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An expanded analysis of local fractional integral inequalities via generalized (s,P)-convexity

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Abstract

This research aims to scrutinize specific parametrized integral inequalities linked to 1, 2, 3, and 4-point Newton-Cotes rules applicable to local fractional differentiable generalized (s,P)-convex functions. To accomplish this objective, we introduce a novel integral identity and deduce multiple integral inequalities tailored to mappings within the aforementioned function class. Furthermore, we present an illustrative example featuring graphical representations and potential practical applications.

Keywords: Newton-Cotes formula; Biparametrized identity; Generalized (s,P)-convex functions; Local fractional integral; Fractal set

1 Introduction

Convexity is a fundamental mathematical concept with many applications in various fields, including optimization, analysis, and geometry [24].

Definition 1.1 If the subsequent inequality is satisfied, then the function ξ defined on $I \subseteq \mathbb{R}$ is said to be convex on I.

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \le \kappa\xi(t_1) + (1-\kappa)\xi(t_2),$$

for all $t_1, t_2 \in I$ and $\kappa \in [0,1]$.

The most well-known and significant inequality related to the notion of convexity is the Hermite-Hadamard inequality, which states that for any convex function ξ defined on the interval $[a,b]$, we have the following inequality [24].

$$\xi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \xi(t)\,dt \le \frac{\xi(a)+\xi(b)}{2}. \tag{1}$$

Convexity has been generalized in multiple ways to include a broader class of mathematical objects beyond traditional convex sets or functions. For instance, one can define h-convex functions [23], which relax the condition of convexity and allow for certain types of nonconvex behavior. Other generalizations include log-convexity [28], which is the class of functions whose natural logarithm is convex, among others. These extensions of convexity have been widely applied in optimization, geometry, and other areas of mathematics [2, 15, 21].

One of the most important generalizations is the class of s-convexity, introduced in [3] as follows:

Definition 1.2 For some fixed $s \in (0,1]$, a function $\xi : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \kappa^s \xi(t_1) + (1-\kappa)^s \xi(t_2)$$

holds for all $t_1, t_2 \in I$ and $\kappa \in [0,1]$.

Moreover, in [5], the authors introduced the class of P-convex functions as follows:

Definition 1.3 A nonnegative function $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-convex on I if the inequality

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \xi(t_1) + \xi(t_2)$$

is satisfied for all $t_1, t_2 \in I$ and $\kappa \in [0,1]$.

The combination of the two previous classes leads to another class of functions known as (s,P)-convex functions, which were introduced by Numan et al. in [22] as follows:

Definition 1.4 The function $\xi : I \subset \mathbb{R} \to \mathbb{R}$ is said to be (s,P)-convex if the subsequent inequality

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \left(\kappa^s + (1-\kappa)^s\right)\left(\xi(t_1) + \xi(t_2)\right)$$

is fulfilled for all $t_1, t_2 \in I$ and $\kappa \in [0,1]$ together with certain fixed $s \in (0,1]$.

Proposition 1.5 [22] *Every nonnegative s-convex function is* (s,P)*-convex.*

Fractal sets have a wide range of applications in modeling natural and artificial phenomena such as coastlines, snowflakes, lightning, and ferns. They are also utilized in image and signal processing, computer graphics, and modeling complex systems. These sets are typically generated using recursive procedures that involve the repeated application of simple rules. The Mandelbrot set [17], Koch snowflake [9], and Sierpinski triangle [29] are some of the well-known examples of fractals. Recently, local fractional calculus has emerged as a new tool for studying fractal sets, providing insights into the intricate behavior of these complex systems exhibiting multiscale behavior [10, 31].

Yang proposed the idea of generalized convexity on fractal sets in 2012 [32]. This concept was introduced as a way to extend the classical notion of convexity to fractal geometry.

Definition 1.6 (Generalized convex function) Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}^{\varsigma}$ with $0 < \varsigma \leq 1$. For all $t_1, t_2 \in I$ and $\kappa \in [0,1]$, if

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \kappa^{\varsigma}\xi(t_1) + (1-\kappa)^{\varsigma}\xi(t_2)$$

holds, then ξ is a generalized convex function on I.

Various researchers have attempted to expand the notion of generalized convexity to encompass a wider range of functions since its inception. Among these extensions, one can cite generalized s-convexity and generalized (s,P)-convexity which are defined as follows:

Definition 1.7 (Generalized s-convex function [20]) Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}^{\varsigma}$ with $0 < \varsigma \leq 1$. For all $t_1, t_2 \in I$ and $\kappa \in [0,1]$, if

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \kappa^{s\varsigma}\xi(t_1) + (1-\kappa)^{s\varsigma}\xi(t_2)$$

holds, then ξ is a generalized s-convex function in the second sense on I.

Definition 1.8 (Generalized (s,P)-convex function [34]) Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}^{\varsigma}$. For any $t_1, t_2 \in I$ and $\kappa \in [0,1]$, if

$$\xi\left(\kappa t_1 + (1-\kappa)t_2\right) \leq \left(\kappa^{s\varsigma} + (1-\kappa)^{s\varsigma}\right)\left(\xi(t_1) + \xi(t_2)\right)$$

holds, then ξ is a generalized (s,P)-convex function on I.

Proposition 1.9 [34] *If a function $\xi : I \subseteq \mathbb{R} \to \mathbb{R}^{\varsigma}$ is nonnegative and generalized s-convex, then it is a generalized (s,P)-convex function.*

Since the introduction of the notion of generalized convexity and its various variants, many researchers have been engaged in establishing error estimates for different quadrature formulas, see [1, 6, 8, 11, 12, 14, 16, 18, 19, 25–27, 30, 35]. These works are of great importance in the development of integral inequalities in the framework of local fractional calculus, but the two landmark works in this context are those of Yu et al. [36] and Du et al. [7], in which the authors conducted a comprehensive analysis of integral inequalities on fractal sets by respectively studying a parametrized three-point Newton-Cotes formula and a biparametrized five-point formula, which allowed them to derive interesting results related to several famous rules.

Motivated by the works of [36] and [7], and with the aim of further expanding the study of Newton-Cotes formulas, this paper introduces a new identity pertaining to a 4-point biparametrized formula, encompassing a broader family of well-known formulas with 1, 2, 3, and 4 points. By utilizing this identity, we establish a wide range of inequalities for functions with generalized (s,P)-convex local fractional derivatives, including both new and previously established results. The study concludes with an example that incorporates 2D and 3D graphical representations, demonstrating the accuracy of the obtained results and providing some practical applications.

Table 1 Derived formulas

x	ρ	$\mathcal{Q}(a,x,b;\rho)$	Formula
$[a,\frac{a+b}{2})$	0	$\frac{\xi(x)+\xi(a+b-x)}{2^{\varsigma}}$	Companion Ostrowski
a	$\rho\in[0,1]$	$\frac{\xi(a)+\xi(b)}{2^{\varsigma}}$	Trapezium
$\frac{a+b}{2}$	0	$\xi(\frac{a+b}{2})$	Midpoint
	$\frac{1}{2}$	$\frac{\xi(a)+\xi(b)}{2^{\varsigma}}$	Trapezium
	$\frac{1}{4}$	$(\frac{1}{4})^{\varsigma}(\xi(a)+2^{\varsigma}\xi(\frac{a+b}{2})+\xi(b))$	Bullen
	$\frac{1}{3}$	$(\frac{1}{6})^{\varsigma}(\xi(a)+4^{\varsigma}\xi(\frac{a+b}{2})+\xi(b))$	Simpson
	$\frac{7}{15}$	$(\frac{1}{30})^{\varsigma}(7^{\varsigma}\xi(a)+16^{\varsigma}\xi(\frac{a+b}{2})+7^{\varsigma}\xi(b))$	Corrected Simpson
$\frac{2a+b}{3}$	$\frac{3}{8}$	$(\frac{1}{8})^{\varsigma}(\xi(a)+\xi(b)+3^{\varsigma}[\xi(\frac{2a+b}{3})+\xi(\frac{a+2b}{3})])$	Simpson 3/8
	$\frac{39}{80}$	$(\frac{1}{80})^{\varsigma}(13^{\varsigma}[\xi(a)+\xi(b)]+27^{\varsigma}[\xi(\frac{2a+b}{3})+\xi(\frac{a+2b}{3})])$	Corrected Simpson 3/8

