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# Fourth order Hankel determinants for certain subclasses of modified sigmoid-activated analytic functions involving the trigonometric sine function

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**Abstract**

The aim of this paper is to introduce two new subclasses $\mathcal{R}_{\sin}^{m}(\Im)$ and $\mathcal{R}_{\sin}(\Im)$ of analytic functions by making use of subordination involving the sine function and the modified sigmoid activation function $\Im(v) = \frac{2}{1+e^{-v}}$, $v \geq 0$ in the open unit disc $E$. Our purpose is to obtain some initial coefficients, Fekete–Szego problems, and upper bounds for the third- and fourth-order Hankel determinants for the functions belonging to these two classes. All the bounds that we will find here are sharp. We also highlight some known consequences of our main results.

**Mathematics Subject Classification:** 30C45; 30C50; 30C80

**Keywords:** Analytic functions; Univalent functions; Salagean differential operator; Bounded turning functions; Sine function; Modified sigmoid activation function; Hankel determinants

## 1 Introduction

Let $\mathcal{A}$ denote the class of functions satisfying the following series form:

$$\chi(z) = z + \sum_{n=2}^{\infty} d_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. The functions $\chi$ having the series form (1.1) are called univalent in $E$ and denoted by $\mathcal{S}$, if it is one to one, that is, for all $z_1, z_2 \in E$, if $\chi(z_1) = \chi(z_2)$ implies $z_1 = z_2$.

In [1] the author defined the class $\mathcal{R}_\alpha$ $(\alpha \geq 0)$ having the functions that satisfy the condition

$$\mathrm{Re}\{\chi'(z) + \alpha z \chi''(z)\} > 0, \quad (z \in E).$$

They also proved that if $\chi \in \mathcal{R}_\alpha$ then $\chi$ is univalent in $E$. Singh and Singh [2] showed that if $\chi \in \mathcal{R}$ ($\alpha = 1$) then $\chi$ is starlike in $E$. Furthermore, they proved in [2] that the class $\mathcal{R}$ is closed under convolution, that is, if $\chi, g \in \mathcal{R}$ then $(\chi * g) \in \mathcal{R}$, where $*$ stands for convolution. Also in [3], Krzyz showed by an example that the class $\mathcal{R}$ is not a subset of the convex functions class $C$. In 2015, Noor et al. [4] generalized the class $\mathcal{R}$ by using the idea of multivalent functions and conic regions. Khan et al. [5] generalized it further in 2021.

An analytic function $w$ under the condition $w(0) = 0$ and $|w(z)| < 1$ is known as a Schwarz function. Consider the functions $\chi$, $g \in \mathcal{A}$, we say that $\chi$ is subordinate to $g$ (indicated by $\chi \prec g$) if $\chi(z) = g(w(z))$. Further, if $g$ is univalent in $E$, then $\chi \prec g \Leftrightarrow \chi(0) = g(0)$ and $\chi(E) \subset g(E)$. The class $\mathcal{P}$ denotes the well-known class of Caratheodory functions [6], which satisfy the conditions $p(0) = 1$ and $\operatorname{Re}(p(z) > 0$, where $z \in E$. Every $p \in \mathcal{P}$, having the series form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.2}$$

Since the nineteenth century, when the geometric function theory was established, coefficient bounds have been continuously important. The Bieberbach conjecture [7], which came to be in 1916, offered an innovative field of study for this field of research. He conjectured that for every $\chi \in \mathcal{S}$ having the series form (1.1), $|d_n| \leq n$, $n \geq 4$. This conjecture was attempted to be proved for a long time by mathematicians until de-Branges [8] proved it in 1985.

In 1992, the authors contributed in their article [9] by revealing the basic structure of families of univalent functions $\mathcal{S}^*(\varphi)$, where $\varphi$ is an analytic function satisfying $\varphi(0) > 0$ and $\operatorname{Re}(\varphi(z)) > 0$ in $E$. When we fix $\varphi(z) = \frac{1+z}{1-z}$, then $\mathcal{S}^*(\varphi) \approxeq \mathcal{S}^*$. Several subfamilies of generalized analytic functions have been studied recently as a specific case of $\mathcal{S}^*(\varphi)$.

