,w}$ *is a quasinorm in* $\Lambda_{\varphi,w}$.

(*iii*) *Let* $E = \lambda_{1,w}$ *and* $\sum_{i=1}^\infty \omega(i) = \infty$. *If* $\alpha_\varphi^0 > 0$, *then the functional* $\rho_\varphi^{\lambda_{1,w}}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_{\varphi,w}$ *is a quasinorm in* $\lambda_{\varphi,w}$.

Remark 5.2 The second conclusion in statement (*iii*) has been proved directly in [15, Proposition 1.3], but the authors did not consider the quasimodular, only the quasinorm.

Applying the above corollary with $\omega \equiv 1$ and Theorem 4.17 with $E = L^1$ we obtain:

Corollary 5.3 (*i*) *Let* $E = L_1(I,\Sigma,\mu)$ *with a finite measure* μ. *If* $\alpha_\varphi^\infty > 0$, *then the functional* $\rho_\varphi^{L_1}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_\varphi$ *is a quasinorm* (*called a Luxemburg–Nakano quasinorm*) *in* L_φ.

(*ii*) *Let* $E = L_1(I,\Sigma,\mu)$ *with an infinite measure* μ. *If* $\alpha_\varphi^a > 0$, *then the functional* $\rho_\varphi^{L_1}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_\varphi$ *is a quasinorm in* L_φ.

(*iii*) *Let* $E = L_1(I,\Sigma,\mu)$ *with an infinite measure* μ. *Assume that the function* φ *is finitely valued and strictly increasing. Then, the functional* $\|\cdot\|_\varphi$ *is a quasinorm in* L_φ *if and only if* $\alpha_\varphi^a > 0$.

(*iv*) *If* $E = l_1$ *and* $\alpha_\varphi^0 > 0$, *then the functional* $\rho_\varphi^{l_1}(\cdot)$ *is a quasimodular and the functional* $\|\cdot\|_\varphi$ *is a quasinorm in* l_φ.

Remark 5.4 The statement (*iii*) has been proved directly in [16, Theorem 1.8].

6 Further research and open ends

6.1 Further research

It is well known that the relations between the modular and the norm play a crucial role in the metric geometry of normed Orlicz spaces (normed Calderón–Lozanovskiĭ spaces). The following basic results have many applications:

Lemma 6.1 *Let E be a Banach ideal space and φ be a convex Orlicz function. Then,*
(i) *if $\rho_\varphi^E(x_n) \to 1$, then $\|x_n\|_\varphi \to 1$ for all sequences (x_n) in E_φ. In particular,*
(ii) *if $\rho_\varphi^E(x) = 1$, then $\|x\|_\varphi = 1$ for every $x \in E_\varphi$.*
(iii) *if $\|x_n\|_\varphi \to 0$, then $\rho_\varphi^E(x_n) \to 0$ for all sequences (x_n) in E_φ.*
Suppose additionally that $\varphi \in \Delta_2^E$. Then,
(iv) *if $\|x_n\|_\varphi \to 1$, then $\rho_\varphi^E(x_n) \to 1$ for all sequences (x_n) in E_φ. In particular,*
(v) *if $\|x\|_\varphi = 1$, then $\rho_\varphi^E(x) = 1$ for each $x \in E_\varphi$.*
Assume additionally that $\varphi \in \Delta_2^E$ and $a_\varphi = 0$. Then,
(vi) *if $\rho_\varphi^E(x_n) \to 0$, then $\|x_n\|_\varphi \to 0$ for all sequences (x_n) in E_φ.*

The known proofs of properties (i)–(vi) cannot be applied for the nonconvex Orlicz function φ. In [7] and [8] we presented new proofs of the above lemma for the quasimodular and the quasinorm. We have shown that for properties (i) and (ii) we need additionally the condition Δ_ε^E, which is a substitute for convexity. Moreover, to show the properties (iv) and (v) the so-called condition Δ_{2-str}^E is required (the condition Δ_{2-str}^E is essentially stronger than Δ_2^E, in general). Finally, the condition (iii) comes from Lemma 3.7 and the condition (vi) is true in the same form as above (see [8]). Next, applying also some new techniques, we described order isomorphic and order linearly isometric copies of l^∞ in the quasinormed Calderón–Lozanovskiĭ spaces E_φ (a number of theorems describe these copies in the natural language of suitable properties of the quasinormed ideal space E and the nondecreasing Orlicz function φ – see [7]). In [8] we characterized the monotonicity properties of quasinormed Calderón–Lozanovskiĭ spaces E_φ, that is, we described the strict monotonicity, the uniform monotonicity, and the respective orthogonal counterparts of the quasinormed Calderón–Lozanovskiĭ spaces E_φ.

6.2 Open ends

The theory of modular spaces has been widely investigated in [26], see also [24]. Next, the modular spaces equipped with the additional measure structure (called modular function spaces) has been studied in [21]. Furthermore, the authors of [18] considered the modular function spaces from the geometry and the fixed-point theory point of view. All the monographs deal with the modulars (convex modulars) that lead to the F-normed (normed) spaces, respectively. It seems natural to study some aspects of the theory developed in the monographs [18] and [21] in the context of quasimodulars.