For $\xi:[a,b]\subset\mathbb{R}\to\mathbb{R}^{\varsigma}$, let us consider the following biparametrized four-point Newton-Cotes formula:

$$\begin{aligned}\mathcal{Q}(a,x,b;\rho)={}&\frac{\rho^{\varsigma}(x-a)^{\varsigma}}{(b-a)^{\varsigma}}\xi(a)+\frac{2^{\varsigma}(1-\rho)^{\varsigma}(x-a)^{\varsigma}+(a+b-2x)^{\varsigma}}{2^{\varsigma}(b-a)^{\varsigma}}\xi(x)\\&+\frac{2^{\varsigma}(1-\rho)^{\varsigma}(x-a)^{\varsigma}+(a+b-2x)^{\varsigma}}{2^{\varsigma}(b-a)^{\varsigma}}\xi(a+b-x)+\frac{\rho^{\varsigma}(x-a)^{\varsigma}}{(b-a)^{\varsigma}}\xi(b),\end{aligned}\qquad(2)$$

with $x\in[a,\frac{a+b}{2}]$ and $\rho\in[0,1]$.

It should be noted that by utilizing formula (2) and selecting specific values for the parameters x and ρ, we can obtain many well-known formulas, see Table 1.

2 Preliminaries

The present section takes another look at the manipulation of real numbers in a fractal set and makes use of the definition of the local fractional derivative and integral put forth by Yang in [32].

If t_1^{ς}, t_2^{ς}, and t_3^{ς} are contained in the set $\mathbb{R}^{\varsigma}$, where $0<\varsigma\le 1$, then the following assertions hold true:

(1) Both $t_1^{\varsigma}+t_2^{\varsigma}$ and $t_1^{\varsigma}t_2^{\varsigma}$ are elements of the set $\mathbb{R}^{\varsigma}$,
(2) $t_1^{\varsigma}+t_2^{\varsigma}=t_2^{\varsigma}+t_1^{\varsigma}=(t_1+t_2)^{\varsigma}=(t_2+t_1)^{\varsigma}$,
(3) $t_1^{\varsigma}+(t_2^{\varsigma}+t_3^{\varsigma})=(t_1+t_2)^{\varsigma}+t_3^{\varsigma}$,
(4) $t_1^{\varsigma}t_2^{\varsigma}=t_2^{\varsigma}t_1^{\varsigma}=(t_1t_2)^{\varsigma}=(t_2t_1)^{\varsigma}$,
(5) $t_1^{\varsigma}(t_2^{\varsigma}t_3^{\varsigma})=(t_1^{\varsigma}t_2^{\varsigma})t_3^{\varsigma}$,
(6) $t_1^{\varsigma}(t_2^{\varsigma}+t_3^{\varsigma})=t_1^{\varsigma}t_2^{\varsigma}+t_1^{\varsigma}t_3^{\varsigma}$,
(7) $t_1^{\varsigma}+0^{\varsigma}=0^{\varsigma}+t_1^{\varsigma}=t_1^{\varsigma}$ and $t_1^{\varsigma}1^{\varsigma}=1^{\varsigma}t_1^{\varsigma}=t_1^{\varsigma}$.

Gao-Yang-Kang were the original proponents of the concept of local fractional derivative and local fractional integral, as described in [32, 33].

Definition 2.1 ([32]) We define a function $\xi:[a,b]\to\mathbb{R}^{\varsigma}$ to be local fractional continuous at $t=t_0$, if for any $\epsilon>0$, the inequality

$$\left|\xi(t)-\xi(t_0)\right|<\epsilon^{\varsigma}$$

holds true for $|t-t_0|<\eta$, with $\eta>0$.

We denote the set of all functions that are local fractional continuous on $[a,b]$ by $C_\varsigma[a,b]$.

Definition 2.2 ([32]) The local fractional derivative of $\xi(t)$ of order ς at $t = t_0$ is defined by

$$\xi^{(\varsigma)}(t_0) = \frac{d^\varsigma \xi(t)}{d\kappa^\varsigma}\Big|_{t=t_0} = \lim_{t\to t_0} \frac{\Delta^\varsigma(\xi(t)-\xi(t_0))}{(t-t_0)^\varsigma},$$

where $\Delta^\varsigma(\xi(t)-\xi(t_0)) \cong \Gamma(\varsigma+1)(\xi(t)-\xi(t_0))$.

We denote the set of all local fractional differentiable functions on $[a,b]$ by $D_\varsigma[a,b]$.

Definition 2.3 ([32]) The local fractional integral of $\xi(t) \in C_\varsigma[a,b]$, is defined by

$${}_aI_b^\varsigma \xi(t) = \frac{1}{\Gamma(\varsigma+1)} \int_a^b \xi(z)(dz)^\varsigma = \frac{1}{\Gamma(\varsigma+1)} \lim_{\Delta z\to 0} \sum_{i=0}^{M-1} \xi(z_i)(\Delta z_i)^\varsigma$$

with $\Delta z_i = z_{i+1} - z_i$ and $\Delta z = \max\{\Delta z_1, \Delta z_2, \dots, \Delta z_{M-1}\}$, where $[z_i, z_{i+1}], i = 0,1,\dots,M-1$ and $a = z_0 < z_1 < \cdots < z_M = b$ is partition of interval $[a,b]$.

It can be inferred that ${}_aI_b^\varsigma \xi(t) = 0$ for $a = b$ and ${}_aI_b^\varsigma \xi(t) = -{}_bI_a^\varsigma \xi(t)$ for $a < b$. If for any $t \in [a,b]$, ${}_aI_b^\varsigma \xi(t)$ exists, then we denoted by $\xi(t) \in I_t^\varsigma[a,b]$.

Lemma 2.4 ([32])

(1) *Suppose that* $\xi(t) = \psi^{(\varsigma)}(t) \in C_\varsigma[a,b]$, *then we have*

$${}_aI_b^\varsigma \xi(t) = \psi(b) - \psi(a).$$

(2) *Suppose that* $\xi, \psi \in D_\varsigma[a,b]$ *and* $\xi^{(\varsigma)}(t), \psi^{(\varsigma)}(t) \in C_\varsigma[a,b]$, *then we have*

$${}_aI_b^\varsigma \xi(t)\psi^{(\varsigma)}(t) = \xi(t)\psi(t)|_a^b - {}_aI_b^\varsigma \xi^{(\varsigma)}(t)\psi(t).$$

Lemma 2.5 ([32]) *For* $\xi(t) = t^{m\varsigma}$, *we have following equations*:

$$\frac{d^\varsigma t^{m\varsigma}}{dt^\varsigma} = \frac{\Gamma(1+m\varsigma)}{\Gamma(1+(m-1)\varsigma)} t^{(m-1)\varsigma},$$

$$\frac{1}{\Gamma(1+\varsigma)} \int_a^b t^{m\varsigma}(dt)^\varsigma = \frac{\Gamma(1+m\varsigma)}{\Gamma(1+(m+1)\varsigma)}\left(b^{(m+1)\varsigma} - a^{(m+1)\varsigma}\right), \quad m \in \mathbb{R}.$$

Lemma 2.6 (Generalized Hölder's inequality [4]) *Let* $\xi, \psi \in C_\varsigma[a,b]$, $p,q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(1+\varsigma)} \int_a^b \left|\xi(t)\psi(t)\right|(dt)^\varsigma \\ &\quad\le \left(\frac{1}{\Gamma(1+\varsigma)} \int_a^b \left|\xi(t)\right|^p (dt)^\varsigma\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\varsigma)} \int_a^b \left|\psi(t)\right|^q (dt)^\varsigma\right)^{\frac{1}{q}}.\end{aligned}$$

