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Optimum solution of $(k,\beth)$-Hilfer FDEs by $\mathcal{A}$-condensing operators and the incorporated measure of noncompactness

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Abstract

The notion of $\mathcal{A}$-condensing operators via the measure of noncompactness is proposed, which retains the existing classes of condensing operators. Results concerning the existence of the best proximity point (pair) of cyclic (noncyclic) $\mathcal{A}$-condensing operators along with the coupled best proximity-point theorem for cyclic $\mathcal{A}$-condensing operators have been formulated. An application to a $(k,\beth)$-Hilfer fractional differential equation of order $2 < p < 3$, type $q \in [0,1]$ satistfying some boundary conditions is presented. The paper is the first to investigate the optimum solution of such a generalized fractional differential equation. The hypothesis involved in the investigation is independent of the incorporated measure of noncompactness, thereby making our result better in application than that present in the literature. Moreover, added numerical examples validate the theoretical conclusions.

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Keywords: Best proximity point (pair); Measure of noncompactness; Hilfer fractional differential equation

1 Introduction

Over the years, the measure of noncompactness ($\mathtt{MNC}$) together with condensing operators has created a remarkable place in the field of Fixed-Point Theory ($\mathtt{FPT}$) [1–3]. Kuratowski initiated the study of $\mathtt{MNC}$, whereas the concept of condensing operators originated in 1955 by Darbo, Schauder's $\mathtt{FPT}$ being the main motivation behind this. In [4], Brouwer proved the following $\mathtt{FPT}$.

Theorem 1.1 *Assume that $\emptyset \neq \Omega \subset \mathbb{R}^n$ is convex and compact. Then, the continuous operator $\Lambda : \Omega \to \Omega$ has a fixed point.*

Schauder then extended this result to infinite-dimensional Banach spaces ($\mathrm{B_S}$) [5]. Meanwhile, Kuratowski [6, 7] stated the notion of $\mathtt{MNC}$ in order to solve certain problems related to general topology and defined it as a real-valued function $\mathcal{K} : B_{\mathcal{Y}} \to [0,\infty)$

such that

$$\mathcal{K}(V) = \inf\left\{\epsilon > 0 : V \subseteq \bigcup_{\alpha=1}^{n} P_\alpha, P_\alpha \subset \mathcal{Y}, \operatorname{diam}(P_\alpha) < \epsilon\right\}, \tag{1}$$

where $B_{\mathcal{Y}}$ is the family of all nonvoid, bounded sets V of the complete metric space $\mathcal{Y}$. Darbo [8] lessened the Schauder's FPT hypothesis by using this MNC $\mathcal{K}$. The beauty of Darbo's theorem lies in the fact that it helped to weaken the compactness assumption, a strong presupposition, by replacing it with an inequality comprising $\mathcal{K}$. This theorem has the following statement.

Theorem 1.2 ([8]) *Let $\Omega \subset \mathcal{Y}$ be nonvoid, convex, closed, and bounded with $\mathcal{Y}$ as a $\mathrm{B_S}$. Then, the continuous map $\Lambda : \Omega \to \Omega$ has a fixed point if $\mathcal{K}(\Lambda C) \le k\mathcal{K}(C)$, for $C \subset \Omega$ with $0 \le k < 1$.*

Fractional calculus deals with the study of differentiation and integration of arbitrary order and thus generalizes the classical structure. This generalization grabbed the focus due to its efficiency in providing a more accurate description for real-world phenomena. For a brief study, one can see [9, 10]. Motivated by the definition of Riemann–Liouville (RL) and Caputo derivatives, the authors [11] proposed the Hilfer derivative and solved an existence–uniqueness problem involving a Hilfer derivative of order between 0 to 1. Later, in 2018, Sousa *et al.* initiated the discussion of the $\daleth$-Hilfer derivative, involving a continuously differentiable increasing function $\daleth$ [12]. In 2023, Haque *et al.* took the $\mathrm{H}\mathbb{BVP}$ ${}^{\mathrm{H}}\mathbb{D}^{p;q}_{a^+}\eta_k + F_k(\nu,\eta) = 0$, with $\eta_k(a) = \eta_k(b) = 0$, for $\nu \in (a,b)$, $k \in \mathbb{N}$, where $1 < p \le 2$, $0 \le q \le 1$, $F_k(\nu,\eta) = c_k(\nu) + f_k(\nu,\eta_1,\eta_2,\eta_3,\dots)$, c_k, f_k for each $k \in \mathbb{N}$ are real-valued continuous functions on $[a,b]$ [13]. Also, in [14], they considered the ψ-$\mathrm{H}\mathbb{FDE}$s with control having the form

$$\begin{cases} {}^{\mathrm{H}}\mathbb{D}^{p,q;\psi}_{0^+}\eta(\nu) = A\eta(\nu) + F(\nu,\eta_\nu) + Bu(\nu), & \nu \in (0,d], \\ \mathbb{I}^{(1-p)(1-q);\psi}\eta(\nu) = \hbar(\nu) \in \mathcal{B}_h, \end{cases} \tag{2}$$

where $0.5 < p < 1$, $0 \le q \le 1$, $\eta(\cdot)$ takes the values in Banach space Z with $\|\cdot\|$, the control function $u(\cdot) \in L^2((0,d],\mathrm{U})$, the Banach space of admissible control functions, with U as a Banach space, $B : L^2((0,d],\mathrm{U}) \to L^2((0,d],\mathrm{Z})$ is a bounded linear operator, and the operator $A : D(A) \subset \mathrm{Z} \to \mathrm{Z}$ is the infinitesimal generator of analytic semigroup $\{T(\nu)\}_{\nu\ge 0}$ on Z. The $\mathbb{DE}$s of the $\mathrm{H}\mathbb{FDE}$, ψ-$\mathrm{H}\mathbb{FDE}$, and $(k,\daleth)$-$\mathrm{H}\mathbb{FDE}$ types, because they provide significant generalizations, are very useful in solving different types of differential equations. For more details of these types of generalizations and results of controllability involving ψ-$\mathrm{H}\mathbb{FDE}$s refer to [15–22] and references therein. The work referenced above inspired us to propose a most generalized version of the Hilfer derivative, the so-called $(k,\daleth)$-Hilfer fractional derivative.

Recently, valuable applications of MNC emphasizing the existence of solutions for a system of fractional differential and integral equations have been presented [23–29]. For example, Patle *et al.* in [25], discussed the existence of optimal solutions of the following system of right-sided ψ-Hilfer fractional differential equations (ψ-$\mathrm{H}\mathbb{FDE}$) of arbitrary order with initial conditions

$$\begin{cases} {}_{\mathrm{H}}\mathbb{D}_{a^+}^{p,q;\daleth}\hat{\eta}(\nu) = \digamma_1(\nu,\hat{\eta}(\nu)), \\ {}_{\mathrm{H}}\mathbb{D}_{a^+}^{p,q;\daleth}\acute{\eta}(\nu) = \digamma_2(\nu,\acute{\eta}(\nu)), \\ \mathcal{I}_{a^+}^{(1-q)(1-p);\daleth}\hat{\eta}(a) = \varsigma_a, \qquad \mathcal{I}_{a^+}^{(1-q)(1-p);\daleth}\acute{\eta}(a) = \acute{\varsigma}_a, \end{cases} \tag{3}$$

for $\nu \in (a,\tau]$, where ${}_{\mathrm{H}}\mathbb{D}_{a^+}^{p,q;\daleth}$ is the left-sided ψ-H$\mathbb{FD}$ operator of order $0 < p < 1$, type $0 < q \leq 1$, $\mathbb{I}_{a^+}^{(1-q)(1-p);\daleth}$ is the RL fractional integral of order $(1-q)(1-p)$; the state $\eta(\cdot)$ takes the values from $\mathcal{X}$, and $\digamma_1 : [a,\tau] \times \mathcal{B}_1 \to \mathcal{X}$ and $\digamma_2 : [a,\tau] \times \mathcal{B}_2 \to \mathcal{X}$, are given mappings satisfying some assumptions.

In this article, the work flow is as follows: We define classes of cyclic and noncyclic $\mathcal{A}$-condensing operators and prove the existence of the best proximity point (bpp) and pair for them, respectively, in the setting of B$_{\mathrm{S}}$s. The consequences of the main results lead to some of the important results in the existing literature, presented as corollaries. Also, we discuss some coupled bpp results. In Sect. 5, the main result is applied to establish the existence of optimum solutions for the class of fractional differential equations involving $(k,\daleth)$-Hilfer derivatives ($(k,\daleth)$-H$\mathbb{FDE}$) of order $2 < p < 3$, type $0 \leq q \leq 1$ in the form

$$\begin{cases} {}^{k,\mathrm{H}}\mathbb{D}^{p,q;\daleth}\eta_1(\nu) = \digamma_1(\nu,\eta_1(\nu)), \\ \eta_1(a) = \eta_1'(a) = 0, \\ \sigma_1\eta_1(b) + \sigma_2\delta_\daleth\eta_1(b) = \sigma_3{}^{k}\mathbb{I}^{\nu;\daleth}\mathfrak{u}_1(\zeta,\eta_1(\zeta)), \end{cases} \tag{4}$$

$$\begin{cases} {}^{k,\mathrm{H}}\mathbb{D}^{p,q;\daleth}\eta_2(\nu) = \digamma_2(\nu,\eta_2(\nu)), \\ \eta_2(a) = 0, \eta_2'(a) = 0, \\ \sigma_1\eta_2(b) + \sigma_2\delta_\daleth\eta_2(b) = \sigma_3{}^{k}\mathbb{I}^{\nu;\daleth}\mathfrak{u}_2(\zeta,\eta_2(\zeta)), \end{cases} \tag{5}$$

for $\nu \in \mathcal{J} = [a,b]$ satisfying the stated boundary conditions. The quantities σ_ι, $\iota = 1,2,3$ are suitable real scalars, functions $\mathfrak{u}_i$, $\digamma_i$, $\daleth$ are all continuous such that $\daleth'(\nu) > 0$ for all $\nu \in \mathcal{J}$ with $\delta_\daleth \equiv \frac{k}{\daleth'(\nu)}\frac{\mathrm{d}}{\mathrm{d}\nu}$, $a < \zeta < b$ and ${}^{k}\mathbb{I}^{\nu;\daleth}$ is the $(k,\daleth)$-RL integral of order $\nu \in (0,\infty)$, $k \in \mathbb{R}$. Finally, we introduce the strengths of the obtained results in future works in the conclusion section, 7.

2 Preliminaries

The compactness of the set or of the operator was not a really big issue, thanks to the Heine–Borel theorem, until the B$_{\mathrm{S}}$s of infinite dimension came into the picture. Justifying its name, MNC is a measure that estimates the degree of noncompactness of a set, a real-valued function that depicts the level of closeness of a set from being compact. The later axiomatic approach is a more convenient form when dealing with MNC and has the following interpretation [1].

Definition 2.1 Let $\mathcal{Y}$, $B_{\mathcal{Y}}$, and V be defined as above. A function $\eta : B_{\mathcal{Y}} \to [0,\infty)$ is said to be an MNC provided

(η1) $\eta(V) = 0 \Longleftrightarrow V$ is precompact (regularity);

(η2) $\eta(V) = \eta(\overline{V})$ (invariance under closure);

(η3) $\eta(V_1 \cup V_2) = \max\{\eta(V_1),\eta(V_2)\}$ (semiadditivity).

From the above three properties, the following conclusions are drawn immediately [1]:

(η4) $V_1 \subset V_2$ implies $\eta(V_1) \leq \eta(V_2)$ (monotonicity);

(η5) $\eta(V_1 \cap V_2) \leq \min\{\eta(V_1), \eta(V_2)\}$;

(η6) For any finite set V, $\eta(V) = 0$ (nonsingularity);

(η7) If the sequence $\{V_n\}_{n=1}^{\infty}$ is decreasing in nature, where each $V_n \in B_{\mathcal{Y}}$ is closed in X and $\lim_{n\to\infty} \eta(V_n) = 0$ then $V_\infty = \bigcap_{n=1}^{\infty} V_n$ is nonvoid as well as compact (generalized Cantor's intersection theorem).

Moreover, if $\mathcal{Y}$ is also a $\mathrm{B_S}$ then MNC η can satisfy the following properties:

(η8) $\eta(wV) = |w|\eta(V)$ for any number w (semihomogeneity);

(η9) $\eta(V_1 + V_2) \leq \eta(V_1) + \eta(V_2)$ (algebraic semiadditivity);

(η10) $\eta(w_0 + V) = \eta(V)$ for any $w_0 \in \mathcal{Y}$ (invariance under translations);

(η11) $\eta(\mathrm{conv}(V)) = \eta(V)$ (invariance under passage to the convex hull).

Another interesting generalization of the Schauder FPT involving cyclic (noncyclic), relatively nonexpansive maps together with a compact (Theorem 2.2) or condensing (Theorem 2.5) operator appeared in [29]. To state this, we need to recall some definitions. For any two nonempty subsets C and D of $\mathcal{Y}$, if $\overline{\Lambda(C)}$ and $\overline{\Lambda(D)}$ are both compact then $\Lambda : C \cup D \to C \cup D$ is called a compact operator. The map Λ is said to be cyclic if $\Lambda(C) \subseteq D$ as well as $\Lambda(D) \subseteq C$ and noncyclic if $\Lambda(C) \subseteq C$ along with $\Lambda(D) \subseteq D$. If $d(\Lambda v, \Lambda w) \leq d(v, w)$, for each $v \in C$, $w \in D$, then Λ is known as relatively nonexpansive. A point $\mathfrak{v}^*_{\mathrm{bpp}} \in C \cup D$ is a bpp of a cyclic map Λ provided

$$\left\| \mathfrak{v}^*_{\mathrm{bpp}} - \Lambda \mathfrak{v}^*_{\mathrm{bpp}} \right\| = \mathrm{dist}(C, D) := \inf\left\{ \|\bar{c} - \bar{d}\| : \bar{c} \in C, \bar{d} \in D \right\}, \tag{6}$$

whereas $(\mathfrak{v}^*_{\mathrm{bpp}}, \mathfrak{w}^*_{\mathrm{bpp}}) \in C \times D$ is a best proximity pair for a noncyclic map Λ if

$$\left\| \mathfrak{v}^*_{\mathrm{bpp}} - \mathfrak{w}^*_{\mathrm{bpp}} \right\| = \mathrm{dist}(C, D), \qquad \mathfrak{v}^*_{\mathrm{bpp}} = \Lambda \mathfrak{v}^*_{\mathrm{bpp}}, \mathfrak{w}^*_{\mathrm{bpp}} = \Lambda \mathfrak{w}^*_{\mathrm{bpp}}. \tag{7}$$

The proximal pair $(C_0, D_0) \subseteq (C, D)$ is given as

$$\begin{aligned} C_0 &= \left\{ \bar{c} \in C | \exists d_0 \in D : \|\bar{c} - d_0\| = \mathrm{dist}(C, D) \right\}, \\ D_0 &= \left\{ \bar{d} \in D | \exists c_0 \in C : \|c_0 - \bar{d}\| = \mathrm{dist}(C, D) \right\}. \end{aligned} \tag{8}$$

The pair (C, D) is called proximinal whenever $C = C_0$ and $D = D_0$. We denote by $\mathcal{M}_\Lambda(C, D)$ the collection of all pairs of subsets $(E, \acute{E})$ inside (C, D) that are nonempty, convex, closed, bounded, proximinal, and Λ-invariant in nature such that $\mathrm{dist}(E, \acute{E}) = \mathrm{dist}(C, D)$. In general, $\mathcal{M}_\Lambda(C, D)$ may be empty, however, if Λ is cyclic (noncyclic), relatively nonexpansive with (C, D) as that nonvoid convex pair that agrees to be weakly compact also inside a $\mathrm{B_S}$ $\mathcal{Y}$, then $(C_0, D_0) \in \mathcal{M}_\Lambda(C, D)$.

We signify some conditions for $\Lambda : C \cup D \to C \cup D$ by the following notations:

(S1) The pair of subsets (C, D) is nonvoid, convex, closed, and bounded in a $\mathrm{B_S}$ $\mathcal{Y}$.

(S2) Λ is relatively nonexpansive.

(S3) Λ is cyclic.

(S4) Λ is noncyclic and $\mathcal{Y}$ is strictly convex.

Theorem 2.2 ([29]) *If $C_0 \neq \emptyset$ and Λ is compact, then Λ possesses a* bpp *and a best proximity pair whenever* (S1), (S2), (S3) *and* (S1), (S2), (S4) *hold, respectively.*

Definition 2.3 ([30]) Let η be an arbitrary MNC and the condition (S1) holds. The operator $\Lambda : C \cup D \to C \cup D$ is called a cyclic (noncyclic), Meir–Keeler condensing (MKC) operator if $\forall \epsilon > 0\ \exists \delta > 0 : \forall (E, \acute{E}) \in \mathcal{M}_\Lambda(C, D)$,

$$\epsilon \leq \eta(E \cup \acute{E}) < \epsilon + \delta \quad \Longrightarrow \quad \eta\big(\Lambda(E) \cup \Lambda(\acute{E})\big) < \epsilon. \tag{9}$$

Definition 2.4 ([31]) A map $\mathcal{L} : [0, \infty) \to [0, \infty)$ is known to be an L-function whenever $\mathcal{L}(0) = 0$ with $\mathcal{L}(\nu) > 0$ for $\nu \in (0, \infty)$ and for any $\nu > 0$ there exists $\delta > 0$ such that $\mathcal{L}(u) \leq \nu$ provided $u \in [\nu, \nu + \delta]$.

