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Solving integral and differential equations via fixed point results involving F-contractions

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Abstract
In this article, we aim to establish fixed point results within the framework of an orthogonal complete metric space by employing an F-weak contraction. Our research extends and generalizes several well-established results found in the existing literature. To substantiate the validity of our findings, we have included illustrative examples. Additionally, our discoveries empower us to ascertain both the existence and uniqueness of solutions for both differential and integral equations.

Mathematics Subject Classification: 47H10; 54H25

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1 Introduction

The theory of fixed point is crucial for solving numerous problems in a variety of study fields. Many scholars investigated the ability to revolutionize the notions of metric and metric spaces. There is a significant amount of literature on several generalizations and developments. Throughout the years, the theory has piqued the interest of several scholars. The Banach contraction theorem has been extended in a variety of ways throughout the years; we suggest the reader [1–8] and references therein.

The notion of an F-contraction, which broadened and expanded the Banach contraction principle, was introduced by D. Wardowski [9] in 2012 as a unique type of contractive mapping. Wardowski demonstrated that if a metric space (E,d) is complete, each F-contraction has a unique fixed point that corresponds to the Picard iteration limit. Just after, Wardowski and Van Dung [10] proposed a weaker notion of an F-contraction, called F-weak contraction, and used it to demonstrate the fixed point theorem. For almost a decade, many scholars have tried to broaden and enhance the survey of F-contractions by generalizing the function F and the spaces with metric form structures, resulting in new Picard mapping group [11–17].

A fascinating idea of orthogonal sets and later orthogonal metric spaces was recently suggested by Gordji et al. [18] They also showed that by applying the Banach fixed point theorem to this newly developed structure, their results may be utilized to ensure the existence and uniqueness of solutions to first-order differential equations. Moreover, the scholars improved the findings in [18] and proved fixed point theorems in the setting of this newly constructed structure [19].

2 Preliminaries

Definition 2.1 ([18]) Consider a set $E \neq \emptyset$ and a binary relation $\curlywedge \subseteq E \times E$. Then $(E, \curlywedge)$ referred to as an orthogonal set if the following criterion is satisfied:

$$\text{for all } \varpi \in E \text{ there exists } \varrho_0 \text{ such that } (\varpi \curlywedge \varrho_0) \text{ or } (\varrho_0 \curlywedge \varpi),$$

and element ϱ_0 is called an orthogonal element.

Definition 2.2 ([18]) Consider a set $E \neq \emptyset$ and a binary relation $\curlywedge \subseteq E \times E$. Any two elements from E are assumed to be orthogonally connected if $\varrho, \varpi \in E$ such that $\varrho \curlywedge \varpi$.

Example 2.1 Let

$$E = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and a binary relation $\curlywedge$ on E defined as $\varrho \curlywedge \varpi$ if $\varrho.\varpi = 0$. Then $(E, \curlywedge)$ is an orthogonal set.

Assume that $(E, \curlywedge)$ is an orthogonal set and with the usual metric d defined on the set E. The triplet $(E, \curlywedge, d)$ is then referred to as an O-metric space (briefly) or orthogonal metric space.

Definition 2.3 ([18]) Let $(E, \curlywedge)$ be a nonempty O-set, then

(i) a sequence $\{\varrho_n\}$ is known as an orthogonal sequence (usually known as an O-sequence) if

$$\varrho_n \curlywedge \varrho_{n+1} \quad \text{or} \quad \varrho_{n+1} \curlywedge \varrho_n, \quad \text{for all } n \in \mathbb{N};$$

(ii) similarly, a sequence $\{\varrho_n\}$ is known as a Cauchy O-sequence if

$$\varrho_n \curlywedge \varrho_{n+1} \quad \text{or} \quad \varrho_{n+1} \curlywedge \varrho_n, \quad \text{for all } n \in \mathbb{N}.$$

Definition 2.4 ([18]) Let $(E, \curlywedge, d)$ be an orthogonal metric space.

(i) A triplet $(E, \curlywedge, d)$ is called an orthogonal complete metric space (briefly $\curlywedge$-complete) if every Cauchy O-sequence converges in E;

(ii) And completeness of a metric space implies O-completeness but the converse is not always true.

Definition 2.5 ([18]) Let $(E, \curlywedge, d)$ be an O-metric space. Then

(i) a mapping $f : E \to E$ is known as orthogonally continuous (briefly O-continuous) if for each O-sequence $\{\varrho_n\}_{n\in\mathbb{N}} \to \varrho$, one has $f(\varrho_n) \to f(\varrho)$ as $n \to \infty$;

(ii) O-continuity is relatively weaker than classical continuity in classical metric spaces.

Definition 2.6 ([18]) Let a pair $(E, \curlywedge)$ be an O-set, where $\curlywedge$ is a binary relation defined on a nonempty set E. A mapping $f : E \to E$ is said to be $\curlywedge$-preserving if $f(\varrho) \curlywedge f(\varpi)$ whenever $\varrho \curlywedge \varpi$ and weakly $\curlywedge$-preserving if $f(\varrho) \curlywedge f(\varpi)$ or $f(\varpi) \curlywedge f(\varrho)$ whenever $\varrho \curlywedge \varpi$.

3 Main results

To continue, we must remember Wardowski's [9] definition of a control function. Let Υ be a set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ that is the following axioms hold:

(F1) for all $a, b \in \mathbb{R}^+$, with $a < b$, $F(a) < F(b)$;

(F2) for all positive sequences $\{\gamma_n\}$, $\lim_{n\to\infty} \gamma_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(\gamma_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\gamma\to 0^+} \gamma^k F(\gamma) = 0$.