3 Main results

Lemma 3.1 *Let $\xi : I \to \mathbb{R}^{\varsigma}$ be a differentiable function on $[a,b]$, and $\xi^{(\varsigma)} \in C_{\varsigma}[a,b]$, then the following equality holds for all $x \in [a, \frac{a+b}{2}]$ and $\rho \in [0,1]$:*

$$\begin{aligned}
&\mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\\
&= \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1(\kappa-\rho)^{\varsigma}\xi^{(\varsigma)}\big((1-\kappa)a+\kappa x\big)(d\kappa)^{\varsigma}\\
&\quad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1(\kappa-1)^{\varsigma}\xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)(d\kappa)^{\varsigma}\\
&\quad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^{\varsigma}\xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)(d\kappa)^{\varsigma}\\
&\quad+\frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1\big(\kappa-(1-\rho)\big)^{\varsigma}\xi^{(\varsigma)}\big((1-\kappa)(a+b-x)+\kappa b\big)(d\kappa)^{\varsigma},
\end{aligned}$$

where $\mathcal{Q}$ is defined as in (2).

Proof Let

$$I = \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}I_1 + \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}I_2 + \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}I_3 + \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}I_4, \tag{3}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1(\kappa-\rho)^{\varsigma}\xi^{(\varsigma)}\big((1-\kappa)a+\kappa x\big)(d\kappa)^{\varsigma},\\
I_2 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1(\kappa-1)^{\varsigma}\xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)(d\kappa)^{\varsigma},\\
I_3 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^{\varsigma}\xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)(d\kappa)^{\varsigma}
\end{aligned}$$

and

$$I_4 = \frac{1}{\Gamma(\varsigma+1)}\int_0^1\big(\kappa-(1-\rho)\big)^{\varsigma}\xi^{(\varsigma)}\big((1-\kappa)(a+b-x)+\kappa b\big)(d\kappa)^{\varsigma}.$$

Using Lemma 2.4, I_1 gives

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1(\kappa-\rho)^{\varsigma}\xi^{(\varsigma)}\big((1-\kappa)a+\kappa x\big)(d\kappa)^{\varsigma}\\
&= \frac{1}{(x-a)^{\varsigma}}(\kappa-\rho)^{\varsigma}\xi\big((1-\kappa)a+\kappa x\big)\big|_0^1\\
&\quad-\frac{1}{(x-a)^{\varsigma}\Gamma(\varsigma+1)}\int_0^1\Gamma(\varsigma+1)\xi\big((1-\kappa)a+\kappa x\big)(d\kappa)^{\varsigma}\\
&= \frac{(1-\rho)^{\varsigma}}{(x-a)^{\varsigma}}\xi(x)+\frac{\rho^{\varsigma}}{(x-a)^{\varsigma}}\xi(a)-\frac{1}{(x-a)^{2\varsigma}}\int_a^x\xi(u)(du)^{\varsigma}.
\end{aligned}\tag{4}$$

Similarly, we get

$$
\begin{aligned}
I_2 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 (\kappa-1)^\varsigma \xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)(d\kappa)^\varsigma \qquad (5)\\
&= \frac{2^\varsigma}{(a+b-2x)^\varsigma}(\kappa-1)^\varsigma \xi\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)\Big|_0^1\\
&\quad - \frac{2^\varsigma}{(a+b-2x)^\varsigma\Gamma(\varsigma+1)}\int_0^1 \Gamma(\varsigma+1)\xi\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)(d\kappa)^\varsigma\\
&= \frac{2^\varsigma}{(a+b-2x)^\varsigma}\xi(x) - \frac{2^{2\varsigma}}{(a+b-2x)^{2\varsigma}}\int_x^{\frac{a+b}{2}}\xi(u)(du)^\varsigma,\\
I_3 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^\varsigma \xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)(d\kappa)^\varsigma \qquad (6)\\
&= \frac{2^\varsigma}{(a+b-2x)^\varsigma}\kappa^\varsigma \xi\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)\Big|_0^1\\
&\quad - \frac{2^\varsigma}{(a+b-2x)^\varsigma\Gamma(\varsigma+1)}\int_0^1 \Gamma(\varsigma+1)\xi\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)(d\kappa)^\varsigma\\
&= \frac{2^\varsigma}{(a+b-2x)^\varsigma}\xi(a+b-x) - \frac{2^{2\varsigma}}{(a+b-2x)^{2\varsigma}}\int_{\frac{a+b}{2}}^{a+b-x}\xi(u)(du)^\varsigma
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \big(\kappa-(1-\rho)\big)^\varsigma \xi^{(\varsigma)}\big((1-\kappa)(a+b-x)+\kappa b\big)(d\kappa)^\varsigma \qquad (7)\\
&= \frac{1}{(x-a)^\varsigma}\big(\kappa-(1-\rho)\big)^\varsigma \xi\big((1-\kappa)(a+b-x)+\kappa b\big)\big|_0^1\\
&\quad - \frac{1}{(x-a)^\varsigma\Gamma(\varsigma+1)}\int_0^1 \Gamma(\varsigma+1)\xi\big((1-\kappa)(a+b-x)+\kappa b\big)(d\kappa)^\varsigma\\
&= \frac{\rho^\varsigma}{(x-a)^\varsigma}\xi(b) + \frac{(1-\rho)^\varsigma}{(x-a)^\varsigma}\xi(a+b-x) - \frac{1}{(x-a)^{2\varsigma}}\int_{a+b-x}^b \xi(u)(du)^\varsigma.
\end{aligned}
$$

Substituting (4)–(7) in (3) and using (2), we get

$$
\begin{aligned}
&\mathcal{Q}(a,x,b;\rho) - \frac{1}{(b-a)^\varsigma}\int_a^b \xi(u)(du)^\varsigma\\
&\quad = \mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_a^b \xi(u)(du)^\varsigma\right)\\
&\quad = \mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^\varsigma}{}_aI_b^\varsigma \xi(t),
\end{aligned}
$$

which is the desired result. □

Theorem 3.2 *Let $\xi:[a,b]\to\mathbb{R}^\varsigma$ be a differentiable function on (a,b) such that $\xi\in D_\varsigma[a,b]$ and $\xi^{(\varsigma)}\in C_\varsigma[a,b]$ with $0\le a<b$. If $|\xi^{(\varsigma)}|$ is generalized (s,P)-convex on $[a,b]$,*

then for all $\rho \in [0,1]$, *we have*

$$\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right| \\
&\quad\le \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\left(2^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\right. \\
&\quad\quad\times \left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^{\varsigma} + \left.\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right) \\
&\quad\quad\times \left(\left|\xi^{(\varsigma)}(a)\right| + \left|\xi^{(\varsigma)}(x)\right| + \left|\xi^{(\varsigma)}(a+b-x)\right| + \left|\xi^{(\varsigma)}(b)\right|\right) \\
&\quad\quad+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\left(\left|\xi^{(\varsigma)}(x)\right| + 2^{\varsigma}\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right| + \left|\xi^{(\varsigma)}(a+b-x)\right|\right)\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)},
\end{aligned}$$

where $\mathcal{Q}(a,x,b;\rho)$ *is defined as in* (2).