Remark 2.1 The mappings L-function characterized (MK) contractions and was further proved to be true for (MKC) operators [31]. A map Λ is a cyclic (noncyclic) L-condensing operator if for an L-function $\mathcal{L}$, we have $\eta(\Lambda E \cup \Lambda \acute{E}) < \mathcal{L}(\eta(E \cup \acute{E}))$, for $(E, \acute{E}) \in \mathcal{M}_\Lambda(C, D)$, whenever $\eta(E \cup \acute{E}) > 0$.

Theorem 2.5 ([30]) *Suppose that* η *is an arbitrary* MNC *and* Λ *is an* MKC *operator such that* $C_0 \neq \emptyset$. *Then,* Λ *has a* bpp *and a best proximity pair whenever* (S1), (S2), (S3) *and* (S1), (S2), (S4) *hold, respectively.*

These were not merely generalizations, rather, they have the potency to show the existence of a bpp (pair). In recent years, very nice works have been done on the existence as well as applications of bpps and pairs. The interested readers are advised to read the articles [23, 26, 28] and references therein. Moving towards the main motivation of this article, Shahzad *et al.* defined a $\mathcal{A}$-contraction [32] using the concept of a Λ-sequence that submerges the class of all R-contractions, Meir–Keeler contractions, $\mathcal{Z}$-contractions, and more. Keeping this in view, we define $\mathcal{A}$-condensing operators in terms of MNC η using the concept of Λ_η-sequence.

3 Best proximity point (pair) results

We now present our notions, namely, the Λ_η-sequence and $\mathcal{A}$-condensing operators. We say that $\{\chi_n\} := \{(\alpha_n, \beta_n)\}$ is a Λ_η-sequence if there exists a sequence of pairs $\{(E_n, \acute{E}_n)\}$ in $\mathcal{M}_\Lambda(C, D)$ such that

$$\alpha_n = \eta(\Lambda E_n \cup \Lambda \acute{E}_n) > 0, \qquad \beta_n = \eta(E_n \cup \acute{E}_n) > 0, \tag{10}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences.

Definition 3.1 Let η be an arbitrary MNC. An operator $\Lambda : C \cup D \to C \cup D$ is $\mathcal{A}$-condensing if one can find a function $\rho : A \times A \to \mathbb{R}$ satisfying the subsequent conditions together with Λ as: (i) $\operatorname{rang}(\eta) \subseteq A \subseteq \mathbb{R}$; (ii) if $\{\chi_n\} \subseteq A^2$ is a Λ_η-sequence such that both $\alpha_n, \beta_n \to \ell$ with $\ell \geq 0$ and verifying $\ell < \alpha_n$ along with $\rho(\alpha_n, \beta_n) > 0$ for every $n \in \mathbb{N}$ then $\ell = 0$; (iii) $\rho(\eta(\Lambda E \cup \Lambda \acute{E}), \eta(E \cup \acute{E})) > 0$, provided $\eta(E \cup \acute{E}) > 0$ and $\eta(\Lambda E \cup \Lambda \acute{E}) > 0$ for every $(E, \acute{E}) \in \mathcal{M}_\Lambda(C, D)$.

It is proved in [32] that not every $\mathcal{A}$-contraction is a Meir–Keeler contraction but the converse is always true. Along the same lines, we note that not every $\mathcal{A}$-condensing operator is MKC but the converse is always true.

Example 3.1 Let $\mathcal{Y} = l_1$ be a nonreflexive $\mathrm{B_S}$ with norm $\|\cdot\|_1$ and $\{e_k\}$ be its standard basis. Define two sets C and D inside $\mathcal{Y}$ as $C = \overline{\mathrm{conv}}(\{e_{2k-1} : k \in \mathbb{N}\})$ and $D = \overline{\mathrm{conv}}(\{e_{2k} : k \in \mathbb{N}\})$. We choose $\mathtt{MNC}$ η operating on any nonempty, bounded subset $V \subseteq \mathcal{Y}$ as $\eta(V) = 0$ whenever V is precompact and $\eta(V) = 1$ elsewhere. Define $\Lambda : C \cup D \to C \cup D$ such that

$$\Lambda\big(\lambda e_{2k-1} + (1-\lambda)e_{2m-1}\big) = e_{2k}, \qquad \Lambda\big(\lambda e_{2k} + (1-\lambda)e_{2m}\big) = e_{2k-1}, \tag{11}$$

where $\lambda \in [0,1]$ and $k, m \in \mathbb{N}$ with $k \leq m$. Note that Λ is cyclic on $C \cup D$. Moreover, $(C,D) \in \mathcal{M}_\Lambda(C,D)$ since (C,D) is proximinal. As, $\eta(\Lambda C \cup \Lambda D) = \eta(C \cup D) > 0$, this means that $\eta(\Lambda C \cup \Lambda D) \not< \eta(C \cup D)$, and hence, Λ cannot be $\mathtt{MKC}$. On the other hand, for any nonzero, positive, and constant function ρ, the condition (A3) is fulfilled. For (A2), let $\{\chi_n\}$ be a Λ_η-sequence with $\alpha_n, \beta_n \to \ell$, $0 \leq \ell < \alpha_n$ and $\rho(\alpha_n, \beta_n) > 0$ for all $n \in \mathbb{N}$. Assume on the contrary that $\ell > 0$. Now, from the definition of a Λ_η-sequence, we have a sequence of pairs $\{(E_n, \acute{E}_n)\}$ in $\mathcal{M}_\Lambda(C,D)$ such that $\alpha_n = \eta(\Lambda E_n \cup \Lambda \acute{E}_n) > 0$ and $\beta_n = \eta(E_n \cup \acute{E}_n) > 0$. This means that $\{\alpha_n\}$ is a constant sequence converging to 1 such that $\alpha_n = \ell$ for infinitely many n, a contradiction. Hence, Λ is $\mathcal{A}$-condensing.

We proceed towards stating our theorems.

Theorem 3.2 *Suppose that* (S1), (S2), *and* (S3) *hold. If* Λ *is a cyclic* $\mathcal{A}$*-condensing operator such that* $\rho(v,w) \leq w - v$ *for all* $v, w \in A \cap (0,\infty)$, *then* Λ *has a* $\mathtt{bpp}$ *provided* $C_0 \neq \emptyset$.

Proof Set $\mathcal{I}_0 = C_0$ with $\mathcal{J}_0 = D_0$ and define

$$\mathcal{I}_n = \overline{\mathrm{conv}}\big(\Lambda(\mathcal{I}_{n-1})\big), \qquad \mathcal{J}_n = \overline{\mathrm{conv}}\big(\Lambda(\mathcal{J}_{n-1})\big), \quad \forall n \in \mathbb{N}. \tag{12}$$

Hence, for

$$\begin{aligned} &n = 1, \quad \mathcal{I}_1 = \overline{\mathrm{conv}}\big(\Lambda(\mathcal{I}_0)\big) = \overline{\mathrm{conv}}\big(\Lambda(C_0)\big) \subseteq D_0 = \mathcal{J}_0, \\ &n = 2, \quad \mathcal{I}_2 = \overline{\mathrm{conv}}\big(\Lambda(\mathcal{I}_1)\big) \subseteq \overline{\mathrm{conv}}\big(\Lambda(\mathcal{J}_0)\big) = \mathcal{J}_1, \\ &\cdots \quad \cdots \\ &\mathcal{I}_{n+1} \subseteq \mathcal{J}_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{13}$$

Analogously, one can derive $\mathcal{J}_{n+1} \subseteq \mathcal{I}_n$, for $n \in \mathbb{N} \cup \{0\}$. Hence, in general, we write

$$\mathcal{I}_{n+2} \subseteq \mathcal{J}_{n+1} \subseteq \mathcal{I}_n \subseteq \mathcal{J}_{n-1}, \quad \forall n \in \mathbb{N}. \tag{14}$$

Hence, the sequence $\{(\mathcal{I}_{2m}, \mathcal{J}_{2m})\}$ of nonempty, convex, and closed pairs in (C_0, D_0) is decreasing in nature. Moreover,

$$\begin{aligned} \Lambda(\mathcal{I}_{2m}) &\subseteq \Lambda(\mathcal{J}_{2m-1}) \subseteq \overline{\mathrm{conv}}\big(\Lambda(\mathcal{J}_{2m-1})\big) = \mathcal{J}_{2m}, \\ \Lambda(\mathcal{J}_{2m}) &\subseteq \Lambda(\mathcal{I}_{2m-1}) \subseteq \overline{\mathrm{conv}}\big(\Lambda(\mathcal{I}_{2m-1})\big) = \mathcal{I}_{2m}, \end{aligned} \tag{15}$$

imply that the pair $(\mathcal{I}_{2m}, \mathcal{J}_{2m})$ is Λ-invariant. Also, for $(\alpha, \beta) \in C_0 \times D_0$ and for all m, we have

$$\mathrm{dist}(\mathcal{I}_{2m}, \mathcal{J}_{2m}) \leq \big\|\Lambda^{2m}\alpha - \Lambda^{2m}\beta\big\| \leq \|\alpha - \beta\| = \mathrm{dist}(C,D). \tag{16}$$

Now, by induction we prove that the pairs $(\mathcal{I}_m, \mathcal{J}_m)$ are proximinal as well. The claim trivially holds for $m = 0$ and so let the pair $(\mathcal{I}_{m-1}, \mathcal{J}_{m-1})$ be proximinal. For $\iota = 1, 2, \dots, \jmath$,

$$\nu = \sum_{\iota=1}^{\jmath} \lambda_\iota \Lambda(\nu_\iota) \in \mathcal{I}_m, \qquad \nu_\iota \in \mathcal{I}_{m-1}, \lambda_\iota \in [0,1], \sum_{\iota=1}^{\jmath\in\mathbb{N}} \lambda_\iota = 1, \tag{17}$$

there exists $w_\iota \in \mathcal{J}_{m-1}$ such that $w = \sum_{\iota=1}^{\jmath} \lambda_\iota \Lambda(w_\iota) \in \mathcal{J}_m$, because of the proximality of the set $\mathcal{I}_{m-1}$, so that

$$\|\nu - w\| = \left\| \sum_{\iota=1}^{\jmath} \lambda_\iota \Lambda(\nu_\iota) - \sum_{\iota=1}^{\jmath} \lambda_\iota \Lambda(w_\iota) \right\| \le \sum_{\iota=1}^{\jmath} \lambda_\iota \|\nu_\iota - w_\iota\| = \operatorname{dist}(C,D), \tag{18}$$

and vice versa. This implies that the pair $(\mathcal{I}_m, \mathcal{J}_m)$ is proximinal and hence $(\mathcal{I}_{2m}, \mathcal{J}_{2m}) \in \mathcal{M}_\Lambda(C,D)$. Let us now consider the following two cases:

Case (i) Suppose there exists $m_0 \in \mathbb{N}$ with $\max\{\eta(\mathcal{I}_{2m_0}), \eta(\mathcal{J}_{2m_0})\} = 0$, then

$$\Lambda : \mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0} \to \mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0}, \tag{19}$$

is a cyclic, relatively nonexpansive map on the compact set $\mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0}$ and hence by Theorem 2.2, Λ will have a bpp.

Case (ii) For every $m \in \mathbb{N}$, let $\max\{\eta(\mathcal{I}_{2m}), \eta(\mathcal{J}_{2m})\} > 0$ and consider

$$\begin{aligned}
\eta(\mathcal{I}_{2m+1} \cup \mathcal{J}_{2m+1}) &= \max\{\eta(\mathcal{I}_{2m+1}), \eta(\mathcal{J}_{2m+1})\} \\
&= \max\{\eta(\overline{\operatorname{conv}}(\Lambda\mathcal{I}_{2m})), \eta(\overline{\operatorname{conv}}(\Lambda\mathcal{J}_{2m}))\} \\
&\le \max\{\eta(\mathcal{I}_{2m}), \eta(\mathcal{J}_{2m})\} = \eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m}) \\
&= \max\{\eta(\overline{\operatorname{conv}}(\Lambda\mathcal{I}_{2m-1})), \eta(\overline{\operatorname{conv}}(\Lambda\mathcal{J}_{2m-1}))\} \\
&= \max\{\eta(\Lambda\mathcal{I}_{2m-1}), \eta(\Lambda\mathcal{J}_{2m-1})\} \\
&\le \max\{\eta(\Lambda\mathcal{I}_{2m-2}), \eta(\Lambda\mathcal{J}_{2m-2})\} \\
&\le \eta(\mathcal{I}_{2m-2} \cup \mathcal{J}_{2m-2}).
\end{aligned} \tag{20}$$

Hence, $\{\eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m})\}_{m=0}^{\infty}$ is a decreasing sequence converging to its infimum, say ℓ. Set $\alpha_m = \eta(\Lambda\mathcal{I}_{2m} \cup \Lambda\mathcal{J}_{2m}) > 0$ and $\beta_m = \eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m}) > 0$, then $\{\chi_n\}$ is a Λ_η-sequence such that $\alpha_m \to \ell$ and $\beta_m \to \ell$ with $\ell \le \alpha_m$. Moreover, $\rho(\alpha_m, \beta_m) > 0$ for all m. Let us suppose for some m that $\ell \not< \alpha_m$ then $\ell = \alpha_k$ for all $k \ge m$. This means that the sequences $\{\alpha_m\}$ and $\{\beta_m\}$ are eventually constant sequences so that $\rho(\alpha_m, \beta_m) \le 0$ for infinitely many m, a contradiction and therefore by (A2) we obtain $\ell = 0$. Hence,

$$\lim_{m\to\infty} \eta(\mathcal{I}_{2m}) = \lim_{m\to\infty} \eta(\mathcal{J}_{2m}) = 0. \tag{21}$$

Define $\mathcal{I}_\infty = \cap\mathcal{I}_{2m}$ and $\mathcal{J}_\infty = \cap\mathcal{J}_{2m}$. Then, the pair $(\mathcal{I}_\infty, \mathcal{J}_\infty)$ is nonvoid, convex, and compact as well as Λ-invariant for which $\operatorname{dist}(\mathcal{I}_\infty, \mathcal{J}_\infty) = \operatorname{dist}(C,D)$. Thus, the application of Theorem 2.2 guarantees that Λ has a bpp. □

Theorem 3.3 *Suppose that* (S1), (S2), *and* (S4) *hold. Then,* Λ *has a best proximity pair provided* $C_0 \neq \emptyset$ *and* Λ *is a noncyclic* $\mathcal{A}$*-condensing operator with* $\rho(v,w) \leq w - v$ *for* $v, w \in A \cap (0,\infty)$.

Proof Set $\mathcal{I}_0 = C_0$ with $\mathcal{J}_0 = D_0$ and define for all $n \in \mathbb{N}$, $\mathcal{I}_n = \overline{\text{conv}}(\Lambda(\mathcal{I}_{n-1}))$ and $\mathcal{J}_n = \overline{\text{conv}}(\Lambda(\mathcal{J}_{n-1}))$. Thus, for

$$\begin{aligned}
&n = 1, \quad \mathcal{I}_1 = \overline{\text{conv}}\big(\Lambda(\mathcal{I}_0)\big) = \overline{\text{conv}}\big(\Lambda(C_0)\big) \subseteq C_0 = \mathcal{I}_0,\\
&n = 2, \quad \mathcal{I}_2 = \overline{\text{conv}}\big(\Lambda(\mathcal{I}_1)\big) \subseteq \overline{\text{conv}}\big(\Lambda(\mathcal{I}_0)\big) = \mathcal{I}_1,\\
&\cdots \quad \cdots\\
&\mathcal{I}_{n+1} \subseteq \mathcal{I}_n, \quad \forall n \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{22}$$

Analogously, one can derive $\mathcal{J}_{n+1} \subseteq \mathcal{J}_n$ for each $n \in \mathbb{N}\cup\{0\}$, so that the sequence $\{(\mathcal{I}_m, \mathcal{J}_m)\}$ of nonempty, convex, and closed pairs in (C_0, D_0) is decreasing in nature. Moreover,

$$\begin{aligned}
&\Lambda(\mathcal{I}_m) \subseteq \Lambda(\mathcal{I}_{m-1}) \subseteq \overline{\text{conv}}\big(\Lambda(\mathcal{I}_{m-1})\big) = \mathcal{I}_m,\\
&\Lambda(\mathcal{J}_m) \subseteq \Lambda(\mathcal{J}_{m-1}) \subseteq \overline{\text{conv}}\big(\Lambda(\mathcal{J}_{m-1})\big) = \mathcal{J}_m,
\end{aligned} \tag{23}$$

imply that the pair $(\mathcal{I}_m, \mathcal{J}_m)$ is Λ-invariant. Also, for $(\alpha, \beta) \in C_0 \times D_0$ and for each m, we obtain

$$\text{dist}(\mathcal{I}_m, \mathcal{J}_m) \leq \big\| \Lambda^m \alpha - \Lambda^m \beta \big\| \leq \|\alpha - \beta\| = \text{dist}(C, D). \tag{24}$$

By induction, one can prove that the pair $(\mathcal{I}_m, \mathcal{J}_m)$ is proximinal, as before, and hence $(\mathcal{I}_m, \mathcal{J}_m)$ in $\mathcal{M}_\Lambda(C, D)$. Let us now consider the following two cases:

Case (i) Suppose there exists $m_0 \in \mathbb{N}$ such that $\max\{\eta(\mathcal{I}_{m_0}), \eta(\mathcal{J}_{m_0})\} = 0$, then $\Lambda : \mathcal{I}_{m_0} \cup \mathcal{J}_{m_0} \to \mathcal{I}_{m_0} \cup \mathcal{J}_{m_0}$ is a noncyclic, relatively nonexpansive map on the compact set $\mathcal{I}_{m_0} \cup \mathcal{J}_{m_0}$ and so Λ will have a best proximity pair.

