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Ruscheweyh-type meromorphic harmonic functions

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Abstract
In this paper, we study classes of meromorphic harmonic functions defined by Ruscheweyh derivatives. In addition to finding certain analytic criteria, we obtain radii of starlikeness and convexity, and some topological properties for the defined classes of functions. Some applications of these results are also given.

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1 Introduction

A complex-valued function f is said to be harmonic in a domain $D \subset \mathbb{C}$ if it has continuous second-order partial derivatives in D that satisfy the Laplace equation

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

If $D = \mathbb{U}(r) := \{z \in \mathbb{C} : 0 < |z| < r\}$, then we say that f is a meromorphic harmonic function in $\mathbb{U}(r)$. We denote by $\mathcal{M}$ the class of all such function with the normalization $f(0) = \infty$

Let a function F be harmonic, orientation-preserving, and univalent in $\mathbb{B} := \{z \in \mathbb{C} : |z| > 1\}$ with $F(\infty) = \infty$. Then, there exists $B \in \mathbb{C}$ and functions

$$\varphi(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \qquad \psi(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \quad \left(0 \le |\alpha| < |\beta|, z \in \mathbb{B}\right),$$

such that

$$F(z) = \varphi(z) + \overline{\psi(z)} + B \log|z|, \quad (z \in \mathbb{B})$$

where $\overline{F_{\bar{z}}}/F_z$ is analytic and bounded by 1 in $\mathbb{E}$ (see, Hengartner and Schober [10]).

Let $f \in \mathcal{M}$ be functions that are univalent and sense-preserving in $\mathbb{U} := \mathbb{U}(1)$. Since the composition of an analytic and harmonic function is the harmonic function, the function

$F = f \circ (\frac{1}{z})$ is orientation-preserving, harmonic, and univalent in $\mathbb{E}$ with $F(\infty) = \infty$. Thus, there exists $B \in \mathbb{C}$ and the functions $h(z) := \varphi(\frac{1}{z})$, $g(z) := \psi(\frac{1}{z})$ such that

$$f(z) = h(z) + \overline{g(z)} - B \log|z| \quad (z \in \mathbb{U}).$$

Let $k \in \mathbb{N} := \mathbb{N}_1$, where $\mathbb{N}_m := \{m, m+1, \ldots\}$. We denote by $\mathcal{M}_{\mathcal{H}}(k)$ the class of functions $f \in \mathcal{M}$ of the form

$$f = h + \overline{g}, \qquad h(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=k}^{\infty} b_n z^n \quad (z \in \mathbb{U}), \tag{1}$$

which are sense-preserving and univalent in $\mathbb{U}$, and let $\mathcal{M}_{\mathcal{H}} := \mathcal{M}_{\mathcal{H}}(1)$.

Recently, classes of meromorphic harmonic functions were intensively studied (see for example [1–11]).

A function $f \in \mathcal{M}_{\mathcal{H}}(k)$ is called meromorphic harmonic starlike in $\mathbb{U}(r)$ if f maps $\partial\mathbb{U}(r)$ onto a curve that is starlike with respect to the origin, *i.e.,*

$$\frac{\partial}{\partial t}\left(\arg f\left(re^{it}\right)\right) < 0 \quad (0 \le t \le 2\pi) \tag{2}$$

or equivalently

$$\operatorname{Re}\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} < 0 \quad \left(|z| = r\right),$$

where

$$\mathcal{D}_{\mathcal{H}} f(z) := -zh'(z) + \overline{zg'(z)} \quad (z \in \mathbb{U}).$$

Let φ and Φ be complex-valued functions in $\mathbb{U}$. If $\varphi(\mathbb{U}) \subset \Phi(\mathbb{U})$, then we say that φ is *weakly subordinate* to Φ, and we write $\varphi(z) \preceq \Phi(z)$ (see Muir [16]).

For functions

$$f_l(z) = \sum_{n=-1}^{\infty} \left(a_{l,n} z^n + \overline{b_{l,n} z^n}\right) \quad (z \in \mathbb{U}, l = 1, 2,)$$

we define the convolution of functions f_1 and f_2 by

$$(f_1 * f_2)(z) = \sum_{n=-1}^{\infty} \left(a_{1,n} a_{2,n} z^n + \overline{b_{1,n} b_{2,n} z^n}\right) \quad (z \in \mathbb{U}).$$

In [17] Ruscheweyh introduced an operator $\mathcal{D}^\lambda$ defined on the class of analytic functions by

$$\mathcal{D}^\lambda g(z) := g(z) * \frac{z}{(z-1)^\lambda} = \frac{z(z^{\lambda-1} g(z))^{(\lambda)}}{\lambda!} \quad (\lambda \in \mathbb{N}_0, z \in \mathbb{U}).$$

Now, we define the Ruscheweyh derivative $\mathcal{D}^\lambda$ on the class of meromorphic harmonic functions. Let $\mathcal{D}_{\mathcal{H}}^\lambda : \mathcal{M}_{\mathcal{H}}(k) \to \mathcal{M}_{\mathcal{H}}(k)$ denote the operator defined for a function $f =$