Let $f : E \to E$ and (E, d) be a metric space. If there exist $\tau > 0$ and $F \in \Upsilon$ such that for all $\varrho, \varpi \in E$,

$$d(f\varrho, f\varpi) > 0 \quad \Longrightarrow \quad \tau + F\bigl(d(f\varrho, f\varpi)\bigr) \le F\bigl(d(\varrho, \varpi)\bigr), \tag{3.1}$$

then f is called an F-contraction mapping.

Example 3.1 ([9]) The following functions belong to Υ:

(i) $F(\varrho) = \frac{-1}{\sqrt{\varrho}}$;

(ii) $F(\varrho) = \ln \varrho$;

(iii) $F(\varrho) = \varrho + \ln \varrho$;

(iv) $F(\varrho) = \ln(\varrho^2 + \varrho)$.

Definition 3.1 ([16]) Consider an O-metric space $(E, \curlywedge, d)$ and a function $f : E \to E$. If there exist $\tau > 0$ and $F \in \Upsilon$ such that for all $\varrho, \varpi \in E$ with $\varrho \curlywedge \varpi$,

$$d(f\varrho, f\varpi) > 0 \quad \Longrightarrow \quad \tau + F\bigl(d(f\varrho, f\varpi)\bigr) \le F\bigl(d(\varrho, \varpi)\bigr), \tag{3.2}$$

then f is called an orthogonal F-contraction mapping or $F_\curlywedge$-contraction.

Remark 1 ([16]) From equation (3.1) and (F1), we obtain

$$d(f\varrho, f\varpi) < d(\varrho, \varpi), \quad \text{for all } \varrho, \varpi \in E \text{ with } \varrho \curlywedge \varpi.$$

In [16], Sawangsup et al. showed that if $(E, \curlywedge, d)$ is an O-complete (not necessarily complete) metric space, $F \in \Upsilon$, and $f : E \to E$ is a an $F_\curlywedge$-contraction, which is $\curlywedge$-continuous and $\curlywedge$-preserving, then f has a unique fixed point in E.

Further, we introduce the notion of an $F_\curlywedge$-weak contraction in the form of the following definition.

Definition 3.2 Consider an O-metric space $(E, \curlywedge, d)$ and a function $f : E \to E$. If there exist $\tau > 0$ and $F \in \Upsilon$ such that

$$\tau + F\bigl(d(f\varrho, f\varpi)\bigr) \le F\bigl(M(\varrho, \varpi)\bigr) \quad \text{when } d(f\varrho, f\varpi) > 0 \text{ and } \varrho \curlywedge \varpi, \tag{3.3}$$

where

$$M(\varrho,\varpi) = \max\left\{d(\varrho,\varpi), d(\varrho,f\varrho), d(\varpi,f\varpi), \frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\},$$

then f is called an $F_{\curlywedge}$-weak contraction.

Example 3.2 Let $E = \{0,1,2,3,4,5,6\}$ be endowed with the usual metric. Let

$$A = \big\{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(3,0),(3,1),(3,4),(3,5),(3,6)\big\}.$$

Define a relation $\curlywedge$ such that $\varrho \curlywedge \varpi$ if and only if $(\varrho,\varpi) \in A$. Clearly, $(E,\curlywedge)$ is an O-set with 0 as an orthogonal element. Define a mapping $f : E \to E$ as

$$f(0) = f(1) = f(2) = f(3) = 0, \qquad f(4) = 3, \qquad f(5) = 2.$$

Let $F(\varrho) = \ln \varrho$, then it can be verified that f is an $F_{\curlywedge}$-weak contraction, however, not an $F_{\curlywedge}$-contraction. Indeed, for $\varrho = 4$ and $\varpi = 3$,

$$\tau + F\big(d(f\varrho,f\varpi)\big) \le F\big(d(\varrho,\varpi)\big)$$

does not hold for any $\tau > 0$.

Remark 2

(i) Every $F_{\curlywedge}$-contraction is an $F_{\curlywedge}$-weak contraction.

(ii) Let f be an $F_{\curlywedge}$-weak contraction. From (3.3), for all $\varrho,\varpi \in E, f\varrho \neq f\varpi$ with $\varrho \curlywedge \varpi$,

$$\begin{aligned} F\big(d(f\varrho,f\varpi)\big) &< \tau + F\big(d(f\varrho,f\varpi)\big) \\ &\le F\left(\max\left\{d(\varrho,\varpi), d(\varrho,f\varrho), d(\varpi,f\varpi), \frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\right). \end{aligned}$$

Then by (F1), we get

$$d(f\varrho,f\varpi) < \max\left\{d(\varrho,\varpi), d(\varrho,f\varrho), d(\varpi,f\varpi), \frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\},$$

for all $\varrho,\varpi \in E, f\varrho \neq f\varpi$, with $\varrho \curlywedge \varpi$.

The converse conclusion of Remark 2 (i) is not valid, as shown in the following example.

Example 3.3 Let $E = [0,1]$ and the metric on E be the Euclidean metric. Suppose $\varrho \curlywedge \varpi$ if $\varrho\varpi \le \{\varrho \text{ or } \varpi\}$. Let $f : E \to E$ be a function defined by

$$f\varrho = \begin{cases} \frac{1}{2}, & \varrho \in [0,1), \\ \frac{1}{4}, & \varrho = 1. \end{cases}$$

By Remark 1, f is not an $F_{\curlywedge}$-contraction because it is not O-continuous. For $\varrho \in [0,1)$ and $\varpi = 1$, we have

$$d(f\varrho,f\varpi) = \frac{1}{4} \quad \text{and} \quad M(\varrho,\varpi) \ge \frac{3}{4}.$$

As a result, by selecting $F(\varrho) = \ln \varrho$, $\varrho \in (0,+\infty)$, and $\tau = \ln 3$, we obtain that f is an $F_{\curlywedge}$-weak contraction.