Proof From Lemma 3.1, properties of modulus, and the generalized (s,P)-convexity of $|\xi^{(\varsigma)}|$, we have

$$\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right| \\
&\quad\le \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1 |\kappa-\rho|^{\varsigma}\left|\xi^{(\varsigma)}\left((1-\kappa)a+\kappa x\right)\right|(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1 (1-\kappa)^{\varsigma}\left|\xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)\right|(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{\varsigma}\left|\xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)\right|(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \left|\kappa-(1-\rho)\right|^{\varsigma}\left|\xi^{(\varsigma)}\left((1-\kappa)(a+b-x)+\kappa b\right)\right|(d\kappa)^{\varsigma} \\
&\quad\le \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)}\int_0^1 |\kappa-\rho|^{\varsigma}\left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)\left(\left|\xi^{(\varsigma)}(a)\right| + \left|\xi^{(\varsigma)}(x)\right|\right)(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)} \\
&\quad\quad\times \int_0^1 (1-\kappa)^{\varsigma}\left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)\left(\left|\xi^{(\varsigma)}(x)\right| + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|\right)(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)} \\
&\quad\quad\times \int_0^1 \kappa^{\varsigma}\left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right| + \left|\xi^{(\varsigma)}(a+b-x)\right|\right)(d\kappa)^{\varsigma} \\
&\quad\quad+ \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\frac{1}{\Gamma(\varsigma+1)} \\
&\quad\quad\times \int_0^1 \left|\kappa-(1-\rho)\right|^{\varsigma}\left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)\left(\left|\xi^{(\varsigma)}(a+b-x)\right| + \left|\xi^{(\varsigma)}(b)\right|\right)(d\kappa)^{\varsigma} \\
&\quad= \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\left(2^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\right.
\end{aligned}$$

$$\times \left(\rho^{s+2} + (1-\rho)^{s+2} - 1\right)^{\varsigma} + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\Bigg)$$
$$\times \left(\left|\xi^{(\varsigma)}(a)\right| + \left|\xi^{(\varsigma)}(x)\right| + \left|\xi^{(\varsigma)}(a+b-x)\right| + \left|\xi^{(\varsigma)}(b)\right|\right)$$
$$+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\left(\left|\xi^{(\varsigma)}(x)\right| + 2^{\varsigma}\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right| + \left|\xi^{(\varsigma)}(a+b-x)\right|\right)\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)},$$

where we have used the facts that

$$\begin{aligned}
&\frac{1}{\Gamma(\varsigma+1)}\int_0^1 |\kappa-\rho|^{\varsigma}\left(\kappa^{s\varsigma} + (1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \left|\kappa-(1-\rho)\right|^{\varsigma}\left(\kappa^{s\varsigma} + (1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^{\rho} (\rho-\kappa)^{\varsigma}\kappa^{s\varsigma}(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_0^{\rho}(\rho-\kappa)^{\varsigma}(1-\kappa)^{s\varsigma}(d\kappa)^{\varsigma} \\
&\quad + \frac{1}{\Gamma(\varsigma+1)}\int_{\rho}^1 (\kappa-\rho)^{\varsigma}\kappa^{s\varsigma}(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_{\rho}^1(\kappa-\rho)^{\varsigma}(1-\kappa)^{s\varsigma}(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^{\rho}\left(\rho^{\varsigma}\kappa^{s\varsigma} - \kappa^{(s+1)\varsigma}\right)(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_{1-\rho}^1\left(\kappa^{(s+1)\varsigma} - (1-\rho)^{\varsigma}\kappa^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&\quad + \frac{1}{\Gamma(\varsigma+1)}\int_{\rho}^1\left(\kappa^{(s+1)\varsigma} - \rho^{\varsigma}\kappa^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&\quad + \frac{1}{\Gamma(\varsigma+1)}\int_0^{1-\rho}\left((1-\rho)^{\varsigma}\kappa^{s\varsigma} - \kappa^{(s+1)\varsigma}\right)(d\kappa)^{\varsigma} \\
&= 2^{\varsigma}\left(\rho^{s+2} + (1-\rho)^{s+2} - 1\right)^{\varsigma}\left[\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right] \\
&\quad + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\varsigma+1)}\int_0^1 (1-\kappa)^{\varsigma}\left(\kappa^{s\varsigma} + (1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{\varsigma}\left(\kappa^{s\varsigma} + (1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{(s+1)\varsigma}(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{\varsigma}(1-\kappa)^{s\varsigma}(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{(s+1)\varsigma}(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_0^1 (1-\kappa)^{\varsigma}\kappa^{s\varsigma}(d\kappa)^{\varsigma} \\
&= \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{(s+1)\varsigma}(d\kappa)^{\varsigma} + \frac{1}{\Gamma(\varsigma+1)}\int_0^1 \left(\kappa^{s\varsigma} - \kappa^{(s+1)\varsigma}\right)(d\kappa)^{\varsigma} \\
&= \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}.
\end{aligned} \tag{9}$$

The proof is completed. □

Corollary 3.3 *In Theorem 3.2, if we take $x = \frac{2a+b}{3}$, we obtain the following inequality*:

$$\begin{aligned}
&\left|\frac{\rho^{\varsigma}}{3^{\varsigma}}\xi(a) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}}\xi\left(\frac{2a+b}{3}\right) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}}\xi\left(\frac{a+2b}{3}\right) + \frac{\rho^{\varsigma}}{3^{\varsigma}}\xi(b) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right| \\
&\le \frac{(b-a)^{\varsigma}}{36^{\varsigma}}\left\{\left(8^{\varsigma}\left(\rho^{s+2} + (1-\rho)^{s+2} - 1\right)^{\varsigma}\left[\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right]\right.\right. \\
&\left. + 4^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right) \\
&\times\left(\left|\xi^{(\varsigma)}(a)\right| + \left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right| + \left|\xi^{(\varsigma)}(b)\right|\right) \\
&\left. + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\left(\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right| + 2^{\varsigma}\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|\right)\right\}.
\end{aligned}$$

Corollary 3.4 *In Corollary 3.3, if we take $\rho = \frac{3}{8}$, then we obtain the following inequality related to Simpson's second formula*:

$$\begin{aligned}
&\left|\frac{1^{\varsigma}}{8^{\varsigma}}\left(\xi(a) + 3^{\varsigma}\xi\left(\frac{2a+b}{3}\right) + 3^{\varsigma}\xi\left(\frac{a+2b}{3}\right) + \xi(b)\right) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right| \\
&\le \frac{(b-a)^{\varsigma}}{(36)^{\varsigma}}\left(\left(8^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\left(\frac{3^{s+2}+5^{s+2}}{8^{s+2}} - 1\right)^{\varsigma}\right.\right. \\
&\left. + 4^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right) \\
&\times\left(\left|\xi^{(\varsigma)}(a)\right| + \left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right| + \left|\xi^{(\varsigma)}(b)\right|\right) \\
&\left. + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\left(\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right| + 2^{\varsigma}\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|\right)\right).
\end{aligned}$$

Remark 3.5 Based on our primary finding, we may identify specific cases that correspond to previously established results. This not only showcases the generality of Theorem 3.2 but also consolidates the foundation laid by earlier studies:

- By setting $\rho = \frac{39}{80}$, Corollary 3.3 will be reduced to Theorem 3 from [13].
- If we set $x = \frac{a+b}{2}$ and $\rho = \frac{1}{4}$ in Theorem 3.2, the outcomes align with Corollary 2.8 by Du et al. in [7], which pertains to the Bullen inequality.

Remark 3.6 Theorem 3.2 leads to a wealth of new discoveries since it allows for the deduction of many outcomes for generalized (s,P)-convex functions. Indeed, if we fix:

1/ $\rho = 0$, then we obtain the companion Ostrowski inequality.

2/ $x = a$, we then obtain the trapezium inequality.

3/ $x = \frac{a+b}{2}$, then we obtain the Simpson-like-type inequality. Moreover, we get

- Midpoint inequality, for $\rho = 0$,
- Trapezium inequality, for $\rho = \frac{1}{2}$,
- Simpson inequality, for $\rho = \frac{1}{3}$,
- Corrected Simpson inequality, for $\rho = \frac{7}{15}$.

