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On the optimal controllability for a class of Katugampola fractional systems

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Abstract

This study is centered on the optimal controllability of differential equations involving fractional derivatives of Katugampola. We derive both necessary and sufficient conditions for optimal controllability by extending Gronwall's inequality with singular kernels. Furthermore, we establish conditions ensuring the existence and uniqueness of mild solutions using the Banach fixed-point theorem and the generalized Laplace transform. To underscore the practical relevance of our findings, we provide an illustrative example.

Mathematics Subject Classification: 26A33; 34A12; 34A08

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1 Introduction

This study discusses the optimal controllability of the following Katugampola-type fractional systems:

$$\begin{cases} {}^{\rho}D^{\alpha}_{a^+}x(t) = Ax(t) + f(t,x(t)) + Bu(t), \quad t \in J' = (a,b] \\ {}^{\rho}I^{1-\alpha}_{a^+}x(a) = D. \end{cases} \tag{1.1}$$

${}^{\rho}D^{\alpha}_{a^+}$ represents the Katugampola (K) fractional derivative of order $\alpha (0 < \alpha < 1)$. ${}^{\rho}I^{1-\alpha}_{a^+}$ is the Katugampola integral of order $1-\alpha$, where $\rho > 0$. $A, B, D \in \mathbb{R}^{n*n}, f(t,x(t)), u(t) \in \mathbb{R}^{n*n}$ are given continuous functions, $u_0(t)$ represents the initial control function.

In recent decades, fractional differential equations have become a focal point of considerable attention due to their efficacy in unraveling the memory and hereditary characteristics present in diverse materials and processes across the realms of physics, mechanics, chemistry, and engineering. Notable monographs by Miller and Ross [1], Podlubny [2], and Kilbas et al. [3], along with their extensive references, serve as valuable resources for delving into the intricacies of fractional calculus theory and expanding our comprehension of this indispensable tool.

Numerous scholars have conducted in-depth investigations into various fractional types, such as the Caputo type, Hadamard type, Riemann–Liouville type, Hilfer type, and others. In 2011, U.N. Katugampola made a substantial contribution by generalizing the Riemann–Liouville fractional operator and the Hadamard-type fractional operator into what is now referred to as the K-type operator. This seminal development was accompanied by a thorough examination of the semigroup characteristics and Merlin transformation associated with K-type fractional calculus. Additionally, he explored the existence and uniqueness of solutions to equations within the framework of K-type fractional calculus [4, 5].

Several other experts have delved into the study of the properties of K-type fractional operators. Utilizing diverse mathematical tools, including fixed point theorems and other theoretical approaches, they have investigated the qualitative theory of K-type fractional differential and integral equations. Noteworthy contributions in this field have been made by Anderson et al. [6], Lupinska et al. [7], Zeng et al. [8], Oliveira [9], Harikrishnan et al. [10, 11], Gou et al. [12], and the papers referenced therein. These collective works substantiate the existence and uniqueness of solutions for the K-type fractional system. Furthermore, stability results are established, leading to broader conclusions that enhance the comprehension of K-type fractional calculus and its diverse applications.

Controllability analysis holds a crucial role in control system design, and in recent years, there has been a growing emphasis on evaluating the controllability of various fractional-order systems. For instance, Ding et al. provided both sufficient and necessary conditions for the optimality of fractional control systems [13]. Mophou applied classical control theory to a fractional diffusion equation within a bounded domain, utilizing the Laplace operator [14–17]. Furthermore, Bahaa extended the findings of [14–17] to address constant variable fractional optimal control problems [18–23]. Notably, Bahaa [24] applied the generalization of the Dubovitskii–Milyutin theorem to interpret the Euler–Lagrange first-order optimality condition, yielding optimal control results for fractional differential systems with the Atangana–Baleanu derivative. Bose et al. [25–27] analyzed the approximate controllability of Hilfer fractional neutral differential equations.

Rohit et al. [28] explored the existence of optimal control for semilinear control systems of fractional order $(1,2]$ within a Hilbert space. It is well-acknowledged that demonstrating the nonsingularity of the Gramian matrix is a central challenge in solving optimal control problems. The works mentioned above, along with the references therein, significantly contribute to unraveling key insights in this field.

However, the optimal controllable result of the evolution equation with the K-type operator is unclear, which prompted us to investigate system (1.1).

Section 2 offers preliminary insights, and in Sect. 3, we present sufficient conditions for the existence and uniqueness of solutions of system (1.1). Moving to Sect. 4, we derive both necessary and sufficient conditions for optimal controllability. This section also outlines the assumptions crucial to the optimal control results. Lastly, Sect. 6 includes an illustrative example designed to showcase the validity of our hypothesis.

