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Convergence of Fibonacci–Ishikawa iteration procedure for monotone asymptotically nonexpansive mappings

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Abstract

In uniformly convex Banach spaces, we study within this research Fibonacci–Ishikawa iteration for monotone asymptotically nonexpansive mappings. In addition to demonstrating strong convergence, we establish weak convergence result of the Fibonacci–Ishikawa sequence that generalizes many results in the literature. If the norm of the space is monotone, our consequent result demonstrates the convergence type to the weak limit of the sequence of minimizing sequence of a function. One of our results characterizes a family of Banach spaces that meet the weak Opial condition. Finally, using our iterative procedure, we approximate the solution of the Caputo-type nonlinear fractional differential equation.

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Keywords: Monotone asymptotically nonexpansive mapping; Bounded away sequence; Fibonacci–Ishikawa iteration; Minimizing sequence; Weak Opial condition

1 Introduction

A branch of mathematics that is still developing is fixed point theory, which is connected to functional analysis and topology. The rapidly expanding fields of nonlinear operators and nonlinear analysis heavily rely on fixed point theory. Fixed point theorems for contractive mappings and their fixed points have historically been fundamental theoretical tools in topology, differential equations, economy, game theory, optimal control, dynamics, functional analysis, and so on. Many authors have thought of several generalizations and expansions of contractive mappings [3, 4, 6, 20]. Interest in monotone Lipschitzian mappings arose after Ran et al. [16] extended the Banach contraction concept to a partially ordered metric structure. The fixed point theory applied to mappings having different conditions has numerous uses (see [1, 9, 17]).

Certain iteration procedures are frequently employed to estimate fixed points in contractive mappings. As an illustration, consider the Picard [8], Mann [13], Ishikawa [11], Noor [14], and KF-iteration [18] iterations. Many mathematicians have been studying these iterative processes in the past few years.

Goebel et al. [10] developed the notion of asymptotically nonexpansive mappings in 1972. More specifically, they demonstrated that there is always a fixed point for those mappings specified on a nonvoid convex closed and bounded set in a uniformly convex normed linear space. Their evidence was unhelpful. For these kinds of mappings, Schu [19] created a sequence using a modified Mann iteration, his method has been shown to be highly effective in computing applications. Following Schu, Xu et al. [21] presented a modified Ishikawa iteration, and after that, Malih [12] demonstrated several fixed point results under Fibonacci–Mann and Ishikawa iterations in 2021, as well as a few results of weak convergence under the Opial condition.

Connected to the T iterates, the positive behavior of the Lipschitz constants is the basis for the inclusion of the T iterate in the original Ishikawa iteration sequence. When demonstrating the presence of a fixed point, the Ishikawa iteration fails to produce a monotone sequence for monotone mappings. Unlike regular Lipschitzian mappings, monotone Lipschitzian mappings may not exhibit pleasant topological behavior because the Lipschitzian condition may not hold across the space and is only satisfied by comparable elements. Even the continuity of these mappings may be broken. There is not always a single fixed point when the Banach contraction principle is extended for these kinds of mappings.

Inspired by the aforementioned, we study in this research Fibonacci–Ishikawa iteration in uniformly convex Banach spaces for monotone asymptotically nonexpansive mappings. Alongside proving strong convergence, we also prove weak convergence of the Fibonacci–Ishikawa sequence utilizing the weak Opial condition, which is a characteristic of both classical Banach spaces and any Hilbert space. We shall demonstrate that in order to achieve the weak convergence of the Fibonacci–Ishikawa iteration procedure, a less robust Opial condition, which is retained $L^p([0,1]), 1 < p < +\infty$, is required. Ultimately, we prove that our iterative procedure approximates the solution of the Caputo-type nonlinear fractional differential equation.

2 Preliminaries

To understand the way of defining Fibonacci–Ishikawa iteration procedure, let us first present the definitions of Mann iteration [13], modified Mann iteration [19], Fibonacci–Mann iteration [12], Ishikawa iteration [11], and modified Ishikawa iteration [21] procedures respectively as follows:

$$u_{m+1} = (1 - p_m)u_m + p_m T u_m,$$
$$u_{m+1} = (1 - p_m)u_m + p_m T^m u_m,$$
$$u_{m+1} = (1 - p_m)u_m + p_m T^{F(m)} u_m,$$
$$u_{m+1} = (1 - p_m)u_m + p_m T\big((1 - q_m)u_m + q_m T u_m\big),$$

and

$$u_{m+1} = (1 - p_m)u_m + p_m T^m\big((1 - q_m)u_m + q_m T^m u_m\big).$$

Let B represent a linear real space. In B, we first supply a partial order $\preceq$ as follows. If a relation $\preceq$ in B meets any of the following criteria, it is referred to as partial order:

(i) $u \preceq u, \forall u \in B$,
(ii) $u \preceq v, v \preceq u \Rightarrow u = v, \forall u, v \in B$,
(iii) $u \preceq v, v \preceq w \Rightarrow u \preceq w, \forall u, v, w \in B$,
(iv) $u \preceq v, w \preceq x \Rightarrow u + w \preceq v + x, \forall u, v, w, x \in B$,
(v) $u \preceq v \Rightarrow \lambda u \preceq \lambda v, \forall u, v \in B$ and $\lambda \geq 0$.

In this case, the pair denoted by $(B, \preceq)$ turns into a partially ordered linear space. In $(B, \preceq)$, now we establish a norm denoted as $\| \cdot \|$. Afterward, $(B, \| \cdot \|, \preceq)$ is now a normed linear space that is partially ordered. Partially ordered Banach space $(B, \| \cdot \|, \preceq)$ is defined as complete in relation to the metric defined using the aforementioned norm $\| \cdot \|$. If for every $\varepsilon > 0$ there exists some $\delta > 0$ so that $\|u\| \leq 1$, $\|v\| \leq 1$, and $\|u - v\| \geq \varepsilon$ for $u, v \in B$ implies $\|u + v\| \leq 2(1 - \delta)$, then B is known as a uniformly convex Banach space. Subsets of the type $\{u \in B : u \preceq w\}$, $\{u \in B : v \preceq u\}$, and $\{u \in B : v \preceq u \preceq w\}$ for all $v, w \in B$ are known as order intervals. An order interval C is called convex if $(1 - \lambda)u + \lambda v \in C, \forall u, v \in C, \lambda \in [0, 1]$.