Case (ii) For every $m \in \mathbb{N}$, let $\max\{\eta(\mathcal{I}_m), \eta(\mathcal{J}_m)\} > 0$ and consider

$$\begin{aligned}
\eta(\mathcal{I}_{m+1} \cup \mathcal{J}_{m+1}) &= \max\big\{\eta(\mathcal{I}_{m+1}), \eta(\mathcal{J}_{m+1})\big\}\\
&= \max\big\{\eta\big(\overline{\text{conv}}(\Lambda\mathcal{I}_m)\big), \eta\big(\overline{\text{conv}}(\Lambda\mathcal{J}_m)\big)\big\}\\
&\leq \max\big\{\eta(\mathcal{I}_m), \eta(\mathcal{J}_m)\big\} = \eta(\mathcal{I}_m \cup \mathcal{J}_m).
\end{aligned} \tag{25}$$

Hence, $\{\eta(\mathcal{I}_m \cup \mathcal{J}_m)\}_{m=0}^{\infty}$ is a decreasing sequence converging to its infimum, say ℓ. Set $\alpha_m = \eta(\Lambda\mathcal{I}_m \cup \Lambda\mathcal{J}_m) > 0$ and $\beta_m = \eta(\mathcal{I}_m \cup \mathcal{J}_m) > 0$, then $\{\chi_m\}$ is a Λ_η-sequence such that $\alpha_m \to \ell$ and $\beta_m \to \ell$ with $\ell \leq \alpha_m$. Moreover, $\rho(\alpha_m, \beta_m) > 0$ for all m. Let us suppose for some m that $\ell \nless \alpha_m$ then $\ell = \alpha_k$ for all $k \geq m$. This means that the sequences $\{\alpha_m\}$ and $\{\beta_m\}$ are eventually constant sequences so that $\rho(\alpha_m, \beta_m) \leq 0$ for infinitely many m, a contradiction and therefore by (A2) we obtain $\ell = 0$. Hence,

$$\lim_{m\to\infty} \eta(\mathcal{I}_m) = \lim_{m\to\infty} \eta(\mathcal{J}_m) = 0. \tag{26}$$

Define $\mathcal{I}_\infty = \bigcap \mathcal{I}_m$ and $\mathcal{J}_\infty = \bigcap \mathcal{J}_m$, then the pair $(\mathcal{I}_\infty, \mathcal{J}_\infty)$ is nonvoid, convex, and compact as well as Λ-invariant for which $\text{dist}(\mathcal{I}_\infty, \mathcal{J}_\infty) = \text{dist}(C, D)$. Thus, the application of Theorem 2.2 guarantees that Λ has a best proximity pair. □

We have the following corollaries as a consequence of our main results: Theorems 3.2 and 3.3.

Corollary 3.1 ([24]) *Suppose the conditions* (S1), (S2), *and* (S3) *hold with* η *as an arbitrary* MNC. *If* $C_0 \neq \emptyset$ *and for every* $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$ *we have*

$$\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi\big(\eta(V_1 \cup V_2)\big) \cdot \eta(V_1 \cup V_2), \tag{27}$$

where $\Psi : [0, \infty) \to [0, 1)$ *is a map satisfying,* $\Psi(\nu_n) \to 1 \implies \nu_n \to 0$, *then* Λ *admits a* bpp.

Proof Set $\tilde{\Psi}(\nu) = \frac{1}{2}(1 + \Psi(\nu))$, for $\nu \in [0, \infty)$. Then, $\tilde{\Psi}(\nu_n) \to 1$ implies $\nu_n \to 0$. Moreover, $\Psi(\nu) < \tilde{\Psi}(\nu) < 1$ for $\nu \in [0, \infty)$, so that

$$\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi\big(\eta(V_1 \cup V_2)\big) \cdot \eta(V_1 \cup V_2) < \tilde{\Psi}\big(\eta(V_1 \cup V_2)\big) \cdot \eta(V_1 \cup V_2). \tag{28}$$

The definition

$$\begin{aligned} &\rho\big(\eta(\Lambda V_1 \cup \Lambda V_2), \eta(V_1 \cup V_2)\big) \\ &\quad = \tilde{\Psi}\big(\eta(V_1 \cup V_2)\big) \cdot \eta(V_1 \cup V_2) - \eta(\Lambda V_1 \cup \Lambda V_2) > 0, \end{aligned} \tag{29}$$

implies that (A3) is satisfied. For (A2), let $\{\chi_n\}$ be a Λ_η-sequence satisfying $\alpha_n \to \ell$ with $0 \leq \ell < \alpha_n$, $\beta_n \to \ell$, and for $n \in \mathbb{N}$, $\rho(\alpha_n, \beta_n) > 0$. Hence,

$$\begin{aligned} \alpha_n &:= \eta(\Lambda E_n \cup \Lambda \acute{E}_n) < \tilde{\Psi}\big(\eta(E_n \cup \acute{E}_n)\big) \cdot \eta(E_n \cup \acute{E}_n) \\ &\quad < \eta(E_n \cup \acute{E}_n) := \beta_n. \end{aligned} \tag{30}$$

Applying $n \to \infty$, we obtain $\tilde{\Psi}(\eta(E_n \cup \acute{E}_n)) \to 1$ and therefore, $\eta(E_n \cup \acute{E}_n) \to 0$ gives $\ell = 0$. Thus, Λ is $\mathcal{A}$-condensing and so Theorem 3.2 concludes the rest. □

The proof of the remaining corollaries can be similarly obtained. However, for more details, one can see [32, 33].

Corollary 3.2 ([24]) *Suppose the conditions* (S1), (S2), *and* (S4) *hold with* η *as an arbitrary* MNC. *If* $C_0 \neq \emptyset$ *and for every* $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$, *we have*

$$\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi\big(\eta(V_1 \cup V_2)\big) \cdot \eta(V_1 \cup V_2), \tag{31}$$

where Ψ *is as in Corollary* 3.1, *then* Λ *admits a best proximity pair.*

Corollary 3.3 *Suppose the conditions* (S1), (S2), *and* (S3) *hold with* η *as an arbitrary* MNC. *If* $C_0 \neq \emptyset$ *and for each* $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$ *we have* $\eta(\Lambda V_1 \cup \Lambda V_2) \leq \nu\eta(V_1 \cup V_2)$ *where* $\nu \in (0, 1)$, *then* Λ *admits a* bpp.

Corollary 3.4 *Suppose the conditions* (S1), (S2), *and* (S4) *hold with* η *as an arbitrary* MNC. *If* $C_0 \neq \emptyset$ *and for every* $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$ *we have* $\eta(\Lambda V_1 \cup \Lambda V_2) \leq \nu\eta(V_1 \cup V_2)$, *where* $0 < \nu < 1$, *then* Λ *admits a best proximity pair.*

Corollary 3.5 Suppose the conditions (S1), (S2), and (S3) hold with η as an arbitrary $\mathtt{MNC}$. If $C_0 \neq \emptyset$ and for every $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$ we have

$$\eta(\Lambda V_1 \cup \Lambda V_2) \leq \eta(V_1 \cup V_2) - \Psi\big(\eta(V_1 \cup V_2)\big), \tag{32}$$

where $\Psi : [0, \infty) \to [0, \infty)$ is lower semicontinuous with $\Psi_2^{-1}(\{0\}) = \{0\}$, then Λ admits a $\mathtt{bpp}$.

Corollary 3.6 *Suppose the conditions* (S1), (S2), *and* (S4) *hold with* η *as an arbitrary* $\mathtt{MNC}$. *If* $C_0 \neq \emptyset$ *and for every* $(V_1, V_2) \in \mathcal{M}_\Lambda(C, D)$, *we obtain*

$$\eta(\Lambda V_1 \cup \Lambda V_2) \leq \eta(V_1 \cup V_2) - \Psi\big(\eta(V_1 \cup V_2)\big), \tag{33}$$

where Ψ *is as in Corollary* 3.5, *then* Λ *admits a best proximity pair.*

4 Coupled best proximity point result

In this section, the coupled $\mathtt{bpp}$ theorem has been developed. For this, we first give the following preliminary concepts.

Suppose (C, D) is a nonvoid pair of subsets inside the metric space $(\mathcal{Y}, d)$ and $\Lambda : (C \times C) \cup (D \times D) \to C \cup D$. The map Λ is called cyclic if $\Lambda(C \times C) \subseteq D$ and $\Lambda(D \times D) \subseteq C$. The point $(\mathfrak{v}^*_{\mathtt{bpp}}, \mathfrak{v}^*_{\mathtt{bpp}}) \in (C \times C) \cup (D \times D)$ is a coupled $\mathtt{bpp}$ of Λ whenever

$$d\big(\mathfrak{v}^*_{\mathtt{bpp}}, \Lambda\big(\mathfrak{v}^*_{\mathtt{bpp}}, \mathfrak{w}^*_{\mathtt{bpp}}\big)\big) = d\big(\mathfrak{w}^*_{\mathtt{bpp}}, \Lambda\big(\mathfrak{w}^*_{\mathtt{bpp}}, \mathfrak{v}^*_{\mathtt{bpp}}\big)\big) = \operatorname{dist}(C, D). \tag{34}$$

Lemma 4.1 ([34]) *If* η_ι *is an* $\mathtt{MNC}$ *on the metric spaces* $\mathcal{Y}_\iota$, $\iota = 1, 2, \ldots, m$, *respectively, then* $\eta(H) = \Phi(\eta_1(H_1), \eta_2(H_2), \ldots, \eta_m(H_m))$, *is also an* $\mathtt{MNC}$ *on* $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_m$, *where* H_ι *stands for the natural projection of* H *into* $\mathcal{Y}_\iota$, *respectively, for* $\iota = 1, 2, \ldots, m$, *provided* $\Phi : [0, \infty)^m \to [0, \infty)$ *is a convex function and* $\Phi(a_1, a_2, \ldots, a_m) = 0$ *iff* $a_\iota = 0$ *for all* $\iota = 1, 2, \ldots, m$.

Lemma 4.2 ([27]) *Let* (C, D) *be a nonvoid pair inside a metric space* $(\mathcal{Y}, d)$ *and the product* $\mathcal{Y} \times \mathcal{Y}$ *be a metric space together with the metric* d_∞ *as*

$$d_\infty\big((v_1, v_2), (w_1, w_2)\big) = \max\big\{d(v_1, w_1), d(v_2, w_2)\big\}, \tag{35}$$

for each $(v_1, v_2), (w_1, w_2) \in \mathcal{Y}^2$. *Then,* (C, D) *is proximinal in* $\mathcal{Y}$ *iff* $(C \times C, D \times D)$ *is proximinal in* $\mathcal{Y}^2$.

Before giving the statement for the coupled $\mathtt{bpp}$ via $\mathcal{A}$-condensing operators, we define a Λ_η-sequence. For two real sequences $\{\alpha_n\}$, $\{\beta_n\}$, we say that $\{\chi_n\} = \{(\alpha_n, \beta_n)\}$ is a Λ_η-sequence if there exists a sequence of nonempty, convex, bounded, closed, and proximinal Λ-invariants pairs $\{(E_n, \acute{E}_n)\}$, $\{(F_n, \acute{F}_n)\}$ in (C, D) with

$$\operatorname{dist}(E_n, \acute{E}_n) = \operatorname{dist}(F_n, \acute{F}_n) = \operatorname{dist}(C, D), \tag{36}$$

such that for each $n \in \mathbb{N}$,

$$\alpha_n = \max\big\{\eta\big(\Lambda(E_n \times F_n) \cup \Lambda(\acute{E}_n \times \acute{F}_n)\big), \eta\big(\Lambda(F_n \times E_n) \cup \Lambda(\acute{F}_n \times \acute{F}_n)\big)\big\} > 0 \tag{37}$$

and $\beta_n = \max\{\eta(E_n \cup F_n), \eta(\acute{E}_n \cup \acute{F}_n)\} > 0$.

Theorem 4.3 *Suppose* (C, D) *is a nonempty, convex, bounded, and closed pair in a* $\mathrm{B_S}$ $\mathcal{Y}$ *with* $C_0 \neq \emptyset$ *and* η *is an* MNC *on* $\mathcal{Y}$. *Let* Λ *be a cyclic map satisfying* (A2) *along with the following conditions*:

(i) *Let* $(E, \acute{E}), (F, \acute{F}) \subseteq (C, D)$ *be nonempty, convex, bounded, closed, and proximinal and* Λ*-invariants pairs with* $\mathrm{dist}(E, \acute{E}) = \mathrm{dist}(F, \acute{F}) = \mathrm{dist}(C, D)$ *such that*

$$\begin{aligned}&\rho\big(\max\big\{\eta\big(\Lambda(E \times F) \cup \Lambda(\acute{E} \times \acute{F})\big), \eta\big(\Lambda(F \times E) \cup \Lambda(\acute{F} \times \acute{F})\big)\big\},\\&\quad\max\big\{\eta(E \cup F), \eta(\acute{E} \cup \acute{F})\big\}\big) > 0.\end{aligned} \tag{38}$$

(ii) *For all pairs* $(v_1, v_2) \in C \times C$ *and* $(w_1, w_2) \in D \times D$, *assume that*

$$d\big(\Lambda(v_1, v_2), \Lambda(w_1, w_2)\big) \leq d_\infty\big((v_1, v_2), (w_1, w_2)\big). \tag{39}$$

(iii) $\rho(v, w) \leq w - v$ *for every* $v, w \in A \cap (0, \infty)$.

Then, Λ *affirms to have a coupled* bpp.

Proof If H_ι stands for the natural projection of H into $\mathcal{Y}_\iota$ for $\iota = 1, 2$, we set $\tilde{\eta}(H) := \max\{\eta(H_1), \eta(H_2)\}$ then $\tilde{\eta}$ becomes an MNC for $\mathcal{Y}^2$. Let us define $\digamma$ by

$$\begin{cases}\digamma : (C \times C) \cup (D \times D) \to (C \times C) \cup (D \times D),\\ \digamma(v, w) = (\Lambda(v, w), \Lambda(w, v)),\end{cases} \tag{40}$$

then $\digamma$ is cyclic on $(C \times C) \cup (D \times D)$. This is because, for any (v, w) in $C \times C$ and together with the cyclic nature of Λ we have $(\Lambda(v, w), \Lambda(w, v)) \in D \times D$, whence $\digamma(C \times C) \subseteq D \times D$. Similarly, one has $\digamma(D \times D) \subseteq C \times C$. To show $\digamma$ is relatively nonexpansive, let $((v_1, v_2), (w_1, w_2)) \in (C \times C) \times (D \times D)$, then

$$\begin{aligned}&d_\infty\big(\digamma(v_1, v_2), \digamma(w_1, w_2)\big)\\&\quad= d_\infty\big(\big(\Lambda(v_1, v_2), \Lambda(v_2, v_1)\big), \big(\Lambda(w_1, w_2), \Lambda(w_2, w_1)\big)\big)\\&\quad= \max\big\{d\big(\Lambda(v_1, v_2), \Lambda(w_1, w_2)\big), d\big(\Lambda(v_2, v_1), \Lambda(w_2, w_1)\big)\big\}\\&\quad\leq \max\big\{d_\infty\big((v_1, v_2), (w_1, w_2)\big), d_\infty\big((w_2, w_1), (v_2, v_1)\big)\big\}\\&\quad= d_\infty\big((v_1, v_2), (w_1, w_2)\big),\end{aligned} \tag{41}$$

is the desired condition. Note that

$$\begin{aligned}&\tilde{\eta}\big(\digamma(E \times F) \cup \digamma(\acute{E} \times \acute{F})\big)\\&\quad= \max\big\{\tilde{\eta}\big(\digamma(E \times F)\big), \tilde{\eta}\big(\digamma(\acute{E} \times \acute{F})\big)\big\}\\&\quad= \max\big\{\tilde{\eta}\big(\Lambda(E \times F) \times \Lambda(F \times E)\big), \tilde{\eta}\big(\Lambda(\acute{E} \times \acute{F}) \times \Lambda(\acute{F} \times \acute{E})\big)\big\}\\&\quad= \max\big\{\max\big\{\eta\big(\Lambda(E \times F)\big), \eta\big(\Lambda(F \times E)\big)\big\},\\&\qquad\max\big\{\eta\big(\Lambda(\acute{E} \times \acute{F})\big), \eta\big(\Lambda(\acute{F} \times \acute{E})\big)\big\}\big\}\end{aligned}$$

$$\begin{aligned} &= \max\{\max\{\eta(\Lambda(E\times F)),\eta(\Lambda(\acute{E}\times\acute{F}))\}, \\ &\quad \max\{\eta(\Lambda(F\times E)),\eta(\Lambda(\acute{F}\times\acute{E}))\}\} \\ &= \max\{\eta(\Lambda(E\times F)\cup\Lambda(\acute{E}\times\acute{F})),\eta(\Lambda(F\times E)\cup\Lambda(\acute{F}\times\acute{E}))\} \end{aligned} \tag{42}$$

and $\tilde{\eta}((E\times F)\cup(\acute{E}\times\acute{F})) = \eta((E\cup F)\cup(\acute{E}\cup\acute{F}))$, so that we obtain

$$\begin{aligned} &\rho(\tilde{\eta}(\digamma(E\times F)\cup\digamma(\acute{E}\times\acute{F})),\tilde{\eta}((E\times F)\cup(\acute{E}\times\acute{F}))) \\ &\quad = \rho(\max\{\eta(\Lambda(E\times F)\cup\Lambda(\acute{E}\times\acute{F})),\eta(\Lambda(F\times E)\cup\Lambda(\acute{F}\times\acute{E}))\}, \\ &\qquad \max\{\eta(E\cup F),\eta(\acute{E}\cup\acute{F})\}) \\ &\quad > 0. \end{aligned} \tag{43}$$

