Optimum solution of \((k, \mathcal{J})\)-Hilfer FDEs by \(A\)-condensing operators and the incorporated measure of noncompactness

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Abstract

The notion of \(A\)-condensing operators via the measure of noncompactness is proposed, which retains the existing classes of condensing operators. Results concerning the existence of the best proximity point (pair) of cyclic (noncyclic) \(A\)-condensing operators along with the coupled best proximity-point theorem for cyclic \(A\)-condensing operators have been formulated. An application to a \((k, \mathcal{J})\)-Hilfer fractional differential equation of order \(2 < p < 3\), type \(q \in [0, 1]\) satisfying some boundary conditions is presented. The paper is the first to investigate the optimum solution of such a generalized fractional differential equation. The hypothesis involved in the investigation is independent of the incorporated measure of noncompactness, thereby making our result better in application than that present in the literature. Moreover, added numerical examples validate the theoretical conclusions.

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1 Introduction

Over the years, the measure of noncompactness (MNC) together with condensing operators has created a remarkable place in the field of Fixed-Point Theory (FPT) \([1–3]\). Kuratowski initiated the study of MNC, whereas the concept of condensing operators originated in 1955 by Darbo, Schauder’s FPT being the main motivation behind this. In \([4]\), Brouwer proved the following FPT.

Theorem 1.1 Assume that \(\emptyset \neq \Omega \subset \mathbb{R}^n\) is convex and compact. Then, the continuous operator \(\Lambda : \Omega \rightarrow \Omega\) has a fixed point.

Schauder then extended this result to infinite-dimensional Banach spaces \((\mathbb{B}_S)\) \([5]\). Meanwhile, Kuratowski \([6, 7]\) stated the notion of MNC in order to solve certain problems related to general topology and defined it as a real-valued function \(K : \mathbb{B}_S \rightarrow [0, \infty)\).
such that

\[ K(V) = \inf \left\{ \epsilon > 0 : V \subseteq \bigcup_{a=1}^{n} P_a, P_a \subset \mathcal{Y}, \text{diam}(P_a) < \epsilon \right\}, \tag{1} \]

where \( B_\mathcal{Y} \) is the family of all nonvoid, bounded sets \( V \) of the complete metric space \( \mathcal{Y} \). Darbo [8] lessened the Schauder’s fixed point hypothesis by using this MNC \( K \). The beauty of Darbo’s theorem lies in the fact that it helped to weaken the compactness assumption, a strong presupposition, by replacing it with an inequality comprising \( K \). This theorem has the following statement.

**Theorem 1.2** ([8]) Let \( \Omega \subset \mathcal{Y} \) be nonvoid, convex, closed, and bounded with \( \mathcal{Y} \) as a \( B_\mathcal{Y} \). Then, the continuous map \( \Lambda : \Omega \to \Omega \) has a fixed point if \( K(\Lambda C) \leq kK(C) \), for \( C \subset \Omega \) with \( 0 \leq k < 1 \).

Fractional calculus deals with the study of differentiation and integration of arbitrary order and thus generalizes the classical structure. This generalization grabbed the focus due to its efficiency in providing a more accurate description for real-world phenomena. For a brief study, one can see [9, 10]. Motivated by the definition of Riemann–Liouville (RL) and Caputo derivatives, the authors [11] proposed the Hilfer derivative and solved an existence–uniqueness problem involving a Hilfer derivative of order between 0 to 1. Later, in 2018, Sousa et al. initiated the discussion of the \( J \)-Hilfer derivative, involving a continuously differentiable increasing function \( \mathcal{J} \) [12]. In 2023, Haque et al. took the H-\( \psi \)-FDEs [13]. Also, in [14], they considered the \( \psi \)-HFDEs with control having the form

\[
\begin{align*}
\mathcal{F}^{\psi}_{0\alpha} \eta(v) &= A\eta(v) + F(v, \eta_v) + Bu(v), \quad v \in (0, d], \\
\|1-p\|^{(1-q)} \psi \eta(v) &= h(v) \in B_\mathcal{U},
\end{align*}
\]

where \( 0 < p < 1, 0 \leq q \leq 1 \), \( \eta(\cdot) \) takes the values in Banach space \( Z \) with \( \| \cdot \| \), the control function \( u(\cdot) \in L^2((0, d], \mathcal{U}) \), the Banach space of admissible control functions, with \( \mathcal{U} \) as a Banach space, \( B : L^1((0, d], \mathcal{U}) \to L^1((0, d], \mathcal{Z}) \) is a bounded linear operator, and the operator \( A : D(A) \subset \mathcal{Z} \to \mathcal{Z} \) is the infinitesimal generator of analytic semigroup \( \{ T(v) \}_{v > 0} \) on \( \mathcal{Z} \). The DEs of the HFDE, \( \psi \)-HFDE, and \( (k, J) \)-HFDE types, because they provide significant generalizations, are very useful in solving different types of differential equations. For more details of these types of generalizations and results of controllability involving \( \psi \)-HFDEs refer to [15–22] and references therein. The work referenced above inspired us to propose a most generalized version of the Hilfer derivative, the so-called \( (k, J) \)-Hilfer fractional derivative.

Recently, valuable applications of MNC emphasizing the existence of solutions for a system of fractional differential and integral equations have been presented [23–29]. For example, Patle et al. in [25], discussed the existence of optimal solutions of the following system of right-sided \( \psi \)-Hilfer fractional differential equations (\( \psi \)-HFDE) of arbitrary order
with initial conditions
\[
\begin{align*}
H_{a}^{p,q,v} \eta(v) &= F_1(v, \tilde{\eta}(v)), \\
H_{a}^{p,q,v} \hat{\eta}(v) &= F_2(v, \tilde{\eta}(v)), \\
J_{a}^{(1-q)(1-p),2} \tilde{\eta}(a) &= \varsigma_{a}, \\
J_{a}^{(1-q)(1-p),2} \hat{\eta}(a) &= \varsigma_{a},
\end{align*}
\]
for \( v \in (a, \tau) \), where \( H_{a}^{p,q,v} \) is the left-sided \( \psi \)-HFDE operator of order \( 0 < p < 1 \), type \( 0 < q \leq 1 \), \( J_{a}^{(1-q)(1-p),2} \) is the RL fractional integral of order \((1-q)(1-p)\); the state \( \eta(\cdot) \) takes the values from \( \mathcal{X} \), and \( F_1 : [a, \tau] \times \mathcal{B}_1 \rightarrow \mathcal{X} \) and \( F_2 : [a, \tau] \times \mathcal{B}_2 \rightarrow \mathcal{X} \), are given mappings satisfying some assumptions.

In this article, the work flow is as follows: We define classes of cyclic and noncyclic \( \mathcal{A} \)-condensing operators and prove the existence of the best proximity point (bpp) and pair for them, respectively, in the setting of BSs. The consequences of the main results lead to some of the important results in the existing literature, presented as corollaries. Also, we discuss some coupled bpp results. In Sect. 5, the main result is applied to establish the existence of optimum solutions for the class of fractional differential equations involving \( (k, \mathcal{I}) \)-Hilfer derivatives \((k, \mathcal{I})\)-HFDE of order \( 2 < p < 3 \), type \( 0 \leq q \leq 1 \) in the form

\[
\begin{align*}
\{k, \mathcal{I}\}^{p,q,v} \eta_1(v) &= F_1(v, \eta_1(v)), \\
\eta_1(a) &= \eta_1'(a) = 0, \\
\sigma_1 \eta_1(b) + \sigma_2 \delta_1 \eta_1(b) &= \sigma_3 \hat{H}^{v-2} u_1(\zeta, \eta_1(\zeta)), \\
\{k, \mathcal{I}\}^{p,q,v} \eta_2(v) &= F_2(v, \eta_2(v)), \\
\eta_2(a) &= 0, \eta_2'(a) = 0, \\
\sigma_1 \eta_2(b) + \sigma_2 \delta_2 \eta_2(b) &= \sigma_3 \hat{H}^{v-2} u_2(\zeta, \eta_2(\zeta)),
\end{align*}
\]

for \( v \in \mathcal{J} = [a, b] \) satisfying the stated boundary conditions. The quantities \( \sigma_i, i = 1, 2, 3 \) are suitable real scalars, functions \( \eta_i, F_i, \mathcal{I} \) are all continuous such that \( \mathcal{I}(v) > 0 \) for all \( v \in \mathcal{J} \) with \( \delta_2 \equiv \frac{k}{k+1}, a < \zeta < b \) and \( k \hat{H}^{v-2} \) is the \((k, \mathcal{I})\)-RL integral of order \( v \in (0, \infty) \), \( k \in \mathbb{R} \). Finally, we introduce the strengths of the obtained results in future works in the conclusion section, 7.

2 Preliminaries

The compactness of the set or of the operator was not a really big issue, thanks to the Heine–Borel theorem, until the BSs of infinite dimension came into the picture. Justifying its name, \( \text{MNC} \) is a measure that estimates the degree of noncompactness of a set, a real-valued function that depicts the level of closeness of a set from being compact. The later axiomatic approach is a more convenient form when dealing with \( \text{MNC} \) and has the following interpretation [1].

Definition 2.1 Let \( \mathcal{Y} \), \( Y \), and \( V \) be defined as above. A function \( \eta : B_Y \rightarrow [0, \infty) \) is said to be an \( \text{MNC} \) provided

- (\eta_1) \( \eta(V) = 0 \iff V \) is precompact (regularity);
- (\eta_2) \( \eta(V) = \eta(\overline{V}) \) (invariance under closure);
- (\eta_3) \( \eta(V \cup V_2) = \max\{\eta(V_1), \eta(V_2)\} \) (semiadditivity).
Another interesting generalization of the Schauder theorem 2.5) operator a appears in [29]. To state this, we need to recall some definitions.

Moreover, if \( Y \) is also a \( B_\delta \) then \( \text{MNC} \eta \) can satisfy the following properties:

\( \eta(wV) = |w|\eta(V) \) for any number \( w \) (semihomogeneity);

\( \eta(V_1 + V_2) \leq \eta(V_1) + \eta(V_2) \) (algebraic semiadditivity);

\( \eta(w_0 + V) = \eta(V) \) for any \( w_0 \in Y \) (invariance under translations);

\( \eta(\text{conv}(V)) = \eta(V) \) (invariance under passage to the convex hull).

Another interesting generalization of the Schauder FPT involving cyclic (noncyclic), relatively nonexpansive maps together with a compact (Theorem 2.2) or condensing (Theorem 2.5) operator appeared in [29]. To state this, we need to recall some definitions. For any two nonempty subsets \( C \) and \( D \) of \( Y \), if \( \Lambda(C) \) and \( \Lambda(D) \) are both compact then \( \Lambda : C \cup D \to C \cup D \) is called a compact operator. The map \( \Lambda \) is said to be cyclic if \( \Lambda(C) \subseteq D \) as well as \( \Lambda(D) \subseteq C \) and noncyclic if \( \Lambda(C) \subseteq C \) along with \( \Lambda(D) \subseteq D \). If \( d(\Lambda v, \Lambda w) \leq d(v, w) \), for each \( v \in C, w \in D \), then \( \Lambda \) is known as relatively nonexpansive.

A point \( v_{\text{bpp}}^* \in C \cup D \) is a \( \text{b bpp} \) of a cyclic map \( \Lambda \) provided

\[
\|v_{\text{bpp}}^* - \Lambda v_{\text{bpp}}^*\| = \text{dist}(C, D) := \inf \{\|\tilde{c} - \tilde{d}\| : \tilde{c} \in C, \tilde{d} \in D\},
\]

whereas \((v_{\text{bpp}}^*, w_{\text{bpp}}^*) \in C \times D\) is a best proximity pair for a noncyclic map \( \Lambda \) if

\[
\|v_{\text{bpp}}^* - w_{\text{bpp}}^*\| = \text{dist}(C, D), \quad v_{\text{bpp}}^* = \Lambda v_{\text{bpp}}^*, w_{\text{bpp}}^* = \Lambda w_{\text{bpp}}^*.
\]

The proximal pair \((C_0, D_0) \subseteq (C, D)\) is given as

\[
C_0 = \{\tilde{c} \in C| \exists \tilde{d}_0 \in D : \|\tilde{c} - \tilde{d}_0\| = \text{dist}(C, D)\},
D_0 = \{\tilde{d} \in D| \exists \tilde{c}_0 \in C : \|\tilde{c}_0 - \tilde{d}\| = \text{dist}(C, D)\}.
\]

The pair \((C, D)\) is called proximinal whenever \( C = C_0 \) and \( D = D_0 \). We denote by \( \mathcal{M}_\Lambda(C, D) \) the collection of all pairs of subsets \((E, \bar{E})\) inside \((C, D)\) that are nonempty, convex, closed, bounded, proximinal, and \( \Lambda \)-invariant in nature such that \( \text{dist}(E, \bar{E}) = \text{dist}(C, D) \). In general, \( \mathcal{M}_\Lambda(C, D) \) may be empty, however, if \( \Lambda \) is cyclic (noncyclic), relatively nonexpansive with \((C, D)\) as that nonvoid convex pair that agrees to be weakly compact also inside a \( B_\delta \) \( \mathcal{Y} \), then \((C_0, D_0) \in \mathcal{M}_\Lambda(C, D) \).

We signify some conditions for \( \Lambda : C \cup D \to C \cup D \) by the following notations:

(S1) The pair of subsets \((C, D)\) is nonvoid, convex, closed, and bounded in a \( B_\delta \) \( \mathcal{Y} \).

(S2) \( \Lambda \) is relatively nonexpansive.

(S3) \( \Lambda \) is cyclic.

