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New L-fuzzy fixed point techniques for studying integral inclusions

Mohammed Shehu Shagari[1], Trad Alotaibi[2], Hassen Aydi[3,4*], Ahmad Aloqaily[5,6] and Nabil Mlaiki[5]

*Correspondence:
hassen.aydi@isima.rnu.tn
[3]Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia
[4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
Full list of author information is available at the end of the article

Abstract

The survey of the available literature shows that a lot of important invariant point problems of Banach and Heilpern types have been examined in both metric and quasimetric spaces. However, a handful of the existing results employed the recent approaches of interpolative contractions. Therefore, based on the new idea of interpolation techniques in fixed point theory, this article studies new notions of L-fuzzy contractions and investigates conditions for the existence of L-fuzzy fixed points for such mappings. On the fact that fixed points of point-to-point mappings satisfying interpolative-type contraction are not always unique, whence making the concepts more fitted for invariant point results of crisp set-valued maps, new multi-valued analogues of the key findings put forward in this work are derived. Comparative illustrations, which indicate the preeminence of the results presented herein, are constructed. From application viewpoint, one of the theorems so obtained is employed to introduce new solvability conditions of Fredholm-type integral inclusions.

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1 Introduction and preliminaries

Throughout this paper, a metric space, a complete metric space, and a fixed point will hereon be written as MS, CMS, and FP, respectively.

Banach [2] introduced one of the most known metric FP concepts, commonly named the Banach contraction principle. Due to the simplicity and usefulness of this principle, it has enjoyed multiple improvements in several directions. In some refinements of Banach invariant point idea, the original inequality is reformulated (see, e.g., [3]), and in others, the topological notions of the space are modified (see [7] and the citations therein). Along the lane, a chief refinement of the contraction technique was brought up by Hardy and Rogers [8]. The paradigm of this result (in [8]) is given hereunder.

Theorem 1.1 [8] *Let (W,d) be a CMS and S be a single-valued mapping on W satisfying*

$$d(Sh,S\mu) \leq a_1 d(h,Sh) + b_1 d(\mu,S\mu) + c_1 d(h,S\mu) + e_1 d(\mu,Sh) + l_1 d(h,\mu),$$

where a_1, b_1, c_1, e_1, l_1 *are nonnegative reals with* $a_1 + b_1 + c_1 + e_1 + l_1 < 1$, *then* S *has a unique FP in* W.

For not too long, Roldán et al. [24] brought up new FP ideas for a host of contractions depending on two functions and a few constants, named multiparametric contractions, and noted out a good number of Hardy–Rogers-type contractions in the frame of metric and quasi-MSs. Of recent as well, Karapinar et al. [14] established some common FP results for interpolative mappings availing Perov-type operators, which fulfil Suzuki-type inequalities. The announcement in Theorem 1.1 has also been moved further by more than a handful of examiners. Related copies of the contraction tools were on several occasions provided by Ćirić [3], Reich [22], and Rus [26].

Definition 1.2 [3, 22, 26]

(i) Rus contraction if we can find $a_1, b_1 \in \mathbb{R}_+$ with $a_1 + b_1 < 1$ such that for all $h, \mu \in W$,

$$d(Sh, S\mu) \leq a_1 d(h, \mu) + b_1 d(\mu, S\mu).$$

(ii) Ćirić–Reich contraction if we can find $a_1, b_1, c \in \mathbb{R}_+$ with $a_1 + b_1 + c < 1$ such that for all $h, \mu \in W$,

$$d(Sh, S\mu) \leq a_1 d(h, \mu) + b_1 d(h, Sh) + c d(\mu, S\mu).$$

A refined copy of the above findings is given hereunder.

Theorem 1.3 [3, 22, 26] *Let* (W, d) *be a CMS and the self-mapping* $S : W \longrightarrow W$ *be a Ćirić–Reich–Rus contraction, that is,*

$$d(Sh, S\mu) \leq c\big[d(h, \mu) + d(h, Sh) + d(\mu, S\mu)\big]$$

for all $h, \mu \in W$, $c \in [0, \frac{1}{3})$. *Then* S *enjoys an FP in* W.

Supported by the interpolation theory, Karapinar et al. [13] initiated the idea of interpolative-type notions as follows.

Definition 1.4 [13] Let (W, d) be an MS. $S : W \longrightarrow W$ is named an interpolative Hardy–Rogers-type contraction if we can find $c \in [0, 1[$ and $a_1, b_1, c_1 \in]0, 1[$ with $a_1 + b_1 + c_1 < 1$:

$$d(Sh, S\mu) \leq c\big[d(h, \mu)\big]^{b_1}\big[d(h, Sh)\big]^{a_1}\big[d(\mu, S\mu)\big]^{c_1}\left[\frac{1}{2}\big(d(h, S\mu) + d(\mu, Sh)\big)\right]^{1-a_1-b_1-c_1} \tag{1.1}$$

for all $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$, where $\mathcal{F}_{ix}(S)$ is the collection of invariant points of S.

For related FP results employing the interpolation approach, the researcher can look up [10–12, 18]. An intersecting behavior of the known FP results of interpolative type mappings is that their FP (if it exists) is not usually one and only one (for reference, see [11, Example 1]). This makes it seemingly clear that FP theorems via the concept of interpolation are more compatible for FP theory of point-to-set-valued maps.

On the flip side, a difficulty in modeling practical problems is connected to the inconclusiveness caused by our incapability to sort events with adequacy. It is a common knowledge that earlier sciences in the bodywork of crisp sets cannot withstand imprecisions efficiently. Therefore, a struggle to ameliorate the aforementioned hurdles led to the launching of fuzzy set by Zadeh [29]. At present, the fundamental ideas of fuzzy sets have been developed and applied in various domains. In 1981, Heilpern [9] employed the idea of fuzzy sets to launch the notion of fuzzy set-valued maps and set-up an FP result for fuzzy mappings, which is a fuzzy improvement of the FP theorems due to Nadler [19] and Banach [2]. Recently, by using the family of Γ functions initiated by Patel and Radenovic [28], Sessa et al. [27] introduced the idea of α_Γ-fuzzy contraction mappings and discussed the existence of FP for such mappings. Meanwhile, variants of Heilpern-type fuzzy invariant point results have been developed (e.g., see [1, 5, 16, 17] and the references therein). A very interesting improvement of fuzzy sets by replacing the interval $[0,1]$ of range set with a complete distributive lattice was brought up by Goguen [6] and is termed L-fuzzy set. Along the line, Rashid et al. [20] came up with the concept of L-fuzzy mappings and obtained common FP theorems through β_{FL}-admissible pair of L-fuzzy mappings. As an improvement of the idea of Hausdorff distance and d_∞-metric for fuzzy sets, Rashid et al. [21] proposed the ideas of $D_{\alpha L}$ and d_L^∞ distances for L-fuzzy sets and deduced some existing FP results for fuzzy set-valued and crisp set-valued maps.

Following the new interpolation approach in the study of FP results launched in [10–13], we noticed that the equivalent concepts with respect to L-fuzzy sets have not yet been studied or, at least, their analogues in fuzzy mathematics are very limited. Hence, this manuscript proposes the idea of interpolative Hardy–Rogers-type and interpolative Reich–Rus–Ćirić-type L-fuzzy contractions in MS and examines new ways for analyzing the L-fuzzy FPs of such contractions. It is worthy to indicate that FP of a single-valued mapping enjoying the interpolative-type expression is not necessarily unique. Therefore, the interpolative techniques are more appropriate for FP theorems of multi-valued maps. On this observation, some new set-valued copies of the L-fuzzy FP theorems studied herewith are discussed.