2 Preliminaries

This section shows some basic well-known definitions. Allow Ω to be a Banach space and the norm is $|\cdot|_{\Omega}$. $C(J,\Omega)$, as usual, represents the Banach space of continuous functions derived from $J=[a,b]$ to Ω, the norm $\|z\|_{C(J,\Omega)}=\sup_{t\in J}\|z(t)\|_{\Omega}$.

Throughout the paper, we denote $g(t) = \frac{t^\rho}{\rho}$ and $\mathcal{G}(t,a) = g(t) - g(a)$. The weighted spaces of continuous functions are denoted by

$$C_{\mu,\rho}[J,\Omega] = \{z : (a,b] \to \mathbb{R} : \mathcal{G}(t,a)^\mu z(t) \in C[J,\Omega]\}, \quad 0 \le \mu \le 1,$$

and the norm

$$\|z\|_{C_{\mu,\rho}} = \|\mathcal{G}(t,a)^\mu z\|_C = \sup_{t\in J}|\mathcal{G}(t,a)^\mu z(t)|.$$

Evidently, $C_{\mu,\rho}[J,\Omega]$ is a Banach space.

We have compiled a list of definitions from Katugampola fractional calculus [4, 5].

Definition 2.1 Let $0 < a < t < \infty$ for the function $z : (a,+\infty) \to R$, the K-type fractional integral of order α is

$$({}^\rho I^\alpha_{a^+} z)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a^+}^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} z(s)\,ds, \quad \alpha > 0, \rho > 0.$$

Definition 2.2 Let $\alpha > 0, \rho > 0, n = [\alpha] + 1$, the function $z : (a,\infty) \to R$, then the K-type fractional derivative can be defined by

$$\begin{aligned}({}^\rho D^\alpha_{a^+} z)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n ({}^\rho I^{n-\alpha}_{a^+} z)(t)\\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_{a^+}^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} z(s)\,ds.\end{aligned}$$

Specially, if $0 < \alpha < 1$, then $({}^\rho D^\alpha_{a^+} z)(t) = \frac{\rho^{\alpha+1}}{\Gamma(-\alpha)} \int_{a^+}^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha+1}} z(s)\,ds$.

Lemma 2.3 *Assume that* $\alpha, \beta > 0, \rho \ge c$, $0 < a < b$, *and* $p \ge 1$ *are the finite real numbers, and* $\rho, c \in R$. *Then, for* $z \in X^p_c(a,b)$, *the following hold*:

$$({}^\rho I^\alpha_{a^+}\, {}^\rho I^\beta_{a^+} z)(t) = ({}^\rho I^{\alpha+\beta}_{a^+} z)(t)$$

and

$$({}^\rho D^\alpha_{a^+}\, {}^\rho I^\alpha_{a^+} z)(t) = z(t).$$

Lemma 2.4 *Let* $0 < \alpha < 1, 0 \le \gamma \le 1$, *if* $f \in C_\gamma$ *and* ${}^\rho I^{1-\alpha}_{a^+} f \in C^1_\gamma[a,b]$, *then*

$$({}^\rho I^\alpha_{a^+}\, {}^\rho D^\alpha_{a^+} f)(t) = f(t) - \frac{({}^\rho I^{1-\alpha}_{a^+} f)(a)}{\Gamma(\alpha)} (g(t) - g(a))^{\alpha-1}.$$

Definition 2.5 [2] The two-parameter Mittag-Leffler function can be defined as

$$\mathcal{E}_{\alpha,\beta}(x) = \sum_{i=0}^\infty \frac{x^i}{\Gamma(\alpha i + \beta)}.$$

Specially, $\beta = 1$, the series for the one-parameter Mittag-Leffler function is as follows:

$$\mathcal{E}_{\alpha}(x) = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(\alpha i + 1)}.$$

Lemma 2.6 [2] *Let* $\alpha \in (0,1]$, $\beta > \alpha$, *and* $z \geq 0$, *then the following inequalities hold*:

$$\frac{1}{\Gamma(\beta) + \Gamma(\beta - \alpha)z} \leq \mathcal{E}_{\alpha,\beta}(-z) \leq \frac{1}{\Gamma(\beta) + \frac{\Gamma(\beta)\Gamma(\beta)}{\Gamma(\beta+\alpha)}z}.$$

Lemma 2.7 [2] *Assume that* $0 < \alpha < 2$, $\beta \in R$, μ *fulfill the inequality* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \alpha\pi\}$, *then there exist* $H > 0, M > 0$, *s.t.*

$$\left|\mathcal{E}_{\alpha,\beta}(z)\right| \leq \frac{H}{1 + |z|} = M, \quad z \in \mathbb{C}, \mu \leq \left|\arg(z)\right| \leq \pi.$$

Lemma 2.8 [12] *Let* $\alpha > 0$, $y(t)$ *and* $a(t)$ *be nonnegative functions, and nondecreasing function* $b(t)$ *be a nonnegative and nondecreasing function for* $t \in [t_0, T]$, $T > 0$, $b(t) \leq K$, *where* K *is a constant. If*

$$y(t) \leq a(t) + b(t)\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}y(s)\,ds, \quad t \in [t_0, T],$$

then

$$y(t) \leq a(t)\mathcal{E}_{\alpha}\big(b(t)\Gamma(\alpha)\mathcal{G}(t,s)^{\alpha}\big),$$

where $g(t) = \frac{t^\rho}{\rho}$ *and* $\mathcal{G}(t,a) = g(t) - g(a)$.