$h+\overline{g}\in\mathcal{M}_{\mathcal{H}}(k)$ by

$$\begin{aligned}\mathcal{D}^{\lambda}_{\mathcal{H}}f(z) &:= \frac{1}{z}+(-1)^{\lambda}\mathcal{D}^{\lambda}\left(h(z)-\frac{1}{z}\right)+\overline{\mathcal{D}^{\lambda}g(z)}\\ &= \frac{1}{z}+\left(f(z)-\frac{1}{z}\right)*\left(\frac{z}{(z-1)^{\lambda}}+\frac{\overline{z}}{(1-\overline{z})^{\lambda}}\right)\\ &= \frac{1}{z}+(-1)^{\lambda}\sum_{n=k}^{\infty}\lambda_n a_n z^n+\sum_{n=k}^{\infty}\lambda_n\overline{b_n}\overline{z}^n \quad (z\in\mathbb{U}),\end{aligned}$$

where

$$\lambda_1=1,\qquad \lambda_n:=\frac{(\lambda+1)\cdot\ldots\cdot(\lambda+n-1)}{(n-1)!}\quad (n=2,3,\ldots). \tag{3}$$

It is clear that $\mathcal{D}^0_{\mathcal{H}}f=f$ and $\mathcal{D}^1_{\mathcal{H}}f=\mathcal{D}_{\mathcal{H}}f$.

Due to Janowski [13] (see also [9]) we define the class $\mathcal{M}^{\lambda}_{\mathcal{H}}(k;M,N)$ of functions $f\in\mathcal{M}_{\mathcal{H}}(k)$ that satisfy the following condition

$$\frac{\mathcal{D}^{\lambda+1}_{\mathcal{H}}f(z)}{\mathcal{D}^{\lambda}_{\mathcal{H}}f(z)}\preceq\frac{1+Mz}{1+Nz},\quad -N\le M<N\le 1. \tag{4}$$

By $\mathcal{W}^{\lambda}_{\mathcal{H}}(k;M,N)$ we denote the class of functions $f\in\mathcal{M}_{\mathcal{H}}(k)$ such that

$$z\mathcal{D}^{\lambda}_{\mathcal{H}}f(z)\preceq\frac{1+Mz}{1+Nz},\quad -N\le M<N\le 1.$$

Moreover, let us denote

$$\begin{aligned}&\mathcal{M}^{*}_{\mathcal{H}}(k;M,N):=\mathcal{M}^{0}_{\mathcal{H}}(k;M,N),\qquad \mathcal{M}^{c}_{\mathcal{H}}(k;M,N):=\mathcal{M}^{1}_{\mathcal{H}}(k;M,N),\\ &\mathcal{M}^{*}_{\mathcal{H}}(\alpha):=\mathcal{M}^{*}_{\mathcal{H}}(1,2\alpha-1,1),\qquad \mathcal{M}^{c}_{\mathcal{H}}(\alpha):=\mathcal{M}^{c}_{\mathcal{H}}(1,2\alpha-1,1).\end{aligned}$$

The classes $\mathcal{M}^{*}_{\mathcal{H}}:=\mathcal{M}^{*}_{\mathcal{H}}(0)$ and $\mathcal{M}^{c}_{\mathcal{H}}:=\mathcal{M}^{c}_{\mathcal{H}}(0)$ were studied in [3] (see also [9]). We see that the function $f\in\mathcal{M}^{*}_{\mathcal{H}}$ is starlike in $\mathbb{U}(r)$ for all $r\in\langle 0,1\rangle$.

In this paper, we obtain some necessary and sufficient conditions for the defined classes of functions. In addition to finding certain analytic criteria, we obtain radii of starlikeness and convexity, and some topological properties for the defined classes of functions. Some applications of these results are also given.

2 Analytic criteria

To obtain the main results we need the following lemma.

Lemma 1 [8] *A complex-valued function φ in $\mathbb{U}$ is weakly subordinate to a complex-valued univalent function Φ in $\mathbb{U}$ if and only if there exists a complex-valued function ω that maps $\mathbb{U}$ into oneself such that $\varphi(z)=\Phi(\omega(z))$, $z\in\mathbb{U}$.*

Theorem 1 *Let $f\in\mathcal{M}$ be of the form* (1) *and*

$$c_n=\lambda_n\big\{n(1+N)+(1+M)\big\},\qquad d_n=\lambda_n\big\{n(1+N)-(1+M)\big\}. \tag{5}$$

Then, $f \in \mathcal{M}^{\lambda}_{\mathcal{H}}(k;M,N)$ *if the condition*

$$\sum_{n=k}^{\infty}\big(c_n|a_n| + d_n|b_n|\big) \le N - M \tag{6}$$

holds true.

Proof It is easy to verify that

$$\frac{c_n}{N-M} \ge n, \qquad \frac{d_n}{N-M} \ge n \quad (n \in \mathbb{N}_k).$$

Thus, by (6) we have

$$\sum_{n=k}^{\infty}\big(n|a_n| + n|b_n|\big) \le 1. \tag{7}$$

It is well known that the Jacobian of f is given by

$$J_f(z) = \big|h'(z)\big|^2 - \big|g'(z)\big|^2 \quad (z \in \mathbb{U}).$$

A function f is locally univalent and sense-preserving if the Jacobian of f is positive in $\mathbb{U}$. Lewy [15] proved that the converse is true for harmonic mappings. Since

$$\begin{aligned}
\big|z^2 J_f(z)\big| &= \big|z^2 h'(z)\big| - \big|z^2 g'(z)\big| \\
&\ge 1 - \sum_{n=k}^{\infty} n|a_n||z|^{n+2} - \sum_{n=k}^{\infty} n|b_n||z|^{n+2} \\
&\ge 1 - |z|\sum_{n=k}^{\infty}\big(n|a_n| + n|b_n|\big) \ge 1 - |z| > 0 \quad (z \in \mathbb{U}),
\end{aligned}$$