We now list a few preliminaries that are specific to our main results. Let (W,d) be an MS and $\mathcal{V}(W)$ be the collation of compact subsets of W. Take $M,U\in\mathcal{V}(W)$ and $r>0$ be arbitrary. Then the sets $N_d(r,M)$ and $E^d_{(M,U)}$ and the distance function $d(M,U)$ are respectively defined as follows:

$$N_d(r,M)=\big\{h\in W: d(h,y)<r,\ \text{for some } y\in M\big\}.$$
$$E^d_{(M,U)}=\big\{r>0: M\subseteq N_d(r,U), U\subseteq N_d(r,M)\big\}.$$
$$d(M,U)=\inf_{h\in M,\mu\in U} d(h,\mu).$$

Then the Hausdorff metric $\widehat{H}$ on $\mathcal{V}(W)$ generated by the metric d is defined as $\widehat{H}(M,U)=\inf E^d_{(M,U)}$ (see [19, P. 3]).

Recall that an ordinary subset M of W is determined by its characteristic function χ_M, defined by $\chi_M : M\longrightarrow\{0,1\}$:

$$\chi_M(h)=\begin{cases}1, & \text{if } h\in M\\ 0, & \text{if } h\notin M.\end{cases}$$

The value $\chi_M(h)$ specifies whether an element belongs to M or not. This idea was employed to define fuzzy sets by permitting an element $h \in A$ to take any value within $[0,1] = I$.

Definition 1.5 [4] A relation $\preceq$ on a nonempty set L is termed a partial order if it is
(i) reflexive;
(ii) antisymmetric;
(iii) transitive.

A set L together with a partial ordering $\preceq$ is named a partially ordered set (poset, for short) and is denoted by $(L, \preceq_L)$. Recall that partial orderings are used to give an order to sets that may not have a natural one. Let L be a nonempty set and $(L, \preceq)$ be a partially ordered set. Then any two elements $\beta, \varrho \in L$ are said to be comparable if either $\beta \preceq \varrho$ or $\varrho \preceq \beta$.

Definition 1.6 [4] A partially ordered set $(L, \preceq_L)$ is named:
(i) a lattice if $\beta \vee \varrho \in L$, $\beta \wedge \varrho \in L$ for any $\beta, \varrho \in L$;
(ii) a complete lattice if $\bigvee \nabla \in L$, $\bigwedge \nabla \in L$ for any $\nabla \subseteq L$;
(iii) distributive lattice if $\beta \vee (\varrho \wedge \xi) = (\beta \vee \varrho) \wedge (\beta \vee \xi)$, $\beta \wedge (\varrho \vee \xi) = (\beta \wedge \varrho) \vee (\beta \wedge \xi)$, for any $\beta, \varrho, \xi \in L$.

A partially ordered set L is named a complete lattice if for every doubleton $\{\beta, \varrho\}$ in L, either $\sup\{\beta, \varrho\} = \beta \bigvee \varrho$ or $\inf\{\beta, \varrho\} = \beta \bigwedge \varrho$ exists.

Definition 1.7 [6] An L-fuzzy set ∇ on a nonempty set W is a function with domain W whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L, respectively.

Remark 1.8 [6] The class of L-fuzzy sets is larger than the class of fuzzy sets as an L-fuzzy set reduces to a fuzzy set if $L = I = [0,1]$.

We denote the class of all L-fuzzy sets on a nonempty set W by L^W.

Definition 1.9 [6]The $\widehat{\tau}_L$-level set of an L-fuzzy set ∇ is denoted by $[\nabla]_{\widehat{\tau} L}$ and is defined as follows:

$$[\nabla]_{\widehat{\tau} L} = \begin{cases} \overline{\{\beta \in W : 0_L \preceq_L \nabla(\beta)\}}, & \text{if } \widehat{\tau}_L = 0 \\ \{\beta \in W : \widehat{\tau}_L \preceq_L \nabla(\beta)\}, & \text{if } \widehat{\tau}_L \in L \setminus \{0_L\}. \end{cases}$$

Definition 1.10 [20] Let W be an arbitrary nonempty set and Y be an MS. A mapping $\widehat{\Psi} : W \longrightarrow L^Y$ is named an L-fuzzy mapping. The function value $\widehat{\Psi}(d)(\varrho)$ is named the degree of membership of ϱ in $\widehat{\Psi}(d)$. For any two L-fuzzy mappings $S, \widehat{\Psi} : W \longrightarrow L^Y$, a point $\mathfrak{o} \in W$ is named an L-fuzzy FP of S if we can find $\widehat{\tau}_L \in L \setminus \{0_L\}$ such that $\mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L}$. A point $\mathfrak{o}$ is known as a common L-fuzzy FP of S and $\widehat{\Psi}$ if $\mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L} \cap [\widehat{\Psi}\mathfrak{o}]_{\widehat{\tau}_L}$.

If we can find $\widehat{\tau}_L \in L \setminus \{0_L\}$ such that $[S]_{\widehat{\tau}_L}, [U]_{\widehat{\tau}_L} \in \mathcal{V}(W)$, then we define

$$p_{\widehat{\tau}_L}(S, U) = \inf_{h \in [S]_{\widehat{\tau}_L}, \mu \in [U]_{\widehat{\tau}_L}} d(h, \mu).$$

$$D_{\widehat{\tau}_L}(S,U) = \widehat{H}\big([S]_{\widehat{\tau}_L}, [U]_{\widehat{\tau}_L}\big).$$

$$p(S,U) = \sup_{\widehat{\tau}_L} p_{\widehat{\tau}_L}(S,U).$$

$$d_L^{\infty}(S,U) = \sup_{\widehat{\tau}_L} D_{\widehat{\tau}_L}(S,U).$$

Observe that $p_{\widehat{\tau}_L}$ is a nondecreasing function of $\widehat{\tau}_L$ (see [9]), d_L^{∞} is a metric on $\mathcal{V}(W)$, and since (W,d) is complete, then $(\mathcal{V}_{\mathcal{F}}(W), d_L^{\infty})$ (see [9]) is also. Adding with, $(W,d) \longmapsto (\mathcal{V}(W), \widehat{H}) \longmapsto (\mathcal{V}_{\mathcal{F}}(W), d_L^{\infty})$ are isometric embeddings under $h \longrightarrow \{h\}$ and $M \longrightarrow \chi_M$, respectively, where

$$\mathcal{V}_{\mathcal{F}}(W) = \big\{S \in L^W : [S]_{\widehat{\tau}_L} \in \mathcal{V}(W), \text{ for each } \widehat{\tau}_L \in L \setminus \{0_L\}\big\}.$$

The following observation made in [19] is useful in discussing our main idea.

Lemma 1.11 [19] *Let S and U be nonempty closed and bounded subsets of an MS W. If $a \in S$, then $d(a,U) \leq \widehat{H}(S,U)$.*

2 Main results

The idea of L-fuzzy contraction of Hardy–Rogers-type is launched in this section, and the corresponding FP results are studied.