Lemma 2.9 [29] *Assume* $0 < \alpha \leq 1$ *and* $0 < a \leq t$, $W(t) : [-a, +\infty) \to R^+$ *is bounded on* $[-a, 0]$ *and continuous. The following generalized Laplace transform holds*:

$$\mathscr{L}\big({}^{c}D_{a^+}^{\alpha,\rho}W(t)\big) = s^{\alpha}\mathscr{L}\big(W(t)\big) - s^{\alpha-1}W(0)$$

and

$$\mathscr{L}\left(\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\beta-1}\mathcal{E}_{\alpha,\beta}\left(-\lambda\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\right)\right) = \frac{s^{\alpha-\beta}}{s^{\alpha} + \lambda}.$$

The following results are based on Sect. 6 in [9].

Theorem 2.10 *The function* $x \in C_{1,\rho}[J, \Omega]$ *is called a solution of* (1.1), *suppose* ${}^{\rho}I_{a^+}^{1-\alpha}x(a) = D$ *such that*

$$x(t) = \frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax(s) + f\big(s, x(s)\big) + Bu(s)\big]\,ds. \tag{2.1}$$

One can rewrite the solution in terms of the Mittag-Leffler function

$$x(t) = D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_{\alpha}\big[A\mathcal{G}(t,a)^{\alpha}\big] \tag{2.2}$$

$$+\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]\big(f\big(s,x(s)\big)+Bu(s)\big)\,ds,$$

where $g(t)=\frac{t^\rho}{\rho}$ *and* $\mathcal{G}(t,a)=g(t)-g(a)$.

Proof Performing ${}^\rho I^\alpha_{a^+}$ to both sides of (1.1) and using Definition 2.2, one can get the following integral equation:

$$x(t)=\frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax(s)+f\big(s,x(s)\big)+Bu(s)\big]\,ds.$$

The sequence can be derived through the method of successive approximations, yielding the following outcome:

$$\begin{aligned}
x_0(t)&=\frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1},\\
x_k(t)&=x_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax_{k-1}(s)+f\big(s,x_{k-1}(s)\big)+Bu_{k-1}(s)\big]\,ds,
\end{aligned}$$

$k\in N, k\geq 1$, then

$$\begin{aligned}
x_1(t)&=x_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax_0(s)+f\big(s,x_0(s)\big)+Bu_0(s)\big]\,ds\\
&=x_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Ax_0(s)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}f\big(s,x_0(s)\big)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Bu_0(s)\,ds\\
&=\frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1}+\frac{A}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1}\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}f\big(s,x_0(s)\big)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Bu_0(s)\,ds\\
&=\frac{D}{\Gamma(\alpha)}\mathcal{G}(t,a)^{\alpha-1}+\frac{DA}{\Gamma(2\alpha)}\mathcal{G}(t,a)^{2\alpha-1}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}f\big(s,x_0(s)\big)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Bu_0(s)\,ds\\
&=D\sum_{i=1}^{2}\frac{A^{i-1}}{\Gamma(\alpha i)}\mathcal{G}(t,a)^{\alpha i-1}+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}f\big(s,x_0(s)\big)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Bu_0(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= D\sum_{i=1}^{2}\frac{A^{i-1}}{\Gamma(\alpha i)}\mathcal{G}(t,a)^{\alpha i-1} + {}^{\rho}I^{\alpha}_{a^+}f(t,x_0) + {}^{\rho}I^{\alpha}_{a^+}(Bu_0).\\
x_2(t) &= x_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax_1(s) + f\big(s,x_1(s)\big) + Bu_1(s)\big]\,ds\\
&= x_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Ax_1(s)\,ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}f\big(s,x_1(s)\big)\,ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}Bu_1(s))\,ds\\
&= D\sum_{i=1}^{3}\frac{A^{i-1}}{\Gamma(\alpha i)}\mathcal{G}(t,a)^{\alpha i-1} + \int_a^t\sum_{i=1}^{2}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}f\big(s,x_{i-1}(s)\big)\,ds\\
&\quad + \int_a^t\sum_{i=1}^{2}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}Bu_{i-1}(s))\,ds.
\end{aligned}
$$