This shows that $\digamma$ satisfies (A3). Hence, $\digamma$ will be $\mathcal{A}$-condensing if $\digamma$ satisfies (A2) too. Let us assume that $\{\chi_n\}$ is an $\digamma_{\tilde{\eta}}$ sequence with $\alpha_n\to\ell$, $\beta_n\to\ell$, $0\le\ell<\alpha_n$ and $\rho(\alpha_n,\beta_n)>0$ for each $n\in\mathbb{N}$, then there exists a sequence $\{(E_n\times F_n,\acute{E}_n\times\acute{F}_n)\}_{n=1}^{\infty}$ such that

$$\begin{aligned} \alpha_n &= \tilde{\eta}(\digamma(E_n\times F_n)\cup\digamma(\acute{E}_n\times\acute{F}_n)) > 0, \\ \beta_n &= \tilde{\eta}((E_n\times F_n)\cup(\acute{E}_n\times\acute{F}_n)) > 0, \end{aligned} \tag{44}$$

so that

$$\alpha_n = \max\{\eta(\Lambda(E_n\times F_n)\cup\Lambda(\acute{E}_n\times\acute{F}_n)),\eta(\Lambda(F_n\times E_n)\cup\Lambda(\acute{F}_n\times\acute{E}_n))\} \tag{45}$$

and $\beta_n = \max\{\eta(E_n\cup F_n),\eta(\acute{E}_n\cup\acute{F}_n)\}$, will be a Λ_η-sequence converging to $\ell=0$. This concludes that $\digamma$ is $\mathcal{A}$-condensing and hence, from Theorem 3.2, $\digamma$ has a $\mathtt{bpp}$ $(\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}})\in(C\times C)\cup(D\times D)$ such that

$$\begin{aligned} \operatorname{dist}(C,D) &= d_\infty((\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}}),\digamma(\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}})) \\ &= d_\infty((\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}}),(\digamma(\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}}),\digamma(\mathfrak{w}^*_{\mathtt{bpp}},\mathfrak{v}^*_{\mathtt{bpp}}))) \\ &= \max\{d(\mathfrak{v}^*_{\mathtt{bpp}},\digamma(\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}})),d(\mathfrak{w}^*_{\mathtt{bpp}},\digamma(\mathfrak{w}^*_{\mathtt{bpp}},\mathfrak{v}^*_{\mathtt{bpp}}))\}. \end{aligned} \tag{46}$$

Therefore, $(\mathfrak{v}^*_{\mathtt{bpp}},\mathfrak{w}^*_{\mathtt{bpp}})$ becomes a coupled $\mathtt{bpp}$ for Λ. □

5 Applications

Various authors using renowned $\mathtt{FPT}$s have shown the existence of solutions to more and more generalized forms of such fractional-order $\mathbb{DE}$s. In recent times, Kucche *et al.* in their paper [35] stated the most general form and the defined $(k,\daleth)$-H$\mathbb{FD}$ operator of order $p\in(0,\infty)$, type $0\le q\le 1$ acting on a function $\hat{\eta}\in C^n[a,b]$ with $n=\lceil\frac{p}{k}\rceil\in\mathbb{N}$ as

$${}^{k,\mathrm{H}}\mathbb{D}^{p,q;\daleth}\hat{\eta}(\nu) = {}^{k}\mathbb{I}^{q(nk-p);\daleth}\left(\frac{k}{\daleth'(\nu)}\frac{\mathrm{d}}{\mathrm{d}\nu}\right)^{n}{}^{k}\mathbb{I}^{(1-q)(nk-p);\daleth}\hat{\eta}(\nu), \tag{47}$$

where $k\in(0,\infty)$, $\daleth\in C^n[a,b]$ is an increasing function with $\daleth'(\nu)\neq 0$ for $\nu\in[a,b]$ and ${}^{k}\mathbb{I}^{p;\daleth}$ is the $(k,\daleth)$-RL integral of order $p\in(0,\infty)$ as

$${}^{k}\mathbb{I}^{p;\daleth}\hat{\eta}(\nu) = \int_a^\nu \daleth'(\varrho)[\breve{\daleth}_\varrho(\nu)]^{p/k-1}\frac{\hat{\eta}(\varrho)}{k\Gamma_k(p)}\,\mathrm{d}\varrho, \tag{48}$$

where $\breve{\beth}_{\varrho}(\nu) = \beth(\nu) - \beth(\varrho)$ and $\Gamma_k(p) = \int_0^{\infty} \varrho^{p-1} e^{-\varrho^k/k}\, d\varrho$ [36]. Here, we consider a pair of $(k,\beth)$-$\mathbb{HFDE}$ (4) and (5) of order $2 < p < 3$, type $0 \le q \le 1$. The following lemma gives an equivalence between the $\mathbb{FDE}$ (4) with the integral equation (49).

Lemma 5.1 *Let $2 < p < 3$, $q \in [0,1]$ and $\vartheta_k = p + q(3k - p)$, then the equivalent integral to the above-mentioned $\mathbb{DE}$ is*

$$\begin{aligned}\eta_1(\nu) = {}^k\mathbb{I}^{p;\beth} F_1\big(\nu,\eta_1(\nu)\big) + \frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}}\big[\sigma_3{}^k\mathbb{I}^{\nu;\beth} u_1\big(\zeta,\eta_1(\zeta)\big)\\ - \sigma_1{}^k\mathbb{I}^{p;\beth} F_1\big(b,\eta_1(b)\big) - \sigma_2{}^k\mathbb{I}^{p-k;\beth} F_1\big(b,\eta_1(b)\big)\big],\end{aligned} \tag{49}$$

where $A = \sigma_1 + \frac{1}{\breve{\beth}_a(b)}\sigma_2(\vartheta_k - k) \neq 0$.

We are now about to show the existence of the optimum solution of the system (4) and (5) for a more general setting. Consider $\mathcal{S} = C(\mathcal{J},\mathcal{Y})$ (here $\mathcal{Y}$ is $\mathrm{B_S}$) with the supremum norm and choose two subsets S_1 and S_2 of $\mathcal{S}$ as

$$\begin{aligned} S_1 &= \big\{\hat{\eta} \in \mathcal{S} : \eta \in C(\mathcal{J},B_1), \hat{\eta}(a) = 0\big\},\\ S_2 &= \big\{\hat{\eta} \in \mathcal{S} : \eta \in C(\mathcal{J},B_2), \hat{\eta}(a) = 0\big\},\end{aligned} \tag{50}$$

where $B_1 = B_{\mathring{\gamma}_1}[p_0]$ and $B_2 = B_{\mathring{\gamma}_2}[q_0]$ represents two closed balls centered at p_0 and q_0 with radius $\mathring{\gamma}_\iota$, $\iota = 1,2$, respectively, in $\mathcal{Y}$. The functions

$$F_1, u_1 : \mathcal{J} \times B_1 \to \mathcal{Y}, \qquad F_2, u_2 : \mathcal{J} \times B_2 \to \mathcal{Y},$$

are all continuous. Clearly, $S_\iota \neq \emptyset$, $\iota = 1,2$ are both bounded, closed, and convex sets in $\mathcal{S}$. Define Λ on $S_1 \cup S_2$ as

$$\Lambda\hat{\eta}(\nu) = \begin{cases} {}^k\mathbb{I}^{p;\beth} F_2(\nu,\hat{\eta}(\nu)) + \frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}}[\sigma_3{}^k\mathbb{I}^{\nu;\beth} u_2(\zeta,\hat{\eta}(\zeta)) & \\ \quad - \sigma_1{}^k\mathbb{I}^{p;\beth} F_2(b,\hat{\eta}(b)) - \sigma_2{}^k\mathbb{I}^{p-k;\beth} F_2(b,\hat{\eta}(b))], & \hat{\eta} \in S_1,\\ {}^k\mathbb{I}^{p;\beth} F_1(\nu,\hat{\eta}(\nu)) + \frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}}[\sigma_3{}^k\mathbb{I}^{\nu;\beth} u_1(\zeta,\hat{\eta}(\zeta)) & \\ \quad - \sigma_1{}^k\mathbb{I}^{p;\beth} F_1(b,\hat{\eta}(b)) - \sigma_2{}^k\mathbb{I}^{p-k;\beth} F_1(b,\hat{\eta}(b))], & \hat{\eta} \in S_2. \end{cases} \tag{51}$$

Lemma 5.2 *The operator Λ, (51) is cyclic on $S_1 \cup S_2$ whenever*

$$\begin{aligned}\left[1 + \frac{|\sigma_1|}{|A|} + \frac{|\sigma_2|}{|A\Upsilon(k,k)|}\right] F_1^\star \Upsilon(p,k) + \frac{|\sigma_3|}{|A|} u_1^\star \Upsilon(\nu,k) + \|p_0\| \le \mathring{\gamma}_1,\\ \left[1 + \frac{|\sigma_1|}{|A|} + \frac{|\sigma_2|}{|A\Upsilon(k,k)|}\right] F_2^\star \Upsilon(p,k) + \frac{|\sigma_3|}{|A|} u_2^\star \Upsilon(\nu,k) + \|q_0\| \le \mathring{\gamma}_2,\end{aligned} \tag{52}$$

where $\Upsilon(\mathrm{r},k) = \frac{1}{\Gamma_k(\mathrm{r}+k)}[\breve{\beth}_a(b)]^{\mathrm{r}/k}$ and

$$F_1^\star = \sup_{\nu\in\mathcal{J}}\big\{\big\|F_1\big(\nu,\hat{\eta}(\nu)\big)\big\| : \hat{\eta} \in S_2\big\},$$

$$u_1^\star = \sup_{\nu\in\mathcal{J}}\big\{\big\|u_1\big(\nu,\hat{\eta}(\nu)\big)\big\| : \hat{\eta} \in S_2\big\},$$

$$F_2^\star = \sup_{\nu\in\mathcal{J}}\{\|F_2(\nu,\hat{\eta}(\nu))\| : \hat{\eta}\in S_1\},$$
$$u_2^\star = \sup_{\nu\in\mathcal{J}}\{\|u_2(\nu,\hat{\eta}(\nu))\| : \hat{\eta}\in S_1\}. \tag{53}$$

Proof Suppose $\eta \in S_1$ and consider

$$\begin{aligned}
\|(\Lambda\hat{\eta})(\nu)\| &\le \|{}^k\mathbb{I}^{p;\beth}F_2(\nu,\hat{\eta}(\nu))\| + \left|\frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}}\right|\Big[\|\sigma_3{}^k\mathbb{I}^{\nu;\beth}u_2(\zeta,\hat{\eta}(\zeta))\| \\
&\quad + \|\sigma_1{}^k\mathbb{I}^{p;\beth}F_2(b,\hat{\eta}(b))\| + \|\sigma_2{}^k\mathbb{I}^{p-k;\beth}F_2(b,\hat{\eta}(b))\|\Big] \\
&\le \frac{F_2^\star}{k\Gamma_k(p)}\int_a^\nu \beth'(\varrho)[\breve{\beth}_\varrho(\nu)]^{\frac{p}{k}-1}\,d\varrho + \frac{1}{|A|}\Big[\frac{|\sigma_3|u_2^\star}{k\Gamma_k(\nu)}\int_a^\zeta \beth'(\varrho)[\breve{\beth}_\varrho(\zeta)]^{\frac{\nu}{k}-1}\,d\varrho \\
&\quad + \frac{|\sigma_1|F_2^\star}{k\Gamma_k(p)}\int_a^b \beth'(\varrho)[\breve{\beth}_\varrho(b)]^{\frac{p}{k}-1}\,d\varrho \\
&\quad + \frac{|\sigma_2|F_2^\star}{k\Gamma_k(p-k)}\int_a^b \beth'(\varrho)[\breve{\beth}_\varrho(b)]^{p-k/k-1}\,d\varrho\Big] \\
&\le \frac{F_2^\star[\breve{\beth}_a(\nu)]^{p/k}}{p\Gamma_k(p)} + \frac{1}{|A|}\Big[\frac{|\sigma_3|u_2^\star}{\nu\Gamma_k(\nu)}[\breve{\beth}_a(\zeta)]^{\nu/k} \\
&\quad + \frac{|\sigma_1|F_2^\star}{p\Gamma_k(p)}[\breve{\beth}_a(b)]^{p/k} + \frac{|\sigma_2|F_2^\star}{(p-k)\Gamma_k(p-k)}[\breve{\beth}_a(b)]^{p-k/k}\Big] \\
&\le \frac{F_2^\star[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)} + \frac{1}{|A|}\Big[\frac{|\sigma_3|u_2^\star[\breve{\beth}_a(b)]^{\nu/k}}{\nu\Gamma_k(\nu)} \\
&\quad + \frac{|\sigma_1|F_2^\star[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)} + \frac{|\sigma_2|F_2^\star[\breve{\beth}_a(b)]^{p-k/k}}{(p-k)\Gamma_k(p-k)}\Big] \\
&\le \mathring{\gamma}_2 - \|q_0\|,
\end{aligned}$$

so that $\|\Lambda\hat{\eta} - q_0\| \le \mathring{\gamma}_2$ and therefore, we obtain $\Lambda\hat{\eta} \in S_2$. Similarly, we can show that $\hat{\eta} \in S_2 \implies \Lambda\hat{\eta} \in S_1$. Hence, Λ is cyclic. □

We now prove the mean-value theorem of integral calculus for a $(k,\beth)$-RL integral. The proof follows a similar technique to that shown in [37].

Lemma 5.3 *If $p,k > 0$ and η is any continuous function then we can find $z \in (a,b)$ such that*

$${}^k\mathbb{I}^{p;\beth}\eta(\nu) = \int_a^\nu \frac{\beth'(\varrho)}{k\Gamma_k(p)}[\breve{\beth}_\varrho(\nu)]^{p/k-1}\eta(\varrho)\,d\varrho = \frac{[\breve{\beth}_a(\nu)]^{p/k}}{p\Gamma_k(p)}\eta(z). \tag{54}$$

Proof Observe that the function

$$\hat{\beth}(\varrho) = \frac{\beth'(\varrho)}{k\Gamma_k(p)}[\breve{\beth}_\varrho(\nu)]^{p/k-1}, \tag{55}$$

is continuous and does not change its sign for the given range. By applying the generalized mean-value theorem of integral calculus, we write

$$ {}^{k}\mathbb{I}^{p;\gimel}\eta(\nu)=\eta(z)\int_a^{\nu}\frac{\gimel'(\varrho)}{k\Gamma_k(p)}\big[\breve{\gimel}_{\varrho}(\nu)\big]^{p/k-1}\,\mathrm{d}\varrho=\frac{[\breve{\gimel}_a(\nu)]^{p/k}}{p\Gamma_k(p)}\eta(z). \tag{56} $$

□

Theorem 5.4 *Along with the assumptions of Lemma 5.2, suppose that there exists positive real scalars λ_ι, $\iota=1,2$ such that for all $\nu\in\mathcal{J}$ and $(\eta_1,\eta_2)\in S_1\times S_2$, we have*

(H1)

$$ \begin{aligned} &\max\big\{\big\|\digamma_2\big(\nu,\eta_1(\nu)\big)-\digamma_1\big(\nu,\eta_2(\nu)\big)\big\|,\big\|\mathfrak{u}_2\big(\nu,\eta_1(\nu)\big)-\mathfrak{u}_1\big(\nu,\eta_2(\nu)\big)\big\|\big\}\\ &\quad\le\lambda_1\big\|\eta_1(\nu)-\eta_2(\nu)\big\|, \end{aligned} \tag{57} $$

where $\lambda_1\Xi\le 1$ with

$$ \Xi=\frac{1}{|A|}\bigg[\bigg(|A|+|\sigma_1|+\frac{|\sigma_2|}{\Upsilon(k,k)}\bigg)\Upsilon(p,k)+|\sigma_3|\Upsilon(\nu,k)\bigg]; \tag{58} $$

(H2)

$$ \begin{aligned} &\max\big\{\big\|\mathfrak{u}_2\big(\nu,\eta_1(\nu)\big)-\mathfrak{u}_2(\nu,0)\big\|,\big\|\digamma_2\big(\nu,\eta_1(\nu)\big)-\digamma_2(\nu,0)\big\|\big\}\le\acute{\lambda}_1\big\|\eta_1(\nu)\big\|,\\ &\max\big\{\big\|\mathfrak{u}_1\big(\nu,\eta_2(\nu)\big)-\mathfrak{u}_1(\nu,0)\big\|,\big\|\digamma_1\big(\nu,\eta_2(\nu)\big)-\digamma_1(\nu,0)\big\|\big\}\le\acute{\lambda}_2\big\|\eta_2(\nu)\big\|, \end{aligned} \tag{59} $$

where $\lambda_2=\max\{\acute{\lambda}_1,\acute{\lambda}_2\}\in(0,1]$ satisfying $\lambda_2\Xi<1$. Then, the system (4) and (5) of $(k,\gimel)$-$\mathbb{HFDE}$ has an optimal solution.