Theorem 3.7 *Assume that all the assumptions of Theorem 3.2 are satisfied. If $|\xi^{(\varsigma)}|^q$ is generalized (s,P)-convex, then for all $\rho \in [0,1]$, the following inequality holds*

$$\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^\varsigma}{}_aI_b^\varsigma \xi(t)\right| \\
&\leq \frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\left((1-\rho)^{p+1}+\rho^{p+1}\right)^{\frac{1}{p}}\right)^\varsigma\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^\varsigma\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\quad\times\left(\left(\left|\xi^{(\varsigma)}(a)\right|^q+\left|\xi^{(\varsigma)}(x)\right|^q\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}(a+b-x)\right|^q+\left|\xi^{(\varsigma)}(b)\right|^q\right)^{\frac{1}{q}}\right) \\
&\quad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^\varsigma\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\quad\times\left(\left(\left|\xi^{(\varsigma)}(x)\right|^q+\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q+\left|\xi^{(\varsigma)}(a+b-x)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where $\mathcal{Q}(a,x,b;\rho)$ is defined as (2), and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 3.1, properties of modulus, the generalized Hölder inequality, and generalized (s,P)-convexity of $|\xi^{(\varsigma)}|^q$, we have

$$\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho) - \frac{\Gamma(\varsigma+1)}{(b-a)^\varsigma}{}_aI_b^\varsigma \xi(t)\right| \\
&\leq \frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^{p\varsigma}(d\kappa)^\varsigma\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\xi^{(\varsigma)}\left((1-\kappa)a+\kappa x\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\quad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1(1-\kappa)^{p\varsigma}(d\kappa)^\varsigma\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\quad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^{p\varsigma}(d\kappa)^\varsigma\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\quad+\frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\kappa-(1-\rho)\right|^{p\varsigma}(d\kappa)^\varsigma\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\xi^{(\varsigma)}\left((1-\kappa)(a+b-x)+\kappa b\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^{p\varsigma}(d\kappa)^\varsigma\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)(d\kappa)^\varsigma\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\left(\left|\xi^{(\varsigma)}(a)\right|^q + \left|\xi^{(\varsigma)}(x)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}(a+b-x)\right|^q + \left|\xi^{(\varsigma)}(b)\right|^q\right)^{\frac{1}{q}}\right) \\
&+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{p\varsigma}(d\kappa)^{\varsigma}\right)^{\frac{1}{p}} \\
&\times \left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma}\right)^{\frac{1}{q}} \\
&\times \left(\left(\left|\xi^{(\varsigma)}(x)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\xi^{(\varsigma)}(a+b-x)\right|^q\right)^{\frac{1}{q}}\right) \\
={}& \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\left(\left((1-\rho)^{p+1}+\rho^{p+1}\right)^{\frac{1}{p}}\right)^{\varsigma}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\times \left(\left(\left|\xi^{(\varsigma)}(a)\right|^q + \left|\xi^{(\varsigma)}(x)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}(a+b-x)\right|^q + \left|\xi^{(\varsigma)}(b)\right|^q\right)^{\frac{1}{q}}\right) \\
&+ \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\times \left(\left(\left|\xi^{(\varsigma)}(x)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\xi^{(\varsigma)}(a+b-x)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where we have used the facts that

$$\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \left(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\right)(d\kappa)^{\varsigma} = 2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}, \tag{10}$$

$$\frac{1}{\Gamma(\varsigma+1)}\int_0^1 \kappa^{p\varsigma}(d\kappa)^{\varsigma} = \frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)} \tag{11}$$

and

$$\frac{1}{\Gamma(\varsigma+1)}\int_0^1 |\kappa-\rho|^{p\varsigma}(d\kappa)^{\varsigma} = \left((1-\rho)^{p+1}+\rho^{p+1}\right)^{\varsigma}\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}. \tag{12}$$

The proof is completed. □

Corollary 3.8 *In Theorem 3.7, if we take $x = \frac{2a+b}{3}$, then we obtain the following inequality*:

$$
\begin{aligned}
&\left|\frac{\rho^{\varsigma}}{3^{\varsigma}}\xi(a) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}}\xi\left(\frac{2a+b}{3}\right) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}}\xi\left(\frac{a+2b}{3}\right) + \frac{\rho^{\varsigma}}{3^{\varsigma}}\xi(b) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right| \\
&\leq \frac{(b-a)^{\varsigma}}{9^{\varsigma}}\left(\left((1-\rho)^{p+1}+\rho^{p+1}\right)^{\frac{1}{p}}\right)^{\varsigma}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\times \left(\left(\left|\xi^{(\varsigma)}(a)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^q + \left|\xi^{(\varsigma)}(b)\right|^q\right)^{\frac{1}{q}}\right) \\
&+ \frac{(b-a)^{\varsigma}}{(36)^{\varsigma}}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}} \\
&\times \left(\left(\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&+ \left.\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Corollary 3.9 *In Corollary 3.8, if we take $\rho = \frac{3}{8}$, then we obtain the following inequality related to Simpson's second formula*:

$$\begin{aligned}
&\left|\frac{1^{\varsigma}}{8^{\varsigma}}\left(\xi(a)+3^{\varsigma}\xi\left(\frac{2a+b}{3}\right)+3^{\varsigma}\xi\left(\frac{2a+b}{3}\right)+\xi(b)\right)-\frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right|\\
&\quad\leq\frac{(b-a)^{\varsigma}}{9^{\varsigma}}\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}\varsigma}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}(a)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^{q}+\left|\xi^{(\varsigma)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&\qquad+\frac{(b-a)^{\varsigma}}{(36)^{\varsigma}}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Remark 3.10 By setting $\rho = \frac{39}{80}$, Corollary 3.8 will be reduced to Theorem 4 from [13].

Theorem 3.11 *Assume that all the assumptions of Theorem 3.2 are satisfied. If $|\xi^{(\varsigma)}|^{q}$ is generalized (s,P)-convex, then for all $\rho \in [0,1]$ and $q > 1$, the following inequality holds*:

$$\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho)-\frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right|\\
&\quad\leq\frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}}\left(\left((1-\rho)^{2}+\rho^{2}\right)^{\varsigma}\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{1-\frac{1}{q}}\\
&\qquad\times\left(\left(2^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}-\frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^{\varsigma}\right.\right.\\
&\qquad\left.\left.+\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)\right)^{\frac{1}{q}}\right.\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}(a)\right|^{q}+\left|\xi^{(\varsigma)}(x)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}(a+b-x)\right|^{q}+\left|\xi^{(\varsigma)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&\qquad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}}\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}(x)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|\xi^{(\varsigma)}(a+b-x)\right|^{q}\right)^{\frac{1}{q}}\right)\Bigg),
\end{aligned}$$

where $\mathcal{Q}(a,x,b;\rho)$ is defined as in (2).

Proof From Lemma 3.1, properties of modulus generalized power mean inequality and generalized (s,P)-convexity of $|\xi^{(\varsigma)}|^q$, we have