Now, we will present the paper's primary conclusion.

Theorem 3.1 *Let $f : E \to E$ be a function and $(E, \curlywedge, d)$ an O-complete (not necessarily complete) metric space, where f is $\curlywedge$-preserving, $F_{\curlywedge}$-weak contraction, and $\curlywedge$-continuous, then f has exactly one fixed point in E. Furthermore, the limit of the iterative sequence $\{f^n \varrho_0\}$ is the fixed point of f.*

Proof By orthogonality of the set, there exists an element $\varrho_0 \in E$ such that

$$(\text{for all } \varpi \in E, \varrho_0 \curlywedge \varpi) \quad \text{or} \quad (\text{for all } \varpi \in E, \varpi \curlywedge \varrho_0).$$

It follows that $(\varrho_0 \curlywedge f\varrho_0)$ or $(f\varrho_0 \curlywedge \varrho_0)$. Let

$$\varrho_1 = f\varrho_0, \varrho_2 = f\varrho_1 = f^2\varrho_0, \dots, \varrho_n = f\varrho_{n-1} = f^n\varrho_0, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Since f is $\curlywedge$-preserving, $\{\varrho_n\}_{n\in\mathbb{N}}$ is an O-sequence. Let $\varrho_{n_0} = \varrho_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then the proof is obvious. Let $\varrho_n \neq \varrho_{n+1}$, for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$, $d(\varrho_{n+1}, \varrho_n) > 0$. Since f is an $F_{\curlywedge}$-weak contraction, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&F\big(d(\varrho_{n+1}, \varrho_n)\big) \\
&\quad = F\big(d(f\varrho_n, f\varrho_{n-1})\big) \\
&\quad \le F\left(\max\left\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, f\varrho_n), d(\varrho_{n-1}, f\varrho_{n-1}), \frac{d(\varrho_n, f\varrho_{n-1}) + d(\varrho_{n-1}, f\varrho_n)}{2}\right\}\right) - \tau \\
&\quad = F\left(\max\left\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1}), d(\varrho_{n-1}, \varrho_n), \frac{d(\varrho_{n-1}, \varrho_{n+1})}{2}\right\}\right) - \tau \\
&\quad = F\left(\max\left\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1}), \frac{d(\varrho_{n-1}, \varrho_{n+1})}{2}\right\}\right) - \tau \\
&\quad \le F\left(\max\left\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1}), \frac{d(\varrho_{n-1}, \varrho_n) + d(\varrho_n, \varrho_{n+1})}{2}\right\}\right) - \tau \\
&\quad = F\big(\max\big\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1})\big\}\big) - \tau.
\end{aligned}$$

Let us assume that there exists an $n \in \mathbb{N}$ such that

$$\max\big\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1})\big\} = d(\varrho_n, \varrho_{n+1}),$$

which implies that

$$F\big(d(\varrho_{n+1}, \varrho_n)\big) \le F\big(d(\varrho_{n+1}, \varrho_n)\big) - \tau,$$

contradicting the assumptions in the definition. Hence,

$$\max\big\{d(\varrho_n, \varrho_{n-1}), d(\varrho_n, \varrho_{n+1})\big\} = d(\varrho_n, \varrho_{n-1}), \quad \text{for all } n \in \mathbb{N},$$

so

$$F\big(d(\varrho_{n+1},\varrho_n)\big) \le F\big(d(\varrho_n,\varrho_{n-1})\big) - \tau, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

From (3.4),

$$F\big(d(\varrho_{n+1},\varrho_n)\big) \le F\big(d(\varrho_1,\varrho_0)\big) - n\tau, \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Letting $n \to \infty$ in the above inequality, we have

$$F\big(d(\varrho_{n+1},\varrho_n)\big) \to -\infty, \tag{3.6}$$

which implies that

$$\lim_{n\to\infty} d(\varrho_{n+1},\varrho_n) = 0 \quad \text{(by property } F2). \tag{3.7}$$

Now, to show that the sequence $\{\varrho_n\}$ is a Cauchy sequence, let us assume there exist $\epsilon > 0$ and two sequences $\{r_n\}$ and $\{s_n\}$, with r_n, s_n both in $\mathbb{N}$, such that

$$r_n > s_n > n, \qquad d(\varrho_{r_n},\varrho_{s_n}) \ge \epsilon, \qquad d(\varrho_{r_{n-1}},\varrho_{s_n}) < \epsilon, \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Thus, $\epsilon \le d(\varrho_{r_n},\varrho_{s_n}) \le d(\varrho_{r_n},\varrho_{r_{n-1}}) + d(\varrho_{r_{n-1}},\varrho_{s_n}) \le d(\varrho_{r_n},\varrho_{r_{n-1}}) + \epsilon = d(\varrho_{r_{n-1}},T\varrho_{r_{n-1}}) + \epsilon$. Considering (3.7) and the preceding inequality, we obtain

$$\lim_{n\to\infty} d(\varrho_{r_n},\varrho_{s_n}) = \epsilon. \tag{3.9}$$

Therefore, from (3.7), there exists $n \in \mathbb{N}$, satisfying

$$d(\varrho_{r_m},f\varrho_{r_m}) < \frac{\epsilon}{3} \quad \text{and} \quad d(\varrho_{s_m},f\varrho_{s_m}) < \frac{\epsilon}{3}, \quad \text{for all } m \ge n. \tag{3.10}$$