(S4) \( \Lambda \) is noncyclic and \( \mathcal{Y} \) is strictly convex.

**Theorem 2.2** ([29]) If \( C_0 \neq \emptyset \) and \( \Lambda \) is compact, then \( \Lambda \) possesses a \( \text{b bpp} \) and a best proximity pair whenever (S1), (S2), (S3) and (S1), (S2), (S4) hold, respectively.
Definition 2.3 ([30]) Let \( \eta \) be an arbitrary \( \text{MNC} \) and the condition (S1) holds. The operator \( \Lambda : C \cup D \to C \cup D \) is called a cyclic (noncyclic), Meir–Keeler condensing (MKC) operator if \( \forall \epsilon > 0 \ \exists \delta > 0 : \forall (E, \hat{E}) \in \mathcal{M}_\Lambda(C, D), \epsilon \leq \eta(E \cup \hat{E}) < \epsilon + \delta \Rightarrow \eta(\Lambda(E) \cup \Lambda(\hat{E})) < \epsilon. \) (9)

Definition 2.4 ([31]) A map \( L : [0, \infty) \to [0, \infty) \) is known to be an \( L \)-function whenever \( L(0) = 0 \) with \( L(v) > 0 \) for \( v \in (0, \infty) \) and for any \( v > 0 \) there exists \( \delta > 0 \) such that \( L(u) \leq v \) provided \( u \in [v, v + \delta] \).

Remark 2.1 The mappings \( L \)-function characterized (MK) contractions and was further proved to be true for (MKC) operators [31]. A map \( \Lambda \) is a cyclic (noncyclic) \( L \)-condensing operator if for an \( L \)-function \( L \), we have \( \eta(\Lambda E \cup \Lambda \hat{E}) < L(\eta(E \cup \hat{E})) \), for \( (E, \hat{E}) \in \mathcal{M}_\Lambda(C, D) \), whenever \( \eta(E \cup \hat{E}) > 0 \).

Theorem 2.5 ([30]) Suppose that \( \eta \) is an arbitrary \( \text{MNC} \) and \( \Lambda \) is an MKC operator such that \( C_0 \neq \emptyset \). Then, \( \Lambda \) is a \( \text{b}\)-\( \text{bpp} \) and a best proximity pair whenever (S1), (S2), (S3) and (S1), (S2), (S4) hold, respectively.

These were not merely generalizations, rather, they have the potency to show the existence of a \( \text{b}\)-\( \text{bpp} \) (pair). In recent years, very nice works have been done on the existence as well as applications of \( \text{b}\)-\( \text{bpp} \) and pairs. The interested readers are advised to read the articles [23, 26, 28] and references therein. Moving towards the main motivation of this article, Shahzad et al. defined a \( \mathcal{A} \)-contraction [32] using the concept of a \( \Lambda \)-sequence that submerges the class of all \( \mathcal{R} \)-contractions, Meir–Keeler contractions, \( \mathcal{Z} \)-contractions, and more. Keeping this in view, we define \( \mathcal{A} \)-condensing operators in terms of \( \text{MNC} \) \( \eta \) using the concept of \( \Lambda \)-sequence.

3 Best proximity point (pair) results

We now present our notions, namely, the \( \Lambda \)-sequence and \( \mathcal{A} \)-condensing operators. We say that \( \{x_n\} := \{(\alpha_n, \beta_n)\} \) is a \( \Lambda \)-sequence if there exists a sequence of pairs \( \{(E_n, \hat{E}_n)\} \) in \( \mathcal{M}_\Lambda(C, D) \) such that

\[
\alpha_n = \eta(\Lambda E_n \cup \Lambda \hat{E}_n) > 0, \quad \beta_n = \eta(E_n \cup \hat{E}_n) > 0,
\]

for each \( n \in \mathbb{N} \), where \( \{\alpha_n\}, \{\beta_n\} \) are two real sequences.

Definition 3.1 Let \( \eta \) be an arbitrary \( \text{MNC} \). An operator \( \Lambda : C \cup D \to C \cup D \) is \( \mathcal{A} \)-condensing if one can find a function \( \rho : A \times A \to \mathbb{R} \) satisfying the subsequent conditions together with \( \Lambda \) as: (i) \( \text{rang}(\eta) \subseteq A \subseteq \mathbb{R} \); (ii) if \( \{x_n\} \subseteq A^2 \) is a \( \Lambda \)-sequence such that both \( \alpha_n, \beta_n \to \ell \) with \( \ell \geq 0 \) and verifying \( \ell < \alpha_n \) along with \( \rho(\alpha_n, \beta_n) > 0 \) for every \( n \in \mathbb{N} \) then \( \ell = 0 \); (iii) \( \rho(\eta(\Lambda E \cup \Lambda \hat{E}), \eta(E \cup \hat{E})) > 0 \), provided \( \eta(E \cup \hat{E}) > 0 \) and \( \eta(\Lambda E \cup \Lambda \hat{E}) > 0 \) for every \( (E, \hat{E}) \in \mathcal{M}_\Lambda(C, D) \).

It is proved in [32] that not every \( \mathcal{A} \)-contraction is a Meir–Keeler contraction but the converse is always true. Along the same lines, we note that not every \( \mathcal{A} \)-condensing operator is MKC but the converse is always true.
Example 3.1 Let $Y = l_1$ be a nonreflexive $B_2$ with norm $\| \cdot \|_1$ and $\{e_k\}$ be its standard basis. Define two sets $C$ and $D$ inside $Y$ as $C = \overline{\text{conv}}(se_{2k-1} : k \in \mathbb{N})$ and $D = \overline{\text{conv}}(se_{2k} : k \in \mathbb{N})$. We choose $\mathcal{M}_N$ operating on any nonempty, bounded subset $V \subseteq Y$ as $\eta(V) = 0$ whenever $V$ is precompact and $\eta(V) = 1$ elsewhere. Define $\Lambda : C \cup D \to C \cup D$ such that

$$\Lambda (\lambda e_{2k-1} + (1-\lambda)e_{2m-1}) = e_{2k}, \quad \Lambda (\lambda e_{2k} + (1-\lambda)e_{2m}) = e_{2k-1},$$

(11)

where $\lambda \in [0,1]$ and $k, m \in \mathbb{N}$ with $k \leq m$. Note that $\Lambda$ is cyclic on $C \cup D$. Moreover, $(C, D) \in \mathcal{M}_N$ since $(C, D)$ is proximinal. As, $\eta(\Lambda C \cup \Lambda D) = \eta(C \cup D) > 0$, this means that $\eta(\Lambda C \cup \Lambda D) < \eta(C \cup D)$, and hence, $\Lambda$ cannot be $\mathcal{M}_N$. On the other hand, for any nonzero, positive, and constant function $\rho$, the condition (A3) is fulfilled. For (A2), let $\{x_n\}$ be a $\Lambda_N$-sequence with $\alpha_n, \beta_n \to \ell$, $0 \leq \ell < \alpha_n$ and $\rho(\alpha_n, \beta_n) > 0$ for all $n \in \mathbb{N}$. Assume on the contrary that $\ell > 0$. Now, from the definition of a $\Lambda_N$-sequence, we have a sequence of pairs $((E_n, \tilde{E}_n))$ in $\mathcal{M} \cup (C, D)$ such that $\alpha_n = \eta(\Lambda E_n \cup \Lambda \tilde{E}_n) > 0$ and $\beta_n = \eta(E_n \cup \tilde{E}_n) > 0$. This means that $\{\alpha_n\}$ is a constant sequence converging to 1 such that $\alpha_n = \ell$ for infinitely many $n$, a contradiction. Hence, $\Lambda$ is $\mathcal{A}$-condensing.

We proceed towards stating our theorems.

Theorem 3.2 Suppose that (S1), (S2), and (S3) hold. If $\Lambda$ is a cyclic $\mathcal{A}$-condensing operator such that $\rho(v, w) \leq w - v$ for all $v, w \in A \cap (0, \infty)$, then $\Lambda$ has a PFP provided $C_0 \neq \emptyset$.

Proof Set $I_0 = C_0$ with $J_0 = D_0$ and define

$$I_n = \overline{\text{conv}}(\Lambda(I_{n-1})), \quad J_n = \overline{\text{conv}}(\Lambda(J_{n-1})), \quad \forall n \in \mathbb{N}. \tag{12}$$

Hence, for

$$n = 1, \quad I_1 = \overline{\text{conv}}(\Lambda(I_0)) = \overline{\text{conv}}(\Lambda(C_0)) \subseteq D_0 = J_0,$$

$$n = 2, \quad I_2 = \overline{\text{conv}}(\Lambda(I_1)) \subseteq \overline{\text{conv}}(\Lambda(J_0)) = J_1,$$

$$\ldots \ldots$$

$$I_{n+1} \subseteq J_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{13}$$

Analogously, one can derive $J_{n+1} \subseteq I_n$, for $n \in \mathbb{N} \cup \{0\}$. Hence, in general, we write

$$I_{n+1} \subseteq J_n \subseteq I_n \subseteq J_{n-1}, \quad \forall n \in \mathbb{N}. \tag{14}$$

Hence, the sequence $(I_{2n}, J_{2n})$ of nonempty, convex, and closed pairs in $(C_0, D_0)$ is decreasing in nature. Moreover,

$$\Lambda(I_{2m}) \subseteq \Lambda(J_{2m-1}) \subseteq \overline{\text{conv}}(\Lambda(J_{2m-1})) = J_{2m}, \tag{15}$$

$$\Lambda(J_{2m}) \subseteq \Lambda(I_{2m-1}) \subseteq \overline{\text{conv}}(\Lambda(I_{2m-1})) = I_{2m},$$

imply that the pair $(I_{2m}, J_{2m})$ is $\Lambda$-invariant. Also, for $(\alpha, \beta) \in C_0 \times D_0$ and for all $m$, we have

$$\text{dist}(I_{2m}, J_{2m}) \leq \| \Lambda^{2m} \alpha - \Lambda^{2m} \beta \| \leq \| \alpha - \beta \| = \text{dist}(C, D). \tag{16}$$
Now, by induction we prove that the pairs \((\mathcal{I}_m, \mathcal{J}_m)\) are proximinal as well. The claim trivially holds for \(m = 0\) and so let the pair \((\mathcal{I}_{m-1}, \mathcal{J}_{m-1})\) be proximinal. For \(i = 1, 2, \ldots, j,\)

\[
v = \sum_{i=1}^{j} \lambda_i A(v_i) \in \mathcal{I}_m, \quad v_i \in \mathcal{I}_{m-1}, \lambda_i \in [0, 1], \sum_{i=1}^{j} \lambda_i = 1,
\]

there exists \(w_i \in \mathcal{J}_{m-1}\) such that \(w = \sum_{i=1}^{j} \lambda_i A(w_i) \in \mathcal{J}_m\), because of the proximality of the set \(\mathcal{I}_{m-1}\), so that

\[
\|v - w\| = \left\| \sum_{i=1}^{j} \lambda_i A(v_i) - \sum_{i=1}^{j} \lambda_i A(w_i) \right\| \leq \sum_{i=1}^{j} \lambda_i \|v_i - w_i\| = \text{dist}(C, D),
\]

and vice versa. This implies that the pair \((\mathcal{I}_m, \mathcal{J}_m)\) is proximinal and hence \((\mathcal{I}_{2m}, \mathcal{J}_{2m}) \in \mathcal{M}_A(C, D)\). Let us now consider the following two cases:

**Case (i)** Suppose there exists \(m_0 \in \mathbb{N}\) with \(\max\{\eta(\mathcal{I}_{2m_0}), \eta(\mathcal{J}_{2m_0})\} = 0\), then

\[
\Lambda : \mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0} \to \mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0},
\]

is a cyclic, relatively nonexpansive map on the compact set \(\mathcal{I}_{2m_0} \cup \mathcal{J}_{2m_0}\) and hence by Theorem 2.2, \(\Lambda\) will have a bpp.

**Case (ii)** For every \(m \in \mathbb{N}\), let \(\max\{\eta(\mathcal{I}_{2m}), \eta(\mathcal{J}_{2m})\} > 0\) and consider

\[
\eta(\mathcal{I}_{2m+1} \cup \mathcal{J}_{2m+1}) = \max\{\eta(\mathcal{I}_{2m+1}), \eta(\mathcal{J}_{2m+1})\} \\
= \max\{\eta(\text{conv}(\Lambda \mathcal{I}_{2m})), \eta(\text{conv}(\Lambda \mathcal{J}_{2m}))\} \\
\leq \max\{\eta(\mathcal{I}_{2m}), \eta(\mathcal{J}_{2m})\} = \eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m}) \\
= \max\{\eta(\text{conv}(\Lambda \mathcal{I}_{2m-1})), \eta(\text{conv}(\Lambda \mathcal{J}_{2m-1}))\} \\
= \max\{\eta(\mathcal{I}_{2m-1}), \eta(\mathcal{J}_{2m-1})\} \\
\leq \max\{\eta(\mathcal{I}_{2m-2}), \eta(\mathcal{J}_{2m-2})\} \\
\leq \eta(\mathcal{I}_{2m-2} \cup \mathcal{J}_{2m-2}).
\]

Hence, \(\eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m})\) is a decreasing sequence converging to its infimum, say \(\ell\). Set \(\alpha_m = \eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m}) > 0\) and \(\beta_m = \eta(\mathcal{I}_{2m} \cup \mathcal{J}_{2m}) > 0\), then \(\{\chi_m\}\) is a \(\Lambda\)-sequence such that \(\alpha_m \to \ell\) and \(\beta_m \to \ell\) with \(\ell \leq \alpha_m\). Moreover, \(\rho(\alpha_m, \beta_m) > 0\) for all \(m\). Let us suppose for some \(m\) that \(\ell \neq \alpha_m\) then \(\ell = \alpha_k\) for all \(k \geq m\). This means that the sequences \(\{\alpha_m\}\) and \(\{\beta_m\}\) are eventually constant sequences so that \(\rho(\alpha_m, \beta_m) \leq 0\) for infinitely many \(m\), a contradiction and therefore by (A2) we obtain \(\ell = 0\). Hence,

\[
\lim_{m \to \infty} \eta(\mathcal{I}_{2m}) = \lim_{m \to \infty} \eta(\mathcal{J}_{2m}) = 0.
\]

Define \(\mathcal{I}_\infty = \cap \mathcal{I}_{2m}\) and \(\mathcal{J}_\infty = \cap \mathcal{J}_{2m}\). Then, the pair \((\mathcal{I}_\infty, \mathcal{J}_\infty)\) is nonvoid, convex, and compact as well as \(\Lambda\)-invariant for which \(\text{dist}(\mathcal{I}_\infty, \mathcal{J}_\infty) = \text{dist}(C, D)\). Thus, the application of Theorem 2.2 guarantees that \(\Lambda\) has a bpp. \(\square\)
Theorem 3.3 Suppose that (S1), (S2), and (S4) hold. Then, $\Lambda$ has a best proximity pair provided $C_0 \neq \emptyset$ and $\Lambda$ is a noncyclic $A$-condensing operator with $\rho(v, w) \leq w - v$ for $v, w \in A \cap (0, \infty)$.