Definition 2.1 Given an MS (W,d), the L-fuzzy set-valued map $S : W \longrightarrow L^W$ is named an interpolative Hardy–Rogers-type (IH-RT) L-fuzzy contraction if we can find a mapping $\widehat{\tau}_L : W \longrightarrow L \setminus \{0_L\}$ and constants $c, a_1, b_1, c_1 \in (0,1)$ with $a_1 + b_1 + c_1 < 1$ such that for all $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$,

$$\begin{aligned} &\widehat{H}\big([Sh]_{\widehat{\tau}_L(h)}, [S\mu]_{\widehat{\tau}_L(\mu)}\big) \\ &\quad \leq c\big[d(h,\mu)\big]^{b_1}\big[d\big(h,[Sh]_{\widehat{\tau}_L(h)}\big)\big]^{a_1}\big[d\big(\mu,[S\mu]_{\widehat{\tau}_L(\mu)}\big)\big]^{c_1} \\ &\qquad \times \left[\frac{1}{2}\big(d\big(h,[S\mu]_{\widehat{\tau}_L(\mu)}\big) + d\big(\mu,[Sh]_{\widehat{\tau}_L(h)}\big)\big)\right]^{1-a_1-b_1-c_1}, \end{aligned} \tag{2.1}$$

where

$$\mathcal{F}_{ix}(S) = \big\{\mathfrak{o} \in W : \mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}, \text{ for some } \widehat{\tau}_L(\mathfrak{o}) \in L \setminus \{0_L\}\big\}.$$

Theorem 2.2 *Let (W,d) be a CMS and $S : W \to L^W$ be an IH-RT L-fuzzy contraction. Suppose that $[Sh]_{\widehat{\tau}_L(h)}$ is a nonempty compact subset of W for each $h \in W$. Then S has an L-fuzzy FP in W.*

Proof Let $h_0 \in W$ be arbitrary. Then, by hypothesis, $[Sh_0]_{\widehat{\tau}_L(h_0)} \in \mathcal{V}(W)$. Choose $h_1 \in [Sh_0]_{\widehat{\tau}_L(h_0)}$, then for this $h_1 \in W$, $[Sh_1]_{\widehat{\tau}_L(h_1)}$ is a nonempty compact subset of W. Therefore, we can find $h_2 \in [Sh_1]_{\widehat{\tau}_L(h_1)}$ such that

$$d(h_1,h_2) = d\big(h_1,[Sh_1]_{\widehat{\tau}_L(h_1)}\big) \leq \widehat{H}\big([Sh_0]_{\widehat{\tau}_L(h_0)}, [Sh_1]_{\widehat{\tau}_L(h_1)}]\big). \tag{2.2}$$

Setting $h = h_0$ and $\mu = h_1$ in (2.1) and using the fact that the function $\varpi(h) = h^{1-a_1-b_1-c_1}$ is nondecreasing yields

$$\begin{aligned}
&\widehat{H}\big([Sh_0]_{\widehat{\tau}_L(h_0)}, [Sh_1]_{\widehat{\tau}_L(h_1)}\big) \\
&\quad \le c\big[d(h_0,h_1)\big]^{b_1}\big[d\big(h_0,[Sh_0]_{\widehat{\tau}_L(h_0)}\big)\big]^{a_1}\big[d\big(h_1,[Sh_1]_{\widehat{\tau}_L(h_1)}\big)\big]^{c_1} \\
&\qquad \cdot \left[\frac{1}{2}\big(d\big(h_0,[Sh_1]_{\widehat{\tau}_L(h_1)}\big) + d\big(h_1,[Sh_0]_{\widehat{\tau}_L(h_0)}\big)\big)\right]^{1-a_1-b_1-c_1} \\
&\quad \le c\big[d(h_0,h_1)\big]^{b_1}\big[d(h_0,h_1)\big]^{a_1} d(h_1,h_2)]^{c_1}\left[\frac{1}{2}\big(d(h_0,h_2) + d(h_1,h_1)\big)\right]^{1-a_1-b_1-c_1} \\
&\quad \le c\big[d(h_0,h_1)\big]^{b_1}\big[d(h_0,h_1)\big]^{a_1} d(h_1,h_2)]^{c_1}\left[\frac{1}{2}\big(d(h_0,h_1) + d(h_1,h_2)\big)\right]^{1-a_1-b_1-c_1}.
\end{aligned} \tag{2.3}$$

Suppose that $d(h_0,h_1) \le d(h_1,h_2)$, then (2.3) produces

$$\begin{aligned}
\widehat{H}\big([Sh_0]_{\widehat{\tau}_L(h_0)}, [Sh_1]_{\widehat{\tau}_L(h_1)}\big) &\le c\big[d(h_1,h_2)\big]^{a_1+b_1+c_1}\big[d(h_1,h_2)\big]^{1-a_1-b_1-c_1} \\
&\le c\big(d(h_1,h_2)\big) \\
&< d(h_1,h_2),
\end{aligned} \tag{2.4}$$

a contradiction. Therefore, $d(h_1,h_2) < d(h_0,h_1)$. Therefore, for $\zeta = \sqrt{c}$ and $\varrho = \zeta d(h_0,h_1)$, (2.3) yields

$$\begin{aligned}
\widehat{H}\big([Sh_0]_{\widehat{\tau}_L(h_0)}, [Sh_1]_{\widehat{\tau}_L(h_1)}\big) &\le\le c\big[d(h_0,h_1)\big]^{a_1+b_1+c_1}\big[d(h_0,h_1)\big]^{1-a_1-b_1-c_1} \\
&\le \ cd(h_0,h_1) \\
&\le \ \varrho.
\end{aligned}$$

It follows that $d(h_1,h_2) < \varrho$ for some $h_2 \in [Sh_1]_{\widehat{\tau}_L(h_1)}$. Thus, $\varrho \in E^d_{([Sh_0]_{\widehat{\tau}_L(h_0)},[Sh_1]_{\widehat{\tau}_L(h_1)})}$. This implies that $[Sh_0]_{\widehat{\tau}_L} \subseteq N_d(\varrho,[Sh_0]_{\widehat{\tau}_L(h_0)})$ and $h_1 \in N_d(\varrho,[Sh_1]_{\widehat{\tau}_L(h_1)})$. On similar steps, we can find $h_2 \in N_d(\zeta d(h_0,h_1),[Sh_2]_{\widehat{\tau}_L(h_2)})$ and $h_3 \in [Sh_2]_{\widehat{\tau}_L(h_2)}$ such that for $\varrho^2 = \zeta^2 d(h_0,h_1)$ we have

$$\begin{aligned}
d(h_2,h_3) &\le \zeta d(h_1,h_2) \\
&\le \varrho^2.
\end{aligned}$$

Therefore, $\varrho^2 \in E^d_{([Sh_1]_{\widehat{\tau}_L(h_1)},[Sh_2]_{\widehat{\tau}_L(h_2)})}$. By induction, we come up with a sequence $\{h_{\mathfrak{r}}\}_{\mathfrak{r}\ge 1}$ in W such that $h_{\mathfrak{r}+1} \in [Sh_{\mathfrak{r}}]_{\widehat{\tau}_L(h_{\mathfrak{r}})}$ and

$$d(h_{\mathfrak{r}},h_{\mathfrak{r}+1}) \le \zeta^{\mathfrak{r}} d(h_0,h_1) \quad \text{for all } \mathfrak{r} \ge 1.$$

We now demonstrate that $\{h_{\mathfrak{r}}\}_{\mathfrak{r}\geq 1}$ is Cauchy in W. So, for all $k \geq 1$,

$$\begin{aligned} d(h_{\mathfrak{r}}, h_{\mathfrak{r}+k}) &\leq d(h_{\mathfrak{r}}, h_{\mathfrak{r}+1}) + d(h_{\mathfrak{r}+1}, h_{\mathfrak{r}+2}) + \cdots + d(h_{\mathfrak{r}+k-1}, h_{\mathfrak{r}+k}) \\ &\vdots \\ &\leq \frac{\zeta^{\mathfrak{r}}}{1-\zeta} d(h_0, h_1). \end{aligned} \tag{2.5}$$

