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An inertial self-adaptive algorithm for solving split feasibility problems and fixed point problems in the class of demicontractive mappings

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Abstract
We propose a hybrid inertial self-adaptive algorithm for solving the split feasibility problem and fixed point problem in the class of demicontractive mappings. Our results are very general and extend several related results existing in the literature from the class of nonexpansive or quasi-nonexpansive mappings to the larger class of demicontractive mappings. Examples to illustrate numerically the effectiveness of the new analytical results are presented.

Keywords: Hilbert space; Bounded linear operator; Split feasibility problem; Fixed point; Variational inequality; Banach contraction; Nonexpansive mapping; Quasi-nonexpansive mapping; Strictly pseudontractive mapping; Demicontractive mapping; Metric projection; Mapping demiclosed at zero; Averaged mapping; Picard iteration; Inertial self-adaptive algorithm; Strong convergence theorem; Nonviscosity type algorithm

1 Introduction

Let H_1, H_2 be real Hilbert spaces, C, Q be nonempty convex closed subsets of H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (*SFP*, for short) is asking to find a point

$$x \in C \quad \text{such that } Ax \in Q. \tag{1}$$

Under the hypothesis that the *SFP* is consistent, i.e., (1) has a solution, this is usually denoted by

$$SFP(C, Q) := \{x \in C \text{ such that } Ax \in Q\} \tag{2}$$

to indicate the two sets involved.

The split feasibility problem includes many important problems in nonlinear analysis modeling a wide range of inverse problems originating in the real world: signal processing, image reconstruction problem of X-ray tomography, statistical learning, etc., a fact that challenged researchers to construct robust and efficient iterative algorithms that solve (1).

Such an algorithm, known under the name of (CQ) algorithm, has been proposed by Byrne [7] (see also [6]), who constructed it by using the fact that *SFP* (1) is equivalent to the following fixed point problem:

$$x = P_C\big((I + \gamma A^*(P_Q - I)A\big)x, \quad x \in C, \tag{3}$$

where P_C and P_Q stand for the orthogonal (metric) projections onto the sets C and Q, respectively, I is the identity map, γ is a positive constant, and A^* denotes the adjoint of A.

By simply applying the Picard iteration corresponding to the fixed point problem (3), we get the (CQ) algorithm, which is thus generated by an initial value $x_1 \in H_1$ and the one step iterative scheme

$$x_{n+1} = P_C\big((I + \gamma_n A^*(P_Q - I)A\big)x_n, \quad n \geq 0, \tag{4}$$

where the step size $\gamma_n \in (0, \frac{2}{\|A\|^2})$.

If, for example, one considers the function

$$f(x) = \frac{1}{2}\big\|(I - P_Q)Ax\big\|^2, \tag{5}$$

then we have

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{6}$$

which indicates the fact that (4) is a particular gradient projection type algorithm. Of course, this is valid in a more general case: if we have a Fréchet differentiable real-valued valued function $f : C \to \mathbb{R}$ and we search for a minimizer of the problem

$$\text{find } \min_{x\in C} f(x), \tag{7}$$

then by means of an equivalent fixed point formulation, i.e.,

$$x = P_C\big(x - \gamma \nabla f(x)\big), \tag{8}$$

we obtain the gradient-projection algorithm

$$x_{n+1} = P_C\big(x_n - \gamma \nabla f(x_n)\big), \quad n \geq 0, \tag{9}$$

which coincides with (4) in the particular case of f given by (5), see [21] for more details.

It is known that when the iteration mapping

$$P_C\big((I + \gamma A^*(P_Q - I)A\big)$$

involved in the (CQ) algorithm (4) is of nonexpansive type, then the (CQ) algorithm converges strongly to a fixed point of it, that is, to a solution of *SFP* (1) (see [7] for more details).

But in applications, there are at least two major difficulties in implementing algorithm (4):

(1) the selection of the step size depends on the operator norm, and its computation is not an easy task at all;

(2) the implementation of the projections P_C and P_Q, depending on the geometry of the two sets C and Q, could be very difficult or even impossible.

To overcome the above-mentioned computational difficulties in a gradient-projection type algorithm, researchers proposed some ways to avoid the calculation of $\|A\|$. Another way to surpass the computation of the norm of A has been suggested by Lopez et al. [12], who proposed the following formula for expressing the step size sequence γ_n:

$$\gamma_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{10}$$

where ρ_n is a sequence of positive real numbers in the interval $(0, 4)$.

Another fixed point approach for solving *SFP* (1) in the class of nonexpansive mappings is due to Qin et al. [15], who considered a viscosity type algorithm given by

$$\begin{cases} x_1 \in C \quad \text{arbitrary} \\ y_n = P_C((1-\delta_n)x_n - \tau_n A^*(I - P_Q)Ax_n) + \delta_n S x_n, \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \geq 1, \end{cases} \tag{11}$$

where $g : C \to C$ is a Banach contraction, $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$, $\{\alpha_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and τ_n are sequences in $(0, 1)$ that satisfy some appropriate conditions, denoted by (C_1)–(C_5).

Under these assumptions, Qin et al. [15] proved that the sequence $\{x_n\}$ generated by algorithm (11) converges strongly to some $x^* \in Fix(T) \cap SFP(C, Q)$ and x^* is the unique solution of the variational inequality

$$\langle x - x^*, g(x^*) - x^* \rangle \leq 0, \quad \forall x \in Fix(T) \cap SFP(C, Q). \tag{12}$$

Subsequently, Kraikaew et al. [11] weakened the assumptions (C_1), (C_2), and (C_4) in Lopez et al. [12] and obtained the same convergence result by a slightly simplified proof.

More recently, Wang et al. [20] extended the previous results in three ways:

(1) by weakening the conditions on the parameters $\{\alpha_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ involved in algorithm (11);

(2) by inserting an inertial term in algorithm (11) in such a way that for choosing the step size there is no more need to calculate the norm of the operator A;

(3) by considering the larger class of quasi-nonexpansive mappings instead of nonexpansive mappings (which were considered in the previous papers).

Starting from the developments presented before, the following question naturally arises:

Question Is it possible to extend the results in Wang et al. [20] to more general classes of mappings that strictly include the class of quasi-nonexpansive mappings?