Continuing with this analytical approach, one can derive

$$
\begin{aligned}
x_k(t) &= x_0(t) + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\big[Ax_{k-1}(s) + f\big(s,x_{k-1}(s)\big) + Bu_{k-1}(s)\big]\,ds\\
&= D\sum_{i=1}^{k+1}\frac{A^{i-1}}{\Gamma(\alpha i)}\mathcal{G}(t,a)^{\alpha i-1} + \int_a^t\sum_{i=1}^{k}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}f\big(s,x_{i-1}(s)\big)\,ds\\
&\quad + \int_a^t\sum_{i=1}^{k}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}Bu_{i-1}(s))\,ds,
\end{aligned}
$$

and $k\to\infty$, we have

$$
\begin{aligned}
x(t) &= D\sum_{i=1}^{\infty}\frac{A^{i-1}}{\Gamma(\alpha i)}\mathcal{G}(t,a)^{\alpha i-1} + \int_a^t\sum_{i=1}^{\infty}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}f\big(s,x_{i-1}(s)\big)\,ds\\
&\quad + \int_a^t\sum_{i=1}^{\infty}\frac{A^{i-1}}{\Gamma(\alpha i)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i-1}Bu_{i-1}(s))\,ds.
\end{aligned}
$$

In view of $i = 0$, we show that

$$
\begin{aligned}
x(t) &= D\sum_{i=0}^{\infty}\frac{A^{i}}{\Gamma(\alpha i+1)}\mathcal{G}(t,a)^{\alpha i} + \int_a^t\sum_{i=0}^{\infty}\frac{A^{i}}{\Gamma(\alpha i+1)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i}f\big(s,x_{i}(s)\big)\,ds\\
&\quad + \int_a^t\sum_{i=0}^{\infty}\frac{A^{i}}{\Gamma(\alpha i+1)}s^{\rho-1}\mathcal{G}(t,s)^{\alpha i}Bu_{i}(s))\,ds.\\
x(t) &= D\mathcal{G}(t,a)^{\alpha-1}\sum_{i=0}^{\infty}\frac{A^{i}}{\Gamma(\alpha i+1)}\mathcal{G}(t,a)^{\alpha i}\\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\sum_{i=0}^{\infty}\frac{A^{i}}{\Gamma(\alpha i+1)}\mathcal{G}(t,s)^{\alpha i}f\big(s,x_0(s)\big)\,ds.
\end{aligned}
$$

The Mittag-Leffler function can be used to express the solution as follows:

$$x(t) = D\mathcal{G}(t,a)^{\alpha-1}\mathscr{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathscr{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]f\big(s,x(s)\big)\,ds. \tag{2.3}$$

In the subsequent steps, we will demonstrate that the solution to (1.1) can be represented using (2.3).

Leveraging the generalized Laplace transform as outlined in Sect. 3.3 of [29] and Sect. 5 of [30], we proceed by applying the transform to both sides of (1.1), resulting in the following expressions:

$$\mathscr{L}\big({}^cD_{a^+}^{\alpha,\rho}x(t)\big) = s^\alpha\mathscr{L}\big(x(t)\big) - D = A\mathscr{L}\big(x(t)\big) + \mathscr{L}\big(f\big(t,x(t)\big)\big)$$

and

$$\mathscr{L}\big(x(t)\big) = \frac{D}{s^\alpha - A} + \frac{1}{s^\alpha - A}\mathscr{L}\big(f\big(t,x(t)\big)\big).$$

In light of Lemma 2.9, one can get

$$x(t) = D\mathcal{G}(t,a)^{\alpha-1}\mathscr{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathscr{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]f\big(s,x(s)\big)\,ds. \tag{2.4}$$

The proof is complete. □

Following that, the well-known Banach fixed point theorem is then reviewed (see [31]).

Theorem 2.11 (*Banach fixed point theorem*). *Assume that* $(X,\mathcal{D})$ *denotes a full metric space and* $f : X \to X$ *for* $0 \le k < 1$ *and all* $x,x' \in X$ *such that* $\mathcal{D}(f(x),f(x')) \le k\mathcal{D}(x,x')$. *Then* f *has a unique fixed point in* X.

3 Qualitative analysis

To investigates the qualitative analysis for (1.1), we suppose that:

$H(1)$: $f : J \times X \to X$ is a function that satisfies:

(i) For any $x \in X$, the function $t \mapsto f(t,x)$ satisfies the measurable condition;

(ii) For any $t \in J$, the function $x \mapsto f(t,x)$ satisfies the Lipschitz condition. It is claimed that there are $x_1,x_2 \in X$ and a constant $L_f < \frac{\rho\alpha}{M(b^\rho - a^\rho)^\alpha}$, $\|f(t,x_1) - f(t,x_2)\|_{C_{1,\rho}} \le L_f\|x_1 - x_2\|_{C_{1,\rho}}$;

(iii) For all $t \in J$ and $x \in C_{1,\rho}[J,\Omega]$, there exist a function $\psi(t) \in C_{1,\rho}[J,I]$ and a constant $\theta > 0$ such that

$$\big\|f(t,x)\big\|_{C_{1,\rho}} = \sup\big\{\|f\|_{C_{1,\rho}}\big\} \le \psi(t) + \theta\|x\|_C.$$

Theorem 3.1 *Given the fulfillment of condition* $H(1)$, *the fractional equation* (1.1) *is ensured to have a unique solution over the interval* J.