Passing to limit in (2.5) as $\mathfrak{r} \longrightarrow \infty$, $\lim_{\mathfrak{r}\to\infty} d(h_{\mathfrak{r}}, h_{\mathfrak{r}+k}) = 0$. Therefore, $\{h_{\mathfrak{r}}\}_{\mathfrak{r}\geq 1}$ is a Cauchy sequence in W. By completeness of W, we can find $\mathfrak{o} \in W$ such that $h_{\mathfrak{r}} \longrightarrow \mathfrak{o}$ as $\mathfrak{r} \longrightarrow \infty$. Now, to show that $\mathfrak{o}$ is an L-fuzzy FP of S, assume that $\mathfrak{o} \notin [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}$ for all $\widehat{\tau}_L \in L$. Then, replacing h, μ with $h_{\mathfrak{r}}$ and $\mathfrak{o}$, respectively, in (2.1) leads to

$$\begin{aligned} & d\big(h_{\mathfrak{r}+1}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) \\ &\quad \leq \widehat{H}\big([Sh_{\mathfrak{r}}]_{\widehat{\tau}_L(h_{\mathfrak{r}})}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) \\ &\quad \leq c\big[d(h_{\mathfrak{r}}, \mathfrak{o})\big]^{b_1}\big[d\big(h_{\mathfrak{r}}, [Sh_{\mathfrak{r}}]_{\widehat{\tau}_L(h_{\mathfrak{r}})}\big)\big]^{a_1}\big[d\big(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big)\big]^{c_1} \\ &\qquad \cdot \left[\frac{1}{2}\big(d\big(h_{\mathfrak{r}}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) + d\big(\mathfrak{o}, [Sh_{\mathfrak{r}}]_{\widehat{\tau}_L(h_{\mathfrak{r}})}\big)\right]^{1-a_1-b_1-c_1} \\ &\quad \leq c\big[d(h_{\mathfrak{r}}, \mathfrak{o})\big]^{b_1}\big[d(h_{\mathfrak{r}}, h_{\mathfrak{r}+1})\big]^{a_1}\big[d\big(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big)\big]^{c_1} \\ &\qquad \cdot \left[\frac{1}{2}\big(d\big(h_{\mathfrak{r}}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) + d(\mathfrak{o}, h_{\mathfrak{r}+1}\big)\right]^{1-a_1-b_1-c_1}. \end{aligned} \tag{2.6}$$

Letting $\mathfrak{r} \longrightarrow \infty$ in (2.6) and using the continuity of d, we obtain $d(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}) = 0$. This proves that $\mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}$ for some $\widehat{\tau}_L(\mathfrak{o}) \in L \setminus \{0_L\}$. □

Example 2.3 Let $L = \{a, b, c, g, s, m, n, v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v$, $a \preceq_L g \preceq_L b \preceq_L v$, $s \preceq_L m \preceq_L v$, $g \preceq_L m \preceq_L v$, $n \preceq_L b \preceq_L v$; and each element of the doubletons $\{c, m\}$, $\{m, b\}$, $\{s, n\}$, $\{n, g\}$ is not comparable. Whence, $(L, \preceq_L)$ is a complete distributive lattice. Let $W = \{2, 6, 7, 12, 20\}$ and define $d : W \times W \longrightarrow \mathbb{R}$ by $d(h, \mu) = |h - \mu|$ for all $h, \mu \in W$. Clearly, (W, d) is a CMS. Define the mapping $S : W \longrightarrow L \setminus \{a\}$ as follows:

$$S(h)(\top) = \begin{cases} v, & \text{if } \top \in \{2, 6\} \\ m, & \text{if } \top = 7 \\ g, & \text{if } \top = 12 \\ b, & \text{if } \top = 20. \end{cases}$$

Suppose that $\widehat{\tau}_L(h) = v \setminus \{a\}$ for all $h \in W$. Then

$$\begin{aligned} [Sh]_v &= \big\{\top \in W : v \preceq S(h)(\top)\big\} \\ &= \{2, 6\}. \end{aligned}$$

To show that S is a Hardy–Rogers-type L-fuzzy contraction, take $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$. Obviously, $h, \mu \in \{12, 20\}$. Thus,

$$\widehat{H}\big([S12]_{\widehat{\tau}_L(12)}, [S20]_{\widehat{\tau}_L(20)}\big) = \widehat{H}\big([S20]_{\widehat{\tau}_L(20)}, [S12]_{\widehat{\tau}_L(12)}\big)$$

$$= \widehat{H}\big(\{2,6\},\{2,6\}\big) = 0.$$

Therefore, all the conditions of Theorem 2.2 are verified. We can see that the set of all L-fuzzy FP of S is given by $\mathcal{F}_{ix}(S) = \{2,6\}$.

Turned on by Theorem 1.3 and the idea of [12, Theorem 4], we investigate the next notion of interpolative Reich–Rus–Ćirić-type (IRRC-T) L-fuzzy contraction and investigate the condition for the existence of L-fuzzy FP for such contraction.

Definition 2.4 Let (W,d) be an MS. An L-fuzzy set-valued map $S : W \longrightarrow L^W$ is named IRRC-T L-fuzzy contraction if we can find a mapping $\widehat{\tau}_L : W \longrightarrow L \setminus \{0_L\}$ and constants $\widehat{\eta} \in [0,1)$, $a_1, b_1 \in (0,1)$ with $a_1 + b_1 < 1$ such that

$$\widehat{H}\big([Sh]_{\widehat{\tau}_L(h)}, [S\mu]_{\widehat{\tau}_L(\mu)}\big) \leq \widehat{\eta}\big[d(h,\mu)\big]^{a_1}\big[d\big(h,[Sh]_{\widehat{\tau}_L(h)}\big)\big]^{b_1}\big[d\big(\mu,[S\mu]_{\widehat{\tau}_L(\mu)}\big)\big]^{1-a_1-b_1} \tag{2.7}$$

for all $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$.

Theorem 2.5 *Let (W,d) be a CMS and $S : W \longrightarrow L^W$ be an IRRC-T L-fuzzy contraction. Suppose further that $[Sh]_{\widehat{\tau}_L(h)}$ is a nonempty compact subset of W for each $h \in W$. Then S has a fuzzy FP in W.*

Proof Let $h_0 \in W$ be given. Then, by hypothesis, we can find $\widehat{\tau}_L(h_0) \in L \setminus \{0_L\}$ such that $[Sh_0]_{\widehat{\tau}_L(h_0)} \in \mathcal{V}(W)$. By compactness of $[Sh_0]_{\widehat{\tau}_L(h_0)}$, we can find $h_1 \in [Sh_0]_{\widehat{\tau}_L(h_0)}$ with $d(h_0,h_1) > 0$ such that $d(h_0,h_1) = d(h_0,[Sh_0]_{\widehat{\tau}_L(h_0)})$. In the same way, by assumption, we can find $\widehat{\tau}_L(h_1) \in L \setminus \{0_L\}$ such that $[Sh_1]_{\widehat{\tau}_L(h_1)}$ is a nonempty compact subset of W. Thus, we can find $h_2 \in [Sh_1]_{\widehat{\tau}_L(h_1)}$ with $d(h_1,h_2) > 0$ such that $d(h_1,h_2) = d(h_1,[Sh_1]_{\widehat{\tau}_L(h_1)})$. In this fashion, we come up with a sequence $\{h_\mathfrak{r}\}_{\mathfrak{r}\geq 1}$ of points of W with $h_{\mathfrak{r}+1} \in [Sh_\mathfrak{r}]_{\widehat{\tau}_L(h_\mathfrak{r})}$, $d(h_\mathfrak{r},h_{\mathfrak{r}+1}) > 0$ such that $d(h_\mathfrak{r},h_{\mathfrak{r}+1}) = d(h_\mathfrak{r},[Sh_\mathfrak{r}]_{\widehat{\tau}_L(h_\mathfrak{r})})$. By Lemma 1.11, we have

$$d(h_\mathfrak{r},h_{\mathfrak{r}+1}) \leq \widehat{H}\big([Sh_{\mathfrak{r}-1}]_{\widehat{\tau}_L(h_{\mathfrak{r}-1})}, [Sh_{\mathfrak{r}+1}]_{\widehat{\tau}_L(h_{\mathfrak{r}+1})}\big). \tag{2.8}$$