For example, Janowski [10] examined the class of Janowski starlike functions $\mathcal{S}^*[L, M]$ ($-1 \leq M < L \leq 1$). Furthermore, by choosing $L = (1 - 2\alpha)$ and $M = -1$, we get $\mathcal{S}^*(\alpha)$ ($0 \leq \alpha < 1$). Sokól and Stankiewicz [11] set $\varphi(z) = \sqrt{1+z}$ and defined the family of class $S^*_{\mathcal{L}}$

$$S^*_{\mathcal{L}} = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec \sqrt{1+z}\right\}.$$

Recently, the authors [12, 13] chose $\varphi(z) = 1 + \sin z$ and $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ and defined the following classes of convex, starlike, and bounded turning functions:

$$\mathcal{C}_{\sin} = \left\{\chi \in \mathcal{S} : 1 + \frac{z\chi''(z)}{\chi'(z)} \prec 1 + \sin z\right\},$$
$$\mathcal{S}^*_{\sin} = \left\{\chi \in \mathcal{A} : \frac{z\chi'(z)}{\chi(z)} \prec 1 + \sin z\right\},$$
$$\mathcal{R}_{\sin} = \left\{\chi \in \mathcal{A} : \chi'(z) + z\chi''(z) \prec 1 + \sin z\right\},$$
$$\mathcal{R}_{\text{card}} = \left\{\chi \in \mathcal{A} : \chi'(z) + z\chi''(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2\right\}.$$

Authors investigated initial bounds, Fekete–Szego problems, and third Hankel determinant for the above mentioned classes. In [13, 14] authors introduced the class of starlike functions whose image under an open unit has a cardioid form. In [15], Mendiratta et al. studied the function class $\mathcal{S}_e^* \equiv \mathcal{S}^*(e^z)$ of starlike functions by using the idea of exponential functions and subordination technique. This class was recently generalized by Srivastava et al. [16], who also found an upper bound for the third-order Hankel determinant.

In 1966 Pommerenke [17] explored research on the Hankel determinants for univalent functions, which were further investigated by Noonan and Thomas [18]. For $\chi \in \mathcal{A}$, the $j$th Hankel determinant is defined by

$$\mathcal{H}_{j,n}(\chi) = \begin{vmatrix} d_n & d_{n+1} & \cdots & d_{n+j-1} \\ d_{n+1} & d_{n+2} & \cdots & d_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n+j-1} & d_{n+j-2} & \dots & d_{n+2j-2} \end{vmatrix}, \tag{1.3}$$

where $n, j \in \mathbb{N}$ and $d_1 = 1$.

For different values of $j$ and $n$, the $j$th Hankel determinant $\mathcal{H}_{j,n}(\chi)$ has a different form. For example, Fekete–Szego functional that is

$$\mathcal{H}_{2,1}(\chi) = \left|d_3 - d_2^2\right| \quad \text{for } j = 2 \text{ and } n = 1,$$

and its modified form is $|d_3 - \mu d_2^2|$, where $\mu \in \mathbb{R}$ (or $\mathbb{C}$) (see [19]). The second Hankel determinant was similarly provided by Janteng [20] in the following form:

$$\mathcal{H}_{2,2}(\chi) = \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} = \left(d_2 d_4 - d_3^2\right),$$

and a number of scholars then looked into it for a few other classes of analytic functions. Further, the third Hankel determinant form is indicated below:

$$\mathcal{H}_{3,1}(\chi) = d_3\left(d_2 d_4 - d_3^2\right) - d_4(d_4 - d_2 d_3) + d_5\left(d_3 - d_2^2\right) \quad \text{for } j = 3 \text{ and } n = 1. \tag{1.4}$$