Proof By Theorem 2.10, for any $x \in C_{1,\rho}[J,\Omega]$, we consider the map $\mathcal{F} : C_{1,\rho}[J,\Omega] \to C_{1,\rho}[J,\Omega]$ as follows:

$$\mathcal{F}(x) = \Big\{ y \in C_{1,\rho}(J,\Omega) : \\ y(t) = D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] \\ + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] f\big(s,x(s)\big)\,ds \\ + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] Bu(s)\,ds \Big\}.$$

We shall show that the map $\mathcal{F}$ has a fixed point based on Theorem 2.11.

First, we show that $\mathcal{F}$ transfers bounded sets into bounded sets in $C_{1,\rho}[J,\Omega]$. Suppose that there exists r such that

$$0 < \frac{M((b^\rho - a^\rho)^{\alpha+1}\|\psi\|_C + \alpha\rho b^{\rho\alpha}(\|D\|_C + \|Bu\|_C))}{\alpha\rho^{\alpha+1} - \theta M(b^\rho - a^\rho)^{\alpha+1}} \le r.$$

For any $x \in B_r = \{x \in C_{1,\rho}[J,\Omega] : \|x\|_{C_{1,\rho}} \le r\}$ and $\varphi \in \mathcal{F}(x)$,

$$\varphi(t) = D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] \\ + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] f\big(s,x(s)\big)\,ds \\ + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] Bu(s)\,ds.$$

By using $H1(iii)$, we can calculate

$$\begin{aligned} &\big\|\varphi(t)\big\|_{C_{1,\rho}} \\ &\quad= \big\|\mathcal{G}(t,a)D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big]\big\|_C \\ &\qquad+ \left\|\mathcal{G}(t,a)\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] f\big(s,x(s)\big)\,ds\right\|_C \\ &\qquad+ \left\|\mathcal{G}(t,a)\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big] Bu(s)\,ds\right\|_C \\ &\quad\le \frac{b^{\rho\alpha}M\|D\|_C}{\rho^\alpha} + \frac{M}{\alpha}\mathcal{G}(t,a)^{\alpha+1}\big(\|\psi\|_C + \theta r\big) + \frac{M}{\alpha}\mathcal{G}(t,a)^{\alpha+1}\|Bu\|_C \\ &\quad\le r. \end{aligned}$$

Therefore $\mathcal{F}(B_r)$ is bounded in $C_{1,\rho}[J,\Omega]$.

Next, we prove that $\{\mathcal{F}(x) : x \in B_r\}$ is contraction.

For any $x, x^* \in B_r$, we get

$$\begin{aligned} &\big\|\mathcal{F}(x) - \mathcal{F}\big(x^*\big)\big\|_{C_{1,\rho}} \\ &\quad= \left\|\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]\big(f\big(s,x(s)\big) - f\big(s,x^*(s)\big)\big)\,ds\right\|_{C_{1,\rho}} \end{aligned}$$

$$= \left\| \mathcal{G}(t,a) \int_a^t s^{\rho-1} \mathcal{G}(t,s)^{\alpha-1} \mathcal{E}_\alpha \left[A\mathcal{G}(t,s)^\alpha \right] \left(f\left(s, x(s)\right) - f\left(s, x^*(s)\right) \right) ds \right\|_C$$
$$\leq ML_f \left\| x - x^* \right\|_{C_{1,\rho}} \int_a^t s^{\rho-1} \mathcal{G}(t,s)^{\alpha-1} ds$$
$$\leq \frac{ML_f (b^\rho - a^\rho)^\alpha}{\rho\alpha} \left\| x - x^* \right\|_{C_{1,\rho}},$$

which yields that $\{\mathcal{F}(x) : x \in B_r\}$ is contraction. According to Theorem 2.11, the map $\mathcal{F}$ has a unique fixed point, implying that equation (1.1) has a unique solution, which completes the proof. □

4 Controllability

Definition 4.1 Suppose that the condition ${}^\rho I_{a^+}^{1-\alpha} x(a) = D$ holds, system (1.1) is controllable, it is said that there exists a control $u \in C_{1,\rho}[J, \Omega]$ such that the solution of system is $x(t_u) = 0$, $t_u \in J$.

Theorem 4.2 *System (1.1) is controllable on $(a, t_u]$ if and only if the Gramian matrix*

$$W_{C_{1,\rho}}(a, t_u] = \int_a^{t_u} (t_u - s)^{\alpha-1} \mathcal{E}_\alpha \left[A\mathcal{G}(t_u, s)^\alpha \right] BB^T \mathcal{E}_\alpha \left[A^T \mathcal{G}(t_u, s)^\alpha \right] ds$$

is nonsingular, where B^T denotes the matrix transpose of B.