Proof Set $\mathcal{I}_0 = C_0$ with $\mathcal{J}_0 = D_0$ and define for all $n \in \mathbb{N}$, $\mathcal{I}_n = \text{conv}(\Lambda(\mathcal{I}_{n-1}))$ and $\mathcal{J}_n = \text{conv}(\Lambda(\mathcal{J}_{n-1}))$. Thus, for

$$n = 1, \quad \mathcal{I}_1 = \text{conv}(\Lambda(\mathcal{I}_0)) = \text{conv}(\Lambda(C_0)) \subseteq C_0 = \mathcal{I}_0,$$

$$n = 2, \quad \mathcal{I}_2 = \text{conv}(\Lambda(\mathcal{I}_1)) \subseteq \text{conv}(\Lambda(\mathcal{I}_0)) = \mathcal{I}_1,$$

$$\ldots \quad \mathcal{I}_{n+1} \subseteq \mathcal{I}_n, \quad \forall n \in \mathbb{N} \cup \{0\}.$$  

Analogously, one can derive $\mathcal{J}_{n+1} \subseteq \mathcal{J}_n$ for each $n \in \mathbb{N} \cup \{0\}$, so that the sequence $\{(\mathcal{I}_n, \mathcal{J}_n)\}$ of nonempty, convex, and closed pairs in $(C_0, D_0)$ is decreasing in nature. Moreover,

$$\Lambda(\mathcal{I}_m) \subseteq \Lambda(\mathcal{I}_{m-1}) \subseteq \text{conv}(\Lambda(\mathcal{I}_{m-1})) = \mathcal{I}_m,$$

$$\Lambda(\mathcal{J}_m) \subseteq \Lambda(\mathcal{J}_{m-1}) \subseteq \text{conv}(\Lambda(\mathcal{J}_{m-1})) = \mathcal{J}_m,$$

imply that the pair $(\mathcal{I}_m, \mathcal{J}_m)$ is $\Lambda$-invariant. Also, for $(\alpha, \beta) \in C_0 \times D_0$ and for each $m$, we obtain

$$\text{dist}(\mathcal{I}_m, \mathcal{J}_m) \leq \|\Lambda^m \alpha - \Lambda^m \beta\| \leq \|\alpha - \beta\| = \text{dist}(C, D).$$

By induction, one can prove that the pair $(\mathcal{I}_m, \mathcal{J}_m)$ is proximinal, as before, and hence $(\mathcal{I}_m, \mathcal{J}_m)$ in $\mathcal{M}_A(C, D)$. Let us now consider the following two cases:

Case (i) Suppose there exists $m_0 \in \mathbb{N}$ such that $\max\{\eta(\mathcal{I}_{m_0}), \eta(\mathcal{J}_{m_0})\} = 0$, then $\Lambda : \mathcal{I}_{m_0} \cup \mathcal{J}_{m_0} \to \mathcal{I}_{m_0} \cup \mathcal{J}_{m_0}$ is a noncyclic, relatively nonexpansive map on the compact set $\mathcal{I}_{m_0} \cup \mathcal{J}_{m_0}$ and so $\Lambda$ will have a best proximity pair.

Case (ii) For every $m \in \mathbb{N}$, let $\max\{\eta(\mathcal{I}_m), \eta(\mathcal{J}_m)\} > 0$ and consider

$$\eta(\mathcal{I}_{m+1} \cup \mathcal{J}_{m+1}) = \max\{\eta(\mathcal{I}_{m+1}), \eta(\mathcal{J}_{m+1})\}$$

$$= \max\{\eta(\text{conv}(\Lambda \mathcal{I}_m)), \eta(\text{conv}(\Lambda \mathcal{J}_m))\}$$

$$\leq \max\{\eta(\mathcal{I}_m), \eta(\mathcal{J}_m)\} = \eta(\mathcal{I}_m \cup \mathcal{J}_m).$$

Hence, $\{\eta(\mathcal{I}_m \cup \mathcal{J}_m)\}_{m=0}^{\infty}$ is a decreasing sequence converging to its infimum, say $\ell$. Set $\alpha_m = \eta(\mathcal{I}_m \cup \Lambda \mathcal{J}_m) > 0$ and $\beta_m = \eta(\mathcal{I}_m \cup \mathcal{J}_m) > 0$, then $\{\chi_m\}$ is a $\Lambda$-sequence such that $\alpha_m \to \ell$ and $\beta_m \to \ell$ with $\ell \leq \alpha_m$. Moreover, $\rho(\alpha_m, \beta_m) > 0$ for all $m$. Let us suppose for some $m$ that $\ell \neq \alpha_m$ then $\ell = \alpha_k$ for all $k \geq m$. This means that the sequences $(\alpha_m)$ and $(\beta_m)$ are eventually constant sequences so that $\rho(\alpha_m, \beta_m) \leq 0$ for infinitely many $m$, a contradiction and therefore by (A2) we obtain $\ell = 0$. Hence,

$$\lim_{m \to \infty} \eta(\mathcal{I}_m) = \lim_{m \to \infty} \eta(\mathcal{J}_m) = 0.$$  

Define $\mathcal{I}_\infty = \bigcap \mathcal{I}_m$ and $\mathcal{J}_\infty = \bigcap \mathcal{J}_m$, then the pair $(\mathcal{I}_\infty, \mathcal{J}_\infty)$ is nonvoid, convex, and compact as well as $\Lambda$-invariant for which $\text{dist}(\mathcal{I}_\infty, \mathcal{J}_\infty) = \text{dist}(C, D)$. Thus, the application of Theorem 2.2 guarantees that $\Lambda$ has a best proximity pair. \qed
We have the following corollaries as a consequence of our main results: Theorems 3.2 and 3.3.

**Corollary 3.1** ([24]) Suppose the conditions (S1), (S2), and (S3) hold with \( \eta \) as an arbitrary MNC. If \( C_0 \neq \emptyset \) and for every \((V_1, V_2) \in \mathcal{M}_\Lambda(C, D)\) we have

\[
\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi(\eta(V_1 \cup V_2)) \cdot \eta(V_1 \cup V_2),
\]

where \( \Psi : [0, \infty) \to [0, 1) \) is a map satisfying, \( \Psi(v_n) \to 1 \implies v_n \to 0 \), then \( \Lambda \) admits a bpp.

**Proof** Set \( \tilde{\Psi}(v) = \frac{1}{2}(1 + \Psi(v)) \), for \( v \in [0, \infty) \). Then, \( \tilde{\Psi}(v_n) \to 1 \) implies \( v_n \to 0 \). Moreover, \( \Psi(v) < \tilde{\Psi}(v) < 1 \) for \( v \in [0, \infty) \), so that

\[
\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi(\eta(V_1 \cup V_2)) \cdot \eta(V_1 \cup V_2) < \tilde{\Psi}(\eta(V_1 \cup V_2)) \cdot \eta(V_1 \cup V_2).
\]

The definition

\[
\rho(\eta(\Lambda V_1 \cup \Lambda V_2), \eta(V_1 \cup V_2)) = \tilde{\Psi}(\eta(V_1 \cup V_2)) \cdot \eta(V_1 \cup V_2) - \eta(\Lambda V_1 \cup \Lambda V_2) > 0,
\]

implies that (A3) is satisfied. For (A2), let \( \{\chi_n\} \) be a \( \Lambda_{\eta} \)-sequence satisfying \( \alpha_n \to \ell \) with \( 0 \leq \ell < \alpha_n, \beta_n \to \ell \), and for \( n \in \mathbb{N} \), \( \rho(\alpha_n, \beta_n) > 0 \). Hence,

\[
\alpha_n := \eta(\Lambda E_n \cup \Lambda \hat{E}_n) < \tilde{\Psi}(\eta(E_n \cup \hat{E}_n)) \cdot \eta(E_n \cup \hat{E}_n) < \eta(E_n \cup \hat{E}_n) := \beta_n.
\]

Applying \( n \to \infty \), we obtain \( \tilde{\Psi}(\eta(E_n \cup \hat{E}_n)) \to 1 \) and therefore, \( \eta(E_n \cup \hat{E}_n) \to 0 \) gives \( \ell = 0 \). Thus, \( \Lambda \) is \( \mathcal{A} \)-condensing and so Theorem 3.2 concludes the rest. \( \square \)

The proof of the remaining corollaries can be similarly obtained. However, for more details, one can see [32, 33].

**Corollary 3.2** ([24]) Suppose the conditions (S1), (S2), and (S4) hold with \( \eta \) as an arbitrary MNC. If \( C_0 \neq \emptyset \) and for every \((V_1, V_2) \in \mathcal{M}_\Lambda(C, D)\), we have

\[
\eta(\Lambda V_1 \cup \Lambda V_2) \leq \Psi(\eta(V_1 \cup V_2)) \cdot \eta(V_1 \cup V_2),
\]

where \( \Psi \) is as in Corollary 3.1, then \( \Lambda \) admits a best proximity pair.

**Corollary 3.3** Suppose the conditions (S1), (S2), and (S3) hold with \( \eta \) as an arbitrary MNC. If \( C_0 \neq \emptyset \) and for each \((V_1, V_2) \in \mathcal{M}_\Lambda(C, D)\) we have \( \eta(\Lambda V_1 \cup \Lambda V_2) \leq v\eta(V_1 \cup V_2) \) where \( v \in (0, 1) \), then \( \Lambda \) admits a bpp.

**Corollary 3.4** Suppose the conditions (S1), (S2), and (S4) hold with \( \eta \) as an arbitrary MNC. If \( C_0 \neq \emptyset \) and for every \((V_1, V_2) \in \mathcal{M}_\Lambda(C, D)\) we have \( \eta(\Lambda V_1 \cup \Lambda V_2) \leq v\eta(V_1 \cup V_2) \), where \( 0 < v < 1 \), then \( \Lambda \) admits a best proximity pair.
Corollary 3.5 Suppose the conditions (S1), (S2), and (S3) hold with \( \eta \) as an arbitrary MNC. If \( C \neq \emptyset \) and for every \( (V_1, V_2) \in \mathcal{M}_\Lambda(C, D) \) we have

\[
\eta(\Lambda V_1 \cup \Lambda V_2) \leq \eta(V_1 \cup V_2) - \Psi(\eta(V_1 \cup V_2)),
\]

where \( \Psi : [0, \infty) \to [0, \infty) \) is lower semicontinuous with \( \Psi^2(0) = 0 \), then \( \Lambda \) admits a bpp.

Corollary 3.6 Suppose the conditions (S1), (S2), and (S4) hold with \( \eta \) as an arbitrary MNC. If \( C \neq \emptyset \) and for every \( (V_1, V_2) \in \mathcal{M}_\Lambda(C, D) \), we obtain

\[
\eta(\Lambda V_1 \cup \Lambda V_2) \leq \eta(V_1 \cup V_2) - \Psi(\eta(V_1 \cup V_2)),
\]

where \( \Psi \) is as in Corollary 3.5, then \( \Lambda \) admits a best proximity pair.

4 Coupled best proximity point result

In this section, the coupled bpp theorem has been developed. For this, we first give the following preliminary concepts.

Suppose \( (C, D) \) is a nonvoid pair of subsets inside the metric space \( \mathcal{Y}, d \) and \( \Lambda : (C \times C) \cup (D \times D) \to C \cup D \). The map \( \Lambda \) is called cyclic if \( \Lambda(C \times C) \subseteq D \) and \( \Lambda(D \times D) \subseteq C \).