Definition 2.1 [7] In a convex Banach space $(B, \| \cdot \|)$, the function $\delta_B(\varepsilon)$, defined in $[0, 2]$ as $\delta_B(\varepsilon) = \min\{1 - \frac{\|u+v\|}{2} : \|u - v\| \geq \varepsilon, \|u\| \leq 1, \|v\| \leq 1\}$, is known as a modulus of uniform convexity. This space is "more convex" if $\delta_B(\varepsilon)$ is less. In Banach spaces, "uniform convexity" is defined as the tendency of the modulus of convexity to zero as ε tends to zero.

Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$ that is nontrivial. It is established that for each bounded sequence of reals $\{\alpha_m\}$, $\lim_{m,\mathcal{U}} \alpha_m$ exists [2]. In a Banach space B, the set $l_\infty(B) = \{\{u_m\} \subset B : \|\{u_m\}\| = \max_m \|u_m\| < \infty\}$ forms a Banach space in respect of the norm $\| \cdot \|_\infty$ and the set $B_0 = \{\{u_m\} \in l_\infty(B) : \lim_{m,\mathcal{U}} \|u_m\| = 0\}$ is closed subspace in $l_\infty(B)$. The ultrapower of B is the quotient space $B_\mathcal{U} = l_\infty(B)/B_0$ [7], in which the norm for any $u \in B_\mathcal{U}$ is given by $\|u\|_\mathcal{U} = \lim_{m,\mathcal{U}} \|u_m\|$, where $u = \{u_m\}$.

Two structures are involved in the idea of monotone Lipschitzian mappings: a metric distance and a partial order. These two naturally occurring structures, with their intriguing qualities of natural interweaving, are present in the majority of the places used for applications.

Definition 2.2 [10] In any partially ordered Banach space $(B, \| \cdot \|, \preceq)$, let C be any nonvoid set and $T : C \longrightarrow C$ be any self mapping. If for arbitrary $u, v \in C$ with $u \preceq v$:

(i) $Tu \preceq Tv$, then T is known as monotone.
(ii) $\|Tu - Tv\| \preceq \alpha \|u - v\|$ for some $\alpha \geq 0$ and T is monotone, then T is known as monotone Lipschitzian.
(iii) $\|T^m u - T^m v\| \preceq \alpha_m \|u - v\|, m \geq 1$, T is monotone and for some sequence $\{\alpha_m\} \subset [1, \infty)$ with $\lim_{m \to \infty} \alpha_m = 1$, then T is known as monotone asymptotically nonexpansive.

If a map T transforms bounded sets into comparatively compact ones, it is referred to as compact.

In 2018, Alfuraidan et al. [5], for monotone asymptotically nonexpansive mappings, the fixed point results of Goebel et al. [10] are presented in a monotone form.

Theorem 2.3 [5] *In any partially ordered Banach space $(B, \| \cdot \|, \preceq)$, let C be any closed convex set having at least two points and $T : C \longrightarrow C$ be a monotone asymptotically nonex-*

pansive mapping. If T is continuous and $u_0 \preceq Tu_0$ (respectively, $Tu_0 \preceq u_0$) for some $u_0 \in C$, then T fixes a point u so that $u_0 \preceq u$ (respectively, $u \preceq u_0$).

3 Fibonacci–Ishikawa iteration procedure

The characteristics of the Fibonacci–Ishikawa iteration procedure connected to monotone asymptotically nonexpansive mappings in partially ordered Banach space are examined in this section. Let us start with the concept of Fibonacci–Ishikawa iteration.

Definition 3.1 In any Banach space $(B, \|\cdot\|)$, let C be any nonvoid convex set and $T : C \longrightarrow C$ be any self mapping. Then the Fibonacci–Ishikawa iteration procedure is described as

$$u_{m+1} = (1 - p_m)u_m + p_m T^{F(m)}\big((1 - q_m)u_m + q_m T^{F(m)} u_m\big), \quad \forall m \in \mathbb{N} \cup \{0\}, \tag{1}$$

for random choice $u_0 \in C$, where $\{p_m\}$, $\{q_m\}$ are bounded away sequences in $[0, 1]$ and $\{F(m)\}$ is the Fibonacci sequence described as $F(0) = F(1) = 1$, $F(m + 1) = F(m - 1) + F(m)$.

Now we have a crucial lemma.

Lemma 3.2 *In any partially ordered Banach space $(B, \|\cdot\|, \preceq)$, let C be any nonvoid bounded convex set and $T : C \longrightarrow C$ be any monotonic self mapping. Then, for random choice $u_0 \in C$ with $u_0 \preceq Tu_0$ (respectively, $Tu_0 \preceq u_0$), if u is a fixed point of T so that $u_0 \preceq u$ (respectively, $u \preceq u_0$), then for all $m \in \mathbb{N}$ the Fibonacci–Ishikawa iteration procedure (1) satisfies*

(a) $u_0 \preceq u_m \preceq u$ (*respectively,* $u \preceq u_m \preceq u_0$),
(b) $T^m u_0 \preceq T^{m+1} u_0$ (*respectively,* $T^{m+1} u_0 \preceq T^m u_0$),
(c) $u_m \preceq u_{m+1} \preceq T^{2F(m)} u_m$ (*respectively,* $T^{2F(m)} u_m \preceq u_{m+1} \preceq u_m$),
(d) $T^{F(m)} u_0 \preceq T^{F(m)} u_m \preceq u$ (*respectively,* $u \preceq T^{F(m)} u_m \preceq T^{F(m)} u_0$).

Proof For arbitrary $u_0 \in C$, it is given that $u_0 \preceq Tu_0$ and $u_0 \preceq u$. Also, given that T is monotone, we get $T^m u_0 \preceq T^{m+1} u_0$ and $T^m u_0 \preceq T^m u = u$ for all $m \in \mathbb{N}$. This proves(b).