Proof Sufficiency. Assume that the matrix $W_{C_{1,\rho}}(a, t_u]$ is nonsingular, then $W_{C_{1,\rho}}^{-1}(a, t_u]$ exists. Set the control $u(t)$ as

$$u(t) = B^T \mathcal{E}_\alpha \left[A^T \mathcal{G}(t_u, s)^\alpha \right] W_{C_{1,\rho}}^{-1}(a, t_u] \Bigg(-D\mathcal{G}(t,a)^{\alpha-1} \mathcal{E}_\alpha \left[A\mathcal{G}(t_u, a)^\alpha \right]$$
$$- \int_a^t s^{\rho-1} \mathcal{G}(t_u, s)^{\alpha-1} \mathcal{E}_\alpha \left[A\mathcal{G}(t_u, s)^\alpha \right] f\left(s, x(s)\right) ds \Bigg).$$

By Theorem 2.10, one can get $x(t_u) = 0$, system (1.1) is controllable on $(a, t_u]$.

Necessity. Assuming system (1.1) is controlled on $(a, tu]$, we shall demonstrate that the Gramian matrix $W_{C_{1,\rho}}(a, t_u]$ is nonsingular. In fact, if $W_{C_{1,\rho}}(a, t_u]$ is singular, then a nonzero vector y_0 exists, such that

$$y_0^T W_{C_{1,\rho}} y_0 = 0.$$

That is,

$$\int_a^{t_u} y_0^T (t_u - s)^{\alpha-1} \mathcal{E}_\alpha \left[A\mathcal{G}(t_u, s)^\alpha \right] BB^T \mathcal{E}_\alpha \left[A^T \mathcal{G}(t_u, s)^\alpha y_0 \right] ds = 0,$$

which yields

$$y_0^T \mathcal{E}_\alpha \left[A\mathcal{G}(t_u, s)^\alpha \right] B = 0.$$

Suppose that system (1.1) is controllable on $(a, t_u]$, and we choose control functions $u_1(t), u_2(t)$ such that

$$x(t_u) = D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,a)^\alpha\big] + \int_a^{t_u} s^{\rho-1}\mathcal{G}(t_u,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,s)^\alpha\big]\big(f\big(s,x(s)\big) + Bu_1(s)\big)\,ds = 0, \tag{4.1}$$

$$y_0 = D\mathcal{G}(t_u,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,a)^\alpha\big] + \int_a^{t_u} s^{\rho-1}\mathcal{G}(t_u,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,s)^\alpha\big]\big(f\big(s,x(s)\big) + Bu_2(s)\big)\,ds \neq 0. \tag{4.2}$$

Inserting (4.1) into (4.2), one can get

$$y_0 = \int_a^t s^{\rho-1}\mathcal{G}(t_u,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,s)^\alpha\big]B\big(\big(u_2(s) - u_1(s)\big)\big)\,ds$$

and

$$y_0^T y_0 = \int_a^t s^{\rho-1}\mathcal{G}(t_u,s)^{\alpha-1}y_0^T\mathcal{E}_\alpha\big[A\mathcal{G}(t_u,s)^\alpha\big]B\big(\big(u_2(s) - u_1(s)\big)\big)\,ds,$$

therefore, $y_0^T y_0 = 0$, which leads to $y_0 = 0$, this result contradicts $y_0 \neq 0$. The proof is finished. □

5 Optimal control results

The set $\mathcal{N}$ is a separable, reflexive Banach space. For $u \in \mathcal{N}$, $\mathcal{C}(\mathcal{N})$ represents a class of nonempty, closed, and convex subsets of $\mathcal{N}$. Given that V is a bounded set within $\mathcal{N}$ and the multifunction $2^{\mathcal{N}} : [0,\infty] \to \mathcal{C}(\mathcal{N})$ is continuous with $2^{\mathcal{N}} \in \mathcal{N}$, the admissible control set $U_{\text{ad}} = u \in C(\mathcal{C}) | u(t) \in 2^{\mathcal{N}}$ ensures the nonempty nature of U_{ad}.