Author contributions

HH initiated the research. PF and PK contributed equally and significantly in this manuscript, and they read and approved the final version.

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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References

1. Bastero, J., Hudzik, H., Steinberg, A.M.: On smallest and largest spaces among rearrangement-invariant p-Banach function spaces ($0 < p < 1$). Indag. Math. **2**, 283–288 (1991)
2. Bennett, C., Sharpley, R.: Interpolation of Operators. Pure and Applied Mathematics, vol. 129. Academic Press, Boston (1988)
3. Birkhoff, G.: Lattice Theory, 3rd edn. American Mathematical Society Colloquium Publications, vol. XXV. Am. Math. Soc., Providence (1967)
4. Cui, Y., Hudzik, H., Kaczmarek, R., Kolwicz, P.: Uniform monotonicity of Orlicz spaces equipped with the Mazur-Orlicz F-norm and dominated best approximation in F-normed Köthe spaces. Math. Nachr. **295**, 487–511 (2022)
5. Dobrakov, I.: On submeasures. I. Diss. Math. **112**, 35 (1974)
6. Foralewski, P., Hudzik, H.: Some basic properties of generalized Calderon-Lozanovskiĭ spaces. Collect. Math. **48**, 523–538 (1997)
7. Foralewski, P., Hudzik, H., Kolwicz, P.: Copies of l^∞ in quasi-normed Calderón–Lozanovskiĭ spaces. Submitted
8. Foralewski, P., Kolwicz, P.: Monotonicities of quasi-normed Calderón–Lozanovskiĭ spaces with applications to approximation problems. Submitted
9. Hudzik, H., Kamińska, A., Mastyło, M.: Monotonicity and rotundity properties in Banach lattices. Rocky Mt. J. Math. **30**, 933–950 (2000)
10. Kalton, N.J.: Convexity conditions for nonlocally convex lattices. Glasg. Math. J. **25**, 141–152 (1984)
11. Kalton, N.J.: Quasi-Banach spaces. In: Handbook of the Geometry of Banach Spaces, vol. 2, pp. 1099–1130. North-Holland, Amsterdam (2003)
12. Kalton, N.J., Peck, N.T., Roberts, J.W.: An *F*-Space Sampler. London Mathematical Society Lecture Note Series, vol. 89. Cambridge University Press, Cambridge (1984)
13. Kamińska, A., Maligranda, L.: Order convexity and concavity of Lorentz spaces $\Lambda_{p,w}, 0 < p < \infty$. Stud. Math. **160**, 267–286 (2004)
14. Kamińska, A., Maligranda, L., Persson, L.E.: Indices, convexity and concavity of Calderón-Lozanovskii spaces. Math. Scand. **92**, 141–160 (2003)
15. Kamińska, A., Raynaud, Y.: Copies of l_p and c_0 in General Quasi-Normed Orlicz-Lorentz Sequence Spaces, Function Spaces. Contemp. Math., vol. 435, pp. 207–227. Am. Math. Soc., Providence (2007)
16. Kamińska, A., Żyluk, M.: Local geometric properties in quasi-normed Orlicz spaces. Banach Cent. Publ. **119**, 197–221 (2019)
17. Kantorovich, L.V., Akilov, G.P.: Functional Analysis, 2nd edn. Pergamon Press, Oxford-Elmsford (1982). Translated from the Russian by, Silcock, Howard L.
18. Khamsi, M.A., Kozlowski, W.M.: Fixed Point Theory in Modular Function Spaces. Springer, Cham (2015). x+245
19. Kiwerski, T., Kolwicz, P., Maligranda, L.: Isomorphic and isometric structure of the optimal domains for Hardy-type operators. Stud. Math. **260**, 45–89 (2021)
20. Kolwicz, P., Leśnik, K., Maligranda, L.: Pointwise multipliers of Calderón-Lozanovskiĭ spaces. Math. Nachr. **286**, 876–907 (2013)
21. Kozlowski, W.M.: Modular Function Spaces. Monographs and Textbooks in Pure and Applied Mathematics, vol. 122. Marcel Dekker, Inc., New York (1988). x+252
22. Lee, H.J.: Complex convexity and monotonicity in quasi-Banach lattices. Isr. J. Math. **159**, 57–91 (2007)
23. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces. II, Function Spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97. Springer, Berlin (1979)
24. Maligranda, L.: Orlicz Spaces and Interpolation. Seminários de Matemática [Seminars in Mathematics], vol. 5. Universidade Estadual de Campinas, Departamento de Matemática, Campinas (1989)
25. Maligranda, L.: Type, cotype and convexity properties of quasi-Banach spaces. In: International Symposium on Banach and Function Spaces, Kitakyushu, 2003, Banach and Function Spaces, pp. 83–120. Yokohama Publ., Yokohama (2004)
26. Musielak, J.: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
27. Raynaud, Y., Tradacete, P.: Calderón-Lozanovskii interpolation on quasi-Banach lattices. Banach J. Math. Anal. **12**, 294–313 (2018)
28. Wnuk, W.: Banach Lattices with Order Continuous Norms. Advanced Topics in Mathematics. Polish Scientific Publishers PWN, Warszawa (1999)

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