$$
\begin{aligned}
&\left|\mathcal{Q}(a,x,b;\rho)-\frac{\Gamma(\varsigma+1)}{(b-a)^\varsigma}{}_aI_b^\varsigma\xi(t)\right| \\
&\quad\le \frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^\varsigma\left|\xi^{(\varsigma)}\big((1-\kappa)a+\kappa x\big)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\qquad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1(1-\kappa)^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1(1-\kappa)^\varsigma\left|\xi^{(\varsigma)}\left((1-\kappa)x+\kappa\frac{a+b}{2}\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\qquad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^\varsigma\left|\xi^{(\varsigma)}\left((1-\kappa)\frac{a+b}{2}+\kappa(a+b-x)\right)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\qquad+\frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\kappa-(1-\rho)\right|^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\left|\kappa-(1-\rho)\right|^\varsigma\left|\xi^{(\varsigma)}\big((1-\kappa)(a+b-x)+\kappa b\big)\right|^q(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\quad\le \frac{(x-a)^{2\varsigma}}{(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1|\kappa-\rho|^\varsigma\big(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\big)\big(\left|\xi^{(\varsigma)}(a)\right|^q+\left|\xi^{(\varsigma)}(x)\right|^q\big)(d\kappa)^\varsigma\right)^{\frac{1}{q}}\right. \\
&\qquad+\left(\frac{1}{\Gamma(\varsigma+1)}\right. \\
&\qquad\left.\left.\times\int_0^1\left|\kappa-(1-\rho)\right|^\varsigma\big(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\big)\big(\left|\xi^{(\varsigma)}(a+b-x)\right|^q+\left|\xi^{(\varsigma)}(b)\right|^q\big)(d\kappa)^\varsigma\right)^{\frac{1}{q}}\right) \\
&\qquad+\frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^\varsigma}\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^\varsigma(d\kappa)^\varsigma\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1(1-\kappa)^\varsigma\big(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\big)\right.\right. \\
&\qquad\left.\times\left(\left|\xi^{(\varsigma)}(x)\right|^q+\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q\right)(d\kappa)^\varsigma\right)^{\frac{1}{q}} \\
&\qquad+\left(\frac{1}{\Gamma(\varsigma+1)}\int_0^1\kappa^\varsigma\big(\kappa^{s\varsigma}+(1-\kappa)^{s\varsigma}\big)\right. \\
&\qquad\left.\left.\times\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q+\left|\xi^{(\varsigma)}(a+b-x)\right|^q\right)(d\kappa)^\varsigma\right)^{\frac{1}{q}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x-a)^{2\varsigma}}{(b-a)^{\varsigma}} \left(\left((1-\rho)^2+\rho^2\right)^{\varsigma} \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\left(2^{\varsigma} \left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)} \right) \left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^{\varsigma} \right.\right. \\
&\quad \left.\left. + \left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} \right) \right) \right)^{\frac{1}{q}} \\
&\quad \times \left(\left(\left|\xi^{(\varsigma)}(a)\right|^q + \left|\xi^{(\varsigma)}(x)\right|^q \right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}(a+b-x)\right|^q + \left|\xi^{(\varsigma)}(b)\right|^q \right)^{\frac{1}{q}} \right) \\
&\quad + \frac{(a+b-2x)^{2\varsigma}}{2^{2\varsigma}(b-a)^{\varsigma}} \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} \right)^{1-\frac{1}{q}} \left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} \right)^{\frac{1}{q}} \\
&\quad \times \left(\left(\left|\xi^{(\varsigma)}(x)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\xi^{(\varsigma)}(a+b-x)\right|^q \right)^{\frac{1}{q}} \right) \Bigg),
\end{aligned}
$$

where we have used (8), (9), and the facts that

$$
\frac{1}{\Gamma(\varsigma+1)} \int_0^1 \kappa^{\varsigma} (d\kappa)^{\varsigma} = \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}
$$

and

$$
\frac{1}{\Gamma(\varsigma+1)} \int_0^1 |\kappa-\rho|^{\varsigma} (d\kappa)^{\varsigma} = \left((1-\rho)^2+\rho^2\right)^{\varsigma} \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}.
$$

The proof is completed. □

Corollary 3.12 *In Theorem 3.11, if we take $x = \frac{2a+b}{3}$, we obtain the following inequality:*

$$
\begin{aligned}
&\left| \frac{\rho^{\varsigma}}{3^{\varsigma}} \xi(a) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}} \xi\left(\frac{2a+b}{3}\right) + \frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}} \xi\left(\frac{a+2b}{3}\right) + \frac{\rho^{\varsigma}}{3^{\varsigma}} \xi(b) - \frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}} {}_aI_b^{\varsigma} \xi(t) \right| \\
&\leq \frac{(b-a)^{\varsigma}}{9^{\varsigma}} \left(\left((1-\rho)^2+\rho^2\right)^{\varsigma} \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\left(2^{\varsigma} \left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)} \right) \left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^{\varsigma} \right.\right. \\
&\quad \left. + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} \right)^{\frac{1}{q}} \\
&\quad \times \left(\left(\left|\xi^{(\varsigma)}(a)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left|\xi^{(\varsigma)}\left(\frac{a+3b}{4}\right)\right|^q + \left|\xi^{(\varsigma)}(b)\right|^q \right)^{\frac{1}{q}} \right) \\
&\quad + \frac{(b-a)^{\varsigma}}{36^{\varsigma}} \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} \right)^{1-\frac{1}{q}} \left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} \right)^{\frac{1}{q}} \\
&\quad \times \left(\left(\left|\xi^{(\varsigma)}(x)\right|^q + \left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^q + \left|\xi^{(\varsigma)}(a+b-x)\right|^q \right)^{\frac{1}{q}} \right) \Bigg).
\end{aligned}
$$

Corollary 3.13 *In Corollary 3.12, if we take $\rho = \frac{3}{8}$, then we obtain the following inequality related to Simpson's second formula.*

$$\begin{aligned}
&\left|\frac{1^{\varsigma}}{8^{\varsigma}}\left(\xi(a)+3^{\varsigma}\xi\left(\frac{2a+b}{3}\right)+3^{\varsigma}\xi\left(\frac{a+2b}{3}\right)+\xi(b)\right)-\frac{\Gamma(\varsigma+1)}{(b-a)^{\varsigma}}{}_aI_b^{\varsigma}\xi(t)\right|\\
&\quad\le\frac{(b-a)^{\varsigma}}{9^{\varsigma}}\left(\left(\frac{17}{32}\right)^{\varsigma}\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{1-\frac{1}{q}}\\
&\qquad\times\left[2^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}-\frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\left(\frac{3^{(s+2)}+5^{(s+2)}}{8^{(s+2)}}-1\right)^{\varsigma}\right.\\
&\qquad\left.+\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right]^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}(a)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^{q}+\left|\xi^{(\varsigma)}(b)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&\qquad+\frac{(b-a)^{\varsigma}}{(36)^{\varsigma}}\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}\left(\frac{2a+b}{3}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\left|\xi^{(\varsigma)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Remark 3.14

- By setting $\rho = \frac{39}{80}$, Corollary 3.12 will be reduced to Theorem 5 from [13].
- Similar to Theorem 3.2, the outcomes related to Theorems 3.7 and 3.11 concerning the quadrature rules in Table 1 can be easily derived.

4 Example and applications

The purpose of this section is to confirm the precision and efficiency of the obtained results. To achieve this, we first present an example with a visual representation that supports the accuracy of our findings. Following that, we provide some practical uses for estimating the error of a specific quadrature formula and some inequalities that involve special means.

4.1 Example supporting study results

To further support and validate the results obtained in this study, we present an illustrative example consisting of multiple cases with 2D and 3D graphical representations that demonstrate the effectiveness and accuracy of our findings.

It should be pointed out that the figures that follow were generated using MATLAB, with the color green representing the Right Hand Side and the color blue representing the Left Hand Side of their corresponding inequalities.

Example 4.1 Let $\xi : [0,1] \to \mathbb{R}^{\varsigma}$ be a function defined for a fixed value $s \in (0,1]$ by $\xi(t) = \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} t^{(s+1)\varsigma}$. Therefore, the derivative $|\xi^{(\varsigma)}(t)| = t^{s\varsigma}$ is a nonnegative generalized s-convex function and hence a generalized (s,P)-convex function according to Proposition 1.9, which satisfies the fundamental condition underlying this study.