Next, we consider the Lagrange problem (P):

Find a control pair $(x^0, u^0) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}}$ such that

$$\mathcal{J}\big(x^0, u^0\big) \le \mathcal{J}\big(x^u, u\big) \quad \text{for all } (x,u) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}},$$

where

$$\mathcal{J}\big(x^u, u\big) := \int_0^b \mathcal{L}\big(t, x^u(t), u(t)\big)\,dt,$$

where x^u denotes the mild solution of system (1.1) associated with the control $u \in U_{\text{ad}}$. We will introduce the following assumption concerning the existence of a solution to problem (P):

$H(2)$: The function $\mathcal{L} : J \times C_{1,\rho}[J,\Omega] \times \mathcal{N} \to R \cup \{\infty\}$ satisfies the following conditions:

(i) The function $\mathcal{L} : J \times C_{1,\rho}[J,\Omega] \times \mathcal{N} \to R \cup \{\infty\}$ is Borel measurable;

(ii) $\mathcal{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $C_{1,\rho}[J,\Omega] \times \mathcal{N}$ for almost all $t \in J$;

(iii) $\mathcal{L}(t,x,\cdot)$ is convex on $\mathcal{N}$ for each $x \in C_{1,\rho}[J,\Omega]$ and almost all $t \in J$;

(iv) There exist constants $c \geq 0$, $d > 0$, φ is nonnegative, and $\varphi \in C_{1,\rho}[J,\Omega]$ such that

$$\mathcal{L}(t,x,u) \geq \varphi(t) + c\|x\|_{C_{1,\rho}} + d\|u\|_{\mathcal{N}}.$$

Following that, we may state the following conclusion on the existence of optimum controls for problem (P).

Theorem 5.1 *If the assumptions of Theorem 4.2 and $H(2)$ are hold, then the Lagrange problem (P) allows at least one optimum pair, implying that there exists an acceptable control pair $(x^0,u^0) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}}$ such that*

$$\mathcal{J}(x^0,u^0) = \int_0^b \mathcal{L}(t,x^0(t),u^0(t))\,dt \leq \mathcal{J}(x^u,u), \quad (x^u,u) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}}.$$

Proof If $\inf\{\mathcal{J}(x^u,u) : (x^u,u) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}}\} = +\infty$, then the conclusion holds.

In our general context, we assume $\inf \mathcal{J}(x^u,u) : (x^u,u) \in C_{1,\rho}[J,\Omega] \times U_{\text{ad}} = \kappa < +\infty$. By virtue of condition $H(2)$, we establish $\kappa > -\infty$. Following the definition of infimum, there exists a viable minimizing sequence pair $\{(x^n,u^n)\} \subset \mathcal{P}_{\text{ad}} \equiv \{(x,u) : x$ is a mild solution of system (1.1) corresponding to $u \in U_{\text{ad}}\}$, such that $\mathcal{J}(x^n,u^n) \to \kappa$ as $n \to +\infty$. Given that $\{u^n\} \subseteq U_{\text{ad}}$ for $n = 1,2,\dots$, the separable reflexive Banach space $\mathcal{N}$ encompasses a bounded subset $\{u^n\}$. Consequently, there exists a subsequence denoted as u^n, and $u^0 \in \mathcal{N}$ such that

$$u^n \quad \stackrel{w}{\to} \quad u^0 \quad \text{in } \mathcal{N}.$$

Given the closed and convex nature of U_{ad}, according to Marzur's lemma, it follows that u_0 is an element of U_{ad}. Consider the sequence of solutions x^n for system (1.1) corresponding to the input sequence u^n, with x^0 denoting the solution of system (1.1) corresponding to u^0. The integral equations below characterize the relationships between x^n and x^0.

$$\begin{aligned}
x^n(t) &= D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] \\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]f\big(s,x^n(s)\big)\,ds \\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]Bu^n(s)\,ds, \\
x^0(t) &= D\mathcal{G}(t,a)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,a)^\alpha\big] \\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]f\big(s,x^0(s)\big)\,ds \\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\big[A\mathcal{G}(t,s)^\alpha\big]Bu^0(s)\,ds.
\end{aligned}$$

It follows from the boundedness of $\{u^n\}$, $\{u^0\}$ and Theorem 4.2 that there exists a positive number λ such that $\|x^n\| \leq \lambda$, $\|x^0\| \leq \lambda$.

For $t \in J$, we obtain

$$\big\|x^n - x^0\big\|_{C_{1,\rho}}$$

$$
\begin{aligned}
&\leq \left\| \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\left[A\mathcal{G}(t,s)^\alpha\right]\left[f\left(s,x^n(s)\right)-f\left(s,x^0(s)\right)\right]ds \right\|_{C_{1,\rho}} \\
&\quad + \left\| \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\left[A\mathcal{G}(t,s)^\alpha\right]\left[Bu^n(s)-Bu^0(s)\right]ds \right\|_{C_{1,\rho}} \\
&\leq \left\| \mathcal{G}(t,a)\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\left[A\mathcal{G}(t,s)^\alpha\right]\left[f\left(s,x^n(s)\right)-f\left(s,x^0(s)\right)\right]ds \right\|_{C} \\
&\quad + \left\| \left(g(t)-g(a)\right)\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\left[A\mathcal{G}(t,s)^\alpha\right]\left[Bu^n(s)-Bu^0(s)\right]ds \right\|_{C} \\
&\leq L_f M \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\left\|x^n-x^0\right\|_{C_{1,\rho}}ds \\
&\quad + \int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\mathcal{E}_\alpha\left[A\mathcal{G}(t,s)^\alpha\right]\left\|Bu^n(s)-Bu^0(s)\right\|_{C_{1,\rho}}ds. \\
&= Q_1(t)+Q_2(t).
\end{aligned}
\tag{5.1}
$$