The aim of this paper is to answer this question in the affirmative, see Theorem 1 below and also its supporting illustration (Example 1). We actually show that we can establish a strong convergence theorem for Algorithm 1, which is obtained from the inertial algorithm (11) used in [20] by inserting an averaged component. We are thus able to show that one can solve the split feasibility problem and the fixed point problem in the class of demicontractive mappings, too.

Our main result (Theorem 1) shows that the new algorithm converges strongly to an element $x^* \in Fix(T) \cap SFP(C, Q)$, which uniquely solves the variational inequality (12).

By doing this, we improve significantly the previous related results in the literature since, by considering averaged mappings in gradient projection type algorithms, one gets important benefits, see the motivation in the excellent paper by Xu [21].

2 Preliminaries

Throughout this section, H denotes a real Hilbert space with the norm and the inner product denoted as usual by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C \subset H$ be a closed and convex set and $T : C \to C$ be a self mapping. Denote by

$$Fix(T) = \{x \in C : Tx = x\}$$

the set of fixed points of T. In the present paper we consider the classes of nonexpansive type mappings introduced by the next definition.

Definition 1 The mapping T is said to be:

1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C; \tag{13}$$

2) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \quad \text{for all } x \in C \text{ and } y \in Fix(T); \tag{14}$$

3) k-strictly pseudocontractive of the Browder–Petryshyn type if there exists $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2, \quad \forall x, y \in C; \tag{15}$$

4) k-demicontractive or quasi k-strictly pseudocontractive (see [5]) if $Fix(T) \neq \emptyset$ and there exists a positive number $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2 \tag{16}$$

for all $x \in C$ and $y \in Fix(T)$.

For the scope of this paper, it is important to note that any quasi-nonexpansive mapping is demicontractive but the reverse is no more true, as shown by the following example.

Example 1 ([4], Example 2.5) Let H be the real line with the usual norm and $C = [0,1]$. Define T on C by $Tx = \frac{7}{8}$ if $0 \leq x < 1$ and $T1 = \frac{1}{4}$. Then: 1) $Fix(T) \neq \emptyset$; 2) T is demicontractive; 3) T is not nonexpansive; 4) T is not quasi-nonexpansive; 5) T is not strictly pseudocontractive.

For more details and a complete diagram of the relationships between the mappings introduced in Definition 1, we also refer to [4].

The following lemmas will be useful in proving our main results in the next section.

Lemma 1 ([13], Lemma 1.1) *For any $x, y \in H$, we have*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall t \in [0,1]$.

Let C be a closed convex subset H. Then the *nearest point* (*metric*) *projection* P_C from H onto C assigns to each $x \in H$ its nearest point in C, denoted by $P_C x$, that is, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } x \in H. \tag{17}$$

The metrical projection has many important properties, of which we collect the following ones.

Lemma 2 ([21], Proposition 3.1) *Given $x \in H$ and $y \in C$, we have*:

(i) $z = P_C x$ *if and only if* $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$;

(ii) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C y\|^2$, $\forall y \in C$;

(iii) $\langle x - y, P_C x - P_C y\rangle \leq \|x - P_C x\|^2 \geq \|P_C x - P_C y\|^2$, $\forall y \in C$.

Remark 2.1 Property *(i)* in Lemma 2 shows that, for any $x \in H$, its projection on the closed convex set C solves the variational inequality $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$;

Property *(ii)* in Lemma 2 expresses the fact that P_C is a *firmly nonexpansive* mapping, while property *(iii)* shows that P_C is 1-inverse strongly monotone.

Lemma 3 ([7]) *Let f be given by* (5). *Then ∇f is $\|A\|^2$-Lipschitzian*.

Denote, as usual, the weak convergence in H by $\rightharpoonup$ and the strong convergence by $\to$. The next concept will be important in our considerations.

Definition 2 A mapping $S : C \to C$ is said to be demiclosed at 0 in $C \subset H$ if, for any sequence $\{x_k\}$ in C, such that $x_k \rightharpoonup x$ and $Su_k \to 0$, we have $Sx = 0$.

Remark 2.2 In the particular case $S = I - T$, then it follows that x in Definition 2 is a fixed point of T.

Lemma 4 ([8], Lemma 7) *Let $\{x_n\}$ be a sequence of nonnegative real numbers, for which we have*

$$x_{n+1} \leq (1-\Gamma_n)x_n + \Gamma_n \Lambda_n, \quad n \geq 1,$$

and

$$x_{n+1} \leq x_n - \Psi_n + \Phi_n, \quad n \geq 1,$$

where $\Gamma_n \in (0,1)$, $\Psi_n \subset [0,\infty)$, and $\{\Lambda_n\}$ and $\{\Phi_n\}$ are two sequences of real numbers with the following properties:

(i) $\sum_{n=1}^{\infty} \Gamma_n = \infty$; (ii) $\lim_{n\to\infty} \Phi_n = 0$; (iii) For any subsequence $\{n_k\}$ of $\{n\}$, $\lim_{k\to\infty} \Psi_{n_k} \leq 0$ implies $\limsup_{k\to\infty} \Lambda_{n_k} \leq 0$.

Then $\lim_{n\to\infty} x_n = 0$.

Lemma 5 ([3], Lemma 3.2) *Let H be a real Hilbert space, $C \subset H$ be a closed and convex set. If $T : C \to C$ is k-demicontractive, then for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive.*

3 Main results

To solve *SFP* (1), we consider the following self-adaptive inertial algorithm.

Algorithm 1

Step 1. Take $x_0, x_1 \in H_1$ arbitrarily chosen; let $n := 1$;

Step 2. Compute x_n by means of the following formulas:

$$\begin{cases} u_n := x_n + \theta_n(x_n - x_{n-1}) \\ y_n := P_C((1-\delta_n)u_n - \tau_n A^*(I-P_Q)Au_n) + \delta_n S_\lambda u_n, \\ x_{n+1} := \alpha_n g(x_n) + \beta_n u_n + \gamma_n y_n, \end{cases} \tag{18}$$

with $S_\lambda = (1-\lambda)I + \lambda S$, $\lambda \in (0,1)$,

$$\theta_n := \begin{cases} \min\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise}, \end{cases} \tag{19}$$

$\theta \geq 0$ is a given number, $\tau_n = \frac{\rho_n f(x_n)}{\|f(u_n)\|^2}$, where f is given by (5), $\rho_n \in (0,4)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(c_1) $\limsup_{n\to\infty} \beta_n < 1$;

(c_2) $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$;

(c_3) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} = +\infty$;

(c_4) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;

(c_5) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$.