Further, we shall prove that

$$d(f\varrho_{r_m},f\varrho_{s_m}) = d(\varrho_{r_{m+1}},\varrho_{s_{m+1}}) > 0, \quad \text{for all } m \ge n. \tag{3.11}$$

For this purpose, suppose there exists $p \ge n$ such that

$$d(\varrho_{r_{p+1}},\varrho_{s_{p+1}}) = 0. \tag{3.12}$$

Using (3.8), (3.10), and (3.14), we have

$$\begin{aligned}
\epsilon &\le d(\varrho_{r_p},\varrho_{s_p}) \\
&\le d(\varrho_{r_p},\varrho_{r_{p+1}}) + d(\varrho_{r_{p+1}},\varrho_{s_p}) \\
&\le d(\varrho_{r_p},\varrho_{r_{p+1}}) + d(\varrho_{r_{p+1}},\varrho_{s_{p+1}}) + d(\varrho_{s_{p+1}},\varrho_{s_p}) \\
&= d(\varrho_{r_p},f\varrho_{r_p}) + d(\varrho_{r_{p+1}},\varrho_{s_{p+1}}) + d(\varrho_{s_p},f\varrho_{s_p}) \\
&< \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.
\end{aligned}$$

This leads to a contradiction, hence (3.11) is true. Therefore, by assumption of the theorem,

$$\tau + F\big(d(f\varrho_{r_m}, f\varrho_{s_m})\big) \leq \alpha F\big(d(\varrho_{r_m}, \varrho_{s_m})\big). \tag{3.13}$$

From (3.9) and (3.13), we get $\tau + F(\epsilon) \leq F(\epsilon)$. That implies that $\{\varrho_n\}$ is a Cauchy O-sequence in E. Then there exists $\varrho^* \in E$ such that $\varrho_n \to \varrho^*$ as $n \to \infty$ because E is O-complete. Then

$$f\big(\varrho^*\big) = \lim_{n\to\infty} f(\varrho_n) = \lim_{n\to\infty} \varrho_{n+1} = \varrho^*$$

because f is $\curlywedge$-continuous. For the uniqueness of ϱ^*, assume that $\varpi^* \in E$ such that $f\varpi^* = \varpi^*$. If $\varrho_n \to \varpi^*$ as $n \to \infty$, then we have $\varrho^* = \varpi^*$. If $\varrho_n \nrightarrow \varpi^*$ as $n \to \infty$, then there exists a subsequence $\{\varrho_{n_k}\}$ of $\{\varrho_n\}$ such that $f\varrho_{n_k} \neq \varpi^*$, for all $k \in \mathbb{N}$. Because of the choice of ϱ_0, specified in the beginning section, we get

$$\big[\varrho_0 \curlywedge \varpi^*\big] \quad \text{or} \quad \big[\varpi^* \curlywedge \varrho_0\big].$$

Because f is $\curlywedge$-preserving and $f^n\varpi^* = \varpi^*$, for all $n \in \mathbb{N}$,

$$\big[f^n\varrho_0 \curlywedge f^n\varpi^*\big] \quad \text{or} \quad \big[f^n\varpi^* \curlywedge f^n\varrho_0\big].$$

Since f is an $F_\curlywedge$-weak contraction and by (3.5), we have

$$\begin{aligned} F\big(d\big(f^{n_k}\varrho_0, \varpi^*\big)\big) &= F\big(d\big(f^{n_k}\varrho_0, f^{n_k}\varpi^*\big)\big) \\ &\leq F\big(d\big(\varrho_0, \varpi^*\big)\big) - n_k\tau, \quad \text{for all } k \in \mathbb{N}, \end{aligned}$$

confirming that $F(d(f^{n_k}\varrho_0, \varpi^*)) \to -\infty$ as $k \to \infty$, and so it follows from (F2) that $d(f^{n_k}\varrho_0, \varpi^*) = 0$ as $k \to \infty$, which is a contradiction. As a result, f has a unique fixed point. □

Example 3.4 Let $E = [0,1] \cap \mathbb{Q}$ be endowed with the usual metric. Define a binary relation $\curlywedge$ on E by $\varrho \curlywedge \varpi$ if $\varrho\varpi = 0$ or ϱ. Clearly, $(E, \curlywedge)$ is an O-set with 0 and 1 as orthogonal elements. Also, E is an O-complete metric space. Define a map

$$f(\varrho) = \begin{cases} \frac{\varrho^2}{4}, & \text{if } \varrho \in [0,1] \cap \mathbb{Q}, \\ 0, & \text{if } \varrho = 1. \end{cases}$$

It is easy to prove that f is $\curlywedge$-continuous and $\curlywedge$-preserving. For $\varrho = 1$ and $\varpi = \frac{3}{4}$, we get $\tau < 0$, which is a contradiction. This implies that f is not an $F_\curlywedge$-contraction. For $\varrho = 1$ and $\varpi \in (0,1) \cap \mathbb{Q}$,

$$\begin{aligned} d(f1, f\varpi) = d\left(0, \frac{\varpi^2}{4}\right) = \frac{\varpi^2}{4} < 1 = d(1, f1) \\ = \max\left\{d(1,\varpi), d(1,f1), d(\varpi, f\varpi), \frac{d(1,f\varpi) + d(\varpi, f1)}{2}\right\}. \end{aligned}$$

Hence, f is an $F_{\curlywedge}$-weak contraction for $F(\xi) = \ln \xi$ with $\tau = \ln 2$. By Theorem 3.1, f has a unique fixed point, namely, $\varrho = 0$.