we have that f is locally univalent and sense-preserving in $\mathbb{U}$. To obtain univalence we assume that $w_1, w_2 \in \mathbb{U}$, $w_1 \ne w_2$. Then,

$$\left|\frac{w_1^n - w_2^n}{w_1 - w_2}\right| = \left|\sum_{l=1}^{n} w_1^{l-1} w_2^{n-l}\right| \le \sum_{l=1}^{n} |w_1|^{l-1}|w_2|^{n-l} \le n \quad (n \in \mathbb{N})$$

and by (7) we obtain

$$\begin{aligned}
\big|f(w_1) - f(w_2)\big| &\ge \big|h(w_1) - h(w_2)\big| - \big|g(w_1) - g(w_2)\big| \\
&= \left|\frac{1}{w_1} - \frac{1}{w_2} - \sum_{n=k}^{\infty} a_n\big(w_1^n - w_2^n\big)\right| - \left|\sum_{n=k}^{\infty}\overline{b_n\big(w_1^n - w_2^n\big)}\right| \\
&\ge \frac{|w_1 - w_2|}{|w_1 w_2|} - \sum_{n=k}^{\infty}|a_n|\big|w_1^n - w_2^n\big| - \sum_{n=k}^{\infty}|b_n|\big|w_1^n - w_2^n\big| \\
&= |w_1 - w_2|\left(\frac{1}{|w_1 w_2|} - \sum_{n=k}^{\infty}|a_n|\left|\frac{w_1^n - w_2^n}{w_1 - w_2}\right| - \sum_{n=k}^{\infty}|b_n|\left|\frac{w_1^n - w_2^n}{w_1 - w_2}\right|\right)
\end{aligned}$$

$$> |w_1 - w_2|\left(1 - \sum_{n=k}^{\infty} n|a_n| - \sum_{n=k}^{\infty} n|b_n|\right) \geq 0.$$

Thus, $f \in \mathcal{M}_{\mathcal{H}}(k)$ and by Lemma 1 we obtain that $f \in \mathcal{M}^*_{\mathcal{H}}(k;M,N)$ if and only if there exists a complex-valued function χ bounded by 1 in $\mathbb{U}$ for which

$$\frac{\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z)}{\mathcal{D}^{\lambda}_{\mathcal{H}} f(z)} = \frac{1 + M\chi(z)}{1 + N\chi(z)} \quad (z \in \mathbb{U}),$$

or equivalently

$$\left|\frac{\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - \mathcal{D}^{\lambda}_{\mathcal{H}} f(z)}{N\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - M\mathcal{D}^{\lambda}_{\mathcal{H}} f(z)(z)}\right| < 1 \quad (z \in \mathbb{U}). \tag{8}$$

Therefore, we need to show that

$$\left|\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - \mathcal{D}^{\lambda}_{\mathcal{H}} f(z)\right| - \left|N\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - \mathcal{D}^{\lambda}_{\mathcal{H}} f(z)\right| < 0 \quad (z \in \mathbb{U}).$$

Putting $|z| = r$ $(0 < r < 1)$ we obtain

$$\begin{aligned}
&\left|\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - \mathcal{D}^{\lambda}_{\mathcal{H}} f(z)\right| - \left|N\mathcal{D}^{\lambda+1}_{\mathcal{H}} f(z) - \mathcal{D}^{\lambda}_{\mathcal{H}} f(z)\right| \\
&\quad = \left|\sum_{n=k}^{\infty} (-1)^{\lambda} \lambda_n (n+1) a_n z^n - \sum_{n=k}^{\infty} \lambda_n (n-1) \overline{b_n} \overline{z}^n\right| \\
&\qquad - \left|(N-M)\frac{1}{z} - \sum_{n=k}^{\infty} (-1)^{\lambda} \lambda_n (Nn+M) a_n z^n + \sum_{n=k}^{\infty} \lambda_n (Nn-M) \overline{b_n} \overline{z}^n\right| \\
&\quad \leq \sum_{n=k}^{\infty} \lambda_n (n+1)|a_n| r^n + \sum_{n=k}^{\infty} \lambda_n (n-1)|b_n| r^n - (N-M)\frac{1}{r} \\
&\qquad + \sum_{n=k}^{\infty} \lambda_n (Nn+M)|a_n| r^n + \sum_{n=k}^{\infty} \lambda_n (Nn-M)|b_n| r^n \\
&\quad \leq \frac{1}{r}\left\{\sum_{n=k}^{\infty} \left(c_n|a_n| + d_n|b_n|\right) r^{n+1} - (N-M)\right\} < 0,
\end{aligned}$$

which implies $f \in \mathcal{M}^{\lambda}_{\mathcal{H}}(k;M,N)$. □

Let $\mathcal{T}^{\lambda}_{\eta}(k)$ be the class of functions $f = h + \overline{g} \in \mathcal{M}(k)$ with varying coefficients (e.g., see [12]) so that