In 2021, for $\chi \in \mathcal{S}$, the authors in [21] found Hankel determinants of second and third order

$$\left|\mathcal{H}_{2,2}(\chi)\right| \le \lambda, \qquad 1 \le \lambda \le \frac{11}{3}$$

and

$$\left|\mathcal{H}_{3,1}(\chi)\right| \le \lambda, \qquad \frac{4}{9} \le \lambda \le \frac{32 + \sqrt{285}}{15}.$$

Recently, different researchers are active to find the sharp bounds of $\mathcal{H}_{j,n}(\chi)$ for a different family of functions. For instance, Cho et al. [22, 23] computed bounds of the second Hankel determinant of the classes of convex, starlike, and bounded turning. Compared to the second and third Hankel determinants, the mathematical computation of the fourth Hankel determinant is significantly more difficult. For specific classes of univalent functions, Babalola [24] calculated the third Hankel determinant in 2010. See the following articles for more details [25–28].

After that, a number of researchers examined the third Hankel determinant for different subclasses of analytic and bi-univalent functions using the same methodology. Zaprawa [29] in 2017 investigated third Hankel determinant for two basic subclasses of univalent functions class as follows:

$$\left|\mathcal{H}_{3,1}(\chi)\right| \le \left\{ \begin{array}{ll} 1 & \text{if } \chi \in \mathcal{S}^* \text{ (class of starlike functions)} \\ \frac{49}{540} & \text{if } \chi \in \mathcal{K} \text{ (class of convex functions)} \end{array} \right\}.$$

But in [30] authors improved the above result in the year 2018 and proved that $|\mathcal{H}_{3,1}(\chi)| \le \frac{8}{9}$, $(\chi \in \mathcal{S}^*)$. In 2021, Zaprawa et al. [31] again improved the result of [30] as follows:

$$\left|\mathcal{H}_{3,1}(\chi)\right| \le \frac{5}{9}, \quad \chi \in \mathcal{S}^*.$$

For the analysis of power series with integral coefficients and singularities, $\mathcal{H}_{j,n}(\chi)$ is very helpful. Numerous technological studies have made use of Hankel determinants, particularly those that depend mainly on mathematical approaches. Readers interested in understanding how the solutions to the above listed problems make use of Hankel determinants ought to read [32–40].

Let $\mathcal{A}_{\Im}$ denote the class of sigmoid functions having the form (see [41])

$$\chi_{\Im}(z) = z + \sum_{n=2}^{\infty} \Im(\nu) d_n z^n, \tag{1.5}$$

where

$$\Im(\nu) = \frac{2}{1 + e^{-\nu}}, \quad \nu \ge 0. \tag{1.6}$$

From (1.6) we see that $\Im(0) = 1$ and $\mathcal{A}_1 = \mathcal{A}$.

**Definition 1.1** [42]. The Sălăgean type differential operator $S^m : \mathcal{A}_{\Im} \to \mathcal{A}_{\Im}$ is defined by

$$\begin{aligned} &S_q^0 \chi_{\Im}(z) = \chi_{\Im}(z), \qquad S^1 \chi(z) = z\chi_{\Im}'(z), \quad \ldots, \\ &S^m \chi_{\Im}(z) = zS\big(S^{m-1}\chi_{\Im}(z)\big), \end{aligned} \tag{1.7}$$

where $\chi_{\Im}(z) \in \mathcal{A}_{\Im}$, $m \in \mathbb{N} \cup \{0\}$. It is easy to prove that if

$$\chi_{\Im}(z) = z + \sum_{n=2}^{\infty} \Im(\nu) d_n z^n \in \mathcal{A}_{\Im},$$

then

$$S^m \chi_{\Im}(z) = z + \sum_{n=2}^{\infty} n^m \Im(\nu) d_n z^n.$$

*Remark* 1.2 When $\Im(\nu) = 1$, we have the Sălăgean differential operator [43].