Example 3.5 Let $E = [0,\infty)$ and $d : E \times E \to [0,\infty)$ be a mapping given by $d(\varrho,\varpi) = |\varrho - \varpi|$ for all $\varrho,\varpi \in E$. Define a sequence $\{W_n\}_{n\in\mathbb{N}}$ as

$$W_n = \frac{n^2(n+1)^2}{4}, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

A relation $\curlywedge$ on the underlying space E is given by $\varrho \curlywedge \varpi \iff \varrho\varpi \in \{\varrho,\varpi\} \subseteq \{W_n\}$. Thus $(E,\curlywedge,d)$ is an O-complete metric space. Now we will define a mapping $f : E \to E$ by

$$f(\varrho) = \begin{cases} W_0, & \text{if } W_0 \le \varrho \le W_1, \\ W_{n-1}, & \text{if } W_n \le \varrho \le W_{n+1}, \text{ for all } n \ge 1. \end{cases}$$

It is straightforward to validate that the mapping f preserves orthogonal continuity. Furthermore, let a function $F \in \Upsilon$ be defined as $F(\alpha) = \alpha + \log(\alpha)$ for all $\alpha > 0$. We argue that f is an $F_{\curlywedge}$-weak contraction with $\tau = 1$. Indeed, let $\varrho,\varpi \in E$ with $\varrho \curlywedge \varpi$ and $d(f\varrho, f\varpi) > 0$. So, we may suppose that $\varrho < \varpi$. Then $\varrho \in \{W_0, W_1\}$ and $\varpi = W_i$ for some $i \in \mathbb{N} \setminus \{1\}$. So, we get

$$\frac{d(f\varrho,f\varpi)}{M(\varrho,\varpi)} e^{[d(f\varrho,f\varpi)-M(\varrho,\varpi)]} = \frac{W_{i-1}-1}{G_i - 1} e^{[W_{i-1}-1-G_i+1]} < e^{-1}.$$

Hence, all the conditions of Theorem 3.1 are satisfied and so f has a unique fixed point $x = W_0$ (see Fig. 1).

Remark 3 By replacing hypothesis $(F3)$ with the following condition, Theorem 3.1 holds:
$(F3)'$ if $\{\varrho_n\}$ is a sequence in E such that $\varrho_n \to \varrho^* \in E$ and $\varrho_n \curlywedge \varrho_{n+1}$ or $\varrho_{n+1} \curlywedge \varrho_n$ for all $n \in \mathbb{N}$, then $\varrho_n \curlywedge \varrho$ or $\varrho \curlywedge \varrho_n$ for all $n \in \mathbb{N}$.

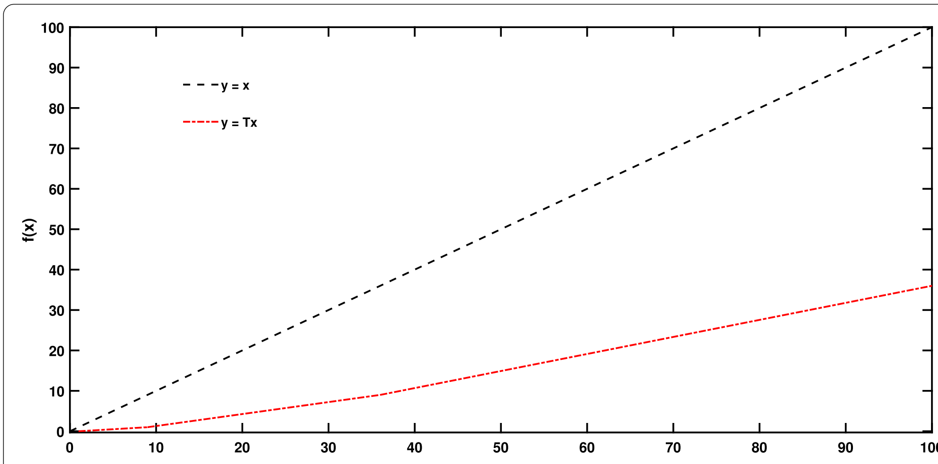

Figure 1 Graphs of $y = x$ and $y = fx$.

Proof By Theorem 3.1, we have previously observed that there exists a point $\varrho^* \in E$ such that $\varrho_n \to \varrho^*$ as $n \to \infty$. Putting $\wp = \{n \in \mathbb{N} | f\varrho_n = f\varrho^*\}$, we consider the following two situations:

Case 1: $\wp$ is not finite. Then there is a subsequence $\{\varrho_{n_k}\}$ of $\{\varrho_n\}$ such that $\varrho_{n_k+1} = f\varrho_{n_k} = f\varrho^*$, for all $n \in \mathbb{N}$. Since $\varrho_n \to \varrho^*$, we get $f\varrho^* = \varrho^*$.