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + (-1)^{\lambda} \sum_{n=k}^{\infty} e^{-i(n+1)\eta} |a_n| z^n - \sum_{n=k}^{\infty} e^{i(n-1)\eta} |b_n| \overline{z}^n \tag{9}$$

and let

$$\mathcal{M}^{\lambda}_{\eta}(k;M,N) := \mathcal{T}^{\lambda}_{\eta}(k) \cap \mathcal{M}^{\lambda}_{\mathcal{H}}(k;M,N), \qquad \mathcal{W}^{\lambda}_{\eta}(k;M,N) := \mathcal{T}^{\lambda}_{\eta}(k) \cap \mathcal{W}^{\lambda}_{\mathcal{H}}(k;M,N),$$

$$\mathcal{M}^{*}_{\eta}(k;M,N) := \mathcal{T}^{0}_{\eta}(k) \cap \mathcal{M}^{*}_{\mathcal{H}}(k;M,N), \qquad \mathcal{M}^{c}_{\eta}(k;M,N) := \mathcal{T}^{1}_{\eta}(k) \cap \mathcal{M}^{c}_{\mathcal{H}}(k;M,N).$$

The sufficient coefficient bound given in Theorem 1 is also necessary for functions to be in the class $\mathcal{M}_\eta^\lambda(k;M,N)$, as stated in the following theorem.

Theorem 2 *Let $f \in \mathcal{T}_\eta^\lambda$ be a function of the form (1). Then, $f \in \mathcal{M}_\eta^\lambda(k;M,N)$ if and only if the condition (6) holds true.*

Proof By Theorem 1 we need to prove the "*only if*" part. Let $f \in \mathcal{M}_\eta^*(k;M,N)$. Then, by (8) we obtain

$$\left|\frac{\sum_{n=k}^{\infty}\lambda_n\{(-1)^\lambda(n+1)a_n z^{n+1}-(n-1)\overline{b_n}z\overline{z}^n\}}{(N-M)-\sum_{n=k}^{\infty}\lambda_n\{(-1)^\lambda(Nn+M)a_n z^{n+1}-(Nn-M)\overline{b_n}\overline{z}\overline{z}^n\}}\right|<1 \quad (z\in\mathbb{U}).$$

Thus, by (9) for $z = re^{i\eta}$ $(0 < r < 1)$, we have

$$\frac{\sum_{n=k}^{\infty}\lambda_n\{(n+1)|a_n|+(n-1)|b_n|\}r^{n+1}}{(N-M)-\sum_{n=k}^{\infty}\lambda_n\{(Nn+M)|a_n|+(Nn-M)|b_n|\}r^{n+1}}<1. \tag{10}$$

The denominator of the left-hand side cannot vanish for $r \in (0,1)$. Also, it is positive for $r = 0$, and in consequence for $r \in (0,1)$. Thus, by (10) we have

$$\sum_{n=k}^{\infty}\big(c_n|a_n|+d_n|b_n|\big)r^{n+1}<N-M \quad (0<r<1). \tag{11}$$

The sequence of partial sums $\{S_n\}$ related to the series $\sum_{n=k}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)$ is a nondecreasing sequence. Moreover, by (11) it is bounded by $N - M$. Hence, the sequence $\{S_n\}$ is convergent and $\sum_{n=k}^{\infty}(\alpha_n|a_n|+\beta_n|b_n|)=\lim_{n\to\infty}S_n \le N-M$, which gives (6). □

Analogously as Theorem 2 we can prove the following theorem.

Theorem 3 *Let $f \in \mathcal{T}_\eta^\lambda(k)$ be a function of the form (1). Then, $f \in \mathcal{W}_\eta^\lambda(k;M,N)$ if and only if*

$$\sum_{n=k}^{\infty}\lambda_n\big(|a_n|+|b_n|\big)\le\frac{N-M}{1+N}. \tag{12}$$

By Theorems 2 and 3 we have the following corollary.

Corollary 1 *Let $a = \frac{1+M}{1+N}$ and*

$$\phi(z)=\frac{1}{z}+\sum_{n=k}^{\infty}\left(\frac{1}{n+a}z^n+\frac{1}{n-a}\overline{z}^n\right) \quad (z\in\mathbb{U}),$$

$$\omega(z)=\frac{1}{z}+\sum_{n=k}^{\infty}\big((n+a)z^n+(n-a)\overline{z}^n\big) \quad (z\in\mathbb{U}).$$

Then,

$$f\in\mathcal{W}_\eta^\lambda(k;M,N) \quad\Leftrightarrow\quad f*\phi\in\mathcal{M}_\eta^\lambda(k;M,N),$$

$$f \in \mathcal{M}_\eta^\lambda(k;M,N) \quad \Leftrightarrow \quad f * \omega \in \mathcal{W}_\eta^\lambda(k;M,N).$$

In particular,

$$\mathcal{W}_\eta^{\lambda+1}(-1,N) = \mathcal{M}_\eta^\lambda(-1,N).$$

Remark 1 If we put $n = 0$ or $n = 1$ in Theorems 1 and 2, then we obtain similar results for the classes $\mathcal{M}_\eta^*(k;M,N)$ and $\mathcal{M}_\eta^c(k;M,N)$.