From Theorem 3.2, we obtain the following multiparametrized four-point Newton-Cotes-type inequalities related to the function under consideration.

$$\begin{aligned}
&\left|\mathcal{Q}(0,x,1;\rho)-\Gamma(\varsigma+1)\,{}_0I_1^{\varsigma}\xi(t)\right| \\
&\leq x^{2\varsigma}\left(2^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}-\frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^{\varsigma}\right. \\
&\quad\left.+\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)\left(x^{s\varsigma}+(1-x)^{s\varsigma}+1^{\varsigma}\right) \\
&\quad+\frac{(1-2x)^{2\varsigma}}{2^{2\varsigma}}\left(x^{s\varsigma}+2^{(1-s)\varsigma}+(1-x)^{s\varsigma}\right)\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)},
\end{aligned}\tag{13}$$

where

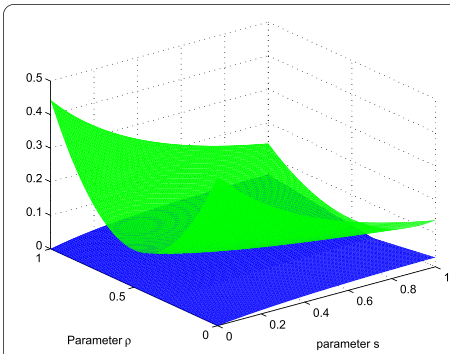
$$\begin{aligned}
&\mathcal{Q}(0,x,1;\rho) \\
&=\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\left[\frac{2^{\varsigma}(1-\rho)^{\varsigma}x^{\varsigma}+(1-2x)^{\varsigma}}{2^{\varsigma}}\left(x^{(s+1)\varsigma}+(1-x)^{(s+1)\varsigma}\right)+\rho^{\varsigma}x^{\varsigma}\right].
\end{aligned}$$

If we set $\varsigma = 1$, it becomes apparent that the outcome is reliant on three parameters. Our subsequent step is to fix one parameter at a time and graph the output as a function of the remaining two parameters.

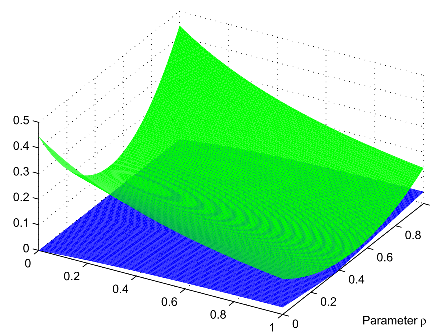
If we fix $x = \frac{2a+b}{3} = \frac{1}{3}$. Consequently, from (13), we obtain

$$\begin{aligned}
&\left|\frac{1}{s+1}\left(\frac{2(1-\rho)+1}{6}\left(\frac{1+2^{s+1}}{3^{s+1}}\right)+\frac{\rho}{3}-\frac{1}{s+2}\right)\right| \\
&\leq\frac{1}{9(s+1)}\left[\left(\frac{2^{\varsigma}(\rho^{s+2}+(1-\rho)^{s+2}-1)^{\varsigma}}{s+2}+1\right)\left(\frac{1+2^{s}+3^{s}}{3^{s}}\right)\right. \\
&\quad\left.+\frac{1}{4}\left(\frac{1+2^{s}}{3^{s}}+2^{1-s}\right)\right].
\end{aligned}\tag{14}$$

The outcome described by Inequality (14) is illustrated in Fig. 1. Conversely, Fig. 2 depicts the special cases of Inequality (14), when $s = \frac{1}{2}$, $\rho \in [0,1]$ and $s \in (0,1]$, $\rho = \frac{3}{8}$, respectively.

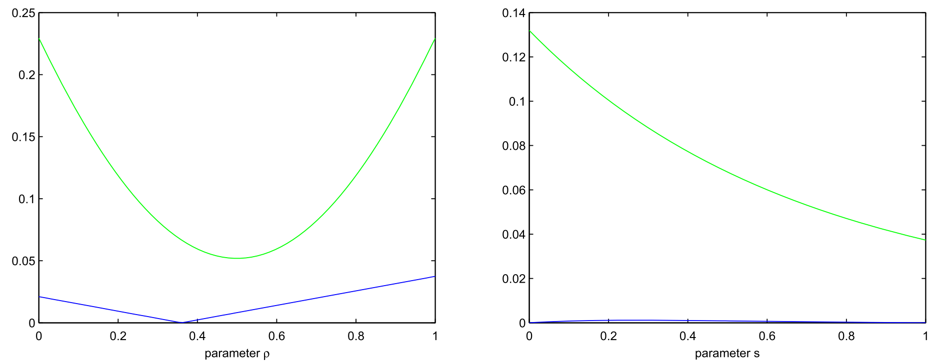


(A) View.1



(B) View.2

Figure 1 $x = \frac{1}{3}$, $\rho \in [0,1]$ and $s \in (0,1]$



(A) $x = \frac{1}{3}$, $s = \frac{1}{2}$ and $\rho \in [0,1]$ (B) $x = \frac{1}{3}$, $\rho = \frac{3}{8}$ and $s \in (0,1]$

Figure 2 Special cases

4.2 Application to quadrature formula

Let Θ be the partition of the points $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a,b]$, and consider the quadrature formula

$$\frac{1}{\Gamma(1+\varsigma)} \int_a^b \xi(u)(du)^\varsigma = \Lambda(\xi,\Theta) + \mathcal{R}(\xi,\Theta),$$

where

$$\begin{aligned}\Lambda(\xi,\Theta) = \sum_{i=0}^{n-1} \frac{(t_{j+1}-t_j)^\varsigma}{\Gamma(\varsigma+1)} \biggl(&\frac{\rho^\varsigma}{3^\varsigma}\xi(t_j) + \frac{(3-2\rho)^\varsigma}{6^\varsigma}\xi\left(\frac{2t_j+t_{j+1}}{3}\right) \\ &+ \frac{(3-2\rho)^\varsigma}{6^\varsigma}\xi\left(\frac{t_j+2t_{j+1}}{3}\right) + \frac{\rho^\varsigma}{3^\varsigma}\xi(t_{j+1})\biggr),\end{aligned}$$

and $\mathcal{R}(\xi,\Theta)$ denotes the associated approximation error.

Proposition 4.2 *Let $n \in \mathbb{N}$, $\rho \in [0,1]$ and $\xi : [a,b] \to \mathbb{R}^\varsigma$ be a differentiable function on (a,b) with $0 \le a < b$ and $\xi^{(\varsigma)} \in C_\varsigma[a,b]$. If $|\xi^{(\varsigma)}|$ is generalized (s,P)-convex function, we have*

$$\begin{aligned}&\left|\mathcal{R}(\xi,\Theta)\right| \\ &\quad\le \sum_{j=0}^{k-1} \frac{(t_{j+1}-t_j)^{2\varsigma}}{9^\varsigma\Gamma(1+\varsigma)} \biggl\{ \biggl(8^\varsigma\left(\rho^{s+2}+(1-\rho)^{s+2}-1\right)^\varsigma \left[\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} - \frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right] \\ &\quad + 4^\varsigma \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\biggr) \\ &\quad \times \left(\left|\xi^{(\varsigma)}(t_j)\right| + \left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right| + \left|\xi^{(\varsigma)}(t_{j+1})\right| \right) \\ &\quad + \frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)} \\ &\quad \times \left(\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right| + 2^\varsigma \left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right| + \left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right| \right) \biggr\}.\end{aligned}$$

Proof Applying Corollary 3.3 on the subintervals $[t_j, t_{j+1}]$ $(j = 0, 1, \ldots, k-1)$ of the partition Θ, we get

$$\begin{aligned}
&\left|\frac{\rho^\varsigma}{3^\varsigma}\big(\xi(t_j)+\xi(t_{j+1})\big)+\frac{(3-2\rho)^\varsigma}{6^\varsigma}\left(\xi\left(\frac{2t_j+t_{j+1}}{3}\right)+\xi\left(\frac{t_j+2t_{j+1}}{3}\right)\right)-\frac{\Gamma(\varsigma+1)}{(b-t_j)^\varsigma}\,{}_{t_j}I^{\varsigma}_{t_{j+1}}\xi(t)\right|\\
&\quad\le\frac{(t_{j+1}-t_j)^\varsigma}{9^\varsigma}\left\{\left(8^\varsigma\big(\rho^{s+2}+(1-\rho)^{s+2}-1\big)^\varsigma\left[\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}-\frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right]\right.\right.\\
&\qquad\left.+4^\varsigma\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)\\
&\qquad\times\left(\left|\xi^{(\varsigma)}(t_j)\right|+\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|+\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|+\left|\xi^{(\varsigma)}(t_{j+1})\right|\right)\\
&\qquad+\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\\
&\qquad\left.\times\left(\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|+2^\varsigma\left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right|+\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|\right)\right\}.
\end{aligned}$$