Case 2: $\wp$ is finite. Then there is $n_0 \in \mathbb{N}$ such that $f\varrho_n \neq f\varrho^*$ for all $n \geq n_0$. Particularly, $\varrho_n \neq \varrho^*$, $d(\varrho_n, \varrho^*) > 0$, and $d(f\varrho_n, f\varrho^*) > 0$ for all $n \in \mathbb{N}$, so we have

$$\begin{aligned}
&\tau + F\big(d\big(f\varrho_n, f\varrho^*\big)\big) \\
&\quad \leq F\left(\max\left\{d\big(\varrho_n, \varrho^*\big), d\big(\varrho^*, f\varrho^*\big), d(\varrho_n, f\varrho_n), \frac{d(\varrho_n, f\varrho^*) + d(\varrho^*, f\varrho_n)}{2}\right\}\right) \\
&\quad \leq F\left(\max\left\{d\big(\varrho_n, \varrho^*\big), d\big(\varrho^*, f\varrho^*\big), d(\varrho_n, \varrho_{n+1}), \frac{d(\varrho_n, \varrho^*) + d(\varrho^*, f\varrho^*) + d(\varrho^*, \varrho_{n+1})}{2}\right\}\right).
\end{aligned}$$

If $d(\varrho^*, f\varrho^*) > 0$, then

$$\lim_{n\to\infty} d\big(\varrho_n, \varrho^*\big) = \lim_{n\to\infty} d\big(\varrho^*, \varrho_{n+1}\big) = 0,$$

and there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, we have

$$\begin{aligned}
&\max\left\{d\big(\varrho_n, \varrho^*\big), d\big(\varrho^*, f\varrho^*\big), d(\varrho_n, \varrho_{n+1}), \frac{d(\varrho_n, \varrho^*) + d(\varrho^*, f\varrho^*) + d(\varrho^*, \varrho_{n+1})}{2}\right\} \\
&\quad = d\big(\varrho^*, f\varrho^*\big).
\end{aligned}$$

Then,

$$\tau + F\big(d\big(f\varrho_n, f\varrho^*\big)\big) \leq F\big(d\big(\varrho^*, f\varrho^*\big)\big) \quad \Rightarrow \quad \tau + F\big(d\big(\varrho_{n+1}, f\varrho^*\big)\big) \leq F\big(d\big(\varrho^*, f\varrho^*\big)\big),$$

for all $n \geq \max\{n_0, n_1\}$. Because of the continuity of F and as $n \to \infty$ in the above inequality, we obtain

$$\tau + F\big(d\big(\varrho^*, f\varrho^*\big)\big) \leq F\big(d\big(\varrho^*, f\varrho^*\big)\big),$$

a contradiction. Therefore, $d(\varrho^*, f\varrho^*) = 0$, i.e., $f\varrho^* = \varrho^*$, and hence ϱ^* is a fixed point. In both situations mentioned above, the function f has a fixed point ϱ^*. The uniqueness of this fixed point follows along similar lines as in the above theorem. □

Remark 4 Take into account f from Example 3.3. Thus it follows naturally that we may check that f has a fixed point $\frac{1}{2}$ and meets all the requirements of Theorem 3.1. However, because f does not represent an orthogonal F-contraction, the requirements of the main result of [16] are not met. Hence, the primary Theorem 3.1 is an extension of the main result of [16].

Corollary 3.1 *Let $(E, \curlywedge, d)$ be an O-complete metric space and suppose $f : E \to E$ satisfies*

$$d(f\varrho, f\varpi) > 0$$

$$\begin{aligned}\implies\quad & \tau + F\big(d(f\varrho,f\varpi)\big)\\ & \leq F\big(\alpha d(\varrho,\varpi)+\beta d(\varrho,f\varrho)+\gamma d(\varpi,f\varpi)+\delta\big[d(\varrho,f\varpi)+d(\varpi,f\varrho)\big]\big),\end{aligned}\tag{3.14}$$

for all $\varrho,\varpi \in E$ with $\varrho \curlywedge \varpi$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma+2\delta < 1$. If F or f is $\curlywedge$-continuous then

(i) *$f\varrho^* = \varrho^*$, ϱ^* is unique;*
(ii) *for all $\varrho \in E$, the sequence $\{f^n\} \to \varrho^*$.*

Proof For all $\varrho,\varpi \in E$, we have

$$\begin{aligned}&\alpha d(\varrho,\varpi)+\beta d(\varrho,f\varrho)+\gamma d(\varpi,f\varpi)+\delta\big[d(\varrho,f\varpi)+d(\varpi,f\varrho)\big]\\ &\quad\leq(\alpha+\beta+\gamma+2\delta)\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\\ &\quad\leq\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varrho,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}.\end{aligned}$$

From (F1), observe that the contractivity condition (3.14) implies the contractivity condition (3.3). Hence the corollary is validated. □

Remark 5 Since the contractivity condition (3.2) implies (3.14) and f is O-continuous, as a result of Remark 1, we find that Corollary 3.1 generalized the main result of [16].

Remark 6 Taking into account the various forms of an $F_\curlywedge$-weak contraction, there is a wide range of known contractions in the literature. For illustration, consider the following:

(1) for all $\varrho,\varpi \in E$ with $\varrho \curlywedge \varpi$ and $\alpha,\beta,\gamma > 0$, $\alpha+\beta+\gamma < 1$, we have that

$$d(f\varrho,f\varpi)\leq\alpha d(\varrho,\varpi)+\beta d(\varrho,f\varrho)+\gamma d(\varpi,f\varpi)$$

implies

$$d(f\varrho,f\varpi)\leq(\alpha+\beta+\gamma)\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}.$$

If $d(f\varrho,f\varpi) > 0$, we get

$$\tau+\ln\big(d(f\varrho,f\varpi)\big)\leq\ln\left(\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\right),$$

where $\tau = \ln\frac{1}{\alpha+\beta+\gamma} > 0$. If we use $F(\varrho) = \ln\varrho$, for all $\varrho > 0$, then Theorem 3.1 is an extension of the primary finding of [6] in the setting of an orthogonal metric space.