Step 3. If $\nabla f(u_n) = 0$, then Stop, otherwise let $n := n+1$ and go to Step 2.

The next technical lemmas will be useful in proving our main result in this paper.

Lemma 6 *Let $S : H_1 \to H_1$ be a k-demicontractive mapping and $\{x_n\}$ be the sequence generated by Algorithm 1. If $x^* \in Fix(S)$, then the sequence $\{\|x_n - x^*\|\}$ is bounded.*

Proof Since S is k-demicontractive, by Lemma 5 we deduce that the averaged mapping $S_\lambda = (1-\lambda)I + \lambda S$ is also quasi-nonexpansive for any $\lambda \in (0, 1-k)$, and that $Fix(S) = Fix(S_\lambda)$ for any $\lambda \in (0, 1]$ (see for example [3]).

In the following, to simplify writing, we shall denote S_λ by T. So, $Fix(T) \neq \emptyset$ and let $x^* \in Fix(T) \cap SFP(C, Q)$. Let y_n be defined by (18).

Then, by using (5) and (6), Lemma 1, and Lemma 2 and exploiting the fact that T is quasi-nonexpansive, we have successively

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C\big((1-\delta_n)\big(u_n - \tau_n A^*(I-P_Q)Au_n\big) + \delta_n Tu_n\big) - x^*\|^2 \\
&\leq \|\big((1-\delta_n)u_n - \tau_n A^*(I-P_Q)Au_n\big) + \delta_n Tu_n) - x^*\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n A^*(I-P_Q)Au_n\big) + \delta_n Tu_n\big)\big\|^2 \\
&= \big\|\delta_n\big(Tu_n - x^*\big) + (1-\delta_n)\big(u_n - \tau_n A^*(I-P_Q)Au_n - x^*\big)\big\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n A^*(I-P_Q)Au_n\big) + \delta_n Tu_n\big)\big\|^2 \\
&\leq \delta_n\|u_n - x^*\|^2 + (1-\delta_n)\|u_n - \tau_n \nabla f(u_n) - x^*\|^2 \\
&\quad - \delta_n(1-\delta_n)\|Tu_n - u_n + \tau_n \nabla f(u_n)\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n \nabla f(u_n)\big) + \delta_n Tu_n\big)\big\|^2 \\
&\leq \delta_n\|u_n - x^*\|^2 + (1-\delta_n)\big(\|u_n - x^*\|^2 + \tau_n^2\|\nabla f(u_n)\|^2 \\
&\quad - 2\tau_n\langle\nabla f(u_n), u_n - x^*\rangle\big) - \delta_n(1-\delta_n)\|Tu_n - u_n + \tau_n A^*(I-P_Q)Au_n\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n \nabla f(u_n)\big) + \delta_n Tu_n\big)\big\|^2.
\end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned}
\langle\nabla f(u_n), u_n - x^*\rangle &= \langle A^*(I-P_Q)Au_n, u_n - x^*\rangle \\
&= \langle(I-P_Q)Au_n - (I-P_C)Ax^*, Au_n - Ax^*\rangle \\
&\geq \|(I-P_Q)Au_n\|^2 = 2f(u_n).
\end{aligned} \tag{21}$$

So, by inserting (21) in (20), we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - 4(1-\delta_n)\tau_n f(x_n) + (1-\delta_n)\tau_n^2\|\nabla f(u_n)\|^2 \\
&\quad - \delta_n(1-\delta_n)\|Tu_n - u_n + \tau_n A^*(I-P_Q)Au_{n_k}\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n \nabla f(u_n)\big) + \delta_n Tu_n\big)\big\|^2. \\
&= \|u_n - x^*\|^2 - (1-\delta_n)\rho_n(4-\rho_n)\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2} \\
&\quad - \delta_n(1-\delta_n)\|Tu_n - u_n + \tau_n A^*(I-P_Q)Au_{n_k}\|^2 \\
&\quad - \big\|(I-P_C)\big((1-\delta_n)\big(u_n - \tau_n \nabla f(u_n)\big) + \delta_n Tu_n\big)\big\|^2.
\end{aligned} \tag{22}$$

Now, having in view that $\rho_n \in (0,4)$ and $\delta_n \in (0,1)$, by (20) and (22) we deduce that

$$\|y_n - x^*\| \leq \|u_n - x^*\|. \tag{23}$$

Denote

$$v_n := \frac{1}{1-\alpha_n}(\beta_n u_n + \gamma_n y_n) \tag{24}$$

and apply Lemma 1, by keeping in mind condition (c_5), to get

$$\begin{aligned}\|v_n - x^*\|^2 &= \left\|\frac{\beta_n}{1-\alpha_n}u_n + \frac{\gamma_n}{1-\alpha_n}y_n - x^*\right\|^2 \\ &= \left\|\frac{\beta_n}{1-\alpha_n}(u_n - x^*) + \frac{\gamma_n}{1-\alpha_n}(y_n - x^*)\right\|^2 \\ &= \frac{\beta_n}{1-\alpha_n}\|u_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|y_n - x^*\|^2 - \frac{\beta_n}{1-\alpha_n}\cdot\frac{\gamma_n}{1-\alpha_n}\|u_n - y_n\|^2. \end{aligned} \tag{25}$$

Now, using assumptions (c_1)–(c_5), from the above inequality we get

$$\|v_n - x^*\|^2 \leq \frac{\beta_n}{1-\alpha_n}\|u_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|y_n - x^*\|^2,$$

which, by using (23) and (22), yields

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \frac{\beta_n}{1-\alpha_n}\|u_n - x^*\|^2 + \frac{\gamma_n}{1-\alpha_n}\|u_n - x^*\|^2 \\ &\quad - (1-\delta_n)\rho_n(4-\rho_n)\cdot\frac{\gamma_n}{1-\alpha_n}\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2} \\ &\quad - \delta_n(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\cdot\|Tu_n - u_n + \tau_n\nabla f(u_n)\|^2 \\ &\quad - \frac{\gamma_n}{1-\alpha_n}\cdot\|(I-P_C)((1-\delta_n)(u_n - \tau_n\nabla f(u_n)) + \delta_n Tu_n)\|^2 \\ &= \|u_n - x^*\|^2 - (1-\delta_n)\rho_n(4-\rho_n)\cdot\frac{\gamma_n}{1-\alpha_n}\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2} \\ &\quad - \delta_n(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\cdot\|Tu_n - u_n + \tau_n\nabla f(u_n)\|^2 \\ &\quad - \frac{\gamma_n}{1-\alpha_n}\cdot\|(I-P_C)((1-\delta_n)(u_n - \tau_n\nabla f(u_n)) + \delta_n Tu_n)\|^2. \end{aligned} \tag{26}$$