Now, we show that $\{h_\mathfrak{r}\}_{\mathfrak{r}\geq 1}$ is a Cauchy sequence in W. Setting $h = h_\mathfrak{r}$ and $\mu = h_{\mathfrak{r}-1}$ in (2.7), we get

$$\begin{aligned} d(h_\mathfrak{r},h_{\mathfrak{r}+1}) &\leq \widehat{H}\big([Sh_\mathfrak{r}]_{\widehat{\tau}_L(h_\mathfrak{r})}, [Sh_{\mathfrak{r}-1}]_{\widehat{\tau}_L(h_{\mathfrak{r}-1})}\big) \\ &\leq \widehat{\eta}\big[d(h_\mathfrak{r},h_{\mathfrak{r}-1})\big]^{a_1}\big[d\big(h_\mathfrak{r},[Sh_\mathfrak{r}]_{\widehat{\tau}_L(h_\mathfrak{r})}\big)\big]^{b_1}\big[d\big(h_{\mathfrak{r}-1},[Sh_{\mathfrak{r}-1}]_{\widehat{\tau}_L(h_{\mathfrak{r}-1})}\big)\big]^{1-a_1-b_1} \\ &\leq \widehat{\eta}\big[d(h_\mathfrak{r},h_{\mathfrak{r}-1})\big]^{a_1}\big[d(h_\mathfrak{r},h_{\mathfrak{r}+1})\big]^{b_1}\big[d(h_{\mathfrak{r}-1},h_\mathfrak{r})\big]^{1-a_1-b_1} \\ &= \widehat{\eta}\big[d(h_\mathfrak{r},h_{\mathfrak{r}-1})\big]^{1-b_1}\big[d(h_\mathfrak{r},h_{\mathfrak{r}+1})\big]^{b_1}. \end{aligned} \tag{2.9}$$

From (2.9), we have

$$d(h_\mathfrak{r},h_{\mathfrak{r}+1}) \leq \widehat{\eta}^{\frac{1}{1-b_1}} d(h_{\mathfrak{r}-1},h_\mathfrak{r}) \quad \text{for all } \mathfrak{r} \in \mathbb{N}. \tag{2.10}$$

We deduce from (2.10) that for all $\mathfrak{r} \in \mathbb{N}$,

$$d(h_\mathfrak{r},h_{\mathfrak{r}+1}) \leq \widehat{\eta} d(h_{\mathfrak{r}-1},h_\mathfrak{r}) \leq \widehat{\eta}^\mathfrak{r} d(h_0,h_1). \tag{2.11}$$

From (2.11), following the proof of Theorem 2.2, we infer that $\{h_{\mathfrak{r}}\}_{\mathfrak{r}\geq 1}$ is a Cauchy sequence in W. The completeness of this space implies that we can find $\mathfrak{o} \in W$ such that $h_{\mathfrak{r}} \longrightarrow \mathfrak{o}$ as $\mathfrak{r} \longrightarrow \infty$. Now, we show that $\mathfrak{o}$ is an L-fuzzy FP of W. For this, replacing h and μ with $h_{\mathfrak{r}}$ and $\mathfrak{o}$, respectively, in (2.7) and using Lemma 1.11 gives

$$\begin{aligned} d\big(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) &\leq d(\mathfrak{o}, h_{\mathfrak{r}+1}) + d\big(h_{\mathfrak{r}+1}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) \\ &\leq d(\mathfrak{o}, h_{\mathfrak{r}+1}) + \widehat{H}\big([Sh_{\mathfrak{r}}]_{\widehat{\tau}_L(h_{\mathfrak{r}})}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big) \\ &\leq d(\mathfrak{o}, h_{\mathfrak{r}+1}) + \widehat{\eta}\big[d(h_{\mathfrak{r}}, u)\big]^{a_1}\big[d(h_{\mathfrak{r}}, h_{\mathfrak{r}+1})\big]^{b_1}\big[d\big(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}\big)\big]^{1-a_1-b_1}. \end{aligned} \tag{2.12}$$

Letting $\mathfrak{r} \longrightarrow \infty$ in (2.12) and using the continuity of the metric d yields $d(\mathfrak{o}, [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}) = 0$. Therefore, $\mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})}$. □

As an extension of the result of Heilpern [9, Theorem 3.1], next we study FP theorems of Hardy–Rogers-type L-fuzzy contraction and Reich–Rus–Ćirić-type L-fuzzy contraction, availing the interpolative technique in association with d_L^∞-metric for L-fuzzy sets. Worthy of note is the fact that L-fuzzy FP results in the setting of d_L^∞-metric are very paramount in computing Hausdorff dimensions. These dimensions aid us to analyze the concept of ε^∞-space, which is of enormous gain in higher energy physics.

Theorem 2.6 *Let (W, d) be a CMS and $S : W \longrightarrow \mathcal{V}_{\mathcal{F}}(W)$ be an L-fuzzy set-valued map. Suppose that the following conditions are satisfied: we can find $c, a_1, b_1, c_1 \in (0, 1)$ with $a_1 + b_1 + c_1 < 1$ such that, for all $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$,*

$$d_L^\infty(Sh, S\mu) \leq c\big[d(h,\mu)\big]^{b_1}\big[p(h, S(h)\big]^{a_1}\big[p(\mu, S\mu)\big]^{c_1}\left[\frac{1}{2}\big(p(h, S\mu) + p(\mu, Sh)\big)\right]^{1-a_1-b_1-c_1} \tag{2.13}$$

Then S has an L-fuzzy FP in W.

Proof Let $h \in W$ be arbitrary, and define the mapping $\widehat{\tau}_L : W \longrightarrow L \setminus \{0_L\}$ by $\widehat{\tau}_L(h) = 1_L$, where 1_L is the top element of L. Then, by hypothesis, $[Sh]_{1_L} \in \mathcal{V}(W)$. Now, for every $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$,

$$\begin{aligned} &D_{1_L}(Sh, S\mu) \\ &\quad\leq d_L^\infty(Sh, S\mu) \\ &\quad\leq c\big[d(h,\mu)\big]^{b_1}\big[p(h, Sh)\big]^{a_1}\big[p(\mu, S\mu)\big]^{c_1}\left[\frac{1}{2}\big(p(h, S\mu) + p(\mu, Sh)\big)\right]^{1-a_1-b_1-c_1}. \end{aligned}$$

Since $[Sh]_{1_L} \subseteq [Sh]_{\widehat{\tau}_L(h)} \in \mathcal{V}(W)$, therefore $d(h, [Sh]_{\widehat{\tau}_L(h)}) \leq d(h, [Sh]_{1_L})$ for each $\widehat{\tau}_L(h) \in L \setminus \{0_L\}$. It follows that $p(h, Sh) \leq d(h, [Sh]_{1_L})$. Thus,

$$\begin{aligned} &\widehat{H}\big([Sh]_{1_L}, [S\mu]_{1_L}\big) \\ &\quad\leq c\big[d(h,\mu)\big]^{b_1}\big[d\big(h, [Sh]_{1_L}\big)\big]^{a_1}\big[d\big(\mu, [S\mu]_{1_L}\big)\big]^{c} \\ &\qquad\times\left[\frac{1}{2}\big(d\big(h, [S\mu]_{1_L}\big), d\big(\mu, [Sh]_{1_L}\big)\big)\right]^{1-a_1-b_1-c}. \end{aligned} \tag{2.14}$$

Therefore, Theorem 2.2 can be invited to find $\mathfrak{o} \in W$ such that $\mathfrak{o} \in [S\mathfrak{o}]_{1_L}$. □

By ignoring some terms in Theorem 2.6, we can obtain the next result using similar arguments.

Theorem 2.7 *Let (W,d) be a CMS and $S: W \longrightarrow \mathcal{V}_{\mathcal{F}}(W)$ be an L-fuzzy set-valued map. Suppose that the following conditions are satisfied: we can find $\widehat{\eta} \in [0,1)$ and $a_1, b_1 \in (0,1)$ with $a_1 + b_1 < 1$ such that, for all $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$,*

$$d_L^{\infty}(Sh, S\mu) \leq \widehat{\eta}\big[d(h,\mu)\big]^{a_1}\big[p(h,S(h)\big]^{b_1}\big[p(\mu,S\mu)\big]^{1-a_1-b_1}. \tag{2.15}$$

Then S has an L-fuzzy FP in W.