3 Radii of convexity and starlikeness of order α

By using condition (2) we generalize the definition of starlikeness of meromorphic harmonic functions. We say that a function $f \in \mathcal{T}_\eta^\lambda(k)$ is starlike of order α in $\mathbb{U}(r)$ if

$$\frac{\partial}{\partial t}\left(\arg f\left(\rho e^{it}\right)\right) < \alpha, \quad 0 < \rho < r < 1, 0 \le t \le 2\pi. \tag{13}$$

Also, a function $f \in \mathcal{T}_\eta^\lambda(k)$ is said to be convex of order α in $\mathbb{U}(r)$ if

$$\frac{\partial}{\partial t}\left(\arg \frac{\partial}{\partial t} f\left(\rho e^{it}\right)\right) < \alpha, \quad 0 < \rho < r < 1, 0 \le t \le 2\pi.$$

It is easy to verify that for a function $f \in \mathcal{T}_\eta^\lambda(k)$ the condition (13) is equivalent to the following

$$\operatorname{Re}\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} > \alpha \quad \left(z \in \mathbb{U}(r)\right)$$

or equivalently

$$\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - f(z)}{\mathcal{D}_{\mathcal{H}} f(z) - (2\alpha - 1) f(z)}\right| < 1 \quad \left(z \in \mathbb{U}(r)\right). \tag{14}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{T}_\eta^\lambda(k)$. We define the radius of starlikeness $R_\alpha^*(\mathcal{B})$ and the radius of convexity $R_\alpha^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$R_\alpha^*(\mathcal{B}) := \inf_{f \in \mathcal{B}}\left(\sup\left\{r \in (0,1] : f \text{ is starlike of order } \alpha \text{ in } \mathbb{U}(r)\right\}\right),$$

$$R_\alpha^c(\mathcal{B}) := \inf_{f \in \mathcal{B}}\left(\sup\left\{r \in (0,1] : f \text{ is convex of order } \alpha \text{ in } \mathbb{U}(r)\right\}\right).$$

Theorem 4

$$R_\alpha^*\left(\mathcal{M}_\eta^\lambda(k;M,N)\right) = \inf_{n \in \mathbb{N}_k}\left(\frac{1-\alpha}{N-M}\min\left\{\frac{c_n}{n+\alpha}, \frac{d_n}{n-\alpha}\right\}\right)^{\frac{1}{n+1}}, \tag{15}$$

where c_n and d_n are defined by (5).

Proof Let $f \in \mathcal{M}_\eta^\lambda(M,N)$ be of the form (1). Then, putting $|z| = r < 1$ we have

$$\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - f(z)}{\mathcal{D}_{\mathcal{H}} f(z) - (2\alpha - 1) f(z)}\right| = \left|\frac{\sum_{n=k}^\infty (n+1) a_n z^n - \sum_{n=k}^\infty (n-1)\overline{b_n \overline{z}^n}}{\frac{2(1-\alpha)}{z} - \sum_{n=k}^\infty (n+2\alpha-1) a_n z^n + \sum_{n=k}^\infty (n-2\alpha+1)\overline{b_n \overline{z}^n}}\right|$$

$$\le \frac{\sum_{n=k}^{\infty}((n+1)|a_n| + (n-1)|b_n|)r^{n+1}}{2(1-\alpha) - \sum_{n=k}^{\infty}((n+2\alpha-1)|a_n| + (n-2\alpha+1)|b_n|)r^{n+1}}.$$

Thus, the condition (14) is true if

$$\sum_{n=k}^{\infty}\left(\frac{n+\alpha}{1-\alpha}|a_n| + \frac{n-\alpha}{1-\alpha}|b_n|\right)r^{n+1} \le 1. \tag{16}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty}\left(\frac{c_n}{N-M}|a_n| + \frac{d_n}{N-M}|b_n|\right) \le 1, \tag{17}$$

where c_n and d_n are defined by (5). Thus, the condition (16) is true if

$$\frac{n+\alpha}{1-\alpha}r^{n+1} \le \frac{c_n}{N-M}, \qquad \frac{n-\alpha}{1-\alpha}r^{n+1} \le \frac{d_n}{N-M} \quad (n \in \mathbb{N}_k),$$

that is, if

$$r \le \left(\frac{1-\alpha}{N-M}\min\left\{\frac{c_n}{n+\alpha}, \frac{d_n}{n-\alpha}\right\}\right)^{\frac{1}{n+1}} \quad (n \in \mathbb{N}_k).$$

It follows that the function f is starlike of order α in the disk $\mathbb{U}(r^*)$, where

$$r^* := \inf_{n\in\mathbb{N}_k}\left(\frac{1-\alpha}{N-M}\min\left\{\frac{c_n}{n+\alpha}, \frac{d_n}{n-\alpha}\right\}\right)^{\frac{1}{n+1}}. \tag{18}$$

The radii of starlikeness $r^*(h_n)$, $r^*(g_n)$ of functions h_n, g_n $(n \in \mathbb{N})$ of the form

$$h_n(z) = \frac{1}{z} + \frac{(-1)^\lambda(N-M)}{c_n e^{i(n+1)\eta}}z^n, \qquad g_n(z) = \frac{1}{z} - \frac{N-M}{d_n e^{i(1-n)\eta}}\overline{z}^n \quad (n \in \mathbb{N}_k, z \in \mathbb{U})$$

are given by

$$r^*(h_n) = \left(\frac{1-\alpha}{n+\alpha}\frac{c_n}{N-M}\right)^{\frac{1}{n+1}}, \qquad r^*(g_n) = \left(\frac{1-\alpha}{n-\alpha}\frac{d_n}{N-M}\right)^{\frac{1}{n+1}}.$$

Therefore, the radius r^* given by (18) cannot be larger. Thus, we have (15). □

The following result may be proved in much the same way as Theorem 4.

Theorem 5 *Let c_n and d_n be defined by (5). Then,*

$$R_\alpha^c\left(\mathcal{M}_\eta^\lambda(k;M,N)\right) = \inf_{n\in\mathbb{N}_k}\left(\frac{1-\alpha}{n(N-M)}\min\left\{\frac{c_n}{n+\alpha}, \frac{d_n}{n-\alpha}\right\}\right)^{\frac{1}{n+1}}.$$

If we put $n = 0$ or $n = 1$ in Theorems 4 and 5 we obtain the following results.