Here we define a new class of bounded turning functions connected with modified sigmoid function and sine functions as follows.

**Definition 1.3** A function $\chi_{\Im} \in \mathcal{R}^m_{\sin}(\Im)$, where $\chi_{\Im}$ is of the form (1.5), if

$$\left(S^m \chi_{\Im}(z)\right)' \prec 1 + \sin(z), \tag{1.8}$$

or it can be defined as

$$\frac{S^{m+1}\chi_{\Im}(z)}{z} \prec 1 + \sin(z).$$

When $\nu = 0$ and $m = 0$ in Definition 1.3, we get a known class proved in [44].
When $m = 0$ in Definition 1.3, we get the following function class.

**Definition 1.4** A function $\chi_{\Im} \in \mathcal{R}_{\sin}(\Im)$, where $\chi_{\Im}$ is of the form (1.5), if

$$\chi'_{\Im}(z) \prec 1 + \sin(z). \tag{1.9}$$

## 2 Set of lemmas

**Lemma 2.1** *Let the function p be of the form* (1.2), *then*

$$|c_n| \leq 2, \quad n \geq 1, \tag{2.1}$$

$$|c_{n+k} - \mu c_n c_k| < 2, \quad \textit{for } 0 \leq \mu \leq 1 \tag{2.2}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad \textit{for } n + m = k + l, \tag{2.3}$$

$$\left|c_{n+2k} - \mu c_n c_k^2\right| \leq 2(1 + 2\mu) \quad \textit{for } \mu \in \mathbb{R}. \tag{2.4}$$

$$\left|c_2 - \frac{c_l^2}{2}\right| \leq 2 - \frac{|c_l^2|}{2}, \tag{2.5}$$

*for complex number* $\mu$, *we have*

$$\left|c_2 - \mu c_2^2\right| \leq 2\max\left\{1, |2\mu - 1|\right\}. \tag{2.6}$$

*For the results in* (2.1), (2.5), (2.2), (2.4), (2.3), *see* [45]. *Also, see* [46] *for inequality* (2.6)

**Lemma 2.2** [47]. *Let the function* $p \in \mathcal{P}$ *be given by* (1.2)), *then*

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \leq 2,$$

*if*

$$0 \leq B \leq 1, \quad \textit{and} \quad B(2B - 1) \leq D \leq B.$$

**Lemma 2.3** [48, 49]. *Let the function* $p \in \mathcal{P}$ *be given by* (1.2)), *then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

*and*

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - \left(4 - c_1^2\right)c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - \left|x^2\right|\right)z,$$

*where* $x, z \in \mathbb{C}$ *with* $|z| \leq 1$ *and* $|x| \leq 1$.

**Lemma 2.4** [50]. *Consider the function* $p \in \mathcal{P}$ *of the form* (1.2), $0 < \gamma < 1$, $0 < \alpha < 1$, *and*

$$\begin{aligned} &8\gamma(1-\gamma)\left\{(\alpha\beta - 2\lambda)^2 + \left(\alpha(\gamma+\alpha) - \beta\right)^2\right\} \\ &\quad + \alpha(1-\alpha)(\beta - 2\gamma\alpha)^2 \\ &\leq 4\alpha^2\gamma(1-\alpha)^2(1-\gamma). \end{aligned} \tag{2.7}$$

*Then*

$$\left|\lambda b_1^4 + \gamma b_2^2 + 2\alpha b_1 b_3 - \frac{3}{2}\beta b_1^2 b_2 - b_4\right| \leq 2. \tag{2.8}$$

We divided our paper into four parts. In Sect. 1, we give some basic definitions including some subclasses of analytic functions, such as starlike, bounded turning, and convex functions, also we give the definitions of the Hankel determinant, the modified sigmoid function, and the sine function. Inspired by the above mentioned work, we create new classes of analytic functions related to modified sigmoid function and sine functions. Also we consider the Salagean type of differential operator and define a new class of analytic functions. In Sect. 2, we use known lemmas to prove our article's main results. For functions $\chi$ in the classes $\mathcal{R}_{\sin}^m(\Im)$ and $\mathcal{R}_{\sin}(\Im)$, we first calculate the initial coefficient bounds, Fekete–Szego problem, and the second, third, and fourth Hankel determinants in Sect. 3 and highlight some known results.