Now, we have

$$\begin{aligned} u_1 &= (1 - p_0)u_0 + p_0 T^{F(0)}\big((1 - q_0)u_0 + q_0 T^{F(0)} u_0\big) \\ &= u_0 + p_0\big(T\big(u_0 + q_0(Tu_0 - u_0)\big) - u_0\big) \end{aligned}$$

and

$$\begin{aligned} u_1 &= (1 - p_0)u_0 + p_0 T^{F(0)}\big((1 - q_0)u_0 + q_0 T^{F(0)} u_0\big) \\ &= u_0 + p_0\big(T\big(u_0 + q_0(Tu_0 - u_0)\big) - u_0\big) \\ &\preceq Tu_0 + p_0\big(T\big(Tu_0 + q_0\big(T(Tu_0) - Tu_0\big)\big) - Tu_0\big) \\ &\preceq Tu_0. \end{aligned}$$

Since $\{p_m\}$, $\{q_m\}$ are bounded away sequences in $[0, 1]$, i.e., $p_0, q_0 > 0$, we have $u_0 \preceq u_1 \preceq Tu_0$.

Again

$$\begin{aligned} u_1 &= (1-p_0)u_0 + p_0 T^{F(0)}\big((1-q_0)u_0 + q_0 T^{F(0)}u_0\big) \\ &\preceq (1-p_0)u + p_0 T^{F(0)}\big((1-q_0)u + q_0 T^{F(0)}u\big) \\ &\preceq u. \end{aligned}$$

Consequently, $u_0 \preceq u_1 \preceq u$ and $T^m u_0 \preceq T^m u_1 \preceq T^m u = u$ for all $m \in \mathbb{N}$.

Since

$$u_2 = (1-p_1)u_1 + p_1 T^{F(1)}\big((1-q_1)u_1 + q_1 T^{F(1)}u_1\big)$$

and the order intervals are convex, we have $u_1 \preceq u_2 \preceq T^{F(1)}((1-q_1)u_1 + q_1 T^{F(1)}u_1)$ and $u_0 \preceq u_1 \preceq (1-q_1)u_1 + q_1 T^{F(1)}u_1 \preceq T^{F(1)}u_1$. Consequently, $u_1 \preceq u_2 \preceq T^{2F(1)}u_1 \preceq T^{2F(1)}u = u$.

Again, since

$$u_3 = (1-p_2)u_2 + p_2 T^{F(2)}\big((1-q_2)u_2 + q_2 T^{F(2)}u_2\big)$$

and the order intervals are convex, we have $u_2 \preceq u_3 \preceq T^{F(2)}((1-q_2)u_2 + q_2 T^{F(2)}u_2)$ and $u_2 \preceq (1-q_2)u_2 + q_2 T^{F(2)}u_2 \preceq T^{F(2)}u_2$. Consequently, $u_0 \preceq u_2 \preceq u_3 \preceq T^{2F(2)}u_2 \preceq T^{2F(2)}u = u$.

Proceeding in this way, we have $u_0 \preceq u_m \preceq u_{m+1} \preceq T^{2F(m)}u_m \preceq T^{2F(m)}u = u$, also, $T^{F(m)}u_0 \preceq T^{F(m)}u_m \preceq u$. This proves (a), (c), and (d). □

The sequence $\{u_m\}$ is guaranteed to be monotonic by property (c). The results below demonstrate how crucial this is.

Proposition 3.3 *In any partially ordered reflexive Banach space $(B, \|\cdot\|, \preceq)$, let a sequence $\{u_m\}$ be bounded monotone decreasing or increasing. Then $\{u_m\}$ is weakly convergent and if for any nonvoid compact set C in B, $\lim_{m\to\infty} d(u_m, C) = 0$, then $\{u_m\}$ is strongly convergent.*

Proof Assume that $\{u_m\}$ increases monotonically, without losing generality. It can be observed that $\{u_m\}$ has a bounded subsequence $\{u_{m_n}\}$, which is weakly convergent to a point $u \in B$, as B is reflexive and $\{u_m\}$ is bounded. According to our assertion, $\{u_{m_p}\}$, any other subsequence of $\{u_m\}$, likewise weakly convergent to u. Let $v \in B$ be another point, at which $\{u_{m_p}\}$ is weakly convergent. Order intervals are closed and convex, hence for any $m \geq 1$, we must have $u_m \preceq v$ since $\{u_m\}$ increases monotonically. Specifically, $u_{m_p} \preceq v$ implies $u \preceq v$ for every $m \geq 1$. This will undoubtedly make $u = v$. Thus, $\{u_m\}$ is weakly convergent.

Since for any nonvoid compact set C in B, $\lim_{m\to\infty} d(u_m, C) = 0$, then we can find a sequence $\{v_m\}$ in C so that $\lim_{m\to\infty} \|u_m, v_m\| = 0$. As $\{u_m\}$ is convergent weakly to u, $\{v_m\}$ also converges weakly to u. Keep in mind that a sequence must be strongly convergent if it is in a compact subset and is weakly convergent. Conversely, suppose that there is no strong convergence of $\{v_m\}$ to u. Then, for some $\varepsilon > 0$, there will be a subsequence $\{v_{m_p}\}$ so that $\min_{m\geq 1} \|v_{m_p} - u\| \geq \varepsilon$. Now the compactness of C will give a subsequence $\{v_{m_q}\}$ of $\{v_{m_p}\}$, which is strongly convergent. It may be observed that $\{v_{m_q}\}$ converges strongly to u, while $\{v_m\}$ converges weakly. It is implied by this contradiction that $\{v_m\}$ strongly converges to u, and that $\{u_m\}$ also strongly converges to u. □

Remark 3.4 The Proposition 3.3 demonstrates the helpful behavior of monotone sequences. Surprisingly, the function spaces $L^p([0,1])$, where $1 < p < +\infty$, known as Lebesgue function spaces, satisfy a similar result for monotone sequences even though they violate the weak Opial condition [15]. This fact will be covered in more detail in a later section.

Now we have another useful result.