The previous inequality implies

$$\|v_n - x^*\| \leq \|u_n - x^*\|, \quad n \geq 1. \tag{27}$$

By (24), (c_5), and the third equation in (18), we obtain

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)v_n, \quad n \geq 1, \tag{28}$$

and so

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n g(x_n)+(1-\alpha_n)v_n - x^*\|\\ &= \|\alpha_n(g(x_n)-x^*)+(1-\alpha_n)(v_n-x^*)\|\\ &\le \alpha_n\|g(x_n)-x^*\|+(1-\alpha_n)\|v_n-x^*\|\\ &\le \alpha_n\|g(x_n)-g(x^*)\|+\alpha_n\|g(x^*)-x^*\|+(1-\alpha_n)\|v_n-x^*\|.\end{aligned}$$

Now, using the fact that g is a c-contraction, we have

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \alpha_n c\|x_n-x^*\|+\alpha_n\|g(x^*)-x^*\|\\ &\quad +(1-\alpha_n)\|x_n-x^*+\theta_n(x_n-x_{n-1})\|\\ &\le \alpha_n c\|x_n-x^*\|+\alpha_n\|g(x^*)-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\ &\quad +(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|\\ &\le (1-\alpha_n(1-c))\|x_n-x^*\|+\alpha_n\|g(x^*)-x^*\|+\theta_n\|x_n-x_{n-1}\|.\end{aligned}$$

Denote $\varepsilon_n := \theta_n\|x_n-x_{n-1}\|$. Then, by the previous inequalities, we get

$$\begin{aligned}\|x_{n+1}-x^*\| &\le (1-\alpha_n(1-c))\|x_n-x^*\|\\ &\quad +\alpha_n(1-c)\left(\frac{\|g(x^*)-x^*\|}{1-c}+\frac{\varepsilon_n}{\alpha_n(1-c)}\right).\end{aligned}\tag{29}$$

Having in mind assumption (c_2), take $M>0$, for which $\frac{\varepsilon_n}{\alpha_n}\le M$ for all $n\ge 1$. Then, by denoting $M_1 := \frac{\|g(x^*)-x^*\|+M}{1-c}$, inequality (29) yields

$$\begin{aligned}\|x_{n+1}-x^*\| &\le (1-\alpha_n(1-c))\|x_n-x^*\|+\alpha_n(1-c)M_1\\ &\le \max\{\|x_n-x^*\|,M_1\},\end{aligned}$$

from which we easily obtain

$$\|x_{n+1}-x^*\|\le \max\{\|x_n-x^*\|,M_1\},\quad n\ge 1,\tag{30}$$

and this shows that $\{\|x_n-x^*\|\}$ is bounded. □

Lemma 7 *Let $S: H_1\to H_1$ be a k-demicontractive mapping such that $I-T$ is demiclosed at zero, $g: H_1\to H_1$ be a c-Banach contraction, and suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $(0,1)$ satisfying conditions (c_1)–(c_5) in Algorithm 1.*

Let $x^*\in Fix(T)\cap SFP(C,Q)$, $\{x_n\}$ be the sequence generated by Algorithm 1, f be defined by (5), and let $\{v_n\}$ be the sequence given by (24). For $n\ge 1$, let us denote

$$\begin{aligned}&\Gamma_n := 2(1-c)\alpha_n;\ \Phi_n := 2\alpha_n\langle g(x_n)-v_n, x_{n+1}-x^*\rangle,\\ &\Lambda_n := \frac{1}{2(1-c)}\Big(\alpha_n\|g(x_n)-x^*\|^2+2\alpha_n\|g(x_n)-x^*\|\|v_n-x^*\|\\ &\qquad +\alpha_n\|x_n-x^*\|^2+\frac{2\epsilon_n}{\alpha_n}\|v_n-x^*\|+2\langle g(x^*)-x^*, v_n-x^*\rangle\Big),\end{aligned}$$

and

$$\Psi_n := (1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho)\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2}$$
$$+\delta_n(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\cdot\left\|Tu_n-u_n+\tau_n\nabla f(u_n)\right\|^2$$
$$+\frac{\gamma_n}{1-\alpha_n}\cdot\left\|(I-P_C)\big((1-\delta_n)\big(u_n-\tau_n\nabla f(u_n)\big)+\delta_n Tu_n\big)\right\|^2. \tag{31}$$

Then, for any subsequence $\{n_k\}$ *of* $\{n\}$, *we have*

$$\limsup_{k\to\infty}\Lambda_{n_k}\le 0, \tag{32}$$

whenever

$$\lim_{k\to\infty}\Psi_{n_k}=0. \tag{33}$$

Proof Assume (33) holds. Then by (31) one deduces that all terms in the expression of Ψ_{n_k} tend to zero as $k\to\infty$. So,

$$\lim_{k\to\infty}\rho_{n_k}(4-\rho_{n_k})\frac{f^2(u_{n_k})}{\|\nabla f(u_{n_k})\|^2}=0$$

and, based on assumptions (c_1)–(c_5), it follows that in fact

$$\lim_{k\to\infty}\frac{f^2(u_{n_k})}{\|\nabla f(u_{n_k})\|^2}=0. \tag{34}$$

On the other hand, since by Lemma 3 $\nabla f(u_{n_k})$ is Lipschitzian, it follows that $\|\nabla f(u_{n_k})\|$ is bounded, and therefore by (34) we deduce that $f(u_{n_k})\to 0$ as $k\to\infty$, which implies that

$$\lim_{k\to\infty}\left\|(I-P_Q)Au_{n_k}\right\|=0.$$

By (33) we also get

$$\lim_{k\to\infty}\left\|Tu_{n_k}-u_{n_k}+\tau_{n_k}A^*(I-P_Q)Au_{n_k}\right\|^2=0, \tag{35}$$