## 3 Main results

Main findings for the function class $\mathcal{R}_{\sin}^m(\Im)$.

**Theorem 3.1** *If* $\chi_\Im \in \mathcal{R}_{\sin}^m(\Im)$ *where* $\chi_\Im$ *has the form* (1.5), *then*

$$|d_2| \leq \frac{1}{2(2^m\Im(\nu))}, \tag{3.1}$$

$$|d_3| \leq \frac{1}{3(3^m\Im(\nu))}, \tag{3.2}$$

$$|d_4| \leq \frac{1}{4(4^m\Im(\nu))}, \tag{3.3}$$

$$|d_5| \leq \frac{1}{5(5^m\Im(\nu))}, \tag{3.4}$$

*where* $\Im(\nu)$ *is given by* (1.6). *The results are sharp.*

*Proof* Let $\chi_\Im \in \mathcal{R}_{\sin}^m(\Im)$, then from relation (1.8) we have

$$\left(S^m\chi_\Im(z)\right)' = 1 + \sin\left(w(z)\right), \tag{3.5}$$

where $w$ is a Schwarz function.

Now consider a function $p$ such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{3.6}$$

then $p \in \mathcal{P}$. From (3.6), a simple computation yields

$$w(z) = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

From (1.8), we can write

$$\begin{aligned}\big(S^m \chi_{\Im}(z)\big)' = {}& 1 + 2\big(2^m \Im(\nu)\big) d_2 z + 3\big(3^m \Im(\nu)\big) d_3 z^2 \\ & + 4\big(4^m \Im(\nu)\big) d_4 z^3 + 5\big(5^m \Im(\nu)\big) d_5 z^4 + \cdots.\end{aligned} \tag{3.7}$$

By using the above values and after simplification, we get

$$\begin{aligned}1 + \sin\big(w(z)\big) = {}& 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right) z^3 \\ & + \left(-\frac{c_1^4}{32} + \frac{5c_1^2 c_2}{16} - \frac{c_1 c_3}{2} - \frac{c_2^2}{2} + \frac{c_4}{2}\right) z^4 + \cdots.\end{aligned} \tag{3.8}$$

From (3.5), (3.7), and (3.8), it follows that

$$d_2 = \frac{c_1}{4(2^m \Im(\nu))}, \tag{3.9}$$

$$d_3 = \frac{1}{3(3^m \Im(\nu))} \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right), \tag{3.10}$$

$$d_4 = \frac{1}{4(4^m \Im(\nu))} \left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right), \tag{3.11}$$

and

$$d_5 = \frac{1}{5(5^m \Im(\nu))} \left(\frac{-1}{32} c_1^4 - \frac{c_2^2}{4} - \frac{c_1 c_3}{2} + \frac{5c_1^2 c_2}{16} + \frac{c_4}{2}\right). \tag{3.12}$$

Applying relation (2.1) in (3.9), we obtain

$$|d_2| \leq \frac{1}{2(2^m \Im(\nu))}.$$

Using relations (2.1) and (2.5) on (3.10), we obtain

$$\begin{aligned}|d_3| &= \frac{1}{6(3^m \Im(\nu))} \left| c_2 - \frac{c_1^2}{2} \right| \\ &\leq \frac{1}{6(3^m \Im(\nu))} \left(2 - \frac{|c_1|^2}{2}\right) \\ &= \frac{1}{3(3^m \Im(\nu))}.\end{aligned}$$