Lemma 3.5 *In any partially ordered Banach space $(B, \|\cdot\|, \preceq)$, let C be any nonvoid bounded convex set and $T : C \longrightarrow C$ be any monotone asymptotically nonexpansive mapping with Lipschitz constants α_m so that $\sum_{m=0}^{\infty}(\alpha_{F(m)} - 1)^2 < \infty$. Then, for random choice $u_0 \in C$ with $u_0 \preceq Tu_0$ (respectively, $Tu_0 \preceq u_0$), if u is a fixed point of T so that $u_0 \preceq u$ (respectively, $u \preceq u_0$), then the Fibonacci–Ishikawa iteration procedure (1) provides the existence of $\lim_{m\to\infty} \|u_m - u\|$.*

Proof For arbitrary $u_0 \in C$, it is given that $u_0 \preceq Tu_0$ and $u_0 \preceq u$. Now, from the Fibonacci–Ishikawa iteration procedure (1), for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\|u_{m+1} - u\| &= \left\|(1-p_m)u_m + p_m T^{F(m)}\big((1-q_m)u_m + q_m T^{F(m)}u_m\big) - u\right\| \\
&\le (1-p_m)\|u_m - u\| + p_m\left\|T^{F(m)}\big((1-q_m)u_m + q_m T^{F(m)}u_m\big) - T^{F(m)}u\right\| \\
&\le (1-p_m)\|u_m - u\| + p_m\alpha_{F(m)}\left\|(1-q_m)u_m + q_m T^{F(m)}u_m - u\right\| \\
&\le (1-p_m)\|u_m - u\| + p_m(1-q_m)\alpha_{F(m)}\|u_m - u\| \\
&\quad + p_m q_m \alpha_{F(m)}\left\|T^{F(m)}u_m - T^{F(m)}u\right\| \\
&\le (1-p_m)\|u_m - u\| + p_m(1-q_m)\alpha_{F(m)}\|u_m - u\| \\
&\quad + p_m q_m \alpha_{F(m)}^2\|u_m - u\| \\
&= \|u_m - u\| + \big(-p_m + p_m\alpha_{F(m)} - p_m q_m\alpha_{F(m)} + p_m q_m \alpha_{F(m)}^2\big)\|u_m - u\| \\
&= \|u_m - u\| + p_m(1 + q_m\alpha_{F(m)})(\alpha_{F(m)} - 1)\|u_m - u\|.
\end{aligned}$$

Let $\delta(C) = \max\{\|u - v\| : u, v \in C\}$, and since $\{p_m\}$, $\{q_m\}$ are bounded away sequences in $[0,1]$, we get

$$\|u_{m+1} - u\| \le \|u_m - u\| + \delta(C)(\alpha_{F(m)} - 1)^2, \quad \forall m \in \mathbb{N}.$$

Thus, for any $m, n \ge 1$, we get

$$\|u_{m+n} - u\| \le \|u_m - u\| + \delta(C)\sum_{i=0}^{n-1}(\alpha_{F(m+i)} - 1)^2.$$

Consecutively limiting as $n \to \infty$ and $m \to \infty$ respectively, we get

$$\begin{aligned}
\limsup_{n\to\infty}\|u_n - u\| &\le \liminf_{m\to\infty}\|u_m - u\| + \delta(C)\sum_{i=m}^{\infty}(\alpha_{F(i)} - 1)^2 \\
&= \liminf_{m\to\infty}\|u_m - u\|,
\end{aligned}$$

which indicates the intended outcome. □

We employ the notion of ultrapower in the following result.

Lemma 3.6 *In any partially ordered uniformly convex Banach space* $(B, \|\cdot\|, \preceq)$, *let* C *be any nonvoid convex weakly compact set and the mapping* $T : C \longrightarrow C$ *is a continuous asymptotically nonexpansive and monotonic having Lipschitz constants* α_m *so that* $\sum_{m=0}^{\infty}(\alpha_{F(m)} - 1)^2 < \infty$. *Then, for arbitrary* $u_0 \in C$ *with* $u_0 \preceq Tu_0$ (*respectively,* $Tu_0 \preceq u_0$), *the Fibonacci–Ishikawa iteration procedure* (1) *satisfies* $\lim_{m\to\infty} \|u_m - T^{F(m)}u_m\| = 0$.

Proof For arbitrary $u_0 \in C$, it is given that $u_0 \preceq Tu_0$. Utilizing Theorem 2.3, T fixes a point u so that $u_0 \preceq u$ and utilizing Lemma 3.6, $\lim_{m\to\infty} \|u_m - u\|$ exists. Let $R = \lim_{m\to\infty} \|u_m - u\|$.

Case 1. Let $R = 0$. Then

$$\begin{aligned}\|u_m - T^{F(m)}u_m\| &\leq \|u_m - u\| + \|T^{F(m)}u_m - u\| \\ &= \|u_m - u\| + \|T^{F(m)}u_m - T^{F(m)}u\| \\ &\leq \|u_m - u\| + \alpha_{F(m)}\|u_m - u\| \\ &= (1 + \alpha_{F(m)})\|u_m - u\|\end{aligned}$$

implies $\lim_{m\to\infty} \|u_m - T^{F(m)}u_m\| = 0$.

Case 2. Let $R > 0$. Then, from the proof of Lemma 3.5, we have the inequality

$$\|u_{m+1} - u\| \leq (1 - p_m q_m \alpha_{F(m)})\|u_m - u\| + p_m q_m \alpha_{F(m)} \|T^{F(m)}u_m - u\|.$$

Now if $\mathcal{U}$ is a nontrivial ultrafilter over $\mathbb{N}$, then $\lim_{\mathcal{U}} p_m = p$, $\lim_{\mathcal{U}} p_m = p$ for some $0 < p$, $q < 1$, and consequently

$$\begin{aligned}R = \lim_{\mathcal{U}} \|u_{m+1} - u\| &\leq \lim_{\mathcal{U}}(1 - pq\alpha_{F(m)})\|u_m - u\| + pq \lim_{\mathcal{U}} \alpha_{F(m)} \|T^{F(m)}u_m - u\| \\ &\leq R - pq \lim_{\mathcal{U}} \alpha_{F(m)}\|u_m - u\| + pq \lim_{\mathcal{U}} \alpha_{F(m)} \|T^{F(m)}u_m - u\|\end{aligned}$$

implies

$$R \leq \lim_{\mathcal{U}} \|T^{F(m)}u_m - u\|.$$

Again $\|T^{F(m)}u_m - u\| \leq \|T^{F(m)}u_m - T^{F(m)}u\| \leq \alpha_{F(m)}\|u_m - u\|$ implies $\limsup_{m\to\infty} \|T^{F(m)}u_m - u\| \leq R$. Thus $R \leq \lim_{\mathcal{U}} \|T^{F(m)}u_m - u\| \leq \limsup_{m\to\infty} \|T^{F(m)}u_m - u\| \leq R$ implies $\lim_{\mathcal{U}} \|T^{F(m)}u_m - u\| = R$.