(2) For all $\varrho,\varpi \in E$ with $\varrho \curlywedge \varpi$ and $k \in [0,1)$, we have that

$$d(f\varrho,f\varpi)\leq ad(\varrho,f\varpi)+bd(\varpi,f\varpi)$$

implies

$$d(f\varrho,f\varpi)\leq k\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}.$$

If $d(f\varrho,f\varpi)>0$, we get

$$\tau+\ln\bigl(d(f\varrho,f\varpi)\bigr)\leq\ln\left(\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\right),$$

where $\tau=\ln\frac{1}{k}>0$. If we use $F(\varrho)=\ln\varrho$, for all $\varrho>0$, then Theorem 3.1 is an extension of the primary finding of [20] in the setting of an orthogonal metric space.

(3) For all $\varrho,\varpi\in E$ with $\varrho\perp\varpi$ and nonnegative numbers $q(\varrho,\varpi)$, $r(\varrho,\varpi)$, $s(\varrho,\varpi)$, and $t(\varrho,\varpi)$, with

$$\sup_{\varrho,\varpi\in E}\bigl\{q(\varrho,\varpi)+r(\varrho,\varpi)+s(\varrho,\varpi)+2t(\varrho,\varpi)\bigr\}=\lambda<1$$

and $d(f\varrho,f\varpi)>0$, we have that

$$\begin{aligned}d(f\varrho,f\varpi)\leq{}& q(\varrho,\varpi)d(\varrho,\varpi)+r(\varrho,\varpi)d(\varrho,f\varrho)+s(\varrho,\varpi)d(\varpi,f\varpi)\\&+t(\varrho,\varpi)\bigl[d(\varpi,f\varrho)+d(\varrho,f\varpi)\bigr]\end{aligned}$$

implies

$$d(f\varrho,f\varpi)\leq\lambda\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}.$$

If $d(f\varrho,f\varpi)>0$, we get

$$\ln\frac{1}{\lambda}+\ln d(f\varrho,f\varpi)\leq\left(\max\left\{d(\varpi,f\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\right),$$

where $\tau=\ln\frac{1}{\lambda}$. If we use $F(\varrho)=\ln\varrho$, for all $\varrho>0$, then Theorem 3.1 is an extension of the primary finding of [3] in the setting of an orthogonal metric space.

(4) For all $\varrho,\varpi\in E$ with $\varrho\perp\varpi$ and nonnegative numbers α, β, γ, δ, ϵ with $\alpha+\beta+\gamma+\delta+\epsilon<1$, we have that

$$d(f\varrho,f\varpi)\leq\frac{\alpha+\beta}{2}\bigl[d(\varrho,f\varrho)+d(\varpi,f\varpi)y\bigr]+\frac{\gamma+\delta}{2}\bigl[d(\varrho,f\varpi)+d(\varpi,f\varrho)\bigr]+\epsilon d(\varrho,\varpi)$$

implies

$$d(f\varrho,f\varpi)\leq(\alpha+\beta+\gamma+\delta+\epsilon)\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}.$$

If $d(f\varrho,f\varpi)>0$, we get

$$\begin{aligned}&\ln\frac{1}{\alpha+\beta+\gamma+\delta+\epsilon}+\ln d(f\varrho,f\varpi)\\&\quad<\ln\left(\max\left\{d(\varrho,\varpi),d(\varrho,f\varrho),d(\varpi,f\varpi),\frac{d(\varrho,f\varpi)+d(\varpi,f\varrho)}{2}\right\}\right),\end{aligned}$$

where $\tau=\ln\frac{1}{\alpha+\beta+\gamma+\delta+\epsilon}>0$. If we use $F(\varrho)=\ln\varrho$, for all $\varrho>0$, then Theorem 3.1 is an extension of the primary finding of [21] in the setting of an orthogonal metric space.

4 Application to differential equations

Recall that, for any $1 \leq q < \infty$, the space $L^q(E,F,\mu)$ (or $L^q(E)$) consists of complex-valued measurable functions β on the underlying space E obeying

$$\int_E \left|\beta(x)\right|^q d\mu(\varrho) < \infty,$$

where μ is the measure and F is a σ-algebra of measurable sets. When $q = 1$, the space $L^1(E)$ is the set of all integrable functions β on the underlying space E and we define the L^1-norm of β by

$$\|\beta\|_1 = \int_E \left|\beta(x)\right| d\mu(\varrho).$$

Theorem 4.1 *Suppose that*

$$\begin{cases} \nu'(t) = f(t,\nu(t)), & t \in I := [0,T], \\ \nu(0) = c, & c \geq 1, \end{cases} \tag{4.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(d1) $f(\xi,q) \geq 0$, *for all* $q \geq 0$ *and* $\xi \in I$;

(d2) *for each* $\varrho,\varpi \in L^1(I)$ *with* $\varrho(\xi)\varpi(\xi) \geq \varrho(\xi)$ *or* $\varrho(\xi)\varpi(\xi) \geq \varpi(\xi)$ *for all* $\xi \in I$, *there exist* $\beta \in L^1(I)$ *and* $\tau > 0$ *such that*

$$\left|f\left(\xi,\varrho(\xi)\right) - f\left(\xi,\varpi(\xi)\right)\right| \leq \frac{\beta(\xi)}{(1+\tau\sqrt{\beta(\xi)})^2}\left|\varrho(\xi) - \varpi(\xi)\right| \tag{4.2}$$

and

$$\left|\varpi(\xi) - \varpi(\xi)\right| \leq \beta(\xi)e^{A(\xi)}$$

for all $\xi \in I$, *where* $A(\xi) := \int_0^\xi |\beta(w)|\,dw$.