Rearranging (3.11) gives

$$\begin{aligned}|d_4| &= \frac{1}{4(4^m\Im(\nu))}\left|\frac{c_3}{2}-\frac{c_1c_2}{2}+\frac{5}{24}c_1^3\right|.\\ &= \frac{1}{8(4^m\Im(\nu))}\left|c_3-2Bc_1c_2+Dc_1^3\right|.\end{aligned}$$

Let

$$B=\frac{1}{2}\quad\text{and}\quad D=\frac{5}{24}.$$

We can observe that $0<B$ and $B(2B-1)<D<B$. Therefore, using Lemma 2.2 leads us to

$$|d_4|\le\frac{1}{4(4^m\Im(\nu))}.$$

From (3.12) it follows that

$$|d_5|=\frac{1}{5(5^m\Im(\nu))}\left|\frac{-1}{32}c_1^4-\frac{c_2^2}{4}-\frac{c_1c_3}{2}+\frac{5c_1^2c_2}{16}+\frac{c_4}{2}\right|$$

or

$$\begin{aligned}|d_5| &= \frac{1}{10(5^m\Im(\nu))}\left|\frac{1}{16}c_1^4+\frac{c_2^2}{2}+2\left(\frac{c_1c_3}{2}\right)-\frac{3}{2}\left(\frac{5c_1^2c_2}{24}\right)-c_4\right|\\ &= \frac{1}{10(5^m\Im(\nu))}\left|\lambda c_1^4+\gamma c_2^2+2\alpha c_1c_3-\frac{3}{2}\beta c_1^2c_2-c_4\right|,\end{aligned}$$

where

$$\lambda=\frac{1}{16},\qquad\gamma=\frac{1}{2},\qquad\alpha=\frac{1}{2},\qquad\beta=\frac{5}{24}.$$

We see that $0<\gamma<1$ and $0<\alpha<1$. Now we calculate inequality (3.1), we see that

$$\begin{aligned}&8\gamma(1-\gamma)\left\{(\alpha\beta-2\lambda)^2+\left(\alpha(\gamma+\alpha)-\beta\right)^2\right\}+\alpha(1-\alpha)(\beta-2\gamma\alpha)^2\\ &\quad\le 4\alpha^2\gamma(1-\alpha)^2(1-\gamma),\end{aligned}$$

where

$$\begin{aligned}&8\gamma(1-\gamma)=2,\qquad(\alpha\beta-2\lambda)^2=\frac{1}{2304},\qquad\left(\alpha(\gamma+\alpha)-\beta\right)^2=\frac{49}{576},\\ &\alpha(1-\alpha)(\beta-2\gamma\alpha)^2=\frac{49}{2304},\qquad 4\alpha^2\gamma(1-\alpha)^2(1-\gamma)=\frac{1}{16}.\end{aligned}$$

By using Lemma 2.4, we get

$$|d_5|\le\frac{1}{5(5^m\Im(\nu))}.$$

For $n = 2,3,4,5$, we take the function $\chi_n(z) = z + \cdots$, such that

$$\left(S^m \chi_{n,\Im}(z)\right)' = 1 + \sin\left(z^{n-1}\right), \quad z \in E,$$

then

$$\left(S^m \chi_{\Im}(z)\right)' \prec 1 + \sin(z)$$

and $\chi_{n,\Im} \in \mathcal{R}^m_{\sin}(\Im)$, where

$$\chi_{2,\Im}(z) = z + \frac{1}{2(2^m \Im(\nu))} z^2 + \cdots, \quad z \in E, \tag{3.13}$$

$$\chi_{3,\Im}(z) = z + \frac{1}{3(3^m \Im(\nu))} z^3 + \cdots, \quad z \in E, \tag{3.14}$$

$$\chi_{4,\Im}(z) = z + \frac{1}{4(4^m \Im(\nu))} z^4 + \cdots, \quad z \in E, \tag{3.15}$$

$$\chi_{5,\Im}(z) = z + \frac{1}{5(5^m \Im(\nu))} z^5 + \cdots, \quad z \in E, \tag{3.16}$$

which shows that the bounds are sharp. □

For $m = 0$ and $\nu = 0$, we get the known sharp result proved in [44].