Corollary 2

$$
\begin{aligned}
&R^*_\alpha\big(\mathcal{M}^*_\eta(k;M,N)\big)\\
&\quad = R^c_\alpha\big(\mathcal{M}^c_\eta(k;M,N)\big)\\
&\quad = \inf_{n\in\mathbb{N}_k}\left(\frac{1-\alpha}{N-M}\min\left\{\frac{n(1+N)+(1+M)}{n+\alpha},\frac{n(1+N)-(1+M)}{n-\alpha}\right\}\right)^{\frac{1}{n+1}},\\
&R^c_\alpha\big(\mathcal{M}^*_\eta(k;M,N)\big)\\
&\quad = \inf_{n\in\mathbb{N}_k}\left(\frac{1-\alpha}{n(N-M)}\min\left\{\frac{n(1+N)+(1+M)}{n+\alpha},\frac{n(1+N)-(1+M)}{n-\alpha}\right\}\right)^{\frac{1}{n+1}},\\
&R^*_\alpha\big(\mathcal{M}^c_\eta(k;M,N)\big)\\
&\quad = \inf_{n\in\mathbb{N}_k}\left(\frac{n(1-\alpha)}{N-M}\min\left\{\frac{n(1+N)+(1+M)}{n+\alpha},\frac{n(1+N)-(1+M)}{n-\alpha}\right\}\right)^{\frac{1}{n+1}}.
\end{aligned}
$$

4 Topological properties

Let us consider the usual topology on $\mathcal{M}_{\mathcal{H}}(k)$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{M}_{\mathcal{H}}(k)$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{B}$ be a subclass of the class $\mathcal{M}_{\mathcal{H}}(k)$. We say that a function $f\in\mathcal{B}$ *is an extreme point of* $\mathcal{B}$ if it cannot be represented as a nondegenerate, convex, and linear combination of two function from $\mathcal{B}$. We denote by $E\mathcal{B}$ the set of all extreme points of $\mathcal{B}$. We have that $E\mathcal{B}\subset\mathcal{B}$.

A class $\mathcal{B}$ is called *convex* if any convex linear combination of two functions from $\mathcal{B}$ belongs to $\mathcal{B}$. We denote by $\overline{co}\mathcal{B}$ the *closed convex* hull of $\mathcal{B}$, i.e., the intersection of all closed, convex subsets of $\mathcal{M}$ that contain $\mathcal{B}$..

A real-valued functional $\mathcal{D}:\mathcal{M}_{\mathcal{H}}(k)\to\mathbb{R}$ is called *convex* on a convex class $\mathcal{B}\subset\mathcal{M}_{\mathcal{H}}(k)$ if for $f,g\in\mathcal{B}$ and $0\le\lambda\le 1$ we have

$$\mathcal{D}\big(\gamma f+(1-\gamma)g\big)\le\gamma\mathcal{D}(f)+(1-\gamma)\mathcal{D}(g).$$

From the Krein–Milman theorem (see [14]) we have the following lemma.

Lemma 2 *Let $\mathcal{B}$ be a nonempty, compact, and convex subclass of the class $\mathcal{M}_{\mathcal{H}}(k)$ and $\mathcal{D}:\mathcal{M}_{\mathcal{H}}(k)\to\mathbb{R}$ be a real-valued, continuous, and convex functional on $\mathcal{B}$. Then,*

$$\mathcal{B}=\overline{co}E\mathcal{B}$$

and

$$\max\big\{\mathcal{D}(f):f\in\mathcal{B}\big\}=\max\big\{\mathcal{D}(f):f\in E\mathcal{B}\big\}.$$

Moreover, from Montel's theorem we obtain the following lemma.

Lemma 3 *A class $\mathcal{B}\subset\mathcal{M}_{\mathcal{H}}(k)$ is compact if and only if $\mathcal{B}$ is closed and locally uniformly bounded.*

Theorem 6 *The class $\mathcal{M}_\eta^\lambda(k;M,N)$ is a compact and convex subclass of $\mathcal{M}_\mathcal{H}(k)$.*

Proof Let $0 \le \lambda \le 1$ and $f_1, f_2 \in \mathcal{M}_\eta^\lambda(k;M,N)$ be functions of the form

$$f_l(z) = \frac{1}{z} + \sum_{n=k}^{\infty}\left(a_{l,n}z^n + \overline{b_{l,n}\overline{z}^n}\right) \quad (z \in \mathbb{U}, l \in \mathbb{N}). \tag{19}$$

Then, we have

$$\begin{aligned}&\lambda f_1(z) + (1-\lambda)f_2(z)\\ &\quad = \frac{1}{z} + \sum_{n=k}^{\infty}\left\{\left(\lambda a_{1,n} + (1-\lambda)a_{2,n}\right)z^n + \overline{\left(\lambda b_{1,n} + (1-\lambda)b_{2,n}\right)z^n}\right\}.\end{aligned}$$

Moreover, by Theorem 2 we obtain

$$\begin{aligned}&\sum_{n=k}^{\infty}\left\{c_n\left|\gamma a_{1,n} + (1-\gamma)a_{2,n}\right| + d_n\left|\gamma b_{1,n} + (1-\gamma)b_{2,n}\right|\right\}\\ &\quad \le \gamma\sum_{n=k}^{\infty}\left\{c_n|a_{1,n}| + d_n|b_{1,n}|\right\} + (1-\gamma)\sum_{n=k}^{\infty}\left\{c_n|a_{2,n}| + d_n|b_{2,n}|\right\}\\ &\quad \le \gamma(N-M) + (1-\gamma)(N-M) = N-M.\end{aligned}$$

Thus, the function $\varphi = \lambda f_1 + (1-\lambda)f_2$ belongs to the class $\mathcal{M}_\eta^\lambda(k;M,N)$ and, in consequence, the class is convex.