and due to the fact that

$$\lim_{k\to\infty}\tau_{n_k}\left\|\nabla f(u_{n_k})\right\|=\lim_{k\to\infty}\frac{\rho_{n_k}f(u_{n_k}}{\|\nabla f(u_{n_k})\|}=0, \tag{36}$$

we obtain

$$\lim_{k\to\infty}\|Tu_{n_k}-u_{n_k}\|=0. \tag{37}$$

On the other hand, by (33) we also obtain

$$\lim_{k\to\infty}\left\|(I-P_C)\big((1-\delta_{n_k})\big(u_{n_k}-\tau_{n_k}\nabla f(u_{n_k})\big)+\delta_{n_k}Tu_{n_k}\big)\right\|=0, \tag{38}$$

which, by using the definition of y_{n_k}, yields

$$\lim_{k\to\infty}\left\|(1-\delta_{n_k})\big(u_{n_k}-\tau_{n_k}\nabla f(u_{n_k})\big)+\delta_{n_k}Tu_{n_k}-y_{n_k}\right\|=0,$$

and this can be written in the expanded form

$$\lim_{k\to\infty}\|(1-\delta_{n_k})u_{n_k}-(1-\delta_{n_k})\tau_{n_k}\nabla f(u_{n_k}))+\delta_{n_k}Tu_{n_k}-y_{n_k}\|=0. \tag{39}$$

By (39) and (36) we get

$$\lim_{k\to\infty}\left\|(1-\delta_{n_k})u_{n_k}+\delta_{n_k}Tu_{n_k}-y_{n_k}\right\|=0,$$

which means that

$$\lim_{k\to\infty}\left\|u_{n_k}-y_{n_k}+\delta_{n_k}(Tu_{n_k}-u_{n_k})\right\|=0. \tag{40}$$

Now, using the fact that

$$\begin{aligned}\|u_{n_k}-y_{n_k}\| &= \left\|u_{n_k}-y_{n_k}+\delta_{n_k}(Tu_{n_k}-u_{n_k})-\delta_{n_k}(Tu_{n_k}-u_{n_k})\right\|\\ &\leq \left\|u_{n_k}-y_{n_k}+\delta_{n_k}(Tu_{n_k}-u_{n_k})\right\|+\delta_{n_k}\|Tu_{n_k}-u_{n_k}\|,\end{aligned}$$

by (37) and (40) we immediately obtain

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \tag{41}$$

By using the definition of v_n in (24), we have

$$\begin{aligned}\|v_{n_k}-u_{n_k}\| &= \left\|\frac{\beta_{n_k}}{1-\alpha_{n_k}}u_{n_k}+\frac{\gamma_{n_k}}{1-\alpha_{n_k}}y_{n_k}-u_{n_k}\right\|\\ &= \left\|\frac{\gamma_{n_k}}{1-\alpha_{n_k}}u_{n_k}+\frac{\gamma_{n_k}}{1-\alpha_{n_k}}y_{n_k}\right\|\\ &= \frac{\gamma_{n_k}}{1-\alpha_{n_k}}\cdot\|y_{n_k}-u_{n_k}\|,\end{aligned}$$

which, by (41), yields

$$\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=0. \tag{42}$$

Since $I-S$ is demiclosed at zero and $T=(1-\lambda)I+\lambda T$, it follows that T is also demiclosed at zero. By means of (37), this implies that $\omega_w(u_{n_k})\subset Fix(T)$.

So, we can choose a subsequence $u_{n_{k_j}}$ of u_{n_k} with the following property:

$$\limsup_{k\to\infty}\big\langle g(x^*)-x^*,u_{n_k}-x^*\big\rangle=\lim_{j\to\infty}\big\langle g(x^*)-x^*,u_{n_{k_j}}-x^*\big\rangle.$$

We can assume, without any loss of generality, that for the above subsequence $u_{n_{k_j}}$ one has $u_{n_{k_j}}\rightharpoonup u'$.

Since $f(u_{n_k}) \to 0$, we obtain

$$0 \le f(u') \le \liminf_{j\to\infty} f(u_{n_{k_j}}) = 0,$$

which implies $f(u') = 0$ and $Au' \in Q$.

Therefore, by (41), $u' \in SFP(C,Q)$ and so $u' \in Fix(T) \cap SFP(C,Q)$. Now, based on (42), we get

$$\begin{aligned}\limsup_{k\to\infty}\langle g(x^*) - x^*, v_{n_k} - x^*\rangle &= \limsup_{k\to\infty}\langle g(x^*) - x^*, u_{n_k} - x^*\rangle \\ &= \lim_{j\to\infty}\langle g(x^*) - x^*, u_{n_{k_j}} - x^*\rangle = \langle g(x^*) - x^*, u' - x^*\rangle \le 0,\end{aligned}$$

which shows that (32) holds. □

Now we are ready to state and prove the main result of our paper.

Theorem 1 *Let $T : H_1 \to H_1$ be a k-demicontractive mapping such that $I - T$ is demiclosed at zero, and let $g : H_1 \to H_1$ be a c-Banach contraction. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $(0,1)$ satisfying conditions (c_1)–(c_5) in Lemma 7.*

If $Fix(T) \cap SFP(C,Q) \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $x^* \in Fix(T) \cap SFP(C,Q)$, which solves uniquely the variational inequality (12).

Proof Using the fact that any metric projection is nonexpansive, on one hand, and that the composition of a nonexpansive mapping and of a contraction is a contraction, on the other hand, it follows that $P_{Fix(T)\cap SFP(C,Q)}g$ is a c-contraction since g is a c-contraction.

Hence $P_{Fix(T)\cap SFP(C,Q)}g$ has a unique fixed point $x^* \in H_1$:

$$x^* = P_{Fix(T)\cap SFP(C,Q)}g(x^*).$$

Moreover, in view of Lemma 2, $x^* \in Fix(T) \cap SFP(C,Q)$ is a solution of the variational inequality (12).

Let $p \in Fix(T) \cap SFP(C,Q)$ be arbitrary. By Lemma 6, it follows that the sequence $\{\|x_n - x^*\|\}$ is bounded.