The point \((w_{\text{bpp}}, w_{\text{bpp}}^\ast) \in (C \times D) \) is a coupled bpp of \( \Lambda \) whenever

\[
d(w_{\text{bpp}}, \Lambda(w_{\text{bpp}}, w_{\text{bpp}}^\ast)) = d(w_{\text{bpp}}^\ast, \Lambda(w_{\text{bpp}}, w_{\text{bpp}}^\ast)) = \text{dist}(C, D).
\]

Lemma 4.1 ([34]) If \( \eta_i \) is an MNC on the metric spaces \( \mathcal{Y}_i \), \( i = 1, 2, \ldots, m \), respectively, then \( \eta(H) = \Phi(\eta_1(H_1), \eta_2(H_2), \ldots, \eta_m(H_m)) \), is also an MNC on \( \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_m \), where \( H_i \) stands for the natural projection of \( H \) into \( \mathcal{Y}_i \), respectively, for \( i = 1, 2, \ldots, m \), provided \( \Phi : [0, \infty)^m \to [0, \infty) \) is a convex function and \( \Phi(a_1, a_2, \ldots, a_m) = 0 \) if \( a_i = 0 \) for all \( i = 1, 2, \ldots, m \).

Lemma 4.2 ([27]) Let \( (C, D) \) be a nonvoid pair inside a metric space \( \mathcal{Y}, d \) and the product \( \mathcal{Y} \times \mathcal{Y} \) be a metric space together with the metric \( d_\infty \) as

\[
d_\infty((v_1, v_2), (w_1, w_2)) = \max\{d(v_1, w_1), d(v_2, w_2)\},
\]

for each \((v_1, v_2), (w_1, w_2) \in \mathcal{Y}^2\). Then, \( (C, D) \) is proximinal in \( \mathcal{Y} \) iff \( (C \times C, D \times D) \) is proximinal in \( \mathcal{Y}^2 \).

Before giving the statement for the coupled bpp via \( \mathcal{A} \)-condensing operators, we define a \( \Lambda_{\eta} \)-sequence. For two real sequences \( \{\alpha_n\}, \{\beta_n\} \), we say that \( \{\chi_n\} = \{(\alpha_n, \beta_n)\} \) is a \( \Lambda_{\eta} \)-sequence if there exists a sequence of nonempty, convex, bounded, closed, and proximinal \( \Lambda \)-invariants pairs \((\{E_n, \hat{E}_n\} \), \( \{F_n, \hat{F}_n\} \) \) in \((C, D) \) with

\[
\text{dist}(E_n, \hat{E}_n) = \text{dist}(F_n, \hat{F}_n) = \text{dist}(C, D),
\]

such that for each \( n \in \mathbb{N} \),

\[
\alpha_n = \max\{\eta(\Lambda(E_n \times F_n) \cup \Lambda(\hat{E}_n \times \hat{F}_n)), \eta(\Lambda(F_n \times E_n) \cup \Lambda(\hat{F}_n \times \hat{E}_n))\} > 0
\]
and \( \beta_n = \max\{\|E_n \cup F_n\|, \|\tilde{E}_n \cup \tilde{F}_n\|\} > 0 \).

**Theorem 4.3** Suppose \( (C,D) \) is a nonempty, convex, bounded, and closed pair in a \( B_S \) \( \mathcal{Y} \) with \( C_0 \neq \emptyset \) and \( \eta \) is an MNC on \( \mathcal{Y} \). Let \( \Lambda \) be a cyclic map satisfying (A2) along with the following conditions:

(i) Let \( (E,\hat{E}), (F,\hat{F}) \subseteq (C,D) \) be nonempty, convex, bounded, closed, and proximinal and \( \Lambda \)-invariants pairs with \( \text{dist}(E,\hat{E}) = \text{dist}(F,\hat{F}) = \text{dist}(C,D) \) such that

\[
\rho\left(\max\left\{\eta(\Lambda(E \times F) \cup \Lambda(\hat{E} \times \hat{F})), \eta(\Lambda(F \times E) \cup \Lambda(\hat{F} \times \hat{E}))\right\}\right),
\]

\[
\max\left\{\eta(E \cup F), \eta(\hat{E} \cup \hat{F})\right\}\right) > 0.
\] (38)

(ii) For all pairs \((v_1, v_2) \in C \times C\) and \((w_1, w_2) \in D \times D\), assume that

\[
d(\Lambda(v_1, v_2), \Lambda(w_1, w_2)) \leq d_\infty((v_1, v_2), (w_1, w_2)).
\] (39)

(iii) \( \rho(v, w) \leq w - v \) for every \( v, w \in A \cap (0, \infty) \).

Then, \( \Lambda \) affirms to have a coupled BPP.

**Proof** If \( H_i \) stands for the natural projection of \( H \) into \( \mathcal{Y} \), for \( i = 1, 2 \), we set \( \tilde{\eta}(H) := \max\{\eta(H_1), \eta(H_2)\} \) then \( \tilde{\eta} \) becomes an MNC for \( \mathcal{Y}^2 \). Let us define \( F \) by

\[
\begin{cases} 
F: (C \times C) \cup (D \times D) \to (C \times C) \cup (D \times D), \\
F(v, w) = (\Lambda(v, w), \Lambda(w, v)),
\end{cases}
\] (40)

then \( F \) is cyclic on \( (C \times C) \cup (D \times D) \). This is because, for any \( (v, w) \) in \( C \times C \) and together with the cyclic nature of \( \Lambda \) we have \( (\Lambda(v, w), \Lambda(w, v)) \in D \times D \), whence \( F(C \times C) \subseteq D \times D \). Similarly, one has \( F(D \times D) \subseteq C \times C \). To show \( F \) is relatively nonexpansive, let \(((v_1, v_2), (w_1, w_2)) \in (C \times C) \times (D \times D)\), then

\[
d_\infty(F(v_1, v_2), F(w_1, w_2))
\]

\[
= d_\infty((\Lambda(v_1, v_2), \Lambda(v_2, v_1)), (\Lambda(w_1, w_2), \Lambda(w_2, w_1)))
\]

\[
= \max\left\{d(\Lambda(v_1, v_2), \Lambda(w_1, w_2)), d(\Lambda(v_2, v_1), \Lambda(w_2, w_1))\right\}\right) \right)
\]

\[
\leq \max\left\{d_\infty((v_1, v_2), (w_1, w_2)), d_\infty((w_1, w_2), (v_2, v_1))\right\}\right) \right)
\]

\[
= d_\infty((v_1, v_2), (w_1, w_2)),
\] (41)

is the desired condition. Note that

\[
\tilde{\eta}(F(E \times F) \cup F(\hat{E} \times \hat{F}))
\]

\[
= \max\{\tilde{\eta}(F(E \times F)), \tilde{\eta}(F(\hat{E} \times \hat{F}))\}
\]

\[
= \max\{\tilde{\eta}(\Lambda(E \times F) \times \Lambda(F \times E)), \tilde{\eta}(\Lambda(\hat{E} \times \hat{F}) \times \Lambda(\hat{F} \times \hat{E}))\}
\]

\[
= \max\{\max\{\eta(\Lambda(E \times F)), \eta(\Lambda(F \times E))\}, \max\{\eta(\Lambda(\hat{E} \times \hat{F})), \eta(\Lambda(\hat{F} \times \hat{E}))\}\}\right) \right)
\]
= \max \{ \max \{ \eta(\Lambda(E \times F), \eta(\Lambda(\hat{E} \times \hat{F})) \}, \\
\max \{ \eta(\Lambda(F \times E), \eta(\Lambda(\hat{F} \times \hat{F})) \} \}
\]

= \max \{ \eta(\Lambda(E \times F) \cup \Lambda(\hat{E} \times \hat{F})), \eta(\Lambda(F \times E) \cup \Lambda(\hat{F} \times \hat{F})) \} \}

(42)

and \( \tilde{\eta}(E \times F) = \eta(E \cup F) \cup (\hat{E} \cup \hat{F}) \), so that we obtain

\[
\rho(\tilde{\eta}(E \times F) \cup F(\hat{E} \times \hat{F})), \tilde{\eta}(E \times F) \cup (\hat{E} \times \hat{F}))
\]

\[
= \rho(\max \{ \eta(\Lambda(E \times F) \cup \Lambda(\hat{E} \times \hat{F})), \eta(\Lambda(F \times E) \cup \Lambda(\hat{F} \times \hat{F})) \}, \\
\max \{ \eta(E \cup F), \eta(\hat{E} \cup \hat{F}) \}) 
\]

> 0. \quad (43)

This shows that \( F \) satisfies (A3). Hence, \( F \) will be \( A \)-condensing if \( F \) satisfies (A2) too. Let us assume that \( \{x_n\} \) is an \( F \)-sequence with \( \alpha_n \to \ell, \beta_n \to \ell, 0 \leq \ell < \alpha_n \) and \( \rho(\alpha_n, \beta_n) > 0 \) for each \( n \in \mathbb{N} \), then there exists a sequence \( \{(E_n \times F_n, \hat{E}_n \times \hat{F}_n)\}_{n=1}^{\infty} \) such that

\[
\alpha_n = \tilde{\eta}(E_n \times F_n) \cup F(\hat{E}_n \times \hat{F}_n) > 0, \\
\beta_n = \tilde{\eta}((E_n \times F_n) \cup (\hat{E}_n \times \hat{F}_n)) > 0,
\]

(44)

so that

\[
\alpha_n = \max \{ \eta(\Lambda(E_n \times F_n) \cup \Lambda(\hat{E}_n \times \hat{F}_n)), \eta(\Lambda(F_n \times E_n) \cup \Lambda(\hat{F}_n \times \hat{E}_n)) \} \}
\]

(45)

and \( \beta_n = \max \{ \eta(E_n \cup F_n), \eta(\hat{E}_n \cup \hat{F}_n) \} \), will be a \( \Lambda_{\eta} \)-sequence converging to \( \ell = 0 \). This concludes that \( F \) is \( A \)-condensing and hence, from Theorem 3.2, \( F \) has a bpp \( (v_{bpp}^*, w_{bpp}^*) \) \( \in (C \times C) \cup (D \times D) \) such that

\[
dist(C, D) = d_{\infty}(\{(v_{bpp}^*, w_{bpp}^*)\}, F(v_{bpp}^*, w_{bpp}^*)) \\
= d_{\infty}(\{(v_{bpp}^*, w_{bpp}^*)\}, F(v_{bpp}^*, w_{bpp}^*) \cup \{(v_{bpp}^*, w_{bpp}^*)\}) \\
= \max \{ d(v_{bpp}^*, F(v_{bpp}^*, w_{bpp}^*)), d(w_{bpp}^*, F(v_{bpp}^*, w_{bpp}^*)) \}. \quad (46)
\]

Therefore, \( (v_{bpp}^*, w_{bpp}^*) \) becomes a coupled bpp for \( \Lambda \).

\( \square \)

5 Applications

Various authors using renowned FPTs have shown the existence of solutions to more and more generalized forms of such fractional-order DDEs. In recent times, Kucche et al. in their paper [35] stated the most general form and the defined \((k, \mathcal{I})\)-Hadamard operator of order \( p \in (0, \infty) \), type \( 0 < q < 1 \) acting on a function \( \hat{\eta} \in C^n[a, b] \) with \( n = \left\lfloor \frac{\ell}{q} \right\rfloor \in \mathbb{N} \) as

\[
k_H^{\mathcal{I}p-q, 2} \hat{\eta}(v) = k_H^{\mathcal{I}(nk-p)2} \left( \frac{k_H^{\mathcal{I}p-q} d}{\mathcal{I}(v)} \right)^n \hat{\eta}(v), \quad (47)
\]

where \( k \in (0, \infty) \), \( \mathcal{I} \in C^n[a, b] \) is an increasing function with \( \mathcal{I}(v) \neq 0 \) for \( v \in [a, b] \) and \( k_H^{\mathcal{I}p} \) is the \((k, \mathcal{I})\)-RL integral of order \( p \in (0, \infty) \) as

\[
k_H^{\mathcal{I}p} \hat{\eta}(v) = \int_a^v \mathcal{I}(\zeta) [\mathcal{I}_a(\zeta)]^{\mathcal{I}p-1} \frac{\hat{\eta}(\zeta)}{k_H^{\mathcal{I}(p)}} d\zeta, \quad (48)
\]
where \( \tilde{\mathcal{A}}_q(v) = \mathcal{I}(v) - \mathcal{I}(q) \) and \( \Gamma_k(p) = \int_0^\infty q^{p-1} e^{-qk} \, dq \) [36]. Here, we consider a pair of \((k,3)\)-HIFDE (4) and (5) of order \( 2 < p < 3 \), type \( 0 \leq q \leq 1 \). The following lemma gives an equivalence between the HIFDE (4) with the integral equation (49).

**Lemma 5.1** Let \( 2 < p < 3, \, q \in [0,1] \) and \( \partial_k = p + q(3k - p) \), then the equivalent integral to the above-mentioned DE is

\[
\eta_1(v) = k^p \eta_2 F_1(v, \eta_1(v)) + \left[ \frac{\tilde{a}_q(v)}{A[a_q(b)]} \right]^{\eta_k - 1} \left[ \sigma_3 k^{\eta_2 - 1} u_1(\xi, \eta_1(\xi)) \right] - \sigma_2 k^{p-\eta_2} F_1(b, \eta_1(b)),
\]

where \( A = \sigma_1 + \frac{1}{\tilde{a}_q(b)} \sigma_2 (\partial_k - k) \neq 0 \).