The class is locally uniformly bounded if for each r, R, $0 < r < R < 1$, there is a real constant $L = L(r,R)$ so that

$$\left|f(z)\right| \le L \quad \left(f \in \mathcal{F}, r \le |z| \le R\right).$$

Let $f \in \mathcal{M}_\eta^\lambda(k;M,N)$, $0 < r \le |z| \le R < 1$. Then, by Theorem 2, we have

$$\left|f(z)\right| \le \frac{1}{r} + \sum_{n=k}^{\infty}\left(|a_n| + |b_n|\right)R^n \le \frac{1}{r} + \sum_{n=k}^{\infty}\left(c_n|a_n| + d_n|b_n|\right) \le \frac{1}{r} + (N-M) =: L.$$

This implies that the class $\mathcal{M}_\eta^\lambda(k;M,N)$ is locally uniformly bounded. Next, we show that it is closed. Let f_l and f be given by (19) and (1), respectively. By Theorem 2 we obtain

$$\sum_{n=k}^{\infty}\left(c_n|a_{l,n}| + d_n|b_{l,n}|\right) \le N-M \quad (l \in \mathbb{N}). \tag{20}$$

If $f_l \to f$, then we obtain that $a_{l,n} \to a_n$ and $b_{l,n} \to b_n$ as $l \to \infty$ $(n \in \mathbb{N}_k)$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=k}^{\infty}(c_n|a_n| + d_n|b_n|)$ is a nondecreasing sequence. Moreover, by (20) it is bounded by $N-M$. Therefore, the sequence $\{S_n\}$ is convergent and $\sum_{n=k}^{\infty}(c_n|a_n| + d_n|b_n|) = \lim_{n\to\infty} S_n \le N-M$. This gives the condition (6), and, in consequence, $f \in \mathcal{M}_\eta^\lambda(k;M,N)$, which completes the proof. □

Theorem 7

$$E\mathcal{M}_{\eta}^{\lambda}(k;M,N)=\{h_n:n\in\mathbb{N}_{k-1}\}\cup\{g_n:n\in\mathbb{N}_k\},$$

where $h_{k-1}(z)=\frac{1}{z}$ *and*

$$h_n(z)=\frac{1}{z}+\frac{(-1)^{\lambda}(N-M)}{c_n e^{i(n+1)\eta}}z^n,\qquad g_n(z)=\frac{1}{z}-\frac{N-M}{d_n e^{i(1-n)\eta}}\overline{z}^n\quad(n\in\mathbb{N}_k,z\in\mathbb{U}).\tag{21}$$

Proof Let $f_1,f_2\in\mathcal{M}_{\eta}^{\lambda}(k;M,N)$ be functions of the form (6), $g_n=\lambda f_1+(1-\lambda)f_2$ with $0<\lambda<1$. Then, by (6) we obtain $|b_{1,n}|=|b_{2,n}|=\frac{N-M}{\beta_n}$. Thus, $a_{1,l}=a_{2,l}=0$ for $l\in\mathbb{N}_k$ and $b_{1,l}=b_{2,l}=0$ for $l\in\mathbb{N}_k\setminus\{n\}$. This means that $g_n=f_1=f_2$, and, in consequence, $g_n\in E\mathcal{M}_{\eta}^{\lambda}(k;M,N)$. In the same way, we show that the functions h_n of the form (21) are the extreme points of the class $\mathcal{M}_{\eta}^{\lambda}(k;M,N)$. Now, let $f\in E\mathcal{M}_{\eta}^{\lambda}(k;M,N)$ be not of the form (21). Then, there exists $r\in\mathbb{N}_k$ such that

$$0<|a_r|<\frac{N-M}{\alpha_r}\quad\text{or}\quad 0<|b_r|<\frac{N-M}{\beta_r}.$$

If $0<|a_r|<\frac{N-M}{\alpha_r}$, then for

$$\lambda=\frac{\alpha_r|a_r|}{N-M},\qquad \varphi=\frac{1}{1-\lambda}(f-\lambda h_r),$$

we have that $0<\lambda<1$, $h_r\neq\varphi$ and $f=\lambda h_r+(1-\lambda)\varphi$. Thus, $f\notin E\mathcal{M}_{\eta}^{\lambda}(k;M,N)$. Analogously, if $0<|b_r|<\frac{N-M}{\beta_n}$, then for

$$\lambda=\frac{\beta_r|b_r|}{N-M},\qquad \phi=\frac{1}{1-\lambda}(f-\lambda g_r),$$

we have that $0<\lambda<1$, $g_r\neq\phi$ and $f=\lambda g_r+(1-\lambda)\phi$. Thus, $f\notin E\mathcal{M}_{\eta}^{\lambda}(k;M,N)$, and the proof is completed. □

5 Applications of extreme points

If the class $\mathcal{B}=\{f_n\in\mathcal{M}_{\mathcal{H}}(k):n\in\mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{co}\mathcal{B}=\left\{\sum_{n=1}^{\infty}\lambda_n f_n:\sum_{n=1}^{\infty}\lambda_n=1,\lambda_n\geq 0(n\in\mathbb{N})\right\}.$$

Thus, by Lemma 2 and Theorem 7 we obtain

Corollary 3

$$\mathcal{M}_{\eta}^{\lambda}(k;M,N)=\left\{\sum_{n=k-1}^{\infty}(\gamma_n h_n+\delta_n g_n):\sum_{n=k-1}^{\infty}(\gamma_n+\delta_n)=1(\delta_{k-1}=0,\gamma_n,\delta_n\geq 0)\right\},$$

where h_n,g_n *are defined by* (21).