Let $\{u_n\}$ be given by the corresponding inertial equation in (18). The we have

$$\|u_n - p\|^2 = \|x_n + \theta_n(x_n - x_{n-1})\|^2,$$

which by applying Lemma 1 yields

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, u_n - p\rangle \\ &\le \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|u_n - p\| \le \|x_n - p\|^2 + 2\varepsilon_n\|u_n - p\|.\end{aligned}$$

Hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + 2\varepsilon_n\|u_n - p\|. \tag{43}$$

As $p \in Fix(T) \cap SFP(C,Q)$ has been taken arbitrarily, we can let it be

$$p := x^* = P_{Fix(T)\cap SFP(C,Q)} g\big(x^*\big).$$

By using (28), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n g(x_n) + (1-\alpha_n)v_n - x^*\|^2 \\
&= \|\alpha_n\big(g(x_n) - x^*\big) + (1-\alpha_n)\big(v_n - x^*\big)\|^2 \le \alpha_n^2\|g(x_n) - x^*\|^2 \\
&\quad + (1-\alpha_n)^2\|v_n - x^*\|^2 + 2\alpha_n\langle g(x_n) - x^*, v_n - x^*\rangle \\
&\quad - 2\alpha_n^2\langle g(x_n) - x^*, v_n - x^*\rangle \\
&\le \alpha_n^2\|g(x_n) - x^*\|^2 + (1-\alpha_n)^2\|v_n - x^*\|^2 + 2\alpha_n\langle g(x_n) - x^*, v_n - x^*\rangle \\
&\quad + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| = \alpha_n^2\|g(x_n) - x^*\|^2 + (1-\alpha_n)^2\|v_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle g(x_n) - g\big(x^*\big), v_n - x^*\rangle + 2\alpha_n\langle g\big(x^*\big) - x^*, v_n - x^*\rangle \\
&\quad + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| \\
&\le \alpha_n^2\|g(x_n) - x^*\|^2 + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| + (1-\alpha_n)^2\|v_n - x^*\|^2 \\
&\quad + \alpha_n \cdot c \cdot \big(\|x_n - x^*\|^2 + \|v_n - x^*\|^2\big) + 2\alpha_n\langle g\big(x^*\big) - x^*, v_n - x^*\rangle. \qquad (44)
\end{aligned}$$

Now, by (23) and (43) we obtain

$$\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 + 2\varepsilon_n\|u_n - x^*\|, \qquad (45)$$

and so, by using (44), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n^2\|g(x_n) - x^*\|^2 + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| \\
&\quad + \alpha_n \cdot c \cdot \big(\|x_n - x^*\|^2 + \|v_n - x^*\|^2\big) + 2\varepsilon_n\|u_n - x^*\| \\
&\quad + 2\alpha_n\langle g\big(x^*\big) - x^*, v_n - x^*\rangle \\
&\quad + (1-\alpha_n)^2 \cdot \big(\|x_n - x^*\|^2 + 2\varepsilon_n\|u_n - x^*\|\big) \\
&= \alpha_n^2\|g(x_n) - x^*\|^2 + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| \\
&\quad + \big(\alpha_n^2 + \big(1 - 2\alpha_n(1-c)\big)\big) \cdot \|x_n - x^*\|^2 + \big(2\varepsilon_n(1-\alpha_n)^2 \\
&\quad + 2\alpha_n c\varepsilon_n\big) \cdot \|u_n - x^*\| \\
&\quad + 2\alpha_n\|g(x_n) - x^*\|\|v_n - x^*\| \le \alpha_n^2\|g(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n^2\|g(x_n) - x^*\|\|v_n - x^*\| \\
&\quad + \alpha_n^2 \cdot \|x_n - x^*\|^2 + \big(1 - 2\alpha_n(1-c)\big)\|x_n - x^*\|^2 \\
&\quad + 4\varepsilon_n \cdot \bigg(\alpha_n\|g(x_n) - x^*\|^2 + 2\alpha_n\|g(x_n) - x^*\|\|v_n - x^*\| + \alpha_n\|x_n - x^*\|^2 \\
&\quad + \frac{4\varepsilon_n}{\alpha_n} \cdot \|u_n - x^*\| + 2\langle g\big(x^*\big) - x^*, v_n - x^*\rangle\bigg) = \big(1 - 2\alpha_n(1-c)\big)\|x_n - x^*\|^2
\end{aligned}$$

$$+ 2\alpha_n(1-c)\frac{1}{2(1-c)}\Bigg[\alpha_n\|g(x_n)-x^*\|^2 + 2\alpha_n\|g(x_n)-x^*\|\,\|v_n-x^*\| + \alpha_n\|x_n-x^*\|^2 + \frac{4\varepsilon_n}{\alpha_n}\cdot\|u_n-x^*\| + 2\langle g(x^*)-x^*, v_n-x^*\rangle\Bigg]. \tag{46}$$

On the other hand, by Lemma 1 and the definition of $\{x_n\}$, we have

$$\|x_{n+1}-x^*\|^2 = \|\alpha_n g(x_n) + (1-\alpha_n)v_n - x^*\|^2 \le \|v_n-x^*\|^2 + 2\alpha_n\langle g(x_n)-v_n, v_n-x^*\rangle. \tag{47}$$

Therefore, by combining (26), (43), and (47) and denoting $S = (1-\lambda)I + \lambda T$, we obtain successively

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \|u_n-x^*\|^2 - (1-\delta_n)\cdot\frac{\gamma_n}{1-\alpha_n}\cdot\rho_n(4-\rho_n)\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2}\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\delta_n(1-\delta_n)\cdot\|Su_n-u_n+\tau_n A^*(I-P_Q)Au_n\|^2\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\|(I-P_C)((1-\delta_n)(u_n-\tau_n A^*(I-P_Q)Au_n)+\delta_n Su_n\|^2\\
&+ 2\alpha_n\cdot\langle g(x_n)-v_n, v_n-x^*\rangle \le \|x_n-x^*\|^2 + 2\varepsilon_n\|u_n-x^*\|^2\\
&- (1-\delta_n)\cdot\frac{\gamma_n}{1-\alpha_n}\cdot\rho_n(4-\rho_n)\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2}\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\delta_n(1-\delta_n)\cdot\|Su_n-u_n+\tau_n A^*(I-P_Q)Au_n\|^2\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\|(I-P_C)((1-\delta_n)(u_n-\tau_n A^*(I-P_Q)Au_n)+\delta_n Su_n\|^2\\
&+ 2\alpha_n\cdot\langle g(x_n)-v_n, v_n-x^*\rangle,
\end{aligned}$$

which yields

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \|x_n-x^*\|^2 + 2\varepsilon_n\|u_n-x^*\|^2 - (1-\delta_n)\cdot\frac{\gamma_n}{1-\alpha_n}\cdot\rho_n(4-\rho_n)\cdot\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2}\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\delta_n(1-\delta_n)\cdot\|Su_n-u_n+\tau_n A^*(I-P_Q)Au_n\|^2\\
&- \frac{\gamma_n}{1-\alpha_n}\cdot\|(I-P_C)((1-\delta_n)(u_n-\tau_n A^*(I-P_Q)Au_n)+\delta_n Su_n\|^2\\
&+ 2\alpha_n\cdot\langle g(x_n)-v_n, v_n-x^*\rangle.
\end{aligned} \tag{48}$$