We are now about to show the existence of the optimum solution of the system (4) and (5) for a more general setting. Consider \( \mathcal{S} = C(J, \mathcal{Y}) \) (here \( \mathcal{Y} \) is \( B_2 \)) with the supremum norm and choose two subsets \( S_1 \) and \( S_2 \) of \( \mathcal{S} \) as

\[
S_1 = \{ \hat{\eta} \in \mathcal{S} : \eta \in C(J, B_1), \hat{\eta}(a) = 0 \},
\]

\[
S_2 = \{ \hat{\eta} \in \mathcal{S} : \eta \in C(J, B_2), \hat{\eta}(a) = 0 \},
\]

where \( B_1 = B_{\hat{\gamma}}[p_0] \) and \( B_2 = B_{\hat{\gamma}}[q_0] \) represents two closed balls centered at \( p_0 \) and \( q_0 \) with radius \( \hat{\gamma}, \, t = 1, 2 \), respectively, in \( \mathcal{Y} \). The functions

\[
F_1, u_1 : J \times B_1 \to \mathcal{Y}, \quad F_2, u_2 : J \times B_2 \to \mathcal{Y},
\]

are all continuous. Clearly, \( S_i \neq \emptyset, \, t = 1, 2 \) are both bounded, closed, and convex sets in \( \mathcal{S} \). Define \( \Lambda \) on \( S_1 \cup S_2 \) as

\[
\Lambda \hat{\eta}(v) = \begin{cases} k^p \eta_2 F_2(v, \hat{\eta}(v)) + \left[ \frac{\tilde{a}_q(0)}{A[a_q(b)]} \right]^{\eta_k - 1} \left[ \sigma_3 k^{\eta_2 - 1} u_2(\xi, \hat{\eta}(\xi)) \right] - \sigma_2 k^{p-\eta_2} F_2(b, \hat{\eta}(b)), \hat{\eta} \in S_1, \\ k^p \eta_2 F_1(v, \hat{\eta}(v)) + \left[ \frac{\tilde{a}_q(b)}{A[a_q(b)]} \right]^{\eta_k - 1} \left[ \sigma_3 k^{\eta_2 - 1} u_1(\xi, \hat{\eta}(\xi)) \right] - \sigma_2 k^{p-\eta_2} F_1(b, \hat{\eta}(b)), \hat{\eta} \in S_2, \end{cases}
\]

**Lemma 5.2** The operator \( \Lambda \), (51) is cyclic on \( S_1 \cup S_2 \) whenever

\[
\begin{bmatrix} 1 + \frac{\lvert \sigma_1 \rvert}{|A|} + \frac{\lvert \sigma_2 \rvert}{|A|^Y(k,k)} \end{bmatrix} F_1^* Y(p,k) + \frac{\lvert \sigma_3 \rvert}{|A|} u_1^* Y(v,k) + \lVert p_0 \rVert \leq \hat{\gamma}_1,
\]

\[
\begin{bmatrix} 1 + \frac{\lvert \sigma_1 \rvert}{|A|} + \frac{\lvert \sigma_2 \rvert}{|A|^Y(k,k)} \end{bmatrix} F_2^* Y(p,k) + \frac{\lvert \sigma_3 \rvert}{|A|} u_2^* Y(v,k) + \lVert q_0 \rVert \leq \hat{\gamma}_2,
\]

where \( Y(r,k) = \frac{1}{r^{1+\partial_k}} \left[ \hat{\gamma}_k(b) \right]^{1/k} \) and

\[
F_1^* = \sup_{v \in J} \left\{ \lVert F_1(v, \hat{\eta}(v)) \rVert : \hat{\eta} \in S_1 \right\},
\]

\[
u_1^* = \sup_{v \in J} \left\{ \lVert u_1(v, \hat{\eta}(v)) \rVert : \hat{\eta} \in S_2 \right\},
\]
\[ F^*_2 = \sup_{v \in \mathcal{F}} \left\{ \| F_2(v, \hat{\eta}(v)) \| : \hat{\eta} \in S_1 \right\}, \]
\[ u^*_2 = \sup_{v \in \mathcal{F}} \left\{ \| u_2(v, \hat{\eta}(v)) \| : \hat{\eta} \in S_1 \right\}. \]

**Proof** Suppose \( \eta \in S_1 \) and consider

\[
\| (\Lambda \hat{\eta})(v) \| \leq \left\|^{k \rho^2 F_2(v, \hat{\eta}(v))} \right\| + \left\|^{\sigma_1} \frac{F_2^*}{k \Gamma_k(p)} \int_a^b \mathcal{I}(\mathcal{Q})\left[ \mathcal{I}_k(v) \right]^{k-1} d\mathcal{Q} \right\| + \left\|^{\sigma_2} \frac{F_2^*}{k \Gamma_k(p-k)} \int_a^b \mathcal{I}(\mathcal{Q})\left[ \mathcal{I}_k(v) \right]^{k-1} d\mathcal{Q} \right\|
\]
\[
\leq \left\|^{k \rho^2 \mathcal{I}_k(v)} \right\| + \left\|^{\sigma_1} \frac{1}{|A|} \left[ \mathcal{I}_k(v) \mathcal{I}_k(\mathcal{Q}) \right]^{k-1} \right\| + \left\|^{\sigma_2} \frac{1}{|A|} \left[ \mathcal{I}_k(v) \mathcal{I}_k(\mathcal{Q}) \right]^{k-1} \right\|
\]
\[
\leq \left\|^{k \rho^2 \mathcal{I}_k(v)} \right\| + \left\|^{\sigma_1} \frac{1}{|A|} \left[ \mathcal{I}_k(v) \mathcal{I}_k(\mathcal{Q}) \right]^{k-1} \right\| + \left\|^{\sigma_2} \frac{1}{|A|} \left[ \mathcal{I}_k(v) \mathcal{I}_k(\mathcal{Q}) \right]^{k-1} \right\|
\]

so that \( \| \Lambda \hat{\eta} - q_0 \| \leq \gamma_2 \) and therefore, we obtain \( \Lambda \hat{\eta} \in S_2 \). Similarly, we can show that \( \hat{\eta} \in S_2 \implies \Lambda \hat{\eta} \in S_1 \). Hence, \( \Lambda \) is cyclic. \( \square \)

We now prove the mean-value theorem of integral calculus for a \((k, \mathcal{I})\)-RL integral. The proof follows a similar technique to that shown in [37].

**Lemma 5.3** If \( p, k > 0 \) and \( \eta \) is any continuous function then we can find \( z \in (a, b) \) such that

\[
^{k \rho^2 \mathcal{I}_k(v)} \mathcal{I}(\mathcal{Q})\left[ \mathcal{I}_k(v) \right]^{k-1} \mathcal{I}(\mathcal{Q}) d\mathcal{Q} = \frac{\mathcal{I}_k(v)}{p \Gamma_k(p)} \mathcal{I}(z). \]

**Proof** Observe that the function

\[
\mathcal{I}(\mathcal{Q}) = \frac{\mathcal{I}(\mathcal{Q}) \mathcal{I}_k(v)}{k \Gamma_k(p)} \mathcal{I}_k(v), \]

(55)
is continuous and does not change its sign for the given range. By applying the generalized mean-value theorem of integral calculus, we write

\[
k_k^p \eta(v) = \eta(z) \int_a^v \frac{\tilde{T}(\xi)}{k! \Gamma(p)} \left[ \tilde{J}_v(\xi) \right]^{p+1} d\xi = \frac{\tilde{J}_v(v)}{p! \Gamma(p)} \eta(z).
\]  

(56)

**Theorem 5.4** Along with the assumptions of Lemma 5.2, suppose that there exists positive real scalars \( \lambda_i, i = 1, 2 \) such that for all \( v \in \mathcal{J} \) and \( (\eta_1, \eta_2) \in S_1 \times S_2 \), we have

\[
\max \left\{ \| F_2(v, \eta_1(v)) - F_1(v, \eta_2(v)) \|, \| u_2(v, \eta_1(v)) - u_1(v, \eta_2(v)) \| \right\} \\
\leq \lambda_1 \| \eta_1(v) - \eta_2(v) \|,
\]

where \( \lambda_1 \in (0, 1) \) with

\[
\Xi = \frac{1}{|A|} \left( |A| + 2 \sigma_2 \| \Upsilon(k, k) \| + 2 \sigma_3 \| \Upsilon(v, k) \| \right).
\]

(57)

(H2)

\[
\max \left\{ \| u_2(v, \eta_1(v)) - u_2(v, 0) \|, \| F_2(v, \eta_1(v)) - F_2(v, 0) \| \right\} \leq \lambda_1 \| \eta_1(v) \|,
\]

\[
\max \left\{ \| u_1(v, \eta_2(v)) - u_1(v, 0) \|, \| F_1(v, \eta_2(v)) - F_1(v, 0) \| \right\} \leq \lambda_2 \| \eta_2(v) \|,
\]

where \( \lambda_2 = \max \{ \lambda_1, \hat{\lambda}_1 \} \in (0, 1) \) satisfying \( \lambda_2 \Xi < 1 \). Then, the system (4) and (5) of (\( k, \mathcal{J} \))-HFE has an optimal solution.

**Proof** Clearly, \( \Lambda \) is cyclic, by its definition. Also, the range of \( \Lambda \) is uniformly bounded, since, for \( \eta \in S_1 \),

\[
\| \Lambda \eta_1(v) \| = \| \Lambda \eta_1(v) - q_0 + q_0 \| \leq \hat{\rho}_2 + \| q_0 \|. \tag{60}
\]

Similar conclusions can be drawn when \( \eta_2 \in S_2 \). We now show that \( \Lambda(S_1) \) is an equicontinuous set in \( S \). For that, let \( v < \hat{\nu} \) and \( \eta_1 \in S_1 \) then

\[
\| \Lambda \eta_1(\hat{\nu}) - \Lambda \eta_1(v) \| \\
= \left\| k_k^p [F_2(\hat{\nu}, \eta_1(\hat{\nu})) - F_2(v, \eta_1(v))] \right\| \\
+ \frac{\tilde{J}_v(\hat{\nu})^{p+2} - \tilde{J}_v(\nu)^{p+1}}{A[\tilde{J}_v(b)]^{p+1}} \left[ \sigma_3 k_k^p u_2(\xi, \eta_1(\xi)) \right] \\
- \sigma_1 k_k^p F_2(b, \eta_1(b)) - \sigma_2 k_k^p \tilde{J}_v(\nu)^{p+1} F_2(b, \eta_1(b)) \right\| \\
\leq \left\| \int_a^\nu \frac{\tilde{T}(\xi)}{k! \Gamma(p)} \left[ \tilde{J}_v(\xi) \right]^{p+1} [F_2(\xi, \eta_1(\xi)) - \eta_1(v)^{p+1}] d\xi \right\| \\
+ \left\| \int_{\nu}^{\hat{\nu}} \frac{\tilde{T}(\xi)}{k! \Gamma(p)} \left[ \tilde{J}_v(\xi) \right]^{p+1} F_2(\xi, \eta_1(\xi)) d\xi \right\|
\]
Indeed, as \( \nu \to \dot{\nu} \), we obtain \( \Lambda_{1_1}(\nu) \to \Lambda_{1_1}(\dot{\nu}) \). Hence, \( \Lambda(S_1) \) is an equicontinuous subset of \( S \). With similar arguments, one can show \( \Lambda(S_2) \) to be equicontinuous too. Hence, by the generalized Arzelá–Ascoli theorem, \((S_1, S_2)\) is relatively compact. In order to show \( \Lambda \) is relatively nonexpansive, let \((\eta_1, \eta_2) \in S_1 \times S_2 \), then for any \( \nu \in J \), we write

\[
\| \Lambda_{1_1}(\nu) - \Lambda_{1_2}(\nu) \| = \left\| \nu \right\|^{\nu} F_2(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_2(\nu, \eta_2(\nu)) \leq \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_1(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_1(\nu, \eta_2(\nu)) \leq \lambda_1 \left\| \eta_1 - \eta_2 \right\| \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_1(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_1(\nu, \eta_2(\nu)) \]
\]

\[
+ \frac{1}{|\mathcal{A}|} \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_2(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_2(\nu, \eta_2(\nu)) \leq \lambda_1 \left\| \eta_1 - \eta_2 \right\| \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_1(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_1(\nu, \eta_2(\nu)) \]
\]

\[
+ \frac{1}{|\mathcal{A}|} \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_2(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_2(\nu, \eta_2(\nu)) \leq \lambda_1 \left\| \eta_1 - \eta_2 \right\| \left\| \nu \right\|^{\nu} \left\| \nu \right\|^{\nu} F_1(\nu, \eta_1(\nu)) + \left\| \nu \right\|^{\nu} F_1(\nu, \eta_2(\nu)) \]
\]
This implies that $\Lambda$ is relatively nonexpansive. Also, $\Lambda$ is $\mathcal{A}$-condensing, because, for $(E, \dot{E}) \in \mathcal{M}_\Lambda(S_1, S_2)$ we have

$$
\eta(\Lambda E \cup \Lambda \dot{E}) = \max \left\{ \eta(\Lambda E), \eta(\Lambda \dot{E}) \right\}
$$

$$
= \max \left\{ \sup_{v \in \mathcal{J}} \left\{ \eta(\Lambda \eta_1(v) : v_1 \in E) \right\}, \sup_{v \in \mathcal{J}} \left\{ \eta(\Lambda \eta_2(v) : v_2 \in \dot{E}) \right\} \right\}
$$

$$
= \max \left\{ \sup_{v \in \mathcal{J}} \left\{ \eta\left( k^{p-2} F_2 (v, \eta_1(v)) + \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_2 (v, \eta_1(v)) \right] : v_1 \in E \right) \right\}, \sup_{v \in \mathcal{J}} \left\{ \eta\left( k^{p-2} F_1 (v, \eta_2(v)) + \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_1 (v, \eta_2(v)) \right] : v_2 \in \dot{E} \right) \right\} \right\}
$$

$$
= \max \left\{ \sup_{v \in \mathcal{J}} \left\{ \eta\left( \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_2 (v, \eta_1(v)) \right] : v_1 \in E \right) \right\}, \sup_{v \in \mathcal{J}} \left\{ \eta\left( \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_1 (v, \eta_2(v)) \right] : v_2 \in \dot{E} \right) \right\} \right\}
$$

$$
= \max \left\{ \sup_{v \in \mathcal{J}} \left\{ \eta\left( \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_2 (v, \eta_1(v)) \right] : v_1 \in E \right) \right\}, \sup_{v \in \mathcal{J}} \left\{ \eta\left( \frac{[\mathcal{A} \mathcal{E}(v)]^{q/k-1}}{p^{1/(k-1)}} \times \left[ \sigma_3 \mathcal{S}_4 \mathcal{U}_2 (\eta_2(v)) - \sigma_1 \mathcal{S}_2 \mathcal{F}_1 (v, \eta_2(v)) \right] : v_2 \in \dot{E} \right) \right\} \right\}
$$

$$
= \eta(\Lambda E) + \eta(\Lambda \dot{E})
$$
In the first example, we examine our results as an application to find a mild solution. The following examples reveal the hidden realities in the application of the results of this section.