Then the differential equation (4.1) *has a unique solution.*

Proof Let $E = \{\varrho \in C(I,\mathbb{R}) : \varrho(t) > 0 \text{ for all } t \in I\}$. Let the orthogonality relation $\curlywedge$ be defined on X as

$$\varrho \curlywedge \varpi \quad \Longleftrightarrow \quad \varrho(t)\varpi(t) \geq \varrho(t) \quad \text{or} \quad \varrho(t)\varpi(t) \geq \varpi(t), \quad \text{for all } t \in I.$$

Since $A(t) = \int_0^t |\beta(s)|\,ds$, we have $A'(t) = |\beta(t)|$ for almost every $t \in I$.

Define a mapping $d : E \times E \to [0,\infty)$ by

$$d(\varrho,\varpi) = \|\varrho - \varpi\|_A = \sup_{t\in I} e^{-A(t)}\left|\varrho(t) - \varpi(t)\right|$$

for all $\varrho,\varpi \in E$. Thus, (E,d) is an O-complete metric space (see [18] for details).

Define a mapping $G : E \to E$ by

$$(G\varrho)(t) = c + \int_0^t f\left(\xi,\varrho(\xi)\right) d\xi.$$

Then, G is $\curlywedge$-continuous.

Now, for each $\varrho, \varpi \in E$ with $\varrho \curlywedge \varpi$ and $t \in I$, we have

$$(G\varrho)(t) = c + \int_0^t f\big(\xi, \varrho(\xi)\big)\, d\xi \geq 1.$$

It follows that $[(G\varrho)(t)][(G\varpi)(t)] \geq (G\varpi)(t)$ and $(G\varrho)(t) \curlywedge (G\varpi)(t)$. Then, G is $\curlywedge$-preserving.

Let a function $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(\varrho) = -\frac{1}{\sqrt{\varrho}}$ for all $\varrho > 0$. It follows that G is an $F_\curlywedge$-weak contraction. Hence the differential equation (4.1) has a unique solution because, by Theorem 3.1, G has a unique fixed point. □

Remark 7 Note that the orthogonality is a necessary condition for the differential equation (4.1) to have a unique solution. Indeed, under assumption (d2) of Theorem 4.1, it is not possible to find the solution of equation (4.1) without orthogonality.

5 Application to integral equations

Let $E = (C[c,d], \mathbb{R})$ be the set of all continuous functions defined on $I = [c,d]$. It is well-accepted that E is equipped with the metric given by $d = \sup_{\varrho \in I} e^{-\tau}|u(\varrho) - v(\varrho)|$. If for all $u, v \in E$ we set $u \curlywedge v$ if $u(\varrho) \leq v(\varrho)$, for all $\varrho \in I$, then the space becomes an O-complete metric space.

Theorem 5.1 *The integral equation*

$$u(\varrho) = \int_c^d H\big(\varrho, \varpi, u(\varpi)\big)\, d\varpi + g(\varrho), \quad \forall \varrho \in [c,d], \tag{5.1}$$

where $t \in I$, $H : I \times I \times \mathbb{R} \to \mathbb{R}$, and $d > c \geq 0$, has unique solution if the following conditions are satisfied:

(i) *$H : [c,d] \times [c,d] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : [c,d] \to \mathbb{R}$;*

(ii) *$H(\varrho, \varpi, \cdot) > 0$ and $\int_c^d H(\varrho, \varpi, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing for all $\varrho, \varpi \in I$;*

(iii) *for all $u, v \in E, \varrho, \varpi \in I$,*

$$\big|H\big(\varrho, \varpi, u(\varpi)\big) - H\big(\varrho, \varpi, v(\varpi)\big)\big| \leq e^{-\varrho-\tau}\big|u(\varrho) - v(\varrho)\big|.$$

Proof Set

$$(fu)(t) = \int_c^d H\big(\varrho, \varpi, u(\varpi)\big)\, d\varpi + g(\varrho), \quad u \in E, \varrho \in [c,d].$$

As $H(\varrho, \varpi,) > 0$, for all $\varrho, \varpi \in [c,d]$, we have

$$\begin{aligned}(fu)(\varrho) &= \int_c^d H\big(\varrho, \varpi, u(\varpi)\big)\, d\varpi + g(\varrho) \\ &\leq \int_c^d H\big(\varrho, \varpi, v(\varpi)\big)\, d\varpi + g(\varrho) \\ &= (fv)(\varpi).\end{aligned}$$

So, f preserves orthogonality. Consider, a Cauchy O-sequence $\{\mu_n\}$ tending to $\mu \in E$. Then

$$\mu_0(\varrho) \leq \mu_1(\varrho) \leq \mu_2(\varrho) \leq \mu_3(\varrho) \leq \cdots \mu_n(\varrho) \leq \cdots \leq \mu(\varrho), \quad \text{for all } \varrho \in I,$$

this implies that $\mu_n \curlywedge \mu$, for all $\varrho \in I$. As f preserves orthogonality, this implies that $f(\mu_n) \to f(\mu)$. Consequently, f is $\curlywedge$-continuous.

Define a function $F : \mathbb{R}^+ \to \mathbb{R}$ as $F(\varrho) = \log \varrho$, for all $\varrho > 0$. Then the conclusion is that f is an $F_\curlywedge$-weak contraction. Hence f has a unique fixed point (by Theorem 3.1), and this implies that the integral equation (5.1) has a unique solution. □

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