Now, for $n \ge 1$, let us denote

$$\begin{aligned}
&\Gamma_n := 2(1-c)\alpha_n;\ \Phi_n := 2\alpha_n\langle g(x_n)-v_n, x_{n+1}-x^*\rangle,\\
&\Lambda_n := \frac{1}{2(1-c)}\Bigg(\alpha_n\|g(x_n)-x^*\|^2 + 2\alpha_n\|g(x_n)-x^*\|\,\|v_n-x^*\|\\
&\quad + \alpha_n\|x_n-x^*\|^2 + \frac{2\epsilon_n}{\alpha_n}\|v_n-x^*\| + 2\langle g(x^*)-x^*, v_n-x^*\rangle\Bigg)
\end{aligned}$$

and

$$\Psi_n := (1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho)\frac{f^2(u_n)}{\|\nabla f(u_n)\|^2} + \delta_n(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\cdot\left\|Tu_n - u_n + \tau_n\nabla f(u_n)\right\|^2 + \frac{\gamma_n}{1-\alpha_n}\cdot\left\|(I-P_C)\big((1-\delta_n)\big(u_n-\tau_n\nabla f(u_n)\big)+\delta_n Tu_n\big)\right\|^2.$$

In view of these notations, inequalities (46) and (48) can be briefly written as

$$\left\|x_{n+1}-x^*\right\|^2 \le (1-\Gamma_n)\left\|x_n - x^*\right\|^2 + \Gamma_n\Lambda_n, \quad n \ge 1,$$
$$\left\|x_{n+1}-x^*\right\|^2 \le \left\|x_n - x^*\right\|^2 - \Psi_n + \Phi_n, \quad n \ge 1,$$

respectively.

By assumptions $(c_1)-(c_5)$ it is easy to deduce that

$$\lim_{n\to\infty}\Gamma_n = 0, \qquad \lim_{n\to\infty}\Phi_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty}\Gamma_n = \infty.$$

In the end, by applying Lemma 7 and Lemma 4, it follows that

$$\lim_{n\to\infty}\left\|x_n - x^*\right\| = 0,$$

which shows that the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to x^*. □

Remark 3.1 We note that the technique of the proof of Theorem 1 is similar to that used in [20] and is based on inserting an averaged component that produces a perturbed version of the inertial algorithm, thus imbedding the demicontractive mappings in the class of quasi-nonexpansive mappings, in view of Lemma 5.

Example 2 Let H be the real line with the usual norm, $C = [0,1]$, and T be the mapping in Example 1. Since T is demicontractive, our Theorem 1 can be applied to solve any consistent split feasibility problem over the set of fixed points of T, whenever $Fix(T) \cap SFP(C,Q) \neq \emptyset$.

We also note that Theorem 2.1 in Qin and Wang [15] cannot be applied to solve consistent split feasibility problems over the set of fixed points of T (because T is not nonexpansive) and also Theorem 1 in Wang et al. [20] cannot be applied to the same problem (because T is not quasi-nonexpansive).

4 Numerical examples

Example 3 We consider the problem given in Example 1 in Wang et al. [20], which is devoted to the solution of a linear system of equations $Ax = b$. We work similarly in $H_1 =$

$H_2 = \mathbb{R}^5$, with the same data, first by taking the mapping S given by

$$S = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and then considering a nonviscosity type algorithm, i.e., taking the contraction mapping g to be the null function $g \equiv 0$. To allow a numerical comparison, we also take

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 5 & -1 \\ 1 & 1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 1 & 5 \\ 2 & 2 & 3 & 6 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \frac{43}{16} \\ 2 \\ \frac{19}{16} \\ \frac{51}{8} \\ \frac{41}{8} \end{pmatrix}.$$

This is a particular example of a split feasibility problem with $C = Fix(S)$ and $Q = \{b\}$.

We performed several numerical experiments in MatLab by using our Algorithm 1 with various particular values on the parameters and compared the obtained results to those presented in Wang et al. [20] (Table 1 and Fig. 1).

By analyzing the diversity of the numerical results thus obtained, we noted a very interesting fact, i.e., that most of the assumptions on the parameters α_n, β_n, δ_n, θ_n, τ_n involved in the iterative process (18) are in fact imposed merely for technical reasons when proving analytically the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 1.

Therefore, some of these assumptions appear to be not necessary for the convergence of the iterative process $\{x_n\}$ in most practical situations, as shown by the numerical results presented in Table 1.

Table 1 Numerical results for the starting point $x = (1, 1, 1, 1, 1)^T$

n	$x_n^{(1)}$	$x_n^{(2)}$	$x_n^{(3)}$	$x_n^{(4)}$	$x_n^{(5)}$
0	1	1	1	1	1
1	0.766667	0.766667	0.766667	0.766667	1
2	0.587778	0.587778	0.587778	0.642222	1
3	0.450630	0.450630	0.463333	0.575852	1
4	0.345483	0.348447	0.381477	0.540454	1
5	0.265562	0.274850	0.329560	0.521576	1
6	0.205764	0.223484	0.297466	0.511507	1
7	0.161887	0.188600	0.278000	0.506137	1
8	0.130347	0.165454	0.266366	0.503273	1
9	0.108124	0.150394	0.259492	0.501746	1
10	0.092758	0.140758	0.255470	0.500931	1
11	0.082315	0.134681	0.253134	0.500497	1
12	0.075327	0.130894	0.251788	0.500265	1
13	0.070716	0.128561	0.251015	0.500141	1
14	0.067713	0.127136	0.250574	0.500075	1
15	0.065779	0.126273	0.250324	0.500040	1
...	...	...	...	...	...
20	0.062790	0.125089	0.250018	0.500002	1
...	...	...	...	...	...
32	0.062501	0.125000	0.250000	0.500000	1
33	0.062500	0.125000	0.250000	0.500000	1

These results were obtained for the same starting point x like the one in Wang et al. [20] but with the following particular values of the involved parameters: $\beta_n = 0$, $\delta_n = 1$ (which do not satisfy all the assumptions in $(c_1) - (c_5)$), $\theta_n = 0$, and $\lambda = 0.5$.