### 6 Illustrative examples

The following examples reveal the hidden realities in the application of the results of this article. In the first example, we examine our results as an application to find a mild solution of the system (4) and (5) for different states of $J$.

#### Example 6.1

According to the system (4) and (5), we consider the system (2.1, 3)-HIDE under conditions for $v \in J = [0, 1]$ in the form

\[
\begin{cases}
2.1.\text{HIDE}_{\mathbb{G},\mathbb{V},\mathbb{C}} \eta_1(v) = \sqrt{2} \exp(\varphi) \sin(\eta_1(v)), \\
\eta_1(0) = \eta_1'(0) = 0, \\
\frac{23}{12} \eta_1(1) + \frac{15}{22} \Delta \eta_1(1) = \frac{32}{65} 2^{1/2} \sqrt{2} \exp(\varphi) \sin(\eta_1(0)) \cdot \sqrt{31},
\end{cases}
\]

\[
\begin{cases}
2.1.\text{HIDE}_{\mathbb{G},\mathbb{V},\mathbb{C}} \eta_2(v) = \sqrt{2} \exp(\varphi) \cos(\eta_2(v)), \\
\eta_2(0) = \eta_2'(0) = 0, \\
\frac{23}{12} \eta_2(1) + \frac{15}{22} \Delta \eta_2(1) = \frac{32}{65} 2^{1/2} \sqrt{2} \exp(\varphi) \cos(\eta_2(0)) \cdot \sqrt{31},
\end{cases}
\]

By choosing $\rho(v, w) = \lambda_1 w \Xi - v$, the operator $\Lambda$ becomes $A$-condensing. Hence, the hypothesis of Theorem 3.2 is fulfilled and hence, the $\text{bpp}$ of $\Lambda$ is the optimum solution of the system (4) and (5). $\blacksquare$

#### 6.1

According to the system (4) and (5), we consider the system (2.1, 3)-HIDE under conditions for $v \in J = [0, 1]$ in the form

\[
\begin{cases}
\frac{3}{2} \eta_1(v) + \frac{5}{8} \eta_1'(v) = \frac{7}{4} 2^{1/2} \sqrt{2} \exp(\varphi) \sin(\eta_1(v)), \\
\eta_1(0) = \eta_1'(0) = 0, \\
\frac{23}{12} \eta_1(1) + \frac{15}{22} \Delta \eta_1(1) = \frac{32}{65} 2^{1/2} \sqrt{2} \exp(\varphi) \sin(\eta_1(0)) \cdot \sqrt{31},
\end{cases}
\]

\[
\begin{cases}
\frac{3}{2} \eta_2(v) + \frac{5}{8} \eta_2'(v) = \frac{7}{4} 2^{1/2} \sqrt{2} \exp(\varphi) \cos(\eta_2(v)), \\
\eta_2(0) = \eta_2'(0) = 0, \\
\frac{23}{12} \eta_2(1) + \frac{15}{22} \Delta \eta_2(1) = \frac{32}{65} 2^{1/2} \sqrt{2} \exp(\varphi) \cos(\eta_2(0)) \cdot \sqrt{31},
\end{cases}
\]

By choosing $\rho(v, w) = \lambda_1 w \Xi - v$, the operator $\Lambda$ becomes $A$-condensing. Hence, the hypothesis of Theorem 3.2 is fulfilled and hence, the $\text{bpp}$ of $\Lambda$ is the optimum solution of the system (4) and (5). $\blacksquare$
with four cases of \( J \) as

\[
J_1(v) = 1.5^v, \quad J_2(v) = v, \quad J_3(v) = \ln\frac{2v + 1}{300}, \quad J_4(v) = \sqrt{v},
\]

(63)

which are in \( C(J) \) and \( J(v) > 0 \) on \( J \), where \( \delta_2 = \frac{1 + \frac{1}{v}}{\frac{\delta_2}{v}} \). Clearly, \( p = \frac{4}{3}, q = \frac{3}{5} \in [0,1] \), and \( v = \frac{2}{5} > 0 \). Then, \( \theta_{21} = p + q(6.3 - p) = \frac{1}{17} \). Also, by using the data, we obtain

\[
A = \sigma_1 + \frac{1}{\lambda_0(b)} \sigma_2(\theta_k - k) = \frac{23}{15} + \frac{45(63/10 - 3)}{\lambda_1(1) - \lambda_2(0)} \approx \begin{pmatrix} 12.0878, & J_1(v) = 1.5^v, & \{ \text{ where } \} \\ 6.8106, & J_2(v) = v, \{ \text{ and } \} \\ 6.3369, & J_3(v) = \ln \frac{2v + 1}{300}, \{ \text{ and } \} \\ 6.8106, & J_4(v) = \sqrt{v}, \{ \text{ and } \} \end{pmatrix} \neq 0.
\]

We define

\[
F_1(v, \eta_1(v)) = \frac{\sqrt{2}\exp(\frac{\pi}{15}) \sin(\eta_1(v))}{\sqrt{65(0.5 + v^2)}}, \quad \quad u_1(v, \eta_1(v)) = \frac{(2 + |v|) \tan(\eta_1(v))}{\sqrt{31\sqrt{5}}},
\]

\[
F_2(v, \eta_2(v)) = \frac{3v^2(1 + |v|) \sin(\eta_2(v))}{\sqrt{27(6 + \cos^2(v)(\sqrt{10} + \sin^2(\eta_2(v)))}}),
\]

\[
u_2(v, \eta_2(v)) = \frac{10v\tan(\eta_2(v))}{\sqrt{30(5 + \tan(\pi/4)) \exp(|v|)(\sqrt{7} + (\tan^{-1}(v))^2)}},
\]

(64)

We observe that,

\[
|F_2(v, \eta_1(v)) - F_1(v, \eta_2(v))| = \left| \frac{3v^2(1 + |v|) \sin(\eta_1(v))}{\sqrt{27(6 + \cos^2(v)(\sqrt{10} + \sin^2(\eta_1(v)))}} - \frac{\sqrt{2}\exp(\pi/15) \sin(\eta_2(v))}{\sqrt{65(0.5 + v^2)}} \right| \leq \frac{\sqrt{2}}{1.5\sqrt{31}} |\eta_1(v) - \eta_2(v)|,
\]

\[
|u_2(v, \eta_2(v)) - u_1(v, \eta_1(v))| = \left| \frac{10v\tan(\eta_2(v))}{\sqrt{30(5 + \tan(\pi/4)) \exp(|v|)(\sqrt{7} + (\tan^{-1}(v))^2))} - \frac{(2 + |v|) \tan(\eta_1(v))}{\sqrt{31\sqrt{5}}} \right| \leq \frac{2\tan(\eta_2(v))}{\sqrt{30\sqrt{7}}} - \frac{2\tan(\eta_1(v))}{\sqrt{30\sqrt{5}}} \leq \frac{2}{\sqrt{30\sqrt{31}}} |\eta_1(v) - \eta_2(v)|.
\]

(65)

Hence, to confirm assumption (H1), from Eq. (57), we have

\[
\max\left\{ \|F_2(v, \eta_1(v)) - F_1(v, \eta_2(v))\|, \|u_2(v, \eta_1(v)) - u_1(v, \eta_2(v))\| \right\} \\
\leq \max\left\{ \frac{\sqrt{2}}{1.5\sqrt{31}} \|\eta_1(v) - \eta_2(v)\|, \frac{2}{\sqrt{30\sqrt{31}}} \|\eta_1(v) - \eta_2(v)\| \right\} \\
\leq \lambda_1 \|\eta_1(v) - \eta_2(v)\|
\]

(66)

where \( \lambda_1 = \frac{4}{\sqrt{30\sqrt{31}}} \). Now, thanks to Eq. (58), we obtain

\[
\Xi = \frac{1}{|A|} \left[ |A| + |\sigma_1| + \frac{|\sigma_2|}{\Upsilon(k, k)} \Upsilon(p, k) + |\sigma_3| \Upsilon(v, k) \right]
\]
\[
\| \frac{1}{|A_i|} \left( \left| A_i \right| + \frac{23}{15} + \frac{|a^{ij}_{22}|}{\Gamma(3,3)} \right) \left( \frac{9}{4},3 \right) + \frac{32}{65} \left( \frac{12}{17},3 \right) \right] \\
= \frac{1}{|A_i|} \left[ \left( \left| A_i \right| + \frac{23}{15} + \frac{|a^{ij}_{22}|}{\Gamma(11,10)} \right) \frac{1}{\Gamma(1)} \left[ \mathcal{J}_1(1) - \mathcal{J}_1(0) \right]^{\gamma_i} \right] \\
+ \frac{32}{65} \left[ \frac{1}{\Gamma(3)} \left[ \mathcal{J}_1(1) - \mathcal{J}_1(0) \right]^{\gamma_i} \right] \\
\begin{cases}
0.4107, & \mathcal{J}_1(v) = 1.5^v, \\
0.8750, & \mathcal{J}_2(v) = v, \\
0.9695, & \mathcal{J}_3(v) = \frac{2v+1}{300}, \\
0.8750, & \mathcal{J}_4(v) = \sqrt{v}.
\end{cases}
\]

Furthermore,
\[
\lambda_1 \Xi \simeq \begin{cases}
0.1169, & \mathcal{J}_1(v) = 1.5^v, \\
0.2491, & \mathcal{J}_2(v) = v, \\
0.2760, & \mathcal{J}_3(v) = \frac{2v+1}{300}, \\
0.2491, & \mathcal{J}_4(v) = \sqrt{v}.
\end{cases} < 1.
\]

Hence, assumption (H1) holds. In Figs. 1 and 2a, 2b, we have plotted the results of \( \Xi \) and \( \lambda_1 \Xi, \lambda_2 \Xi \) for the system (61) and (62). Also, these results are shown in Tables 1 and 2. To check the next assumption (H2), we take help from relations (59). Hence,
\[
\left| u_2(v, \eta_1(v)) - u_2(v, 0) \right| \\
= \frac{10v \tan(\eta_1(v))}{\sqrt{30(5 + \tan(\pi/4))\exp(|v|)(\sqrt{7} + (\tan^{-1}(v))^2)}} \leq \frac{2}{\sqrt{30\sqrt{7}}} |\eta_1(v)|, 
\]

Figure 1: Representation of \( \Xi \) in assumption (H1) for four cases of \( \Xi \) in the system (61) and (62) in Example 6.1.
Figure 2 2D plot of $\lambda_1 \Xi$ and $\lambda_2 \Xi$ in assumption (H2) for four cases of $\mathcal{G}$ in the system (61) and (62) in Example 6.1

Table 1 Numerical values of $\Xi$, and $\lambda_1 \Xi$ and $\lambda_2 \Xi$ in Example 6.1 when $\mathcal{J}(v) = 1.5^v, v$

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\mathcal{J}(v) = 1.5^v$</th>
<th>$\lambda_1 \Xi &lt; 1$</th>
<th>$\lambda_2 \Xi &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>0.0224</td>
<td>0.0064</td>
<td>0.0219</td>
</tr>
<tr>
<td>0.17</td>
<td>0.0483</td>
<td>0.0137</td>
<td>0.0473</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0763</td>
<td>0.0217</td>
<td>0.0748</td>
</tr>
<tr>
<td>0.33</td>
<td>0.1063</td>
<td>0.0303</td>
<td>0.1042</td>
</tr>
<tr>
<td>0.42</td>
<td>0.1382</td>
<td>0.0393</td>
<td>0.1354</td>
</tr>
<tr>
<td>0.50</td>
<td>0.1717</td>
<td>0.0489</td>
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</tr>
<tr>
<td>0.58</td>
<td>0.2070</td>
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<td>0.2029</td>
</tr>
<tr>
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<td>0.0695</td>
<td>0.2392</td>
</tr>
<tr>
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<td>0.0806</td>
<td>0.2773</td>
</tr>
<tr>
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<td>0.3237</td>
<td>0.0922</td>
<td>0.3171</td>
</tr>
<tr>
<td>0.92</td>
<td>0.3662</td>
<td>0.1043</td>
<td>0.3588</td>
</tr>
<tr>
<td>1.00</td>
<td>0.4107</td>
<td>0.1169</td>
<td>0.4024</td>
</tr>
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<table>
<thead>
<tr>
<th>$v$</th>
<th>$\mathcal{J}(v) = \ln 2^{v/300}, \sqrt{v}$</th>
<th>$\lambda_1 \Xi &lt; 1$</th>
<th>$\lambda_2 \Xi &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>0.1139</td>
<td>0.0324</td>
<td>0.1116</td>
</tr>
<tr>
<td>0.17</td>
<td>0.2247</td>
<td>0.0640</td>
<td>0.2202</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3267</td>
<td>0.0930</td>
<td>0.3201</td>
</tr>
<tr>
<td>0.33</td>
<td>0.4204</td>
<td>0.1197</td>
<td>0.4119</td>
</tr>
<tr>
<td>0.42</td>
<td>0.5067</td>
<td>0.1443</td>
<td>0.4965</td>
</tr>
<tr>
<td>0.50</td>
<td>0.5866</td>
<td>0.1670</td>
<td>0.5747</td>
</tr>
<tr>
<td>0.58</td>
<td>0.6609</td>
<td>0.1882</td>
<td>0.6475</td>
</tr>
<tr>
<td>0.67</td>
<td>0.7303</td>
<td>0.2079</td>
<td>0.7156</td>
</tr>
<tr>
<td>0.75</td>
<td>0.7954</td>
<td>0.2265</td>
<td>0.7793</td>
</tr>
<tr>
<td>0.83</td>
<td>0.8567</td>
<td>0.2439</td>
<td>0.8394</td>
</tr>
<tr>
<td>0.92</td>
<td>0.9146</td>
<td>0.2604</td>
<td>0.8961</td>
</tr>
<tr>
<td>1.00</td>
<td>0.9695</td>
<td>0.2760</td>
<td>0.9499</td>
</tr>
</tbody>
</table>