Proof Clearly, Λ is cyclic, by its definition. Also, the range of Λ is uniformly bounded, since, for $\eta\in S_1$,

$$ \big\|\Lambda\eta_1(\nu)\big\|=\big\|\Lambda\eta_1(\nu)-q_0+q_0\big\|\le\mathring{\gamma}_2+\|q_0\|. \tag{60} $$

Similar conclusions can be drawn when $\eta_2\in S_2$. We now show that $\Lambda(S_1)$ is an equicontinuous set in S. For that, let $\nu<\acute{\nu}$ and $\eta_1\in S_1$ then

$$ \begin{aligned} &\big\|\Lambda\eta_1(\acute{\nu})-\Lambda\eta_1(\nu)\big\|\\ &\quad=\bigg\|{}^{k}\mathbb{I}^{p;\gimel}\big[\digamma_2\big(\acute{\nu},\eta_1(\acute{\nu})\big)-\digamma_2\big(\nu,\eta_1(\nu)\big)\big]\\ &\qquad+\frac{[\breve{\gimel}_a(\acute{\nu})]^{\vartheta_k/k-1}-[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\big[\sigma_3{}^{k}\mathbb{I}^{\nu;\gimel}\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)\\ &\qquad-\sigma_1{}^{k}\mathbb{I}^{p;\gimel}\digamma_2\big(b,\eta_1(b)\big)-\sigma_2{}^{k}\mathbb{I}^{p-k;\gimel}\digamma_2\big(b,\eta_1(b)\big)\big]\bigg\|\\ &\quad\le\bigg\|\int_a^{\nu}\frac{\gimel'(\varrho)}{k\Gamma_k(p)}\big[\big[\breve{\gimel}_{\varrho}(\acute{\nu})\big]^{p/k-1}-\big[\breve{\gimel}_{\varrho}(\nu)\big]^{p/k-1}\big]\digamma_2\big(\varrho,\eta_1(\varrho)\big)\,\mathrm{d}\varrho\bigg\|\\ &\qquad+\bigg\|\int_{\nu}^{\acute{\nu}}\frac{\gimel'(\varrho)}{k\Gamma_k(p)}\big[\breve{\gimel}_{\varrho}(\acute{\nu})\big]^{p/k-1}\digamma_2\big(\varrho,\eta_1(\varrho)\big)\,\mathrm{d}\varrho\bigg\| \end{aligned} $$

$$
\begin{aligned}
&+\frac{[\breve{\gimel}_a(\acute{\nu})]^{\vartheta_k/k-1}-[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{|A|[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\big\|\sigma_3{}^k\mathbb{I}^{\nu;\gimel}\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)\\
&-\sigma_1{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)-\sigma_2{}^k\mathbb{I}^{p-k;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)\big\|\\
\leq{}&\mathcal{F}_2^\star\int_a^\nu\frac{\gimel'(\varrho)}{k\Gamma_k(p)}\big[\big[\breve{\gimel}_\varrho(\acute{\nu})\big]^{p/k-1}-\big[\breve{\gimel}_\varrho(\nu)\big]^{p/k-1}\big]\,\mathrm{d}\varrho\\
&+\mathcal{F}_2^\star\int_\nu^{\acute{\nu}}\frac{\gimel'(\varrho)}{k\Gamma_k(p)}\big[\breve{\gimel}_\varrho(\acute{\nu})\big]^{p/k-1}\,\mathrm{d}\varrho\\
&+\frac{[\breve{\gimel}_a(\acute{\nu})]^{\vartheta_k/k-1}-[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{|A|[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\big\|\sigma_3{}^k\mathbb{I}^{\nu;\gimel}\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)\\
&-\sigma_1{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)-\sigma_2{}^k\mathbb{I}^{p-k;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)\big\|\\
\leq{}&\frac{\mathcal{F}_2^\star}{p\Gamma_k(p)}\big[\big[\breve{\gimel}_a(\acute{\nu})\big]^{p/k}-\big[\breve{\gimel}_\nu(\acute{\nu})\big]^{p/k}-\big[\breve{\gimel}_a(\nu)\big]^{p/k}+\big[\breve{\gimel}_\nu(\acute{\nu})\big]^{p/k}\big]\\
&+\frac{[\breve{\gimel}_a(\acute{\nu})]^{\vartheta_k/k-1}-[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{|A|[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\big\|\sigma_3{}^k\mathbb{I}^{\nu;\gimel}\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)\\
&-\sigma_1{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)-\sigma_2{}^k\mathbb{I}^{p-k;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)\big\|.
\end{aligned}
$$

Indeed, as $\nu\to\acute{\nu}$, we obtain $\Lambda\eta_1(\nu)\to\Lambda\eta_1(\acute{\nu})$. Hence, $\Lambda(S_1)$ is an equicontinuous subset of S. With similar arguments, one can show $\Lambda(S_2)$ to be equicontinuous too. Hence, by the generalized Arzelá–Ascoli theorem, (S_1,S_2) is relatively compact. In order to show Λ is relatively nonexpansive, let $(\eta_1,\eta_2)\in S_1\times S_2$, then for any $\nu\in\mathcal{J}$, we write

$$
\begin{aligned}
\big\|\Lambda\eta_1(\nu)-\Lambda\eta_2(\nu)\big\|={}&\bigg\|{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_2\big(\nu,\eta_1(\nu)\big)+\frac{[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\\
&\times\big[\sigma_3{}^k\mathbb{I}^{\nu;\gimel}\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)-\sigma_1{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)\\
&-\sigma_2{}^k\mathbb{I}^{p-k;\gimel}\mathcal{F}_2\big(b,\eta_1(b)\big)\big]-{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_1\big(\nu,\eta_2(\nu)\big)\\
&-\frac{[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\big[\sigma_3{}^k\mathbb{I}^{\nu;\gimel}\mathfrak{u}_1\big(\zeta,\eta_2(\zeta)\big)\\
&-\sigma_1{}^k\mathbb{I}^{p;\gimel}\mathcal{F}_1\big(b,\eta_2(b)\big)-\sigma_2{}^k\mathbb{I}^{p-k;\gimel}\mathcal{F}_1\big(b,\eta_2(b)\big)\big]\bigg\|\\
\leq{}&{}^k\mathbb{I}^{p;\gimel}\big\|\mathfrak{u}_2\big(\nu,\eta_1(\nu)\big)-\mathcal{F}_1\big(\nu,\eta_2(\nu)\big)\big\|\\
&+\frac{1}{|A|}\big\{|\sigma_3|{}^k\mathbb{I}^{\nu;\gimel}\big\|\mathfrak{u}_2\big(\zeta,\eta_1(\zeta)\big)-\mathfrak{u}_1\big(\zeta,\eta_2(\zeta)\big)\big\|\\
&+|\sigma_1|{}^k\mathbb{I}^{p;\gimel}\big\|\mathcal{F}_2\big(b,\eta_1(b)\big)-\mathcal{F}_1\big(b,\eta_2(b)\big)\big\|\\
&+|\sigma_2|{}^k\mathbb{I}^{p-k;\gimel}\big\|\mathcal{F}_2\big(b,\eta_1(b)\big)-\mathcal{F}_1\big(b,\eta_2(b)\big)\big\|\big\}\\
\leq{}&\lambda_1\|\eta_1-\eta_2\|\bigg\{\frac{1}{k\Gamma_k(p)}\int_a^\nu\gimel'(\varrho)\big[\breve{\gimel}_\varrho(\nu)\big]^{p/k-1}\,\mathrm{d}\varrho\\
&+\frac{1}{|A|}\bigg[\frac{|\sigma_3|}{k\Gamma_k(\nu)}\int_a^\zeta\gimel'(\varrho)\big[\breve{\gimel}_\varrho(\zeta)\big]^{\nu/k-1}\,\mathrm{d}\varrho\\
&+\frac{|\sigma_1|}{k\Gamma_k(p)}\int_a^b\gimel'(\varrho)\big[\breve{\gimel}_\varrho(b)\big]^{p/k-1}\,\mathrm{d}\varrho
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\sigma_2|}{k\Gamma_k(p-k)}\int_a^b \beth'(\varrho)\big[\breve{\beth}_\varrho(b)\big]^{p-k/k-1}\,\mathrm{d}\varrho\Bigg]\Bigg\} \\
&\le \lambda_1\|\eta_1-\eta_2\|\Bigg\{\frac{[\breve{\beth}_a(\nu)]^{p/k}}{p\Gamma_k(p)}+\frac{1}{|A|}\Bigg[\frac{|\sigma_3|[\breve{\beth}_a(\zeta)]^{q/k}}{\nu\Gamma_k(\nu)} \\
&\quad+\frac{|\sigma_1|[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)}+\frac{|\sigma_2|[\breve{\beth}_a(b)]^{p-k/k}}{(p-k)\Gamma_k(p-k)}\Bigg]\Bigg\} \\
&\le \lambda_1\|\eta_1-\eta_2\|\Bigg\{\frac{[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)}+\frac{1}{|A|}\Bigg[\frac{|\sigma_3|[\breve{\beth}_a(b)]^{\nu/k}}{\nu\Gamma_k(\nu)} \\
&\quad+\frac{|\sigma_1|[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)}+\frac{|\sigma_2|[\breve{\beth}_a(b)]^{p-k/k}}{(p-k)\Gamma_k(p-k)}\Bigg]\Bigg\} \\
&= \lambda_1\|\eta_1-\eta_2\|\Bigg\{\Upsilon(p,k)+\frac{1}{|A|}\big[|\sigma_3|\Upsilon(\nu,k) \\
&\quad+|\sigma_1|\Upsilon(p,k)+|\sigma_2|\Upsilon(p-k,k)\big]\Bigg\} \\
&\le \|\sigma_1-\sigma_2\|.
\end{aligned}
$$

This implies that Λ is relatively nonexpansive. Also, Λ is $\mathcal{A}$-condensing, because, for $(E,\acute{E})\in\mathcal{M}_\Lambda(S_1,S_2)$ we have

$$
\begin{aligned}
\eta(\Lambda E\cup\Lambda\acute{E}) &= \max\big\{\eta(\Lambda E),\eta(\Lambda\acute{E})\big\} \\
&= \max\Big\{\sup_{\nu\in\mathcal{J}}\big\{\eta\big(\{\Lambda\eta_1(\nu):\eta_1\in E\}\big)\big\},\sup_{\nu\in\mathcal{J}}\big\{\eta\big(\{\Lambda\eta_2(\nu):\eta_2\in\acute{E}\}\big)\big\}\Big\} \\
&= \max\Bigg\{\sup_{\nu\in\mathcal{J}}\Bigg\{\eta\Bigg(\Bigg\{{}^k\mathbb{I}^{p;\beth}\digamma_2\big(\nu,\eta_1(\nu)\big)+\frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}} \\
&\quad\times\big[\sigma_3{}^k\sigma_3^{\nu;\beth}\mathrm{u}_2\big(\zeta,\eta_1(\zeta)\big)-\sigma_1{}^k\mathbb{I}^{p;\beth}\digamma_2\big(b,\eta_1(b)\big) \\
&\quad-\sigma_2{}^k\mathbb{I}^{p-k;\beth}\digamma_2\big(b,\eta_1(b)\big)\big]:\eta_1\in E\Bigg\}\Bigg)\Bigg\}, \\
&\quad\sup_{\nu\in\mathcal{J}}\Bigg\{\eta\Bigg(\Bigg\{{}^k\mathbb{I}^{p;\beth}\digamma_1\big(\nu,\eta_2(\nu)\big)+\frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}} \\
&\quad\times\big[\sigma_3{}^k\mathbb{I}^{\nu;\beth}\mathrm{u}_1\big(\zeta,\eta_2(\zeta)\big)-\sigma_1{}^k\mathbb{I}^{p;\beth}\digamma_1\big(b,\eta_2(b)\big) \\
&\quad-\sigma_2{}^k\mathbb{I}^{p-k;\beth}\digamma_1\big(b,\eta_2(b)\big)\big]:\eta_2\in\acute{E}\Bigg\}\Bigg)\Bigg\}\Bigg\} \\
&= \max\Bigg\{\sup_{\nu\in\mathcal{J}}\Bigg\{\eta\Bigg(\Bigg\{\frac{[\breve{\beth}_a(\nu)]^{p/k}\digamma_2(z,\eta_1(z))}{p\Gamma_k(p)} \\
&\quad+\frac{[\breve{\beth}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\beth}_a(b)]^{\vartheta_k/k-1}}\Bigg[\frac{\sigma_3[\breve{\beth}_a(\zeta)]^{\nu/k}}{\nu\Gamma_k(\nu)}\mathrm{u}_2\big(z,\eta_1(z)\big)-\frac{\sigma_1[\breve{\beth}_a(b)]^{p/k}}{p\Gamma_k(p)}\digamma_2\big(z,\eta_1(z)\big) \\
&\quad-\frac{\sigma_2[\breve{\beth}_a(b)]^{p-k/k}}{p\Gamma_k(p)}\digamma_2\big(z,\eta_1(z)\big)\Bigg]:\text{for some } z\in\mathcal{J}\Bigg\}\Bigg)\Bigg\}, \\
&\quad\sup_{\nu\in\mathcal{J}}\Bigg\{\eta\Bigg(\Bigg\{\frac{[\breve{\beth}_a(\nu)]^{p/k}\digamma_1(z,\eta_2(z))}{p\Gamma_k(p)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{[\breve{\gimel}_a(\nu)]^{\vartheta_k/k-1}}{A[\breve{\gimel}_a(b)]^{\vartheta_k/k-1}}\left[\frac{\sigma_3[\breve{\gimel}_a(\zeta)]^{\nu/k}}{\nu\Gamma_k(\nu)}\mathfrak{u}_1\big(z,\eta_2(z)\big)-\frac{\sigma_1[\breve{\gimel}_a(b)]^{p/k}}{p\Gamma_k(p)}\digamma_1\big(z,\eta_2(z)\big)\right.\\
&\left.-\frac{\sigma_2[\breve{\gimel}_a(b)]^{p-k/k}}{p\Gamma_k(p)}\digamma_1\big(z,\eta_2(z)\big)\right]:\text{for some } z\in\mathcal{J}\bigg\}\bigg)\bigg\}\bigg\}\\
<{}&\max\bigg\{\frac{[\breve{\gimel}_a(b)]^{p/k}}{p\Gamma_k(p)}\eta\big(\big\{\digamma_2\big(z,\eta_1(z)\big)+\digamma_2(z,0)-\digamma_2(z,0)\big\}\big)\\
&+\frac{1}{|A|}\left[\frac{|\sigma_3|[\breve{\gimel}_a(b)]^{\nu/k}}{\nu\Gamma_k(\nu)}\eta\big(\big\{\mathfrak{u}_2\big(z,\eta_1(z)\big)+\mathfrak{u}_2(z,0)-\mathfrak{u}_2(z,0)\big\}\big)\right.\\
&+\frac{|\sigma_1|[\breve{\gimel}_a(b)]^{p/k}}{p\Gamma_k(p)}\eta\big(\big\{\digamma_2\big(z,\eta_1(z)\big)+\digamma_2(z,0)-\digamma_2(z,0)\big\}\big)\\
&\left.+\frac{|\sigma_2|[\breve{\gimel}_a(b)]^{p-k/k}}{(p-k)\Gamma_k(p-k)}\eta\big(\big\{\digamma_2\big(z,\eta_1(z)\big)+\digamma_2(z,0)-\digamma_2(z,0)\big\}\big)\right],\\
&\frac{[\breve{\gimel}_a(b)]^{p/k}}{p\Gamma_k(p)}\eta\big(\big\{\digamma_1\big(z,\eta_2(z)\big)+\digamma_1(z,0)-\digamma_1(z,0)\big\}\big)\\
&+\frac{1}{|A|}\left[\frac{|\sigma_3|[\breve{\gimel}_a(b)]^{\nu/k}}{\nu\Gamma_k(\nu)}\eta\big(\big\{\mathfrak{u}_1\big(z,\eta_2(z)\big)+\mathfrak{u}_1(z,0)-\mathfrak{u}_1(z,0)\big\}\big)\right.\\
&+\left(\frac{|\sigma_1|L^{p/k}}{p\Gamma_k(p)}+\frac{|\sigma_2|[\breve{\gimel}_a(b)]^{p-k/k}}{(p-k)\Gamma_k(p-k)}\right)\\
&\left.\times\eta\big(\big\{\digamma_1\big(z,\eta_2(z)\big)+\digamma_1(z,0)-\digamma_1(z,0)\big\}\big)\right]\bigg\}\\
\le{}&\max\big\{\lambda_2\Xi\eta(E),\lambda_2\Xi\eta(\acute{E})\big\}=\lambda_2\Xi\eta(E\cup\acute{E}).
\end{aligned}
$$

By choosing $\rho(\nu,w)=\lambda_2 w\Xi-\nu$, the operator Λ becomes $\mathcal{A}$-condensing. Hence, the hypothesis of Theorem 3.2 is fulfilled and hence, the bpp of Λ is the optimum solution of the system (4) and (5). □

6 Illustrative examples

The following examples reveal the hidden realities in the application of the results of this article. In the first example, we examine our results as an application to find a mild solution of the system (4) and (5) for different states of $\gimel$.