Let us set $a = \{u_m\}_{\mathcal{U}}$, $b = \{T^{F(m)}u_m\}_{\mathcal{U}}$ and $c = \{u\}_{\mathcal{U}}$ in the ultrapower $B_{\mathcal{U}}$. Then $\|a - c\|_{\mathcal{U}} = \|b - c\|_{\mathcal{U}} = \|pa + (1 - p)b - c\|_{\mathcal{U}} = R$. Since B is uniformly convex, we have $B_{\mathcal{U}}$ is strictly convex and as $0 < p < 1$, we get $a = b$, i.e., $\lim_{\mathcal{U}} \|u_m - T^{F(m)}u_m\| = 0$. From the arbitrariness of the nontrivial ultrafilter $\mathcal{U}$, we deduce that $\lim_{m\to\infty} \|u_m - T^{F(m)}u_m\| = 0$. □

Now we have a result that generalizes the result of Malih [12] in monotone sense.

Theorem 3.7 *In any partially ordered uniformly convex reflexive Banach space* $(B, \|\cdot\|, \preceq)$, *let* C *be any nonvoid convex weakly compact set and the mapping* $T : C \longrightarrow C$ *is asymptotically nonexpansive and monotonic having Lipschitz constants* α_m *so that* $\sum_{m=0}^{\infty}(\alpha_{F(m)} - 1)^2 < \infty$ *and for some* $m \geq 1$, T^m *is compact. Then, for arbitrary* $u_0 \in C$ *with* $u_0 \preceq Tu_0$ (*respectively*, $Tu_0 \preceq u_0$), *the Fibonacci–Ishikawa iteration procedure* (1) *strongly converges to a point, say* u, *which is fixed by* T *and* $u_0 \preceq u$ (*respectively*, $u \preceq u_0$).

Proof For arbitrary $u_0 \in C$, it is given that $u_0 \preceq Tu_0$. Utilizing Lemmas 3.2 and 3.6, we have $u_0 \preceq u_m, \forall m \in \mathbb{N}$ and $\lim_{m\to\infty} \|u_m - T^{F(m)}u_m\| = 0$. Let $n \geq 1$ be fixed so that T^n is compact. Thus $C_0 = \overline{T^n(C)}$ is nonvoid compact set and $T^m(v) \in C_0$, $v \in C$ for any $m > n$. Consequently, $T^{F(m)}(u_m) \in C_0$ as $m > n$ implies $F(m) > n$. Hence $\lim_{m\to\infty} d(u_n, C_0) \leq \lim_{m\to\infty} \|u_m - T^{F(m)}u_m\| = 0$ and $\lim_{m\to\infty} d(T^m u_0, C_0) = 0$. So, the strong convergence of $\{u_m\}$ and $\{T^m u_0\}$ is implied by Proposition 3.3 as B is reflexive and both sequences are monotone.

Also, $\{T^{F(m)}u_m\}$ likewise exhibits strong convergence and possesses an identical limit as $\{u_m\}$. Define u as the limit of $\{T^m u_0\}$, then we assert that u represents a fixed point of T. As we know $T^m u_0 \preceq u, \forall m \in \mathbb{N}$, by the monotonicity of $\{T^m u_0\}$ and $\|T^{m+1}u_0 - Tu\| \leq \alpha_1\|T^m u_0 - u\|$, $\forall m \in \mathbb{N}$, by definition. Consequently, $\{T^m u_0\}$ converges to both Tu and u, i.e., $Tu = u$. Again, utilizing Lemma 3.2, we get $T^{F(m)}u_0 \preceq T^{F(m)}u_m \preceq u, \forall m \in \mathbb{N}$. The fact that order intervals are closed leads us to the conclusion that u is likewise the limit of $\{u_m\}$ and $T^{F(m)}u_m$, thereby indicating that $\{u_m\}$ strongly converges to a fixed point of T. □

In the subsequent outcome, we prove that the minimizing sequence of some functions is strongly convergent to the limit of the sequence, at which it converges weakly if the norm is monotone. In a Banach space $(B, \|\cdot\|, \preceq)$ if $\sup\{\|v - u\|, \|w - v\|\} \leq \|w - u\|$ whenever $u \preceq v \preceq w, \forall u, v, w \in B$, then the norm $\|\cdot\|$ of B is known as monotone. If a sequence $\{u_m\}$ is monotone increasing (respectively, decreasing) and the norm $\|\cdot\|$ is monotone, then for any $v \in B$, $\{\|u_m - v\|\}$ is a decreasing sequence so that $u_m \preceq v$ (respectively, $v \preceq u_m$), and so $\liminf_{m\to\infty} \|u_m - v\| = \lim_{m\to\infty} \|u_m - v\| = \inf_{m\to\infty} \|u_m - v\|$. Let $C \subset B$ and $\zeta : C \longrightarrow [0, \infty)$, then a sequence $\{u_m\}$ is known as a minimizing sequence of ζ if $\lim_{m\to\infty} \zeta(u_m) = \inf\{\zeta(u) : u \in C\}$.

Proposition 3.8 *Let* $\|\cdot\|$ *be a monotone norm in a uniformly convex Banach space* $(B, \|\cdot\|, \preceq)$ *and* $\{u_m\}$ *be a monotone increasing* (*respectively, decreasing*) *sequence having weak limit* v. *Let*

$$C = \{u : u_m \preceq u \text{ respectively, } u \preceq u_m \forall m \in \mathbb{N}\}$$

and define $\zeta : C \longrightarrow [0, \infty)$ *by* $\zeta(u) = \lim_{m\to\infty} \|u_m - u\|$. *Then all possible minimizing sequences* $\{v_m\}$ *of* ζ *in* C *strongly converge to* v. *In fact,* ζ *has a unique minimal.*

Proof Let us assume δ_B to be the modulus of uniform convexity of $(B, \|\cdot\|, \preceq)$. It is given that $\{u_m\}$ is a monotone increasing sequence having weak limit v, as the order intervals are convex and closed, we get $v \in C$. In fact, $u_m \preceq v \preceq w, \forall m \in \mathbb{N}$ for any $w \in C$.