Example 2.8 Let $W = \{\wp_{\mathfrak{r}} = \frac{\mathfrak{r}(\mathfrak{r}+1)}{5} : \mathfrak{r} = 1,2,\ldots\} \cup \{0\}$ and $d(h,\mu) = |h-\mu|$ for all $h, \mu \in W$. Then (W,d) is a CMS. Let $L = \{a,b,c,g,s,m,n,v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v$, $a \preceq_L g \preceq_L b \preceq_L v$, $s \preceq_L m \preceq_L v$, $g \preceq_L m \preceq_L v$, $n \preceq_L b \preceq_L v$; and each element of the doubletons $\{c,m\}$, $\{m,b\}$, $\{s,n\}$, $\{n,g\}$ is not comparable. Then $(L,\preceq_L)$ is a complete distributive lattice. Define an L-fuzzy set-valued map $S: W \longrightarrow \mathcal{V}_{\mathcal{F}}(W)$ as follows:

For $h = 0$,

$$S(0)(\mathsf{T}) = \begin{cases} b, & \text{if } \mathsf{T} = 0 \\ v, & \text{if } \mathsf{T} = \wp_1 \\ g, & \text{if } \mathsf{T} = \wp_2 \\ s, & \text{if } \mathsf{T} = \wp_{\mathfrak{r}}, \mathfrak{r} \geq 3, \end{cases}$$

and for $h \in W \setminus \{0\}$,

$$S(h)(\mathsf{T}) = \begin{cases} c, & \text{if } \mathsf{T} = \wp_1 \\ m, & \text{if } \mathsf{T} = \wp_2 \\ v, & \text{if } \mathsf{T} \in \{\wp_3, \wp_4, \ldots, \wp_{\mathfrak{r}-1}\}, \mathfrak{r} \geq 3. \end{cases}$$

Define the mapping $\widehat{\tau}_L : W \longrightarrow L \setminus \{a\}$ by $\widehat{\tau}_L(h) = v$ for all $h \in W$. Then

$$[Sh]_{\widehat{\tau}_L(h)} = \begin{cases} \{\wp_1\}, & \text{if } h = 0 \\ \{\wp_3, \wp_4, \ldots, \wp_{\mathfrak{r}-1}\}, & \text{if } h \neq 0, \mathfrak{r} \geq 3. \end{cases}$$

Now, to see that the contractive condition (2.15) holds, let $h, \mu \in W \setminus \mathcal{F}_{ix}(S)$. Clearly, $h = \mu = \wp_1$. Therefore,

$$d_{\infty}\big(S(h), S(\mu)\big) = 0 \leq \widehat{\eta}\big[d(h,\mu)\big]^{a_1}\big[p\big(h,S(h)\big)\big]^{b_1}\big[p\big(\mu,S(\mu)\big)\big]^{1-a_1-b_1}$$

for all $\widehat{\eta} \in (0,1)$. This shows that (2.15) holds for all $h, \mu \in W$. Therefore, all the assumptions of Theorem 2.7 are satisfied. We see that S has many L-fuzzy FP in W.

In contrast, S is not a fuzzy set-valued contraction in the sense of Heilpern [9]. To show this, take $h = 0$ and $\mu = \wp_{\mathfrak{r}-1}$, $\mathfrak{r} \geq 3$, we have

$$\begin{aligned}\sup_{\mathfrak{r}\geq 3}\frac{\widehat{H}([S0]_\nu,[S\wp_{\mathfrak{r}-1}]_\nu)}{d(0,\wp_{\mathfrak{r}-1})} &= \sup_{\mathfrak{r}\geq 3}\frac{\wp_{\mathfrak{r}-1}-1}{\wp_{\mathfrak{r}-1}}\\ &= \sup_{\mathfrak{r}\geq 3}\frac{\frac{\mathfrak{r}(\mathfrak{r}-1)}{5}-1}{\frac{\mathfrak{r}(\mathfrak{r}-1)}{5}}\\ &= \sup_{\mathfrak{r}\geq 3}\left[1-\frac{5}{\mathfrak{r}(\mathfrak{r}-1)}\right] = 1.\end{aligned}$$

Therefore, the main result of Heilpern [9] is inapplicable to this example.

3 Applications to crisp set-valued and single-valued mappings

Let (W,d) be an MS, $CB(W)$ and $\mathcal{N}(W)$ be the classes of nonempty closed and bounded and nonempty subsets of W, respectively. A mapping $\mho : W \longrightarrow \mathcal{N}(W)$ is named a multi-valued contraction (see [19]) if we can find a constant $c \in (0,1)$ such that $\widehat{H}(Fh,F\mu) \leq cd(h,\mu)$ for all $h,\mu \in W$. A point $\mathfrak{o} \in W$ is termed an FP of $\mho$ if $\mathfrak{o} \in F\mathfrak{o}$. Nadler [19, Theorem 5] noted that each multi-valued contraction on a CMS enjoys an FP. Among the extensions of multi-valued contractions in the sense of Nadler that we are concerned with here are the ones studied by Reich [23] and Rus [25].

Theorem 3.1 (*see Rus* [25]) *Let (W,d) be a CMS and $\mho : W \longrightarrow CB(W)$ be a multi-valued mapping. Suppose that we can find $a_1, b_1 \in \mathbb{R}_+$ with $a_1 + b_1 < 1$ such that, for all $h,\mu \in W$,*

$$\widehat{H}(Fh,F\mu) \leq a_1 d(h,\mu) + b_1 d(\mu,F\mu).$$

Then we can find $\mathfrak{o} \in W$ such that $\mathfrak{o} \in F\mathfrak{o}$.

Theorem 3.2 (*see Reich* [23]) *Let (W,d) be a CMS and $\mho : W \longrightarrow CB(W)$ be a multi-valued mapping. Suppose that we can find $a_1, b_1 \in \mathbb{R}_+$ with $a_1 + b_1 + c < 1$ such that, for all $h,\mu \in W$,*

$$\widehat{H}(Fh,F\mu) \leq a_1 d(h,\mu) + b_1 d(h,Fh) + cd(\mu,F\mu).$$

Then we can find $\mathfrak{o} \in W$ such that $\mathfrak{o} \in F\mathfrak{o}$.

Herewith, we come up with some consequences and equivalent results of our main theorems in the framework of both single-valued and multi-valued mappings. First, we present multi-valued analogues of Theorems 2.2 and 2.5. They are also crisp set-valued refinements of the recently established FP theorems due to Karapinar et al. [13, Theorem 4] and Karapinar et al. [11, Corollary 1], respectively.

Corollary 3.3 *Let (W,d) be a CMS and $F : W \longrightarrow \mathcal{V}(W)$ be a multi-valued mapping. Suppose that we can find $c, a_1, b, c, \in (0,1]$ with $a_1 + b + c < 1$ such that, for all $h,\mu \in W \setminus$*

$\mathcal{F}_{ix}(F)$,

$$\widehat{H}(Fh,F\mu) \le c\big[d(h,\mu)\big]^{b}\big[d(h,Fh)\big]^{a_1}\big[d(\mu,F\mu)\big]^{c}\left[\frac{1}{2}\big(d(h,F\mu)+d(\mu,Fh)\big)\right]^{1-a_1-b-c}. \quad (3.1)$$

Then we can find $\mathfrak{o} \in W$ *such that* $\mathfrak{o} \in F\mathfrak{o}$.