Example 6.1 According to the system (4) and (5), we consider the system $(2.1,\gimel)$-$\mathbb{HFDE}$ under conditions for $\nu\in\mathcal{J}=[0,1]$ in the form

$$
\begin{cases}
{}^{2.1,\mathrm{H}}\mathbb{D}^{9/4,3/5;\gimel}\eta_1(\nu)=\frac{\sqrt{2}\exp(\nu/8)\sin(\eta_1(\nu))}{\sqrt[3]{65}(0.5+\nu^2)},\\
\eta_1(0)=\eta_1'(0)=0,\\
\frac{23}{15}\eta_1(1)+\frac{45}{22}\delta_\gimel\eta_1(1)=\frac{32}{65}\,{}^{2.1}\mathbb{I}^{12/17;\gimel}\big(\frac{(2+|\nu|)\tan(\eta_1(\nu))}{\sqrt[3]{31}\sqrt{5}}\big),
\end{cases}
\tag{61}
$$

$$
\begin{cases}
{}^{2.1,\mathrm{H}}\mathbb{D}^{9/4,3/5;\gimel}\eta_2(\nu)=\frac{3\nu^2(1+|\nu|)\sin(\eta_2(\nu))}{\sqrt[4]{27}(6+\cos^2\nu)(\sqrt{10}+\sin^2(\eta_2(\nu)))},\\
\eta_2(0)=\eta_2'(0)=0,\\
\frac{23}{15}\eta_2(1)+\frac{45}{22}\delta_\gimel\eta_2(1)=\frac{32}{65}\,{}^{2.1}\mathbb{I}^{12/17;\gimel}\big[\frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5+\tan(\pi\nu/4))\exp(|\nu|)[\sqrt{7}+(\tan^{-1}(\nu))^2]}\big],
\end{cases}
\tag{62}
$$

with four cases of $\gimel$ as

$$\gimel_1(\nu) = 1.5^{\nu}, \qquad \gimel_2(\nu) = \nu, \qquad \gimel_3(\nu) = \ln \frac{2\nu + 1}{300}, \qquad \gimel_4(\nu) = \sqrt{\nu}, \tag{63}$$

which are in $C(\mathcal{J})$ and $\gimel_i'(\nu) > 0$ on $\mathcal{J}$, where $\delta_\gimel = \frac{2.1}{\gimel'(\nu)}\frac{d}{d\nu}$. Clearly, $p = \frac{9}{4} \in (2,3)$, $q = \frac{3}{5} \in [0,1]$, and $\nu = \frac{12}{17} > 0$. Then, $\vartheta_{2.1} = p + q(6.3 - p) = \frac{117}{25}$. Also, by using the data, we obtain

$$A = \sigma_1 + \frac{1}{\tilde{\gimel}_a(b)}\sigma_2(\vartheta_k - k) = \frac{23}{15} + \frac{\frac{45}{22}(\frac{63}{10} - 3)}{\gimel_i(1) - \gimel_i(0)} \simeq \begin{Bmatrix} 12.0878, & \gimel_1(\nu) = 1.5^{\nu}, \\ 6.8106, & \gimel_2(\nu) = \nu, \\ 6.3369, & \gimel_3(\nu) = \ln \frac{2\nu+1}{300}, \\ 6.8106, & \gimel_4(\nu) = \sqrt{\nu}, \end{Bmatrix} \neq 0.$$

We define

$$\begin{aligned} &\digamma_1\big(\nu, \eta(\nu)\big) = \frac{\sqrt{2}\exp(\frac{\nu}{8})\sin(\eta_1(\nu))}{\sqrt[3]{65}(0.5 + \nu^2)}, \qquad \mathfrak{u}_1\big(\nu, \eta_1(\nu)\big) = \frac{(2 + |\nu|)\tan(\eta_1(\nu))}{\sqrt[3]{31}\sqrt{5}}, \\ &\digamma_2\big(\nu, \eta_2(\nu)\big) = \frac{3\nu^2(1 + |\nu|)\sin(\eta_2(\nu))}{\sqrt[4]{27}(6 + \cos^2\nu)(\sqrt{10} + \sin^2(\eta_2(\nu)))}, \\ &\mathfrak{u}_2\big(\nu, \eta_2(\nu)\big) = \frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5 + \tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7} + (\tan^{-1}(\nu))^2)}. \end{aligned} \tag{64}$$

We observe that,

$$\begin{aligned} &\big|\digamma_2\big(\nu, \eta_1(\nu)\big) - \digamma_1\big(\nu, \eta_2(\nu)\big)\big| \\ &\quad = \left| \frac{3\nu^2(1 + |\nu|)\sin(\eta_1(\nu))}{\sqrt[4]{27}(6 + \cos^2\nu)(\sqrt{10} + \sin^2(\eta_1(\nu)))} - \frac{\sqrt{2}\exp(\nu/8)\sin(\eta_2(\nu))}{\sqrt[3]{65}(0.5 + \nu^2)} \right| \\ &\quad \leq \left| \frac{\sin(\eta_1(\nu))}{\sqrt{10}\sqrt[4]{27}} - \frac{\sqrt{2}\sin(\eta_2(\nu))}{1.5\sqrt[3]{65}} \right| \leq \frac{\sqrt{2}}{1.5\sqrt[3]{35}}\big|\eta_1(\nu) - \eta_2(\nu)\big|, \\ &\big|\mathfrak{u}_2\big(\nu, \eta_2(\nu)\big) - \mathfrak{u}_1\big(\nu, \eta_1(\nu)\big)\big| \\ &\quad = \left| \frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5 + \tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7} + (\tan^{-1}(\nu))^2)} - \frac{(2 + |\nu|)\tan(\eta_1(\nu))}{\sqrt[3]{31}\sqrt{5}} \right| \\ &\quad \leq \left| \frac{2\tan(\eta_2(\nu))}{\sqrt{7}\sqrt[3]{30}} - \frac{2\tan(\eta_1(\nu))}{\sqrt{5}\sqrt[3]{31}} \right| \leq \frac{2}{\sqrt{5}\sqrt[3]{31}}\big|\eta_1(\nu) - \eta_2(\nu)\big|. \end{aligned} \tag{65}$$

Hence, to confirm assumption (H1), from Eq. (57), we have

$$\begin{aligned} &\max\big\{\big\|\digamma_2\big(\nu, \eta_1(\nu)\big) - \digamma_1\big(\nu, \eta_2(\nu)\big)\big\|, \big\|\mathfrak{u}_2\big(\nu, \eta_1(\nu)\big) - \mathfrak{u}_1\big(\nu, \eta_2(\nu)\big)\big\|\big\} \\ &\quad \leq \max\left\{ \frac{\sqrt{2}}{1.5\sqrt[3]{65}}\big\|\eta_1(\nu) - \eta_2(\nu)\big\|, \frac{2}{\sqrt{5}\sqrt[3]{31}}\big\|\eta_1(\nu) - \eta_2(\nu)\big\| \right\} \\ &\quad \leq \lambda_1\big\|\eta_1(\nu) - \eta_2(\nu)\big\|, \end{aligned} \tag{66}$$

where $\lambda_1 = \frac{2}{\sqrt{5}\sqrt[3]{31}}$. Now, thanks to Eq. (58), we obtain

$$\Xi = \frac{1}{|A|}\left[\left(|A| + |\sigma_1| + \frac{|\sigma_2|}{\Upsilon(k,k)}\right)\Upsilon(p,k) + |\sigma_3|\Upsilon(\nu,k)\right]$$

$$= \frac{1}{|A_i|}\left[\left(|A_i| + \left|\frac{23}{15}\right| + \frac{|45/22|}{\Upsilon(3,3)}\right)\Upsilon\left(\frac{9}{4},3\right) + \left|\frac{32}{65}\right|\Upsilon\left(\frac{12}{17},3\right)\right]$$

$$= \frac{1}{|A_i|}\left[\left(|A_i| + \left|\frac{23}{15}\right| + \frac{|45/22|}{\frac{1}{\Gamma(6)}[\gimel_i(1) - \gimel_i(0)]}\right)\frac{1}{\Gamma(\frac{21}{4})}\left[\gimel_i(1) - \gimel_i(0)\right]^{3/4}\right.$$

$$\left. + \left|\frac{32}{65}\right|\frac{1}{\Gamma(63/17)}\left[\gimel_i(1) - \gimel_i(0)\right]^{4/17}\right]$$

$$\simeq \begin{cases} 0.4107, & \gimel_1(\nu) = 1.5^{\nu}, \\ 0.8750, & \gimel_2(\nu) = \nu, \\ 0.9695, & \gimel_3(\nu) = \frac{2\nu+1}{300}, \\ 0.8750, & \gimel_4(\nu) = \sqrt{\nu}. \end{cases} \tag{67}$$

Furthermore,

$$\lambda_1 \Xi \simeq \left\{\begin{matrix} 0.1169, & \gimel_1(\nu) = 1.5^{\nu}, \\ 0.2491, & \gimel_2(\nu) = \nu, \\ 0.2760, & \gimel_3(\nu) = \frac{2\nu+1}{300}, \\ 0.2491, & \gimel_4(\nu) = \sqrt{\nu}. \end{matrix}\right\} < 1. \tag{68}$$

Hence, assumption (H1) holds. In Figs. 1 and 2a, 2b, we have plotted the results of Ξ and $\lambda_1 \Xi$, $\lambda_2 \Xi$ for the system (61) and (62). Also, these results are shown in Tables 1 and 2. To check the next assumption (H2), we take help from relations (59). Hence,

$$\begin{aligned} &\left|\mathfrak{u}_2(\nu, \eta_1(\nu)) - \mathfrak{u}_2(\nu, 0)\right| \\ &= \left|\frac{10\nu \tan(\eta_1(\nu))}{\sqrt[3]{30}(5 + \tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7} + (\tan^{-1}(\nu))^2)}\right| \leq \frac{2}{\sqrt[3]{30}\sqrt{7}}\left|\eta_1(\nu)\right|, \end{aligned} \tag{69}$$

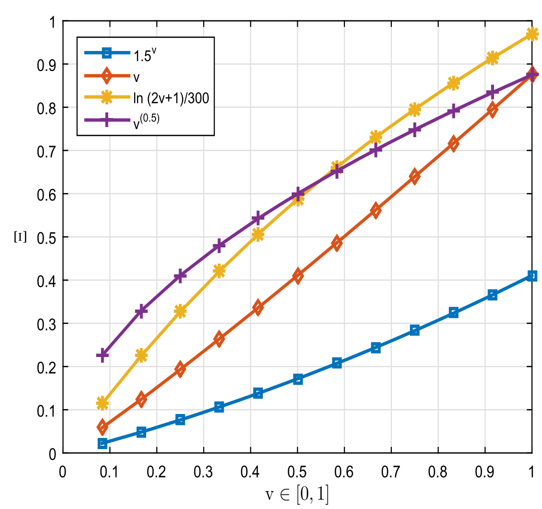


Figure 1 Representation of Ξ in assumption (H1) for four cases of $\gimel$ in the system (61) and (62) in Example 6.1

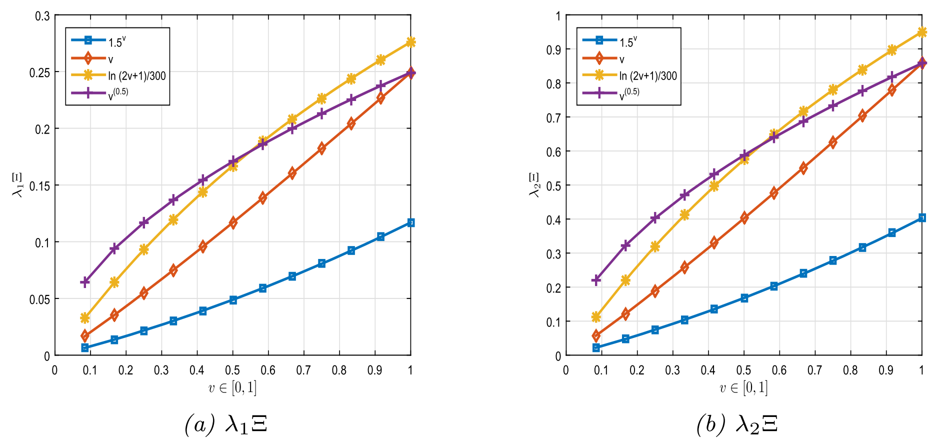


Figure 2 2D plot of $\lambda_1\,\Xi$ and $\lambda_2\,\Xi$ in assumption (H2) for four cases of $\gimel$ in the system (61) and (62) in Example 6.1

Table 1 Numerical values of Ξ, and $\lambda_1\,\Xi$ in Example 6.1 when $\gimel(v) = 1.5^v, v$

v	$\gimel_1(v) = 1.5^v$			$\gimel_2(v) = v$		
	Ξ	$\lambda_1\,\Xi < 1$	$\lambda_2\,\Xi < 1$	v	$\lambda_1\,\Xi < 1$	$\lambda_2\,\Xi < 1$
0.08	0.0224	0.0064	0.0219	0.0584	0.0166	0.0572
0.17	0.0483	0.0137	0.0473	0.1240	0.0353	0.1215
0.25	0.0763	0.0217	0.0748	0.1928	0.0549	0.1889
0.33	0.1063	0.0303	0.1042	0.2639	0.0751	0.2585
0.42	0.1382	0.0393	0.1354	0.3366	0.0958	0.3298
0.50	0.1717	0.0489	0.1683	0.4107	0.1169	0.4024
0.58	0.2070	0.0589	0.2029	0.4859	0.1384	0.4761
0.67	0.2441	0.0695	0.2392	0.5622	0.1601	0.5508
0.75	0.2830	0.0806	0.2773	0.6393	0.1820	0.6264
0.83	0.3237	0.0922	0.3171	0.7172	0.2042	0.7027
0.92	0.3662	0.1043	0.3588	0.7958	0.2266	0.7797
1.00	0.4107	0.1169	0.4024	0.8750	0.2491	0.8573