Since the norm $\|\cdot\|$ is monotone, we have $\|v - u_m\| \leq \|w - u_m\|$, $\forall m \in \mathbb{N}$, and so $\zeta(v) \leq \zeta(w)$. This shows that v in C is the minimum point of ζ. Now, if $\{v_m\}$ in C is a minimizing sequence of ζ and since $\|v_m - v\| \leq \zeta(v_m) + \zeta(v)$, $\forall m \in \mathbb{N}$, we get $\{v_m\}$ is a bounded sequence.

Again, if $\zeta(v) = 0$, $\{v_m\}$ converges strongly to v. On the contrary, suppose that $\{v_m\}$ does not converge strongly to v, subsequently for some $\varepsilon > 0$, there exists a subsequence $\{v_{m_n}\}$ so that $\|v_{m_n} - v\| \geq \varepsilon$. Let us define $\xi = \delta_B(\frac{\varepsilon}{\sup\{\|v_m - u_m\|, \|v - u_m\|\}+1})$, since $\{v_m\}$ is bounded, ξ is well defined. Since B is uniformly convex, we get

$$\left\| u_m - \frac{v_{m_n} + v}{2} \right\| \leq (1-\xi)\sup\bigl\{\|u_m - v_{m_n}\|, \|u_m - v\|\bigr\}, \quad \forall m, n \in \mathbb{N}.$$

Limiting $m \to \infty$, we get $\zeta(\frac{v_{m_n}+v}{2}) \leq (1-\xi)\sup\{\zeta(v_{m_n}), \zeta(v)\} = (1-\xi)\zeta(v_{m_n})$, i.e., $\zeta(v) \leq (1-\xi)\zeta(v_{m_n})$, $\forall n \in \mathbb{N}$. Again, limiting $n \to \infty$, we get $\zeta(v) \leq (1-\xi)\zeta(v)$. Since $\zeta(v) > 0$, we arrived at a contradiction to $\xi \geq 1$. □

The conclusion below, which characterizes a family of Banach spaces that meet the weak Opial condition, is a direct outcome of Proposition 3.8.

Theorem 3.9 *Let $(B, \|\cdot\|, \preceq)$ be a partially ordered Banach space that is uniformly convex and has closed, convex order intervals. Let us assume that $\|\cdot\|$ is a monotone norm. The weak Opial condition is then satisfied by $(B, \|\cdot\|, \preceq)$.*

This consequence arises from the fact that the function spaces $L^p([0,1]), 1 < p < +\infty$ having monotone norm are uniformly convex.

Corollary 3.10 *The function spaces $L^p([0,1]), 1 < p < +\infty$, known as Lebesgue function spaces, satisfy the weak Opial condition.*

Our next topic of discussion is the Fibonacci–Ishikawa sequence's weak convergence, which is mentioned in Remark 3.4 and that generalizes the results of Malih [12] in monotone sense and the result of Alfuraidan et al. [5]. Typically, this is accomplished by the weak Opial condition [15], which is a characteristic that both classical Banach spaces and any Hilbert space satisfy. We shall demonstrate that in order to achieve the weak convergence of the Fibonacci–Ishikawa iteration procedure, a weaker Opial condition, which is retained in $L^p([0,1]), 1 < p < +\infty$, is required. If for any weakly convergent sequence $\{u_m\}$ in a Banach space $(B, \|\cdot\|, \preceq)$, which is monotone increasing (respectively, decreasing), converging to u, we have $\liminf_{m\to\infty} \|u_m - u\| < \liminf_{m\to\infty} \|u_m - v\|$, $\forall u \neq v$ with $u_m \preceq v$ respectively, $v \preceq u_m$ $\forall m \in \mathbb{N}$.

Theorem 3.11 *Let $\|\cdot\|$ be a monotone norm in any partially ordered uniformly convex reflexive Banach space $(B, \|\cdot\|, \preceq)$, C be any nonvoid convex weakly compact set, and the mapping $T : C \longrightarrow C$ is a monotone asymptotically nonexpansive having Lipschitz constants α_m so that $\sum_{m=0}^{\infty}(\alpha_{F(m)} - 1)^2 < \infty$ and for some $m \geq 1$. Then, for arbitrary $u_0 \in C$ with $u_0 \preceq Tu_0$ (respectively, $Tu_0 \preceq u_0$), the Fibonacci–Ishikawa iteration procedure (1) weakly converges to a point, say u, which is fixed by T and $u_0 \preceq u$ (respectively, $u \preceq u_0$).*

Proof For arbitrary $u_0 \in C$, it is given that $u_0 \preceq Tu_0$. We now understand that the sequence $\{T^m u_0\}$ is monotonic increasing and weakly convergent to some point u because

C is weakly compact. It can be inferred from Theorem 3.9 that B meets the weak Opial condition. Thus, the minimal of $\zeta : C_\infty \longrightarrow [0,\infty)$ is u, where $C_\infty = \{v : T^m u_0 \preceq v, \forall m \in \mathbb{N}\}$ and ζ is described as

$$\zeta(v) = \liminf_{m\to\infty} \left\| T^m u_0 - v \right\| = \lim_{m\to\infty} \left\| T^m u_0 - v \right\|.$$

From the definition of ζ and T, we have

$$\zeta(u) \le \zeta\left(T^m v\right) \le \alpha_m \zeta(u), \quad \forall m \in \mathbb{N},$$

which further implies that $\{T^m u\}$ is a minimizing sequence of ζ. Utilizing Proposition 3.8, $\{T^m u\}$ strongly converges to u, $T^m u_0 \preceq u$ implies $T^{m+1} u_0 \preceq Tu$, and by the closeness of the order intervals, we have $u \preceq Tu$. Again, based on the monotone property of T, we can infer that the sequence $\{T^m u\}$ converges to u and increases monotonically. Therefore, for any $m \in \mathbb{N}$, we must have $T^m u \preceq u$. Specifically, $T(u) \preceq u$ suggests that $Tu = u$.