By

$$u^n \xrightarrow{w} u^0 \quad \text{in } C_{1,\rho},$$

we have

$$\left\|Bu^n(s)-Bu^0(s)\right\|_{C_{1,\rho}} \to 0, \quad \text{as } n\to\infty,$$

it is said $Q_2(t)\to 0$. Then we can get

$$\left\|x^n-x^0\right\|_{C_{1,\rho}} \leq Q_2(t)+L_f M\int_a^t s^{\rho-1}\mathcal{G}(t,s)^{\alpha-1}\left\|x^n-x^0\right\|_{C_{1,\rho}}ds.$$

By employing the generalized form of Gronwall's inequality with singular kernels (refer to Lemma 2.8), we infer that

$$\left\|x^n-x^0\right\|_{C_{1,\rho}} \leq Q_2(t)E_\alpha\left(ML_f\Gamma(\alpha)\mathcal{G}(t,s)^\alpha\right).$$

This yields that

$$x^n \xrightarrow{s} x^0 \quad \text{in } C_{1,\rho}[J,\Omega] \text{ as } n\to\infty.$$

Note that $H(2)$ indicates that all of the assumptions of the Balder theorem are true. We can conclude that $(x,u)\to\int_0^b \mathcal{L}(t,x(t),u(t))\,dt$ is sequentially lower semicontinuous in the strong topology of $C_{1,\rho}[J,\Omega]$. Since $C_{1,\rho}[J,\Omega]\subset L^1(J,Y)$, $\mathcal{J}$ is weakly lower semicontinuous on $C_{1,\rho}[J,\Omega]$, and since by $H(2)$(iv), $\mathcal{J}>-\infty$, $\mathcal{J}$ reaches its infimum at $u_0\in U_{\mathrm{ad}}$, that is,

$$\kappa = \lim_{n\to\infty}\int_0^b \mathcal{L}\left(t,x^n(t),u^m(t)\right)dt \geq \int_0^b \mathcal{L}\left(t,x^0(t),u^0(t)\right)dt = \mathcal{J}\left(x^0,u^0\right)\geq \kappa.$$

The proof is completed. □

6 An example

In this section, we illustrate our results through an example involving linear Katugampola fractional systems. Consider the following fractional system (6.1):

$$\begin{cases} {}^1D^{\frac{1}{2}}_{1^+}x(t) = Ax(t) + f(t,x(t)) + Bu(t), \quad t \in J' = (0,1], \\ {}^1I^{\frac{1}{2}}_{1^+}x(1) = D, \end{cases} \tag{6.1}$$

where $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, we choose $a = 0, t_u = 1$, one can get $A^2 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $A^3 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $BB^T = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

By Theorem 4.2, we will prove that the following Gramian matrix is nonsingular:

$$W_{C_{1,\rho}}(0,1] = \int_0^1 (1-s)^{\frac{1}{2}} \mathcal{E}_{\frac{1}{2}}\left[A\mathcal{G}(1,s)^{\frac{1}{2}}\right] BB^T \mathcal{E}_{\frac{1}{2}}\left[A^T\mathcal{G}(1,s)^{\frac{1}{2}}\right] ds.$$

The Mittag-Leffler function is

$$\mathcal{E}_{\frac{1}{2}}\left[A\mathcal{G}(1,s)^{\frac{1}{2}}\right] = \mathcal{E}_{\frac{1}{2}}\left[A(1-s)^{\frac{1}{2}}\right] = I + \frac{A(1-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{A^2(1-s)}{\Gamma(2)},$$

and by computation, we can obtain

$$W_{C_{1,\rho}}(0,1] = \begin{pmatrix} 0.6667 & 1.1289 & 1.2093 \\ 0 & 0.6667 & 0 \\ 0 & 0 & 0.6667 \end{pmatrix}.$$

Therefore the matrix $W_{C_{1,\rho}}(0,1]$ is nonsingular, then system (6.1) is controllable.

7 Conclusions

This study employs the fixed point theorem and the generalized Laplace transform to establish crucial conditions for both the existence and uniqueness of solutions within a class of K-type fractional-order systems. This investigation extends and advances our understanding of the subject matter. Furthermore, the study presents both sufficient and necessary conditions for the existence of optimal controllability by extending Gronwall's inequality with singular kernels, laying a robust foundation for future research in this field.

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Author contributions
Xianghu Liu researched and analyzed the model. Yanfang Li conceived and designed the experiments. All authors read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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