Table 2 Numerical values of Ξ, and $\lambda_1\,\Xi$ $\lambda_1\,\Xi$ in Example 6.1 when $\gimel(v) = \ln\frac{2v+1}{300}, \sqrt{v}$

v	$\gimel_3(v) = \ln\frac{2v+1}{300}$			$\gimel_4(v) = \sqrt{v}$		
	Ξ	$\lambda_1\,\Xi < 1$	$\lambda_2\,\Xi < 1$	v	$\lambda_1\,\Xi < 1$	$\lambda_2\,\Xi < 1$
0.08	0.1139	0.0324	0.1116	0.2256	0.0642	0.2210
0.17	0.2247	0.0640	0.2202	0.3292	0.0937	0.3225
0.25	0.3267	0.0930	0.3201	0.4107	0.1169	0.4024
0.33	0.4204	0.1197	0.4119	0.4805	0.1368	0.4708
0.42	0.5067	0.1443	0.4965	0.5427	0.1545	0.5318
0.50	0.5866	0.1670	0.5747	0.5995	0.1707	0.5874
0.58	0.6609	0.1882	0.6475	0.6521	0.1857	0.6389
0.67	0.7303	0.2079	0.7156	0.7014	0.1997	0.6872
0.75	0.7954	0.2265	0.7793	0.7479	0.2130	0.7328
0.83	0.8567	0.2439	0.8394	0.7922	0.2255	0.7762
0.92	0.9146	0.2604	0.8961	0.8344	0.2376	0.8176
1.00	0.9695	0.2760	0.9499	0.8750	0.2491	0.8573

$$
\begin{aligned}
\left|F_2\big(\nu,\eta_1(\nu)\big) - F_2(\nu,0)\right| &= \left|\frac{3\nu^2(1+|\nu|)\sin(\eta_1(\nu))}{\sqrt[4]{27}(6+\cos^2\nu)(\sqrt{10}+\sin^2(\eta_1(\nu)))}\right| \\
&\le \frac{1}{\sqrt[4]{27}}\left|\frac{\sin(\eta_1(\nu))}{\sqrt{10}+\sin^2(\eta_1(\nu))}\right| \le \frac{1}{\sqrt{10}\sqrt[4]{27}}\left|\eta_1(\nu)\right| \qquad (70)
\end{aligned}
$$

and

$$\begin{aligned}&\max\left\{\left\|\mathrm{u}_2\big(\nu,\eta_1(\nu)\big)-\mathrm{u}_2(\nu,0)\right\|,\left\|\digamma_2\big(\nu,\eta_1(\nu)\big)-\digamma_2(\nu,0)\right\|\right\}\\&\quad=\max\left\{\frac{2}{\sqrt[3]{30}\sqrt{7}}\left\|\eta_1(\nu)\right\|,\frac{1}{\sqrt{10}\sqrt[4]{27}}\left\|\eta_1(\nu)\right\|\right\}\le\acute{\lambda}_1\left\|\eta_1(\nu)\right\|,\end{aligned}\tag{71}$$

where $\acute{\lambda}_1=\frac{2}{\sqrt[3]{30}\sqrt{7}}$. In addition,

$$\begin{aligned}\left|\mathrm{u}_1\big(\nu,\eta_2(\nu)\big)-\mathrm{u}_1(\nu,0)\right|&=\left|\frac{(2+|\nu|)\tan(\eta_2(\nu))}{\sqrt[3]{31}\sqrt{5}}\right|\\&\le\frac{3}{\sqrt{5}\sqrt[3]{31}}\left|\tan\big(\eta_2(\nu)\big)\right|\le\frac{3}{\sqrt{5}\sqrt[3]{31}}\left|\eta_2(\nu)\right|,\\\left|\digamma_1\big(\nu,\eta_2(\nu)\big)-\digamma_1(\nu,0)\right|&=\left|\frac{\sqrt{2}\exp(\nu/8)\sin(\eta_2(\nu))}{\sqrt[3]{65}(0.5+\nu^2)}\right|\\&\le\frac{e^{1/8}\sqrt{2}}{0.5\sqrt[3]{65}}\left|\sin\big(\eta_2(\nu)\big)\right|\le\frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{65}}\left|\eta_2(\nu)\right|\end{aligned}\tag{72}$$

and

$$\begin{aligned}&\max\left\{\left\|\mathrm{u}_1\big(\nu,\eta_2(\nu)\big)-\mathrm{u}_1(\nu,0)\right\|,\left\|\digamma_1\big(\nu,\eta_2(\nu)\big)-\digamma_1(\nu,0)\right\|\right\}\\&\quad=\max\left\{\frac{3}{\sqrt{5}\sqrt[3]{31}}\left\|\eta_2(\nu)\right\|,\frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{65}}\left\|\eta_2(\nu)\right\|\right\}\le\acute{\lambda}_2\left\|\eta_2(\nu)\right\|,\end{aligned}\tag{73}$$

where $\acute{\lambda}_2=\frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{35}}$. Therefore,

$$\lambda_2=\max\{\acute{\lambda}_1,\acute{\lambda}_2\}=\left\{\frac{2}{\sqrt[3]{30}\sqrt{7}},\frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{35}}\right\}=\frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{35}}\in(0,1]\tag{74}$$

and

$$\lambda_2\,\Xi\simeq\left\{\begin{array}{ll}0.4024, & \beth_1(\nu)=1.5^{\nu},\\0.8573, & \beth_2(\nu)=\nu,\\0.9499, & \beth_3(\nu)=\frac{2\nu+1}{300},\\0.8573, & \beth_4(\nu)=\sqrt{\nu}.\end{array}\right\}<1.\tag{75}$$

Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (61) and (62) of $(2.1,\beth)$-$\mathrm{H}\mathbb{FDE}$ has an optimal solution.

In the next example, we can see the correctness of the results for the existence of an optimal solution of the system (4) and (5) for different fractional derivatives of order p.

Example 6.2 We consider the system of $(2.1,\beth)$-$\mathrm{H}\mathbb{FDE}$ (61) and (62) in Example 6.1 for $\nu\in\mathcal{J}=[0,1]$ in the form

$$\begin{cases}{}^{2.1,\mathrm{H}}\mathbb{D}^{p,3/5;\nu/2}\eta_1(\nu)=\frac{\sqrt{2}e^{\nu/8}\sin(\eta_1(\nu))}{\sqrt[3]{65}(0.5+\nu^2)},\\{}^{2.1,\mathrm{H}}\mathbb{D}^{p,3/5;\nu/2}\eta_2(\nu)=\frac{3\nu^2(1+|\nu|)\sin(\eta_2(\nu))}{\sqrt[4]{27}(6+\cos^2\nu)(\sqrt{10}+\sin^2(\eta_2(\nu)))},\end{cases}\tag{76}$$

for the following three cases of p: $p = \{\frac{9}{4}, \frac{5}{2}, \frac{23}{8}\}$, under the same conditions

$$\begin{cases} \eta_1(0) = \eta_1'(0) = 0, \qquad \eta_2(0) = \eta_2'(0) = 0, \\ \frac{23}{15}\eta_1(1) + \frac{45}{22}\delta_{\nu/2}\eta_1(1) = \frac{32}{65}{}^{2.1}\mathbb{I}^{12/17;\nu/2}\left(\frac{(2+|\nu|)\tan(\eta_1(\nu))}{\sqrt[3]{31}\sqrt{5}}\right), \\ \frac{23}{15}\eta_2(1) + \frac{45}{22}\delta_{\nu/2}\eta_2(1) = \frac{32}{65}{}^{2.1}\mathbb{I}^{12/17;\nu/2}\left(\frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5+\tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7}+(\tan^{-1}(\nu))^2)}\right), \end{cases} \tag{77}$$

for $\nu \in \mathcal{J}$ and $\beth(\nu) = \frac{1}{2}\nu \in C(\mathcal{J})$. Clearly, $q = \frac{3}{5} \in [0,1]$ and $\nu = \frac{12}{17} > 0$. Then,

$$\vartheta_{2.1} = p_i + q(6.3 - p_i) = \begin{cases} 4.68, & p = \frac{9}{4}, \\ 4.78, & p = \frac{5}{2}, \\ 4.93, & p = \frac{23}{8} \end{cases} \tag{78}$$

and

$$A = \sigma_1 + \frac{\sigma_2(\vartheta_{2.1i} - k)}{\beth(1) - \beth(0)} \simeq \left\{ \begin{array}{ll} 12.0878, & p = \frac{9}{4}, \\ 12.4969, & p = \frac{5}{2}, \\ 13.1106, & p = \frac{23}{8}, \end{array} \right\} \neq 0. \tag{79}$$

By choosing the same defined functions $\digamma_1(\nu,\eta(\nu))$, $\mathfrak{u}_1(\nu,\eta_1(\nu))$, $\digamma_2(\nu,\eta_2(\nu))$, and $\mathfrak{u}_2(\nu,\eta_2(\nu))$, again, assumptions (H1) and (H2) are valid with

$$\begin{aligned} &\left\|\digamma_2(\nu,\eta_1(\nu)) - \digamma_1(\nu,\eta_2(\nu))\right\| \le \frac{\sqrt{2}}{1.5\sqrt[3]{35}}\left\|\eta_1(\nu) - \eta_2(\nu)\right\|, \\ &\left\|\mathfrak{u}_2(\nu,\eta_2(\nu)) - \mathfrak{u}_1(\nu,\eta_1(\nu))\right\| \le \frac{2}{\sqrt{5}\sqrt[3]{31}}\left\|\eta_1(\nu) - \eta_2(\nu)\right\| \end{aligned} \tag{80}$$

and $\lambda_1 = \frac{2}{\sqrt{5}\sqrt[3]{31}} \in (0,1)$, $\lambda_2 = \frac{2\sqrt{2}}{\sqrt[3]{35}}e^{1/8} \in (0,1]$. In addition, Eq. (58) implies that

$$\Xi = \frac{1}{|A_i|}\left[\left(|A_i| + |\sigma_1| + \frac{|\sigma_2|}{\Upsilon(k,k)}\right)\Upsilon(p_i,k) + |\sigma_3|\Upsilon(\nu,k)\right] \simeq \begin{cases} 0.4106, & p = \frac{9}{4}, \\ 0.3263, & p = \frac{5}{2}, \\ 0.2300, & p = \frac{23}{8}. \end{cases}$$

In Fig. 3, we have plotted the results of Ξ for the system (76) and (77). Also, these results are shown in Table 3. Furthermore,

$$\lambda_1\Xi \simeq \left\{ \begin{array}{ll} 0.1169, & p = \frac{9}{4}, \\ 0.0929, & p = \frac{5}{2}, \\ 0.0655, & p = \frac{23}{8}, \end{array} \right\} < 1, \qquad \lambda_2\Xi \simeq \left\{ \begin{array}{ll} 0.4024, & p = \frac{9}{4}, \\ 0.3198, & p = \frac{5}{2}, \\ 0.2254, & p = \frac{23}{8}, \end{array} \right\} < 1. \tag{81}$$

In Figs. 4a and 4b, we have plotted the results of $\lambda_1\Xi$ and $\lambda_2\Xi$ for the system (76) and (77). Also, these results are shown in Table 3. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (76) and (77) of $(2.1,\beth)$-$\mathbb{HFDE}$ has an optimal solution.

The next example examines the results for different values of type q.

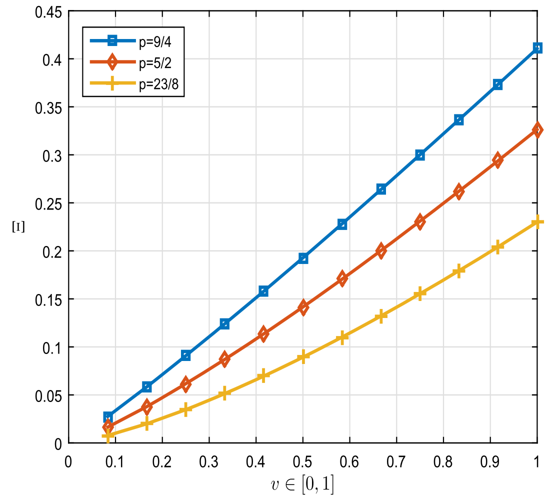


Figure 3 Representation of Ξ in assumption (H1) for three cases of order p in the system (76) and (77) in Example 6.2

Table 3 Numerical values of Ξ, and λ_1 Ξ λ_2 Ξ in Example 6.2 when $p = \frac{9}{4}, \frac{5}{2}, \frac{23}{8}$

v	$p = \frac{9}{4}$			$p = \frac{5}{2}$			$p = \frac{23}{8}$		
	Ξ	λ_1 Ξ < 1	λ_2 Ξ < 1	Ξ	λ_1 Ξ < 1	λ_2 Ξ < 1	Ξ	λ_1 Ξ < 1	λ_2 Ξ < 1
0.08	0.0275	0.0078	0.0270	0.0164	0.0047	0.0161	0.0077	0.0022	0.0076
0.17	0.0584	0.0166	0.0572	0.0378	0.0108	0.0370	0.0199	0.0057	0.0195
0.25	0.0907	0.0258	0.0889	0.0615	0.0175	0.0603	0.0347	0.0099	0.0340
0.33	0.1240	0.0353	0.1215	0.0870	0.0248	0.0852	0.0513	0.0146	0.0503
0.42	0.1581	0.0450	0.1549	0.1138	0.0324	0.1115	0.0696	0.0198	0.0682
0.50	0.1928	0.0549	0.1889	0.1417	0.0403	0.1388	0.0893	0.0254	0.0875
0.58	0.2281	0.0650	0.2235	0.1706	0.0486	0.1671	0.1102	0.0314	0.1080
0.67	0.2639	0.0751	0.2585	0.2003	0.0570	0.1963	0.1322	0.0377	0.1296
0.75	0.3000	0.0854	0.2940	0.2308	0.0657	0.2262	0.1553	0.0442	0.1522
0.83	0.3366	0.0958	0.3298	0.2621	0.0746	0.2568	0.1793	0.0511	0.1757
0.92	0.3735	0.1063	0.3659	0.2939	0.0837	0.2880	0.2043	0.0582	0.2001
1.00	0.4107	0.1169	0.4024	0.3264	0.0929	0.3198	0.2300	0.0655	0.2254

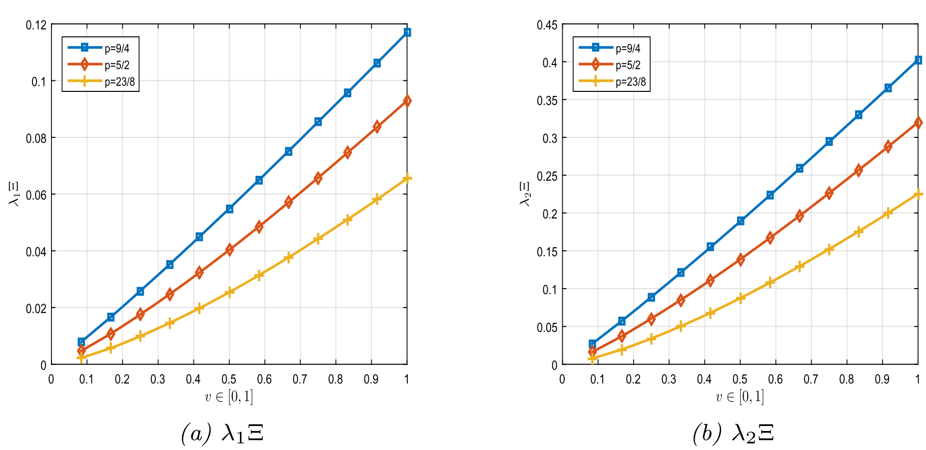


Figure 4 2D plot of λ_1 Ξ and λ_2 Ξ in assumption (H2) for three cases of p in the system (76) and (77) in Example 6.2

Example 6.3 Consider the system (61) and (62) in Example 6.1 for $\nu \in \mathcal{J} = [0,1]$ in the form

$$\begin{cases} {}^{2.1,\mathrm{H}}\mathbb{D}^{11/4,q;\sqrt{2\nu}}\eta_1(\nu) = \frac{\sqrt{2}e^{\nu/4}\sin(\eta_1(\nu))}{(0.5+\nu^2)\sqrt[3]{65}}, \\ {}^{2.1,\mathrm{H}}\mathbb{D}^{11/4,q;\sqrt{2\nu}}\eta_2(\nu) = \frac{3\nu^2(1+|\nu|)\sin(\eta_2(\nu))}{\sqrt[4]{27}(6+\cos^2\nu)(\sqrt{10}+\sin^2(\eta_2(\nu)))}, \end{cases} \tag{82}$$

for the following four cases of q: $q = \{\frac{8}{11}, \frac{9}{11}, \frac{10}{11}, 1\}$, with conditions

$$\begin{cases} \eta_1(0) = \eta_1'(0) = 0, \qquad \eta_2(0) = \eta_2'(0) = 0, \\ \frac{23}{15}\eta_1(1) + \frac{45}{22}\delta_{\sqrt{2\nu}}\eta_1(1) = \frac{32}{65}{}^{2.1}\mathbb{I}^{12/17;\sqrt{2\nu}}(\frac{(2+|\nu|)\tan(\eta_1(\nu))}{\sqrt{5}\sqrt[3]{31}}), \\ \frac{23}{15}\eta_2(1) + \frac{45}{22}\delta_{\sqrt{2\nu}}\eta_2(1) \\ \quad = \frac{32}{65}{}^{2.1}\mathbb{I}^{12/17;\sqrt{2\nu}}[\frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5+\tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7}+(\tan^{-1}(\nu))^2)}], \end{cases} \tag{83}$$

for $\nu \in \mathcal{J}$ and $\daleth(\nu) = \sqrt{2\nu} \in C(\mathcal{J})$. Clearly, $p = \frac{11}{4} \in (2,3)$ and $\nu = \frac{12}{17} > 0$. Then,

$$\vartheta_{2.1} = p + q_i(6.3 - p) = \begin{cases} 5.3318, & q = \frac{8}{11}, \\ 5.6545, & q = \frac{9}{11}, \\ 5.9772, & q = \frac{10}{11}, \\ 6.3000, & q = 1 \end{cases} \tag{84}$$

and

$$A = \sigma_1 + \frac{\sigma_2(\vartheta_{2.1_i} - k)}{\daleth(1) - \daleth(0)} \simeq \begin{Bmatrix} 6.2076, & q = \frac{8}{11}, \\ 6.6744, & q = \frac{9}{11}, \\ 7.1412, & q = \frac{10}{11}, \\ 7.6080, & q = 1, \end{Bmatrix} \neq 0. \tag{85}$$

We checked the correctness of assumptions (H1) and (H2) in Example 6.1, only here we present new numerical results for different values of q. We saw that $\lambda_1 = \frac{2}{\sqrt{5}\sqrt[3]{31}} \in (0,1]$ and $\lambda_2 = \frac{2e^{1/8}\sqrt{2}}{\sqrt[3]{35}} \in (0,1]$. Now, by employing Eq. (58), we obtain

$$\Xi = \frac{1}{|A_i|}\left[\left(|A_i| + |\sigma_1| + \frac{|\sigma_2|}{\Upsilon(k,k)}\right)\Upsilon(p_i,k) + |\sigma_3|\Upsilon(\nu,k)\right] \simeq \begin{cases} 0.9594, & q = \frac{8}{11}, \\ 0.9278, & q = \frac{9}{11}, \\ 0.9004, & q = \frac{10}{11}, \\ 0.8763, & q = 1. \end{cases}$$