It is easy to show that the following real-valued functionals are convex and continuous on $\mathcal{M}_{\mathcal{H}}(k)$:

$$\mathcal{D}(f) = a_n, \qquad \mathcal{D}(f) = b_n, \qquad \mathcal{D}(f) = \left|f(z)\right|, \qquad \mathcal{D}(f) = \left|\mathcal{D}_{\mathcal{H}}^{\lambda} f(z)\right|,$$
$$\mathcal{D}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\gamma} d\theta\right)^{1/\gamma} \quad \left(f \in \mathcal{M}_{\mathcal{H}}(k)\right).$$

for each fixed value of $n \in \mathbb{N}_k$, $z \in \mathbb{U}$, $\gamma \geq 1$, $0 < r < 1$. Thus, by Lemma 2 and Theorem 7 we have the following results.

Corollary 4 *Let $f \in \mathcal{M}_{\eta}^{\lambda}(k;M,N)$ be a function of the form* (1), $0 < r < 1$, $\gamma \geq 1$. *Then,*

$$|a_n| \leq \frac{N-M}{c_n}, \qquad |b_n| \leq \frac{N-M}{d_n} \quad (n \in \mathbb{N}_k),$$
$$\frac{1}{r} - \frac{N-M}{d_k} r^k \leq \left|f(z)\right| \leq \frac{1}{r} + \frac{N-M}{d_k} r^k \quad \left(|z| = r\right),$$
$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\gamma} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|h_1\left(re^{i\theta}\right)\right|^{\gamma} d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_{\mathcal{H}}^{\lambda} f\left(re^{i\theta}\right)\right|^{\gamma} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_{\mathcal{H}}^{\lambda} h_1\left(re^{i\theta}\right)\right|^{\gamma} d\theta,$$

where c_n, d_n are defined by (5). *The results are sharp with extremal functions h_n, g_n of the form* (21).

Corollary 5 *If $f \in \mathcal{M}_{\eta}^{\lambda}(k;M,N)$, then*

$$\mathbb{U}(r) \subset f(\mathbb{U}),$$

where

$$r = 1 - \frac{N-M}{k^{\lambda+1}(1+N) - k^{\lambda}(1+M)}.$$

Remark 2 If we put $n = 0$ or $n = 1$ in Corrolaries 3 and 4 we obtain similar results for the classes $\mathcal{M}_{\eta}^{*}(k;M,N)$ and $\mathcal{M}_{\eta}^{c}(k;M,N)$.

By using Corollary 1 and the results above we obtain the corollaries listed below.

Corollary 6 *The class $\mathcal{W}_{\eta}(k;M,N)$ is a convex and compact subset of $\mathcal{M}_{\mathcal{H}}(k)$. Moreover,*

$$E\mathcal{W}_{\eta}(k;M,N) = \{h_n : n \in \mathbb{N}_{k-1}\} \cup \{g_n : n \in \mathbb{N}_k\}$$

and

$$\mathcal{W}_{\eta}(k;M,N) = \left\{\sum_{n=1}^{\infty}(\gamma_n h_n + \delta_n g_n) : \sum_{n=1}^{\infty}(\gamma_n + \delta_n) = 1, \delta_1 = 0, \gamma_n, \delta_n \geq 0 (n \in \mathbb{N})\right\},$$

where $h_{k-1}(z) = z$ *and*

$$h_n(z) = z + \frac{N-M}{(1+N)n} z^n, \qquad g_n(z) = z - \frac{N-M}{(1+N)n} \overline{z}^n \quad (z \in \mathbb{U}). \tag{22}$$

Corollary 7 *If* $f \in \mathcal{W}_\eta(k;M,N)$ *is of the form* (1), *then*

$$\begin{aligned}
&|a_n| \le \frac{N-M}{(1+N)n}, \qquad |b_n| \le \frac{N-M}{(1+N)n} \quad (n \in \mathbb{N}),\\
&\frac{1}{r} - \frac{N-M}{(1+N)k} r^k \le \left|f(z)\right| \le \frac{1}{r} + \frac{N-M}{(1+N)k} r^k \quad \left(|z| = r < 1\right),\\
&\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\gamma d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|h_2\left(re^{i\theta}\right)\right|^\lambda d\theta,\\
&\frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_{\mathcal{H}}^\lambda f\left(re^{i\theta}\right)\right|^\gamma d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{D}_{\mathcal{H}}^\lambda h_2\left(re^{i\theta}\right)\right|^\gamma d\theta.
\end{aligned}$$

The results are sharp with extremal functions h_n, g_n *of the form* (22).

Corollary 8 *If* $f \in \mathcal{W}_\eta(k;M,N)$, *then*

$$\mathbb{U}\left(1 - \frac{N-M}{(1+N)k}\right) \subset f(\mathbb{U}).$$

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