Proof Consider a mapping $\vartheta : W \longrightarrow L\setminus\{0_L\}$ and an L-fuzzy set-valued map $S : W \longrightarrow L^W$ defined by

$$S(h)(\top) = \begin{cases} \vartheta h, & \text{if } \top \in Fh \\ 0_L, & \text{if } \top \notin Fh. \end{cases}$$

Taking $\widehat{\tau}_L(h) = \vartheta(h)$ for all $h \in W$ leads to

$$[Sh]_{\widehat{\tau}_L(h)} = \big\{\top \in W : \widehat{\tau}_L(h) \preceq S(h)(\top)\big\} = Fh.$$

Therefore, Theorem 2.2 can be used to find $\mathfrak{o} \in W$ such that $\mathfrak{o} \in F\mathfrak{o} = [S\mathfrak{o}]_{\widehat{\tau}_L}$. □

Example 3.4 Let $W = [1,5]$ and $d(h,\mu) = |h-\mu|$ for all $h,\mu \in W$. Then (W,d) is a CMS. Define $\mho : W \longrightarrow \mathcal{V}(W)$ by

$$Fh = \begin{cases} [1,2], & \text{if } 1 \le h < 2 \\ [3,5], & \text{if } 2 \le h \le 5. \end{cases}$$

Let $h,\mu \in W\setminus\mathcal{F}_{ix}(F)$. Clearly, $h,\mu \in (1,2)$ and

$$\begin{aligned} \widehat{H}(Fh,F\mu) &= \widehat{H}\big([1,2],[1,2]\big) = 0 \\ &\le c\big[d(h,\mu)\big]^{b_1}\big[d(h,Fh)\big]^{a_1}\big[d(\mu,F\mu)\big]^{c_1}\left[\frac{1}{2}\big(d(h,F\mu)+d(\mu,Fh)\big)\right]^{1-a_1-b_1-c_1}. \end{aligned}$$

Therefore, all the assumptions of Corollary 3.3 are satisfied. We see that $\mho$ has many FP in W.

In contrast, $\mho$ is not a multi-valued contraction since for $h = 1$ and $\mu = 2$ we have

$$\begin{aligned} \widehat{H}(F1,F2) &= \widehat{H}\big([1,2],[3,5]\big) \\ &= 3 > c(1) = cd(1,2) \end{aligned}$$

for all $c \in (0,1)$. Therefore, the result of Nadler [19, Theorem 5] is not applicable in this example to obtain an FP of $\mho$. In the same way, since $\mho 1 = [1,2]$ and $\mho 2 = [3,5]$, we have

$$d(1,F1) = \inf_{\varrho\in[1,2]} d(1,\varrho) = 0,$$

$$d(2,F2) = \inf_{\xi\in[3,5]} d(2,\xi) = 1.$$

Therefore,

$$\begin{aligned}\widehat{H}(F1,F2) &= \widehat{H}\big([1,2],[3,5]\big)\\ &= 3 > a_1 + b_1 = a_1(1) + b_1(1)\\ &= a_1 d(1,2) + b_1 d(2,F2)\end{aligned}$$

for all $a_1, b_1 \in \mathbb{R}_+$ satisfying $a_1 + b_1 < 1$. That is to say, Theorem 3.1 due to Rus [25] is inapplicable to this example to find an FP of $\mho$.

In like manner,

$$\begin{aligned}\widehat{H}(F1,F2) &= \widehat{H}\big([1,2],[3,5]\big)\\ &= 3 > a_1 + c_1 = a_1(1) + b_1(0) + c_1(1)\\ &= a_1 d(1,2) + b_1 d(1,F1) + c_1 d(2,F2)\end{aligned}$$

for all $a_1, b_1, c_1 \in \mathbb{R}_+$ with $a_1 + b_1 + c_1 < 1$. Therefore, Theorem 3.2 due to Reich [23] is not applicable in this case to locate any FP of $\mho$.

Corollary 3.5 (*see Karapinar et al.* [13, *Theorem* 4]) *Let* (W,d) *be a CMS and* $f : W \to W$ *be a single-valued mapping. Suppose that we can find* $c, a_1, b_1, c_1 \in (0,1)$ *with* $a_1 + b_1 + c_1 < 1$ *such that, for all* $h, \mu \in W \setminus \mathcal{F}_{ix}(f)$, *we have*

$$d(fh,f\mu) \leq c\big[d(h,\mu)\big]^{b_1}\big[d(h,fh)\big]^{a_1}\big[d(\mu,f\mu)\big]^{c_1}\left[\frac{1}{2}\big(d(h,f\mu) + d(\mu,fh)\big)\right]^{1-a_1-b_1-c_1}. \quad (3.2)$$

Then we can find $\mathfrak{o} \in W$ *such that* $f\mathfrak{o} = \mathfrak{o}$.

Proof Let $\widehat{\tau}_L : W \longrightarrow L \setminus \{0_L\}$ be a mapping, and define an L-fuzzy set-valued map $S : W \longrightarrow L^W$ as follows:

$$S(h)(\mathsf{T}) = \begin{cases}\widehat{\tau}_L(h), & \text{if } t = fh\\ 0_L, & \text{if } \mathsf{T} \neq fh.\end{cases}$$

Then

$$[Sh]_{\widehat{\tau}_L(h)} = \big\{\mathsf{T} \in W : \widehat{\tau}_L(h) \preceq S(h)(\mathsf{T})\big\} = \{fh\}.$$

Clearly, $\{fh\} \in \mathcal{V}(W)$ for all $h \in W$. Note that in this case $\widehat{H}([Sh]_{\widehat{\tau}_L(h)}, [Sh]_{\widehat{\tau}_L(\mu)}) = d(fh,f\mu)$ for all $h,\mu \in W$. Therefore, Theorem 2.2 can be invited to find $\mathfrak{o} \in W$ such that $\mathfrak{o} \in [S\mathfrak{o}]_{\widehat{\tau}_L(\mathfrak{o})} = \{f\mathfrak{o}\}$, which signifies further that $\mathfrak{o} = f\mathfrak{o}$. □

By maintaining the procedure for determining Corollary 3.5, additionally, we can arrive at the following.

Corollary 3.6 (*see Karapinar et al.* [11, *Corollary* 1]) *Let* (W,d) *be a CMS and* $f : W \to W$ *be a single-valued mapping. Suppose that we can find* $c, a_1, b_1 \in (0,1)$ *with* $a_1 + b_1 < 1$ *such* *that, for all* $h,\mu \in W \setminus \mathcal{F}_{ix}(f)$, *we have*

$$d(fh,f\mu) \leq c\big[d(h,\mu)\big]^{a_1}\big[d(h,fh)\big]^{b_1}\big[d(\mu,f\mu)\big]^{1-a_1-b_1}. \tag{3.3}$$

Then we can find $\mathfrak{o} \in W$ *such that* $f\mathfrak{o} = \mathfrak{o}$

4 Applications to Fredholm-type integral inclusions

Herewith, we apply Theorem 2.2 to investigate new conditions for the existence of solutions to a Fredholm-type integral inclusion, given as

$$h(\top) \in \left[\widehat{\tau}(\top) + \int_d^j \mathcal{L}\big(\top,t,h(t)\big)\,dt, \top \in [d,j]\right], \tag{4.1}$$

where $h \in C([d,j],\mathbb{R})$ is an unknown real-valued continuous function defined on $[d,j]$, $\widehat{\tau}$ is a given real-valued continuous function, and L is a given set-valued map. The family of nonempty compact and convex subsets of $\mathbb{R}$ is denoted by $\digamma_{cv}(\mathbb{R})$.

Now, we study the existence of solutions of (4.1) under the following assumptions.

Theorem 4.1 *Let* $W = C([d,j],\mathbb{R})$ *and suppose that*:

(C_1) *the set-valued map* $L : [d,j] \times [d,j] \times \mathbb{R} \longrightarrow \digamma_{cv}(\mathbb{R})$ *is such that, for every* $h \in W$, *the map* $L_h(\top,t) := \mathcal{L}(\top,t,h(t))$ *is lower semicontinuous*;

(C_2) $\widehat{\tau} \in C([d,j],\mathbb{R})$;

(C_3) *we can find a function* $\xi : (0,\infty) \longrightarrow \mathbb{R}$ *such that, for all* $h,\mu \in W$,

$$\widehat{H}\big(L_h(\top,t),L_\mu(\top,t)\big) \leq \pi(\top,t)\xi(\top)\big(\big|h(t)-\mu(t)\big|\big)^r$$

for each $\top,t \in [d,j]$, *where* $\sup_{d\leq t\leq j}(\int_d^j \pi(\top,t)\,ds) \leq 1$, $\pi(\top,.) \in L^1[d,j]$ *and* $r \in (0,1)$. *Then the integral inclusion* (4.1) *has at least one solution in* W.