Multiplying both sides of above inequality by $\frac{1}{\Gamma(1+\varsigma)}(t_{j+1}-t_j)^\varsigma$, then summing the obtained inequalities for all $j = 0, 1, \ldots, k-1$, and using the triangular inequality, we get the desired result. □

Proposition 4.3 *Let $n \in \mathbb{N}$, $\rho \in [0,1]$ and $\xi : [a,b] \to \mathbb{R}^\varsigma$ be a differentiable function on (a,b) with $0 \le a < b$ and $\xi^{(\varsigma)} \in C_\varsigma[a,b]$. If $|\wp^{(\varsigma)}|^q$ is generalized (s,P)-convex function, we have*

$$\begin{aligned}
&\left|\mathcal{R}(\xi,\Theta)\right|\\
&\quad\le\sum_{j=0}^{k-1}\frac{(t_{j+1}-t_j)^{2\varsigma}}{9^\varsigma\Gamma(1+\varsigma)}\Big(\big((1-\rho)^{p+1}+\rho^{p+1}\big)^{\frac{1}{p}}\Big)^\varsigma\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\\
&\qquad\times\left(2^\varsigma\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\left(\left(\left|\xi^{(\varsigma)}(t_j)\right|^q+\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|^q+\left|\xi^{(\varsigma)}(t_{j+1})\right|^q\right)^{\frac{1}{q}}\right)\\
&\qquad+\frac{(t_{j+1}-t_j)^{2\varsigma}}{(36)^\varsigma\Gamma(1+\varsigma)}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^\varsigma\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|^q+\left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right|^q+\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof Applying Corollary 3.8 on the subintervals $[t_j, t_{j+1}]$ $(j = 0, 1, \dots, k-1)$ of the partition Θ, we get

$$\begin{aligned}
&\left|\frac{\rho^{\varsigma}}{3^{\varsigma}}\big(\xi(t_j)+\xi(t_{j+1})\big)+\frac{(3-2\rho)^{\varsigma}}{6^{\varsigma}}\left(\xi\left(\frac{2t_j+t_{j+1}}{3}\right)+\xi\left(\frac{t_j+2t_{j+1}}{3}\right)\right)-\frac{\Gamma(\varsigma+1)}{(b-t_j)^{\varsigma}}\,{}_{t_j}I^{\varsigma}_{t_{j+1}}\xi(t)\right|\\
&\quad\le\frac{(t_{j+1}-t_j)^{\varsigma}}{9^{\varsigma}}\Big(\big((1-\rho)^{p+1}+\rho^{p+1}\big)^{\frac{1}{p}}\Big)^{\varsigma}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\big|\xi^{(\varsigma)}(t_j)\big|^{q}+\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|^{q}+\big|\xi^{(\varsigma)}(t_{j+1})\big|^{q}\right)^{\frac{1}{q}}\right)\\
&\qquad+\frac{(t_{j+1}-t_j)^{\varsigma}}{(36)^{\varsigma}}\left(\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\right)^{\frac{1}{p}}\left(2^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)^{\frac{1}{q}}\\
&\qquad\times\left(\left(\left|\xi^{(\varsigma)}\left(\frac{2t_j+t_{j+1}}{3}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\left|\xi^{(\varsigma)}\left(\frac{t_j+t_{j+1}}{2}\right)\right|^{q}+\left|\xi^{(\varsigma)}\left(\frac{t_j+2t_{j+1}}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Multiplying both sides of above inequality by $\frac{1}{\Gamma(1+\varsigma)}(t_{j+1}-t_j)^{\varsigma}$, then summing the obtained inequalities for all $j = 0, 1, \dots, k-1$, and using the triangular inequality, we get the desired result. □

4.3 Application to special means

For arbitrary real numbers a, b we have:

The generalized Arithmetic mean: $A(a,b) = \frac{a^{\varsigma}+b^{\varsigma}}{2^{\varsigma}}$.

The generalized p-Logarithmic mean: $L_p(a,b) = \big[\frac{\Gamma(1+p\varsigma)}{\Gamma(1+(p+1)\varsigma)}\big(\frac{b^{(p+1)\varsigma}-a^{(p+1)\varsigma}}{(b-a)^{\varsigma}}\big)\big]^{\frac{1}{p}}$, $a, b \in \mathbb{R}$, $a \neq b$ and $p \in \mathbb{Z}\backslash\{-1, 0\}$.

Proposition 4.4 *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $n \geq 2$, then we have*

$$\begin{aligned}
&\big|A^{(s+1)\varsigma}(a,b)+3^{\varsigma}A\big(A^{(s+1)\varsigma}(a,a,b),A^{(s+1)\varsigma}(a,b,b)\big)-4^{\varsigma}\Gamma(\varsigma+1)L^{s+1}_{s+1}(a,b)\big|\\
&\quad\le\frac{(b-a)^{\varsigma}}{9^{\varsigma}}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)\\
&\qquad\times\left(\left(8^{\varsigma}\left(\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}-\frac{\Gamma(1+(s+1)\varsigma)}{\Gamma(1+(s+2)\varsigma)}\right)\left(\frac{3^{(s+2)}+5^{(s+2)}}{8^{(s+2)}}-1\right)^{\varsigma}\right.\right.\\
&\qquad\left.+4^{\varsigma}\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right)\left(a^{s\varsigma}+\left(\frac{2a+b}{3}\right)^{s\varsigma}+\left(\frac{a+2b}{3}\right)^{s\varsigma}+b^{s\varsigma}\right)\\
&\qquad\left.+\left(\left|\left(\frac{2a+b}{3}\right)^{s\varsigma}\right|+2^{\varsigma}\left(\frac{a+b}{2}\right)^{s\varsigma}+\left(\frac{a+2b}{3}\right)^{s\varsigma}\right)\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}\right).
\end{aligned}$$

Proof This follows from Corollary 3.3 with $\rho = \frac{39}{80}$ applied to the function $\xi(t) = t^{(s+1)\varsigma}$, where $\xi : (0, +\infty) \to \mathbb{R}^{\varsigma}$. □

5 Conclusion

In conclusion, this study makes significant contributions to the field of integral inequalities in local fractional calculus by examining particular parametrized integral inequalities for local fractional differentiable generalized (s,P)-convex functions. The introduction of a novel integral identity has allowed for the derivation of multiple integral inequalities for a broader family of well-known Newton-Cotes formulas with 1, 2, 3, and 4 points. These inequalities build upon previous works and include both new and previously established results, further enriching this area of research. The illustrative example, along with 2D and 3D graphical representations, provides strong evidence for the accuracy of the obtained results and highlights potential practical applications. Overall, this study expands the scope of understanding and paves the way for future research in the development of integral inequalities in the context of local fractional calculus.

Author contributions

Conceptualization: H.L., A.L. and F.J. Methodology: H.L., H.Y.X. and B.M. Validation: F.J. and B.M Investigation: H.L., A.L. and H.Y.X. Writing—original draft preparation: H.L. and A.L. Writing—review and editing: H.L., H.Y.X. and B.M. Visualization: A.L. and F.J. Supervision: F.J. and B.M. Project administration: A.L. and H.Y.X. All authors have read and agreed to the final version of the manuscript.

Data availability

Data sharing is not relevant to this article, as there was no generation or analysis of new data during the course of this study.

Declarations

Competing interests

The authors declare no competing interests.

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