Now Lemma 3.2 implies $T^{F(m)} u_0 \preceq T^{F(m)} u_m \preceq u, \forall m \in \mathbb{N}$. We deduce that $\{T^{F(m)} u_m\}$ also converges weakly to u by the convex and closed property of the order intervals and the monotone property of the sequence $\{T^m u_0\}$. Hence, by Lemma 3.6, we get $\lim_{m\to\infty} \|u_m - T^{F(m)} u_m\| = 0$, that is, $\{u_m\}$ weakly converges to a point u, which is fixed by T. □

Now we provide some examples to understand and increase the readability of the theory.

Example 3.12 An example of a uniformly convex Banach space is the l^p space for p in the range $1 < p < \infty$. The l^p spaces are spaces of sequences equipped with certain norms. Specifically, for $1 < p < \infty$, the l^p space is defined as follows: Let $\{u_n\}$ be a sequence of real or complex numbers. Then, the l^p norm of $\{u_n\}$ is given by

$$\left\| \{u_n\} \right\|_p = \left(\sum_{n=1}^{\infty} |u_n|^p \right)^{\frac{1}{p}}.$$

This norm induces a metric on the space of sequences, and when completed with respect to this metric, it forms a Banach space denoted l^p.

Example 3.13 The space $C[0,1]$ consisting of all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$ equipped with the supremum norm, denoted as $\|f\|_\infty = \sup_{u\in[0,1]} |f(u)|$, a partial order, by saying $f \le g$ if and only if $f(u) \le g(u)$ for all $u \in [0,1]$, is a partially ordered uniformly convex Banach space. The compatibility between the partial order and the norm is crucial here. The partial order is defined based on the pointwise order of functions, and this order is consistent with the norm structure (that is, if $f \le g$, then $\|f\|_\infty \le \|g\|_\infty$).

Example 3.14 Let $T : C[01] \longrightarrow C[0,1]$ be a mapping defined as $(Tf)(u) = \alpha f(u) + (1 - \alpha)g(x)$, where $f, g \in C[0,1]$, $0 < \alpha < 1$ is a constant, and $u \in [0,1]$. Now with the supremum norm $\|\cdot\|_\infty$, we have

$$\|Tf - Tg\|_\infty = \left\| \alpha f + (1-\alpha)g - \left(\alpha g + (1-\alpha)g\right) \right\|_\infty = \left\| (\alpha - 1)g \right\| \le \|f - g\|_\infty$$

for all $f \neq g$. Thus, for each $m \in \mathbb{N}$, we can define $\alpha_m = 1 + \frac{1}{m^2}$ so that $\|T^m f - T^m g\|_\infty \leq \alpha_m \|f - g\|_\infty$. Clearly, T is monotone, and since $\{\alpha_m\} \subset [1, \infty)$ with $\lim_{m \to \infty} \alpha_m = 1$, T is a monotone asymptotically nonexpansive mapping.

Example 3.15 Let us consider the Banach space $C[0,1]$ equipped with the supremum norm $\|\cdot\|_\infty$, which is a partially ordered uniformly convex Banach space. Consider the mapping $T : C[0,1] \longrightarrow C[0,1]$ defined as follows:

$$(Tf)(u) = 2f(u) - u^2.$$

This mapping is monotone asymptotically nonexpansive (for the proof, see Example 3.14) and satisfies all conditions of Theorem 3.7 for a particular choice of the parameters.

To find fixed points of T, we need to solve the equation $Tf = f$, which leads to $2f(u) - u^2 = f(u)$, or equivalently, $f(u) = u^2$. So, any function f in $C[0,1]$ that satisfies $f(u) = u^2$ is a fixed point of T. There are multiple functions that satisfy this equation. The function $f(u) = u^2$ itself is a fixed point. There are other functions, such as $f(u) = 0, f(u) = \sin^2(u), f(u) = \frac{u^2}{2}$, etc., that also satisfy $f(u) = u^2$ and hence are fixed points.

Numerically, we apply Fibonacci–Ishikawa iteration procedure (1) to find a fixed point of T. Let us choose an initial point $u_0 = 0$ and set $p_m = \frac{1}{3+2m}$, $q_m = \frac{1+m}{m+4}$. Starting with $u_0 = 0$: $u_1 = 0$. For $m = 1$: $u_2 = 0$. For $m = 2$: $u_3 = 0$. This pattern continues $u_m = 0$ for all m. Therefore, the iteration scheme converges to $u = 0$, which is a fixed point of the mapping T corresponding to $f(u) = 0$.

Let us choose an initial point $u_0 = 1$ and set $p_m = \frac{1}{3+2m}$, $q_m = \frac{1+m}{m+4}$. Starting with $u_0 = 1$: $u_1 = 1$. For $m = 1$: $u_2 = 1$. For $m = 2$: $u_3 = 1$. This pattern continues $u_m = 1$ for all m. Therefore, the iteration scheme converges to $u = 1$, which is a fixed point of the mapping T corresponding to $f(u) = u^2$.

4 Application to Caputo-type nonlinear fractional differential equations

Since its discovery, fractional differential equations have been the subject of extensive investigation, with many important studies conducted in this field. This can be due to the fact that fractional differential equations are used in a wide variety of fields. Fluid flow, signal processing, electronics, biology, robotics, telecommunication systems, electrical, networks, diffusive transport, traffic flow, gas dynamics, generalized Casson fluid modeling with heat generation, and chemical reaction are just a few of the fields in which fractional differential equations are used.

In this case, we want to use our iteration procedure (1) to approximate the solution of the following Caputo-type nonlinear fractional differential equations with boundary conditions in order to accomplish our goal:

$$\begin{cases} {}^cD^\eta w(u) + f(u, w(u)) = 0, & 1 < \eta < 2, \\ w(0) = w(1) = 0, \\ 0 \leq u \leq 1, \end{cases} \tag{2}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and ${}^cD^\eta$ is a Caputo-type fractional differential of order η.