Proof Define $d : W \times W \longrightarrow \mathbb{R}$ by

$$d(h,\mu) = \max_{d\leq\top\leq j}\big|h(h)-\mu(\top)\big| \quad \text{for all } h,\mu \in W,$$

then (W,d) is a complete MS. Let $S : W \longrightarrow L^W$ be an L-fuzzy set-valued map. Consider the $\widehat{\tau}_L$-level set of S defined as

$$[Sh]_{\widehat{\tau}_L(h)} = \left\{\mu \in W : \mu(\top) \in \widehat{\tau}(\top) + \int_d^j \mathcal{L}\big(\top,t,h(t)\big)\,ds, \top \in [d,j]\right\}.$$

Obviously, the set of solutions of equation (4.1) coincides with the set of L-fuzzy FP of the set-valued map S. Therefore, we need to show that under the given assumptions, S has at least one L-fuzzy FP in W. To do this, we will verify that all the assumptions of Theorem 2.2 are satisfied.

Let $h \in W$ be arbitrary. Since the set-valued map $L_h : [d,j]^2 \longrightarrow \digamma_{cv}(\mathbb{R})$ is lower semicontinuous, it follows from Michael's selection theorem ([15, Theorem 1]) that we can find a continuous map $\rho_h : [d,j]^2 \longrightarrow \mathbb{R}$ such that $\rho_h(\top,\rho_h(\top,t) \in L_h(\top,t)$ for each $(\top,t) \in [d,j]^2$. Thus, $\widehat{\tau}(\top) + \int_d^j \rho_h(\top,t)\,ds \in [Sh]_{\widehat{\tau}_L(h)}$. So $[Sh]_{\widehat{\tau}_L(h)}$ is nonempty. One can easily see that

$[Sh]_{\widehat{\tau}_L(h)}$ is a compact subset of W. In other words, given that $\widehat{\tau} \in C([d,j])$ and $L_h(\top,t)$ is continuous on $[d,j]^2$, their range sets are also continuous for each $h \in W$.

Take $h_1, h_2 \in W$. Then we can find $\widehat{\tau}_L(h_1), \widehat{\tau}_L(h_2) \in L \setminus \{0_L\}$ such that $[Sh_1]_{\widehat{\tau}_L(h_1)}$ and $[Sh_2]_{\widehat{\tau}_L(h_2)}$ are nonempty compact subsets of W. Then we can find an arbitrary point $\mu_1 \in [Sh_1]_{\widehat{\tau}_L(h_1)}$ with

$$\mu_1(\top) \in \widehat{\tau}(\top) + \int_d^j \mathcal{L}\big(\top, t, h_1(t)\big)\, ds, \quad \top \in [d,j].$$

This means that for each $(\top, t) \in [d,j]^2$ there exists $\rho_{h_1} \in L_{h_1}(\top,t)$ such that

$$\mu_1(\top) = \widehat{\tau}(\top) + \int_d^j \rho_{h_1}(\top,t)\, ds, \quad \top \in [d,j].$$

Since from (C_2)

$$\widehat{H}\big(\mathcal{L}\big(\top,t,h_1(t)\big), \mathcal{L}\big(\top,t,h_2(t)\big)\big) \le \pi(\top,t)\xi(\top)\big(\big|h_1(t) - h_2(t)\big|\big)^r$$

for each $\top, t \in [d,j]$ and $r \in (0,1)$, we can find $\rho_{h_2} \in L_{h_2}(\top,t)$ such that

$$\big|\rho_{h_1}(\top,t) - \rho_{h_2}(\top,t)\big| \le \pi(\top,t)\xi(\top)\big(\big|h_1(t) - h_2(t)\big|\big)^r$$

for all $(\top,t) \in [d,j]^2$.

Now, consider the set-valued map $\mathfrak{M}$ defined by

$$\mathfrak{M}(\top,t) = L_{h_2}(\top,t) \cap \big\{\varrho \in \mathbb{R} : \big|\rho_{h_1}(\top,t) - \varrho\big| \le \pi(\top,t)\xi(\top)\big(\big|h_1(t) - h_2(t)\big|\big)^r\big\}.$$

Taking into account the fact that from (C_1), $\mathfrak{M}$ is lower semicontinuous, we can find a continuous map $\rho_{h_2} : [d,j]^2 \longrightarrow \mathbb{R}$ such that $\rho_{h_2}(\top,t) \in \mathfrak{M}(\top,t)$ for all $(\top,t) \in [d,j]^2$. Then

$$\begin{aligned}\mu_2(\top) &= \widehat{\tau}(\top) + \int_d^j \rho_{h_1}(\top,t)\, ds \\ &\in \widehat{\tau}(\top) + \int_d^j \mathcal{L}\big(\top,t,h_2(t)\big)\, ds, \quad \top \in [d,j].\end{aligned}$$

Thus, $\mu_2 \in [Sh_2]_{\widehat{\tau}_L(h_2)}$ and

$$\begin{aligned}\big|\mu_1(\top) - \mu_2(\top)\big| &\le \left(\int_d^j \big|\rho_{h_1}(\top,t) - \rho_{h_2}(\top,t)\big|\, ds\right) \\ &\le \sup\left(\int_d^j \pi(\top,t)\, ds\right)\xi(\top)\big(\big|h_1(t) - h_2(t)\big|\big)^r \\ &\le \xi(\top)\big(\big|h_1(t) - h_2(t)\big|\big)^r.\end{aligned}$$

Therefore,

$$\widehat{H}\big([Sh_1]_{\widehat{\tau}_L(h_1)}, [Sh_2]_{\widehat{\tau}_L(h_2)}\big) \le \xi(\top)\big(d(h_1,h_2)\big)^r. \tag{4.2}$$

Therefore, taking $h = h_1$ and $\mu = h_2$ in (4.2) yields

$$\widehat{H}\big([Sh]_{\widehat{\tau}_L(h)}, [S\mu]_{\widehat{\tau}_L(\mu)}\big) \leq \xi(\top)\big(d(h,\mu)\big)^r.$$

Therefore, all the conditions of Theorem 2.2 are fulfilled with $\xi(\top) = c\top$ for all $\top > 0$ and $c \in (0,1)$. As a result, the conclusion of Theorem 4.1 holds good. □

5 Conclusion

New invariant point results of L-fuzzy maps were introduced and conditions under which such mappings possess FPs were studied (see Theorems 2.2, 2.5, 2.6, 2.7) in this paper. The presented results are refinements of some already announced ideas in [9–13, 19, 23, 25]. Comparative examples (Examples 2.8 and 3.4) were constructed to support the theoretical assumptions of the proposed concepts. From the usability point of consideration, Theorem 2.2 was applied to set up new condition for analyzing the existence of solutions to a Fredholm-type integral inclusion.

The findings of this work widened up the coverage of nonclassical mathematics by incorporating interpolation approaches in L-fuzzy FP theory. However, the ideas presented herein, being set up in the framework of MS, are rudimentary. Whence, they can be fine-tuned when studied in the setting of some generalized MS or quasi-MS.

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

Author details

[1]Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria. [2]Department of Mathematics and Statistics, Taif University, Taif, Saudi Arabia. [3]Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia. [4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa. [5]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia. [6]School of computer, Data and Mathematical Sciences, Western Sydney University, Sydney- 2150, Australia.

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