In Fig. 5, we have plotted the results of Ξ for the system (82) and (83). Also, these results are shown in Table 4. Furthermore,

$$\lambda_1\Xi \simeq \begin{Bmatrix} 0.2732, & q = \frac{8}{11}, \\ 0.2642, & q = \frac{9}{11}, \\ 0.2564, & q = \frac{10}{11}, \\ 0.2495, & q = 1, \end{Bmatrix} < 1, \qquad \lambda_2\Xi \simeq \begin{Bmatrix} 0.9401, & q = \frac{8}{11}, \\ 0.9092, & q = \frac{9}{11}, \\ 0.8823, & q = \frac{10}{11}, \\ 0.8586, & q = 1, \end{Bmatrix} < 1. \tag{86}$$

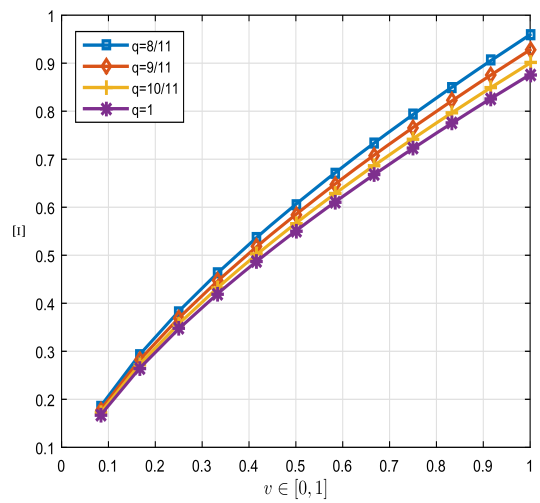
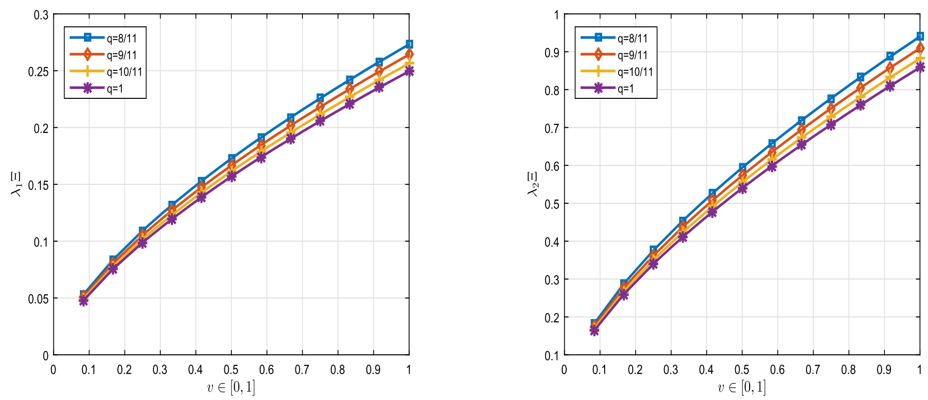


Figure 5 Representation of Ξ in assumption (H1) for four cases of type q in the system (82) and (83) in Example 6.3

Table 4 Numerical values of Ξ, and $\lambda_1\Xi$ $\lambda_2\Xi$ in Example 6.3 when $q = \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, 1$

v	Ξ	$\lambda_1\Xi < 1$	$\lambda_2\Xi < 1$	Ξ	$\lambda_1\Xi < 1$	$\lambda_2\Xi < 1$
	$q = \frac{8}{11}$			$q = \frac{9}{11}$		
0.08	0.1856	0.0528	0.1819	0.1784	0.0508	0.1748
0.17	0.2932	0.0835	0.2873	0.2822	0.0803	0.2765
0.25	0.3833	0.1091	0.3755	0.3692	0.1051	0.3617
0.33	0.4635	0.1320	0.4542	0.4468	0.1272	0.4378
0.42	0.5373	0.1530	0.5264	0.5182	0.1475	0.5077
0.50	0.6062	0.1726	0.5939	0.5849	0.1665	0.5731
0.58	0.6713	0.1911	0.6578	0.6480	0.1845	0.6350
0.67	0.7334	0.2088	0.7186	0.7083	0.2017	0.6940
0.75	0.7929	0.2258	0.7769	0.7660	0.2181	0.7506
0.83	0.8502	0.2421	0.8331	0.8217	0.2340	0.8051
0.92	0.9057	0.2579	0.8874	0.8756	0.2493	0.8579
1.00	0.9595	0.2732	0.9401	0.9279	0.2642	0.9092
	$q = \frac{10}{11}$			$q = 1$		
0.08	0.1724	0.0491	0.1689	0.1672	0.0476	0.1638
0.17	0.2728	0.0777	0.2673	0.2648	0.0754	0.2594
0.25	0.3571	0.1017	0.3499	0.3467	0.0987	0.3397
0.33	0.4325	0.1231	0.4237	0.4200	0.1196	0.4116
0.42	0.5018	0.1429	0.4916	0.4875	0.1388	0.4777
0.50	0.5666	0.1613	0.5552	0.5507	0.1568	0.5396
0.58	0.6280	0.1788	0.6153	0.6105	0.1738	0.5982
0.67	0.6866	0.1955	0.6727	0.6676	0.1901	0.6542
0.75	0.7428	0.2115	0.7278	0.7225	0.2057	0.7079
0.83	0.7970	0.2269	0.7809	0.7754	0.2208	0.7597
0.92	0.8495	0.2419	0.8323	0.8266	0.2354	0.8099
1.00	0.9004	0.2564	0.8823	0.8763	0.2495	0.8586

In Figs. 6a and 6b, we have plotted the results of $\lambda_1\Xi$ and $\lambda_2\Xi$ for the system (82) and (83). Also, these results are shown in Table 4. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (82) and (83) of $(2.1, \beth)$-$\mathbb{HFDE}$ has an optimal solution.



(a) $\lambda_1 \Xi$ *(b)* $\lambda_2 \Xi$

Figure 6 2D plot of $\lambda_1\,\Xi$ and $\lambda_2\,\Xi$ in assumption (H2) for four cases of type q in the system (82) and (83) in Example 6.3

We have discussed the results of the article by changing the order of the fractional integral ν in the next example.

Example 6.4 We consider the same system in Example 6.1 for $\nu \in \mathcal{J} = [0,1]$ in the form

$$\begin{cases} {}^{2.1,H}\mathbb{D}^{8/3,7/12;1.5^\nu}\eta_1(\nu) = \frac{\sqrt{2}e^{\nu/8}\sin(\eta_1(\nu))}{(0.5+\nu^2)\sqrt[3]{65}}, \\ {}^{2.1,H}\mathbb{D}^{8/3,7/12;1.5^\nu}\eta_2(\nu) = \frac{3\nu^2(1+|\nu|)\sin(\eta_2(\nu))}{\sqrt[4]{27}(6+\cos^2\nu)(\sqrt{10}+\sin^2(\eta_2(\nu)))}, \end{cases} \tag{87}$$

for the following four cases of $\nu \in (0,\infty)$: $\nu = \{\frac{12}{17}, \frac{24}{17}, \frac{35}{17}, \frac{47}{17}\}$, under the same conditions

$$\begin{cases} \eta_1(0) = \eta_1'(0) = 0, \qquad \eta_2(0) = \eta_2'(0) = 0, \\ \frac{23}{15}\eta_1(1) + \frac{45}{22}\delta_{1.5^\nu}\eta_1(1) = \frac{32}{65}\,{}^{2.1}\mathbb{I}^{\nu;1.5^\nu}\Big(\frac{(2+|\nu|)\tan(\eta_1(\nu))}{\sqrt[3]{31}\sqrt{5}}\Big), \\ \frac{23}{15}\eta_2(1) + \frac{45}{22}\delta_{1.5^\nu}\eta_2(1) = \frac{32}{65}\,{}^{2.1}\mathbb{I}^{\nu;1.5^\nu}\Big(\frac{10\nu\tan(\eta_2(\nu))}{\sqrt[3]{30}(5+\tan(\pi\nu/4))\exp(|\nu|)(\sqrt{7}+(\tan^{-1}(\nu))^2)}\Big), \end{cases} \tag{88}$$

for $\nu \in \mathcal{J}$ with $\gimel(\nu) = 1.5^\nu \in C(\mathcal{J})$. Clearly, $p = \frac{8}{3} \in (2,3)$ and $p = \frac{7}{12} \in [0,1]$. Then,

$$\vartheta_{2.1} = p + q(6.3 - p) = 4.7861, \qquad A = \sigma_1 + \frac{\sigma_2(\vartheta_{2.1} - k)}{\gimel(1) - \gimel(0)} \simeq 12.5219 \neq 0.$$

We checked the correctness of assumptions (H1) and (H2) in Example 6.1, only here we present new numerical results for different values of fractional integral order ν. As in the previous examples, Eq. (58) implies

$$\Xi = \frac{1}{|A|}\bigg[\bigg(|A| + |\sigma_1| + \frac{|\sigma_2|}{\Upsilon(k,k)}\bigg)\Upsilon(p,k) + |\sigma_3|\Upsilon(\nu_i,k)\bigg] \simeq \begin{cases} 0.2822, & \nu = \frac{12}{17}, \\ 0.2716, & \nu = \frac{24}{17}, \\ 0.2647, & \nu = \frac{35}{17}, \\ 0.2601, & \nu = \frac{47}{17}. \end{cases}$$

In Fig. 7, we have plotted the results of Ξ for the system (87) and (88). Also, these results are shown in Table 5. Furthermore,

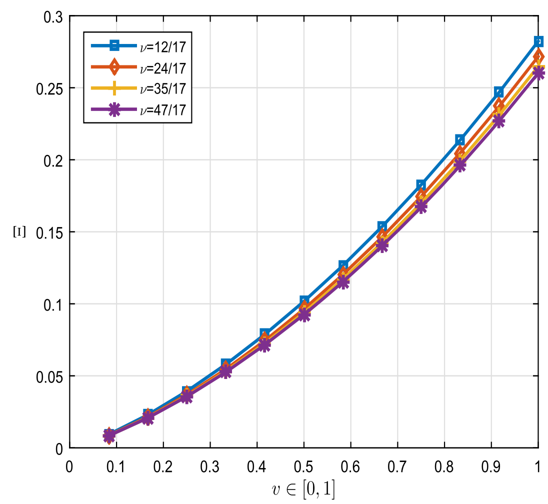


Figure 7 Representation of Ξ in assumption (H1) for four cases of fractional integral order ν in the system (87) and (88) in Example 6.4

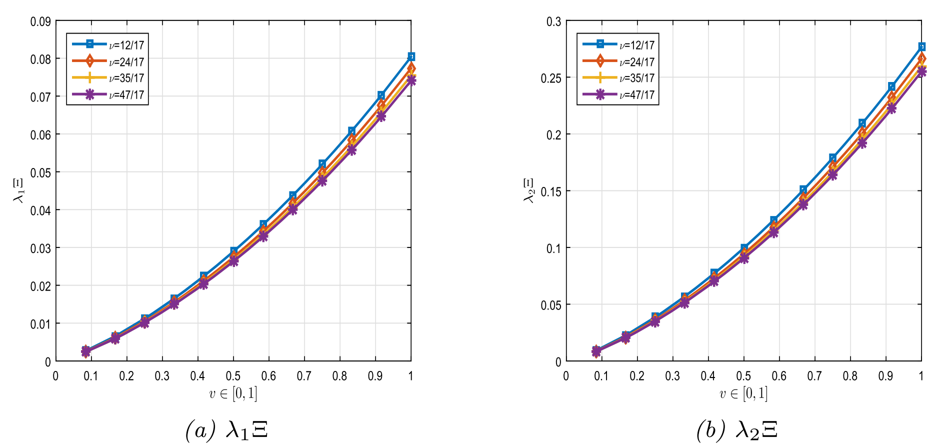


Figure 8 2D plot of $\lambda_1\Xi$ and $\lambda_2\Xi$ in assumption (H2) for four cases of fractional integral order ν in the system (87) and (88) in Example 6.4

$$\lambda_1\Xi \simeq \begin{Bmatrix} 0.0804, & \nu = \frac{12}{17}, \\ 0.0773, & \nu = \frac{24}{17}, \\ 0.0754, & \nu = \frac{35}{17}, \\ 0.0741, & \nu = \frac{47}{17}, \end{Bmatrix} < 1, \qquad \lambda_2\Xi \simeq \begin{Bmatrix} 0.2766, & \nu = \frac{12}{17}, \\ 0.2662, & \nu = \frac{24}{17}, \\ 0.2594, & \nu = \frac{35}{17}, \\ 0.2549, & \nu = \frac{47}{17}, \end{Bmatrix} < 1. \tag{89}$$

In Figs. 8a and 8b, we have plotted the results of $\lambda_1\Xi$ and $\lambda_2\Xi$ for the system (87) and (88). Also, these results are shown in Table 5. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (87) and (88) of $(2.1, \daleth)$-$\mathbb{HFDE}$ has an optimal solution.

7 Conclusion

In this work, we define a class of cyclic and noncyclic $\mathcal{A}$-condensing operators and prove the existence of a $\mathtt{bpp}$ and pair for them in the setting of $\mathrm{B_Ss}$. Also, our main results lead to some of the important results in the existing literature, presented as corollaries. In ad-

Table 5 Numerical values of Ξ, and $\lambda_1 \Xi$ $\lambda_2 \Xi$ in Example 6.4 when $\nu = \frac{12}{17}, \frac{24}{17}, \frac{35}{17}, \frac{47}{17}$

ν	Ξ	$\lambda_1 \Xi < 1$	$\lambda_2 \Xi < 1$	Ξ	$\lambda_1 \Xi < 1$	$\lambda_2 \Xi < 1$
	$\nu = \frac{12}{17}$			$\nu = \frac{24}{17}$		
0.08	0.0093	0.0026	0.0091	0.0086	0.0025	0.0084
0.17	0.0229	0.0065	0.0224	0.0214	0.0061	0.0210
0.25	0.0393	0.0112	0.0385	0.0369	0.0105	0.0361
0.33	0.0580	0.0165	0.0568	0.0546	0.0155	0.0535
0.42	0.0788	0.0224	0.0772	0.0744	0.0212	0.0729
0.50	0.1017	0.0290	0.0997	0.0964	0.0274	0.0944
0.58	0.1266	0.0361	0.1241	0.1203	0.0343	0.1179
0.67	0.1536	0.0437	0.1505	0.1463	0.0417	0.1434
0.75	0.1826	0.0520	0.1789	0.1744	0.0497	0.1709
0.83	0.2137	0.0608	0.2093	0.2046	0.0583	0.2005
0.92	0.2469	0.0703	0.2419	0.2370	0.0675	0.2322
1.00	0.2823	0.0804	0.2766	0.2717	0.0773	0.2662
	$\nu = \frac{35}{17}$			$\nu = \frac{47}{17}$		
0.08	0.0085	0.0024	0.0083	0.0084	0.0024	0.0082
0.17	0.0209	0.0060	0.0205	0.0208	0.0059	0.0204
0.25	0.0360	0.0102	0.0352	0.0356	0.0101	0.0349
0.33	0.0532	0.0152	0.0521	0.0526	0.0150	0.0516
0.42	0.0725	0.0206	0.0711	0.0716	0.0204	0.0702
0.50	0.0938	0.0267	0.0919	0.0925	0.0264	0.0907
0.58	0.1171	0.0334	0.1148	0.1154	0.0329	0.1131
0.67	0.1425	0.0406	0.1396	0.1403	0.0399	0.1374
0.75	0.1698	0.0484	0.1664	0.1671	0.0476	0.1637
0.83	0.1993	0.0567	0.1953	0.1960	0.0558	0.1920
0.92	0.2309	0.0658	0.2263	0.2270	0.0646	0.2224
1.00	0.2648	0.0754	0.2594	0.2601	0.0741	0.2549

dition, we discuss some coupled bpp results. The main result is applied to establish the existence of optimum solutions for the class of a system of $(k, \beth)$-$\mathbb{HFDE}$ of order $2 < p < 3$, type $0 \leq q \leq 1$ under integral and initial conditions. In the final step, we designed examples, and obtained numerical results of the system that well confirm the assumptions used. The technique used in this article can be used as a generalization in the area of solutions of nonlinear fractional and q-fractional differential equations via the bpp theory. The results of this research can establish more capabilities in the articles, such as [18, 38–43], that are presented about the existence and uniqueness of fractional differential equations and inclusions.

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Author contributions
GKK: Actualization, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. *DKP*: Validation, methodology, formal analysis, investigation, initial draft, and was a major contributor in writing the manuscript. *PRP*: Actualization, validation, methodology, formal analysis, investigation, and initial draft. *MES*: Formal analysis, validation, methodology, software, simulation and was a major contributor in writing the revision manuscript. All authors read and approved the final manuscript.

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Consent for publication
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Competing interests
The authors declare no competing interests.

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