Table 2 Numerical values of $\Xi$, and $\lambda_1 \Xi ; \lambda_1 \Xi$ in Example 6.1 when $\mathcal{J}(v) = \ln 2^{v/300}, \sqrt{v}$

<table>
<thead>
<tr>
<th>$v$</th>
<th>$\mathcal{J}(v) = \ln 2^{v/300}$</th>
<th>$\lambda_1 \Xi &lt; 1$</th>
<th>$\lambda_2 \Xi &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>0.1139</td>
<td>0.0324</td>
<td>0.1116</td>
</tr>
<tr>
<td>0.17</td>
<td>0.2247</td>
<td>0.0640</td>
<td>0.2202</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3267</td>
<td>0.0930</td>
<td>0.3201</td>
</tr>
<tr>
<td>0.33</td>
<td>0.4204</td>
<td>0.1197</td>
<td>0.4119</td>
</tr>
<tr>
<td>0.42</td>
<td>0.5067</td>
<td>0.1443</td>
<td>0.4965</td>
</tr>
<tr>
<td>0.50</td>
<td>0.5866</td>
<td>0.1670</td>
<td>0.5747</td>
</tr>
<tr>
<td>0.58</td>
<td>0.6609</td>
<td>0.1882</td>
<td>0.6475</td>
</tr>
<tr>
<td>0.67</td>
<td>0.7303</td>
<td>0.2079</td>
<td>0.7156</td>
</tr>
<tr>
<td>0.75</td>
<td>0.7954</td>
<td>0.2265</td>
<td>0.7793</td>
</tr>
<tr>
<td>0.83</td>
<td>0.8567</td>
<td>0.2439</td>
<td>0.8394</td>
</tr>
<tr>
<td>0.92</td>
<td>0.9146</td>
<td>0.2604</td>
<td>0.8961</td>
</tr>
<tr>
<td>1.00</td>
<td>0.9695</td>
<td>0.2760</td>
<td>0.9499</td>
</tr>
</tbody>
</table>

\[
|F_2(v, \eta_1(v)) - F_2(v, 0)| \
\leq \frac{3v^2(1 + |v|) \sin(\eta_1(v))}{\sqrt{27}(6 + \cos^2 v)(\sqrt{10 + \sin^2(\eta_1(v)))}} \\
\leq \frac{1}{\sqrt{27}} \frac{\sin(\eta_1(v))}{\sqrt{10 + \sin^2(\eta_1(v))}} \leq \frac{1}{\sqrt{10} \sqrt{27}} |\eta_1(v)|
\]
and
\[
\max\left\{ \|u_2(v, \eta_1(v)) - u_2(v, 0)\|, \|F_2(v, \eta_1(v)) - F_2(v, 0)\| \right\} = \max\left\{ \frac{2}{\sqrt[3]{30}}, \frac{1}{\sqrt[10]{27}} \|\eta_1(v)\| \right\} \leq \dot{\lambda}_1 \|\eta_1(v)\|, \tag{71}
\]
where \(\dot{\lambda}_1 = \frac{2}{\sqrt[3]{30}}\). In addition,
\[
|u_1(v, \eta_2(v)) - u_1(v, 0)| = \left| \frac{(2 + |v|) \tan(\eta_2(v))}{\sqrt[3]{30}} \right| \leq \left| \frac{3}{\sqrt[5]{31}} \tan(\eta_2(v)) \right| \leq \frac{3}{\sqrt[5]{31}} |\eta_2(v)|, \\
|F_1(v, \eta_2(v)) - F_1(v, 0)| = \left| \frac{\sqrt[3]{2} \exp(\eta_2(v)) \sin(\eta_2(v))}{\sqrt[65]{65(0.5 + v^2)}} \right| \leq \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{0.5 \sqrt[65]{65}} |\sin(\eta_2(v))| \leq \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[65]{65}} |\eta_2(v)|, \tag{72}
\]
and
\[
\max\left\{ \|u_1(v, \eta_2(v)) - u_1(v, 0)\|, \|F_1(v, \eta_2(v)) - F_1(v, 0)\| \right\} = \max\left\{ \frac{3}{\sqrt[5]{31}} \|\eta_2(v)\|, \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[65]{65}} \|\eta_2(v)\| \right\} \leq \dot{\lambda}_2 \|\eta_2(v)\|, \tag{73}
\]
where \(\dot{\lambda}_2 = \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[3]{35}}\). Therefore,
\[
\dot{\lambda}_2 = \max\{\dot{\lambda}_1, \dot{\lambda}_2\} = \left\{ \frac{2}{\sqrt[3]{30}}, \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[3]{35}} \right\} = \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[3]{35}} \in (0, 1] \tag{74}
\]
and
\[
\lambda_2 \triangleq \dot{\lambda}_2 \dot{\lambda}_1 = \left\{ \begin{array}{ll}
0.4024, & \text{if } \eta_1(v) = 1.5v, \\
0.8573, & \text{if } \eta_2(v) = v, \\
0.9499, & \text{if } \eta_3(v) = \frac{2v + 1}{300}, \\
0.8573, & \text{if } \eta_4(v) = \sqrt{v}. \\
\end{array} \right. < 1. \tag{75}
\]

Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (61) and (62) of \((2.1, \mathcal{J})-\text{HFDE}\) has an optimal solution.

In the next example, we can see the correctness of the results for the existence of an optimal solution of the system (4) and (5) for different fractional derivatives of order \(p\).

**Example 6.2** We consider the system of \((2.1, \mathcal{J})-\text{HFDE}\) (61) and (62) in Example 6.1 for \(v \in \mathcal{J} = [0, 1]\) in the form
\[
\begin{align*}
\text{H}_{\alpha, \beta, \eta_1(v)}^{p} \eta_1(v) &= \frac{2e^{\frac{1}{28} \sqrt[3]{2}}}{\sqrt[65]{65(0.5 + v^2)}} \sin(\eta_1(v)), \\
\text{H}_{\alpha, \beta, \eta_2(v)}^{p} \eta_2(v) &= \frac{3v^2(1 + \eta_2(v)) \sin(\eta_2(v))}{\sqrt[27]{27(6 + \cos^2 v)(\sqrt[10]{10} + \sin^2(\eta_2(v)))}},
\end{align*}
\tag{76}
\]
for the following three cases of $p$: $p = \left\{ \frac{9}{4}, \frac{5}{2}, \frac{23}{8} \right\}$, under the same conditions

\[
\begin{cases}
\eta_1(0) = \eta'_1(0) = 0, & \eta_2(0) = \eta'_2(0) = 0, \\
\frac{23}{15} \eta_1(1) + \frac{45}{22} \delta_0 \eta_1(1) = \frac{32}{65} \frac{2}{1} \tan(\eta_1(v)), \\
\frac{23}{15} \eta_2(1) + \frac{45}{22} \delta_0 \eta_2(1) = \frac{32}{65} \frac{2}{1} \tan(\eta_2(v)),
\end{cases}
\] (77)

for $v \in \mathcal{J}$ and $\mathcal{J}(\mathcal{J}) = \frac{1}{2}v \in C(\mathcal{J})$. Clearly, $q = \frac{2}{3} \in [0, 1]$ and $v = \frac{12}{17} > 0$. Then,

\[
\vartheta_{2,1} = p_i + q(6.3 - p_i) = \begin{cases} 
4.68, & p = \frac{9}{4}, \\
4.78, & p = \frac{5}{2}, \\
4.93, & p = \frac{23}{8},
\end{cases}
\] (78)

and

\[
A = \sigma_1 + \frac{\sigma_2(\vartheta_{2,1} - k)}{2(1) - 2(0)} \simeq \begin{cases} 
12.0878, & p = \frac{9}{4}, \\
12.4969, & p = \frac{5}{2}, \\
13.1106, & p = \frac{23}{8},
\end{cases}
\] (79)

By choosing the same defined functions $F_1(v, \eta(v)), u_1(v, \eta_1(v)), F_2(v, \eta_2(v))$, and $u_2(v, \eta_2(v))$, again, assumptions (H1) and (H2) are valid with

\[
\|F_2(v, \eta_1(v)) - F_1(v, \eta_2(v))\| \leq \frac{\sqrt{2}}{1.5\sqrt{35}} \|\eta_1(v) - \eta_2(v)\|,
\]

\[
\|u_2(v, \eta_2(v)) - u_1(v, \eta_1(v))\| \leq \frac{2}{\sqrt{5\sqrt{31}}} \|\eta_1(v) - \eta_2(v)\|
\] (80)

and $\lambda_1 = \frac{2}{\sqrt{5\sqrt{31}}} \in (0, 1)$, $\lambda_2 = \frac{2\sqrt{2}}{\sqrt{35}} e^{\frac{1}{3}} \in (0, 1]$. In addition, Eq. (58) implies that

\[
\Xi = \frac{1}{|A|} \left[ \left| A_1 \right| + |\sigma_1| + \frac{|\sigma_2|}{\gamma(k,k)} \gamma(p_i,k) + |\sigma_3| \gamma(v,k) \right] \simeq \begin{cases} 
0.4106, & p = \frac{9}{4}, \\
0.3263, & p = \frac{5}{2}, \\
0.2300, & p = \frac{23}{8},
\end{cases}
\]

In Fig. 3, we have plotted the results of $\Xi$ for the system (76) and (77). Also, these results are shown in Table 3. Furthermore,

\[
\lambda_1 \Xi \simeq \begin{cases} 
0.1169, & p = \frac{9}{4}, \\
0.0929, & p = \frac{5}{2}, \\
0.0655, & p = \frac{23}{8},
\end{cases} < 1, \quad \lambda_2 \Xi \simeq \begin{cases} 
0.4024, & p = \frac{9}{4}, \\
0.3198, & p = \frac{5}{2}, \\
0.2254, & p = \frac{23}{8},
\end{cases} < 1.
\] (81)

In Figs. 4a and 4b, we have plotted the results of $\lambda_1 \Xi$ and $\lambda_2 \Xi$ for the system (76) and (77). Also, these results are shown in Table 3. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (76) and (77) of (2.1, $\mathcal{J}$)-HFDE has an optimal solution.

The next example examines the results for different values of type $q$. 
Figure 3 Representation of $\Xi$ in assumption (H1) for three cases of order $p$ in the system (76) and (77) in Example 6.2.

Table 3 Numerical values of $\Xi$, and $\lambda_1 \Xi, \lambda_2 \Xi$ in Example 6.2 when $p = \frac{9}{2}, \frac{5}{2}, \frac{23}{8}$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$p = \frac{9}{2}$</th>
<th>$p = \frac{5}{2}$</th>
<th>$p = \frac{23}{8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_1 \Xi &lt; 1$</td>
<td>$\lambda_2 \Xi &lt; 1$</td>
<td>$\lambda_1 \Xi &lt; 1$</td>
</tr>
<tr>
<td>0.08</td>
<td>0.0275</td>
<td>0.0078</td>
<td>0.0270</td>
</tr>
<tr>
<td>0.17</td>
<td>0.0584</td>
<td>0.0166</td>
<td>0.0572</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0907</td>
<td>0.0258</td>
<td>0.0889</td>
</tr>
<tr>
<td>0.33</td>
<td>0.1240</td>
<td>0.0353</td>
<td>0.1215</td>
</tr>
<tr>
<td>0.42</td>
<td>0.1581</td>
<td>0.0450</td>
<td>0.1549</td>
</tr>
<tr>
<td>0.50</td>
<td>0.1928</td>
<td>0.0549</td>
<td>0.1889</td>
</tr>
<tr>
<td>0.58</td>
<td>0.2281</td>
<td>0.0650</td>
<td>0.2235</td>
</tr>
<tr>
<td>0.67</td>
<td>0.2639</td>
<td>0.0751</td>
<td>0.2585</td>
</tr>
<tr>
<td>0.75</td>
<td>0.3000</td>
<td>0.0854</td>
<td>0.2940</td>
</tr>
<tr>
<td>0.83</td>
<td>0.3366</td>
<td>0.0958</td>
<td>0.3298</td>
</tr>
<tr>
<td>0.92</td>
<td>0.3735</td>
<td>0.1063</td>
<td>0.3659</td>
</tr>
<tr>
<td>1.00</td>
<td>0.4107</td>
<td>0.1169</td>
<td>0.4024</td>
</tr>
</tbody>
</table>