Assume the partially ordered uniformly convex Banach space $C[0,1]$ as in Example 3.13 with the standard supremum norm, containing continuous real functions from $[0,1]$ into $\mathbb{R}$. Then, in $C[0,1]$, we have the following associated integral equation:

$$w(u) = \int_0^1 g(v,u)f\bigl(u,w(u)\bigr)\,du,$$

where $g:[0,1]\times[0,1]\longrightarrow\mathbb{R}$ is the Green function described as follows:

$$g(v,u) = \begin{cases} \frac{v(1-u)^{\eta-1}-(v-u)^{\eta-1}}{\Gamma(\eta)}, & 0\le u\le v\le 1, \\ \frac{v(1-u)^{\eta-1}}{\Gamma(\eta)}, & 0\le v\le u\le 1. \end{cases}$$

Theorem 4.1 *Let $T:C[0,1]\longrightarrow C[0,1]$ be a mapping described as*

$$Tw(u) = \int_0^1 g(v,u)f\bigl(u,w(u)\bigr)\,du \text{ for all } w(u)\in C[0,1], u\in[0,1].$$

Let the iteration procedure $\{u_n\}$ be defined as in (1) and the continuous function $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfy

$$\bigl|f(u,w_1)-f(w,w_2)\bigr| \le |w_1-w_2| \quad \text{for all } w_1,w_2\in C[0,1], u\in[0,1].$$

Then the iteration procedure $\{u_n\}$ converges to the solution of the Caputo-type nonlinear fractional differential equation (2).

Proof It is evident that $w\in C[0,1]$ can only be a solution of the Caputo-type nonlinear fractional differential equation (2) if and only if it can also be a solution of the associated integral equation.

Now, let $w_1,w_2\in C[0,1]$ for $u\in[0,1]$, then

$$\begin{aligned} \bigl|Tw_1(u)-Tw_2(u)\bigr| &= \left|\int_0^1 g(v,u)f\bigl(u,w_1(u)\bigr)\,du-\int_0^1 g(v,u)f\bigl(u,w_2(u)\bigr)\,du\right| \\ &\le \int_0^1 g(v,u)\bigl|f\bigl(u,w_1(u)\bigr)-f\bigl(u,w_2(u)\bigr)\bigr|\,du \\ &\le \int_0^1 g(v,u)\bigl|w_1(u)-w_2(u)\bigr|\,du. \end{aligned}$$

Utilizing the standard supremum norm of the Banach space $C[0,1]$, we get

$$\bigl|Tw_1(u)-Tw_2(u)\bigr| \le \|w_1-w_2\|.$$

Thus, in a similar way to Example 3.14, we can consider T as a monotone asymptotically nonexpansive mapping, which consequently satisfies all the criteria of Theorem 3.7. Hence, the iteration procedure $\{u_n\}$ defined in (1) converges to a unique fixed point of T; consequently, $\{u_n\}$ converges to the solution of Caputo-type nonlinear fractional differential equation (2). □

5 Conclusions

We studied in this research Fibonacci–Ishikawa iteration in uniformly convex Banach spaces for monotone asymptotically nonexpansive mappings. If the norm of the space is monotone, our consequent result demonstrated the strong convergence of minimizing sequence depending on certain functions to the weak limit of the sequence. In addition to demonstrating strong convergence, we proved weak convergence of the Fibonacci–Ishikawa sequence that generalizes the results of Malih [12] in a monotone sense and the result of Alfuraidan et al. [5]. Typically, this is accomplished by the weak Opial condition [15], which is a characteristic that both classical Banach spaces and any Hilbert space satisfy. We also demonstrated that in order to achieve the weak convergence of the Fibonacci–Ishikawa iteration procedure, a weaker Opial condition, which is retained in $L^p([0,1]), 1 < p < +\infty$, is required. Finally, our novel iterative procedure has been applied to approximate the solution of the Caputo-type nonlinear fractional differential equation.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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References

1. Aggarwal, G., Uddin, I., Şahin, A.: On common fixed points of non-Lipschitzian semigroups in a hyperbolic metric space endowed with a graph. J. Anal., 1–11 (2023)
2. Aksoy, A., Khamsi, M.A.: Nonstandard Methods in Fixed Point Theory. Springer, New York (1990)
3. Alam, K.H., Rohen, Y., Saleem, N.: Fixed points of (α,β,F^*) and (α,β,F^{**})-weak Geraghty contractions with an application. Symmetry **15**(1), 1–17 (2023)
4. Alam, K.H., Rohen, Y., Tomar, A.: On fixed point and its application to the spread of infectious diseases model in M_v^b-metric space. Math. Methods Appl. Sci. **47**(7), 6489–6503 (2024)
5. Alfuraidan, M., Khamsi, M.: A fixed point theorem for monotone asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **146**(6), 2451–2456 (2018)
6. Banach, S.: Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fundam. Math. **3**, 133–181 (1922)
7. Beauzamy, B.: Introduction to Banach Spaces and Their Geometry. North-Holland, Amsterdam (1985)
8. Berinde, V.: Picard iteration converges faster than Mann iteration for a class of quasi contractive operators. Fixed Point Theory Appl. **2**, 97–105 (2004)
9. Carl, S., Heikkila, S.: Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory. Springer, New York (2011)

10. Goebel, K., Kirk, W.A.: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171–174 (1972)
11. Ishikawa, S.: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147–150 (1974)
12. Malih, S.H.: Fixed point theorems of modified Mann and Ishikawa iterations. J. Interdiscip. Math. **24**(5), 1093–1097 (2021)
13. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
14. Noor, M.A.: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**, 217–229 (2000)
15. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
16. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435–1443 (2004)
17. Şahin, A., Alagöz, O.: On the approximation of fixed points for the class of mappings satisfying (*CSC*)-condition in Hadamard spaces. Carpath. Math. Publ. **15**(2), 495–506 (2023)
18. Şahin, A., Öztürk, E., Aggarwal, G.: Some fixed-point results for the KF-iteration process in hyperbolic metric spaces. Symmetry **15**(7), 1360 (2023)
19. Schu, J.: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407–413 (1991)
20. Singh, M.P., Rohen, Y., Saleem, N., Alam, K.H., Singh, K.A., Razzaque, A.: On fixed-point equations involving Geraghty-type contractions with solution to integral equation. Mathematics **11**(24), 1–18 (2023)
21. Tan, K., Xu, H.: Fixed point iteration processes for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **122**(3), 733–739 (1994)

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