It is also worth mentioning that we obtained the exact solution $x^* = (\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1)$ of the problem after $n = 33$ iterations.

If we compare our numerical results to the results given in Table 1 and Figure in Wang et al. [20], where the authors have taken the values $\alpha_n = \frac{1}{10n}$, $\beta_n = 0.5$, $\delta_n = 0.5, \dots$, and the same starting point $x = (1, 1, 1, 1, 1)^T$ for Algorithm (9), we observe that the exact solution was not obtained even after 10,000 iterations…

In fact, both algorithms considered in Wang et al. [20], i.e., algorithms (7) and (9), are extremely slow: even after performing 10,000 iterations the exact solution x^* is obtained with an error of 9.4925×10^{-5}.

In our opinion, this is because any inertial type algorithm (i.e., with $\theta_n \neq 0$) is usually slower than the noninertial ones, as illustrated by the numerical examples in Table 1, see also the results reported in Berinde [2], but for a slightly different context.

5 Conclusions

1. We introduced a hybrid inertial self-adaptive algorithm for solving the split feasibility problem and fixed point problem in the class of demicontractive mappings.

2. As shown by Example 2, our theoretical results extend several related results existing in the literature from the class of nonexpansive or quasi-nonexpansive mappings to the larger class of demicontractive mappings.

3. We performed numerical experiments, see Example 3, designed to compare our results to those presented in Wang et al. [20]. The numerical results presented in Table 1 clearly illustrate the superiority of our results over the related existing ones in the literature. These numerical results also naturally raise an open problem: find weaker conditions on the parameters α_n, β_n, δ_n, θ_n, τ_n such that the iterative process (18) still converges to an element $x^* \in Fix(T) \cap SFP(C, Q)$.

4. For other related works that allow similar developments to the ones in the current paper, we refer the readers to Berinde [1], Hu et al. [9], Kingkam and Nantadilok [10], Padcharoen et al. [14], Sharma and Chandok [16], Shi et al. [17], Tiammee and Tiamme [18], Uba at al. [19], etc.

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Declarations

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References

1. Berinde, V.: Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition. Carpath. J. Math. **36**(1), 27–34 (2020)
2. Berinde, V.: A modified Krasnoselskii-Mann iterative algorithm for approximating fixed points of enriched nonexpansive mappings. Symmetry **14**(1), 123 (2022). https://doi.org/10.3390/sym14010123
3. Berinde, V.: Approximating fixed points of demicontractive mappings via the quasi-nonexpansive case. Carpath. J. Math. **39**(1), 73–85 (2023)
4. Berinde, V.: On a useful lemma that relates quasi-nonexpansive and demicontractive mappings in Hilbert spaces. Creative Math. Inform. **33**(1), 7–21 (2024)
5. Berinde, V., Petruşel, A., Rus, I.A.: Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces. Fixed Point Theory **24**(2), 525–540 (2023)
6. Byrne, C.: Iterative oblique projection onto convex set and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
7. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103–120 (2004)
8. He, S., Yang, C.: Solving the variational inequality problem defined on intersection of finite level sets. Abstr. Appl. Anal. **2013**, 942315 (2013)
9. Hu, R., Sofonea, M., Xiao, Y-B.: A fixed point approach of variational-hemivariational inequalities. Carpathian J. Math. **38** (3), 573–581 (2022)
10. Kingkam, P., Nantadilok, J.: Iterative process for finding fixed points of quasi-nonexpansive multimaps in CAT(0) spaces. Korean J. Math. **31**(1), 35–48 (2023)
11. Kraikaew, R., Saejung, S.: A simple look at the method for solving split feasibility problems in Hilbert spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(3), 117 (2020)
12. López, G., Martín-Márquez, V., Wang, F.H., Xu, H.-K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**(8), 085004 (2012)
13. Marino, G., Xu, H.-K.: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**(1), 336–346 (2007)
14. Padcharoen, A., Sokhuma, K., Abubakar, J.: Projection methods for quasi-nonexpansive multivalued mappings in Hilbert spaces. AIMS Math. **8**(3), 7242–7257 (2023)
15. Qin, X.L., Wang, L.: A fixed point method for solving a split feasibility problem in Hilbert spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(1), 315–325 (2019)
16. Sharma, S., Chandok, S.: Split fixed point problems for quasi-nonexpansive mappings in Hilbert spaces. Sci. Bull. "Politeh." Univ. Buchar., Ser. A, Appl. Math. Phys. **86**(1), 109–118 (2024)
17. Shi, Y., Zhu, Z., Zhang, Y.: New algorithm for non-monotone and non-Lipschitz equilibrium problem over fixed point set of a quasi-nonexpansive non-self mapping in Hilbert space. Appl. Set-Valued Anal. Optim. **7**(3), 279–290 (2023)
18. Tiammee, S., Tiammee, J.: Weak convergence of inertial proximal point algorithm for a family of nonexpansive mappings in Hilbert spaces. Carpath. J. Math. **40**(1), 173–185 (2024)
19. Uba, M.O., Onyido, M.A., Udeani, C.I., Nwokoro, P.U.: A hybrid scheme for fixed points of a countable family of generalized nonexpansive-type maps and finite families of variational inequality and equilibrium problems, with applications. Carpath. J. Math. **39**(1), 281–292 (2023)
20. Wang, Y.Q., Xu, T.T., Yao, Y.-C., Jiang, B.N.: Self-adaptive method and inertial modification for solving the split feasibility problem and fixed-point problem of quasi-nonexpansive mapping. Mathematics **10**, 1612 (2022). https://doi.org/10.3390/math10091612
21. Xu, H.-K.: Averaged mappings and the gradient-projection algorithm. J. Optim. Theory Appl. **150**(2), 360–378 (2011)

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