Figure 4 2D plot of $\lambda_1 \Xi$ and $\lambda_2 \Xi$ in assumption (H2) for three cases of $p$ in the system (76) and (77) in Example 6.2.
Example 6.3 Consider the system (61) and (62) in Example 6.1 for \( v \in J = [0,1] \) in the form

\[
\begin{align*}
\eta_1(v) = & \frac{\sqrt{2} v^{11} \sin(\eta_1(v))}{(0.5 + v^{2})^{1/4}}, \\
\eta_2(v) = & \frac{3 \sqrt{2} v^{11} \sin(\eta_2(v))}{(0.5 + v^{2})^{1/4}},
\end{align*}
\]

(82)

for the following four cases of \( q \): \( q = \{ 8, 9, 10, 1 \} \), with conditions

\[
\begin{align*}
\eta_1(0) = & \eta_1'(0) = 0, \\
\eta_2(0) = & \eta_2'(0) = 0,
\end{align*}
\]

(83)

for \( v \in J \) and \( \sqrt{2} v \in C(J) \). Clearly, \( p = \frac{11}{4} \in (2,3) \) and \( v = \frac{12}{17} > 0 \). Then,

\[
\theta_{2.1} = p + q(6.3 - p) = \begin{cases} 5.3318, & q = \frac{8}{17} \\ 5.6545, & q = \frac{9}{17} \\ 5.9772, & q = \frac{10}{17} \\ 6.3000, & q = 1 \end{cases}
\]

(84)

and

\[
A = \sigma_1 + \frac{\sigma_2 (\theta_{2.1} - k)}{2(1 - J(0))} \simeq \begin{cases} 6.2076, & q = \frac{8}{17} \\ 6.6744, & q = \frac{9}{17} \\ 7.1412, & q = \frac{10}{17} \\ 7.6080, & q = 1 \end{cases}
\]

(85)

We checked the correctness of assumptions (H1) and (H2) in Example 6.1, only here we present new numerical results for different values of \( q \). We saw that \( \lambda_1 = \frac{2^{1/8} \sqrt{3}}{\sqrt{53}} \in (0,1] \) and \( \lambda_2 = \frac{2^{1/8} \sqrt{3}}{\sqrt{53}} \in (0,1] \). Now, by employing Eq. (58), we obtain

\[
\Xi = \frac{1}{|A_1|} \left[ (|A_1| + |\sigma_1| + \frac{|\sigma_2|}{Y(k,k)}) Y(p_{2.1}, k) + |\sigma_1| Y(v, k) \right] \simeq \begin{cases} 0.9594, & q = \frac{8}{17} \\ 0.9278, & q = \frac{9}{17} \\ 0.9004, & q = \frac{10}{17} \\ 0.8763, & q = 1 \end{cases}
\]

(86)

In Fig. 5, we have plotted the results of \( \Xi \) for the system (82) and (83). Also, these results are shown in Table 4. Furthermore,
Figure 5  Representation of $X_1$ in assumption (H1) for four cases of type $q$ in the system (82) and (83) in Example 6.3

In Figs. 6a and 6b, we have plotted the results of $\lambda_1 X_1$ and $\lambda_2 X_2$ for the system (82) and (83). Also, these results are shown in Table 4. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (82) and (83) of (2.1, $\mathcal{J}$)-HFDE has an optimal solution.

Table 4 Numerical values of $X_1$ and $\lambda_1 X_1$, $\lambda_2 X_2$ in Example 6.3 when $q = \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, 1$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\lambda_1 X_1 &lt; 1$</th>
<th>$\lambda_2 X_2 &lt; 1$</th>
<th>$X_1$</th>
<th>$\lambda_1 X_1 &lt; 1$</th>
<th>$\lambda_2 X_2 &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{8}{11}$</td>
<td>0.1856 0.0528</td>
<td>0.1819</td>
<td>0.1784 0.0508</td>
<td>0.1748</td>
<td></td>
</tr>
<tr>
<td>$\frac{9}{11}$</td>
<td>0.2932 0.0835</td>
<td>0.2873</td>
<td>0.2822 0.0803</td>
<td>0.2765</td>
<td></td>
</tr>
<tr>
<td>$\frac{10}{11}$</td>
<td>0.3833 0.1091</td>
<td>0.3755</td>
<td>0.3692 0.1051</td>
<td>0.3617</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.4635 0.1320</td>
<td>0.4542</td>
<td>0.4468 0.1272</td>
<td>0.4378</td>
<td></td>
</tr>
<tr>
<td>$\frac{8}{11}$</td>
<td>0.5373 0.1530</td>
<td>0.5264</td>
<td>0.5182 0.1475</td>
<td>0.5077</td>
<td></td>
</tr>
<tr>
<td>$\frac{9}{11}$</td>
<td>0.6062 0.1726</td>
<td>0.5939</td>
<td>0.5849 0.1665</td>
<td>0.5731</td>
<td></td>
</tr>
<tr>
<td>$\frac{10}{11}$</td>
<td>0.6713 0.1911</td>
<td>0.6578</td>
<td>0.6480 0.1845</td>
<td>0.6350</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.7334 0.2088</td>
<td>0.7186</td>
<td>0.7083 0.2017</td>
<td>0.6940</td>
<td></td>
</tr>
<tr>
<td>$\frac{8}{11}$</td>
<td>0.7929 0.2258</td>
<td>0.7769</td>
<td>0.7660 0.2181</td>
<td>0.7506</td>
<td></td>
</tr>
<tr>
<td>$\frac{9}{11}$</td>
<td>0.8502 0.2421</td>
<td>0.8331</td>
<td>0.8217 0.2340</td>
<td>0.8051</td>
<td></td>
</tr>
<tr>
<td>$\frac{10}{11}$</td>
<td>0.9057 0.2579</td>
<td>0.8874</td>
<td>0.8756 0.2493</td>
<td>0.8579</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.9595 0.2732</td>
<td>0.9401</td>
<td>0.9279 0.2642</td>
<td>0.9092</td>
<td></td>
</tr>
</tbody>
</table>
We have discussed the results of the article by changing the order of the fractional integral $\nu$ in the next example.

**Example 6.4** We consider the same system in Example 6.1 for $v \in J = [0, 1]$ in the form

\[
\begin{align*}
2.1D^{\nu, \lambda}_{\alpha} \eta_1(v) &= \frac{\sqrt{2}v^{8/3}\sin(\eta_1(v))}{(0.5v+1)^\sqrt{65}}, \\
2.1D^{\nu, \lambda}_{\alpha} \eta_2(v) &= \frac{3v^2(1+|v|)\sin(\eta_2(v))}{\sqrt{27}(6+\cos(2v))} \left(\sqrt{10}+\sin(\eta_2(v))\right),
\end{align*}
\]

for the following four cases of $v \in (0, \infty)$: $v = \left\{\frac{12}{17}, \frac{24}{17}, \frac{35}{17}, \frac{47}{17}\right\}$, under the same conditions

\[
\begin{align*}
\eta_1(0) &= \eta_1'(0) = 0, & \eta_2(0) &= \eta_2'(0) = 0, \\
\frac{23}{15} \eta_1(1) + \frac{45}{22} \eta_1'(1) &= \frac{32}{65} \eta_1(v), & \frac{32}{65} \eta_1(v) &= \frac{10v\tan(\eta_2(v))}{\sqrt{31}v^3} \left(\frac{5v+10\tan(\eta_2(v))\exp(|v|)}{\sqrt{7+\tan^{-1}(\eta_2(v))^2}}\right),
\end{align*}
\]

for $v \in J$ with $\mathcal{J}(v) = 1.5^v \in C(J)$. Clearly, $p = \frac{8}{3} \in (2, 3)$ and $q = \frac{7}{12} \in [0, 1]$. Then,

\[
\vartheta_{2.1} = p + q(6.3 - p) = 4.7861, \quad A = \sigma_1 + \frac{\sigma_2(\vartheta_{2.1} - k)}{\mathcal{J}(1) - \mathcal{J}(0)} \approx 12.5219 \not= 0.
\]

We checked the correctness of assumptions (H1) and (H2) in Example 6.1, only here we present new numerical results for different values of fractional integral order $v$. As in the previous examples, Eq. (58) implies

\[
\Xi = \frac{1}{|A|} \left( |A| + |\sigma_1| + \frac{|\sigma_2|}{\mathcal{J}(k,k)} \Gamma(p,k) + |\sigma_3| \Gamma(v,k) \right) \approx \left\{\begin{array}{c}
0.2822, \quad v = \frac{12}{17}, \\
0.2716, \quad v = \frac{24}{17}, \\
0.2647, \quad v = \frac{35}{17}, \\
0.2601, \quad v = \frac{47}{17}.
\end{array}\right.
\]

In Fig. 7, we have plotted the results of $\Xi$ for the system (87) and (88). Also, these results are shown in Table 5. Furthermore,
In Figs. 8a and 8b, we have plotted the results of $\lambda_1 \Xi$ and $\lambda_2 \Xi$ for the system (87) and (88). Also, these results are shown in Table 5. Then, all the conditions of Theorem 5.4 are satisfied. Hence, the system (87) and (88) of $(2,1,\mathcal{I})$-HFDE has an optimal solution.

7 Conclusion

In this work, we define a class of cyclic and noncyclic $\mathcal{A}$-condensing operators and prove the existence of a $\mathcal{BPP}$ and pair for them in the setting of $\mathcal{B}_S$s. Also, our main results lead to some of the important results in the existing literature, presented as corollaries. In ad-
Table 5 Numerical values of $\Xi$, and $\lambda_1$, $\lambda_2$ in Example 6.4 when $\nu = 12/17, 24/17, 35/17, 47/17$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\Xi$</th>
<th>$\lambda_1 \Xi &lt; 1$</th>
<th>$\lambda_2 \Xi &lt; 1$</th>
<th>$\Xi$</th>
<th>$\lambda_1 \Xi &lt; 1$</th>
<th>$\lambda_2 \Xi &lt; 1$</th>
</tr>
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<tr>
<td>$\nu = 12/17$</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>0.08</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0086</td>
<td>0.0025</td>
<td>0.0084</td>
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<tr>
<td>0.17</td>
<td>0.0229</td>
<td>0.0065</td>
<td>0.0224</td>
<td>0.0214</td>
<td>0.0061</td>
<td>0.0210</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0393</td>
<td>0.0112</td>
<td>0.0385</td>
<td>0.0369</td>
<td>0.0105</td>
<td>0.0361</td>
</tr>
<tr>
<td>0.33</td>
<td>0.0580</td>
<td>0.0165</td>
<td>0.0568</td>
<td>0.0546</td>
<td>0.0155</td>
<td>0.0535</td>
</tr>
<tr>
<td>0.42</td>
<td>0.0788</td>
<td>0.0224</td>
<td>0.0772</td>
<td>0.0744</td>
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<td>0.0997</td>
<td>0.0964</td>
<td>0.0274</td>
<td>0.0944</td>
</tr>
<tr>
<td>0.58</td>
<td>0.1266</td>
<td>0.0361</td>
<td>0.1241</td>
<td>0.1203</td>
<td>0.0343</td>
<td>0.1179</td>
</tr>
<tr>
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<td>0.1536</td>
<td>0.0437</td>
<td>0.1505</td>
<td>0.1463</td>
<td>0.0417</td>
<td>0.1434</td>
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<tr>
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<td>0.1826</td>
<td>0.0520</td>
<td>0.1789</td>
<td>0.1744</td>
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</tr>
<tr>
<td>0.83</td>
<td>0.2137</td>
<td>0.0608</td>
<td>0.2093</td>
<td>0.2046</td>
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<td>0.2005</td>
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<tr>
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<td>0.2469</td>
<td>0.0703</td>
<td>0.2419</td>
<td>0.2370</td>
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<td>0.2322</td>
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<tr>
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<td>0.2823</td>
<td>0.0804</td>
<td>0.2766</td>
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<td>0.0773</td>
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<tr>
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<td>0.0083</td>
<td>0.0084</td>
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<td>0.0102</td>
<td>0.0352</td>
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<td>0.0150</td>
<td>0.0516</td>
</tr>
<tr>
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<td>0.0206</td>
<td>0.0711</td>
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<td>0.0702</td>
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<td>0.0264</td>
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</tr>
<tr>
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<td>0.0334</td>
<td>0.1148</td>
<td>0.1154</td>
<td>0.0329</td>
<td>0.1131</td>
</tr>
<tr>
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<td>0.1396</td>
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</tr>
<tr>
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<td>0.0484</td>
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<tr>
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<td>0.92</td>
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<td>0.0658</td>
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<td>0.2549</td>
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</table>

dition, we discuss some coupled $\mathcal{P}_p$ results. The main result is applied to establish the existence of optimum solutions for the class of a system of $(k, \mathcal{J})$-$HFDE$s of order $2 < p < 3$, type $0 \leq q \leq 1$ under integral and initial conditions. In the final step, we designed examples, and obtained numerical results of the system that well confirm the assumptions used. The technique used in this article can be used as a generalization in the area of solutions of nonlinear fractional and $q$-fractional differential equations via the $\mathcal{P}_p$ theory. The results of this research can establish more capabilities in the articles, such as [18, 38–43], that are presented about the existence and uniqueness of fractional differential equations and inclusions.

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Author contributions
GKK: Actualization, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. DKP: Validation, methodology, formal analysis, investigation, initial draft, and was a major contributor in writing the manuscript. PRP: Actualization, validation, methodology, formal analysis, investigation, and initial draft. MES: Formal analysis, validation, methodology, software, simulation and was a major contributor in writing the revision manuscript. All authors read and approved the final manuscript.

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Data availability
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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