**Conjecture 3.2** *If a function $\chi_{\Im} \in \mathcal{R}^m_{\sin}(\Im)$ is of the form (1.8), then*

$$|d_n| \le \frac{1}{n(n^m \Im(\nu))} \quad \text{for } n \ge 6, \tag{3.17}$$

*where $\Im(\nu)$ is given by (1.6).*

**Theorem 3.3** *If a function $\chi_{\Im}$ given in (1.5) belongs to the class $\mathcal{R}_{\sin}(\Im)$, then*

$$|d_2| \le \frac{1}{2\Im(\nu)}, \qquad |d_3| \le \frac{1}{3\Im(\nu)}, \qquad |d_4| \le \frac{1}{4\Im(\nu)}, \qquad |d_5| \le \frac{1}{5\Im(\nu)},$$

*where $\Im(\nu)$ is given by (1.6). The results are sharp for functions (3.13) to (3.16).*

*Proof* Using the same procedure as we adopted in Theorem 3.2, we obtain the required result of Theorem 3.3. □

**Conjecture 3.4** *If a function $\chi_{\Im} \in \mathcal{R}_{\sin}(\Im)$ is of the form (1.9), then*

$$|d_n| \le \frac{1}{n\Im(\nu)}, \quad n \ge 6,$$

*where $\Im(\nu)$ is given by (1.6).*

**Theorem 3.5** *Let $\chi_{\Im} \in \mathcal{R}^m_{\sin}(\Im)$. Then, for a complex number $\rho$,*

$$\left|d_3 - \rho d_2^2\right| \le \frac{1}{3(3^m \Im(\nu))} \max\left\{1, \left|\frac{3\rho(3^m \Im(\nu))}{4(2^m \Im(\nu))^2}\right|\right\}, \tag{3.18}$$

*where $\Im(\nu)$ is given by (1.6). The result is sharp.*

*Proof* Using (3.9) and (3.10), one may write

$$\left|d_3-\rho d_2^2\right| = \frac{1}{6(3^m\Im(\nu))}\left|c_2-\left(\frac{4(2^m\Im(\nu))^2+3\rho(3^m\Im(\nu))}{8(2^m\Im(\nu))^2}\right)c_1^2\right|.$$

Application of relation (2.6) gives

$$\left|d_3-\rho d_2^2\right| \le \frac{1}{3(3^m\Im(\nu))}\max\left\{1,\left|\frac{3\rho(3^m\Im(\nu))}{4(2^m\Im(\nu))^2}\right|\right\}.$$

For the sharpness of (3.18), we consider (3.13) with $n=2$

$$\chi_{2,\Im}(z) = z+\frac{1}{2(2^m\Im(\nu))}z^2+\cdots,\quad z\in E,$$

which gives equality in (3.18) when $|\rho|\ge\frac{4(2^m\Im(\nu))^2}{3(3^m\Im(\nu))}$, namely

$$\left|d_3-\rho d_2^2\right| = \left|\rho d_2^2\right| = \frac{|\rho|}{4(2^m\Im(\nu))^2}.$$

For $|\rho|\le\frac{4(2^m\Im(\nu))^2}{3(3^m\Im(\nu))}$, consider

$$\chi_3(z) = z+\frac{1}{3(3^m\Im(\nu))}z^3+\cdots,\quad z\in E,$$

which gives

$$\left|d_3-\rho d_2^2\right| = |d_3| = \frac{1}{3(3^m\Im(\nu))} = \frac{1}{3(3^m\Im(\nu))}\max\left\{1,\left|\frac{3\rho(3^m\Im(\nu))}{4(2^m\Im(\nu))^2}\right|\right\}.$$ □

**Corollary 3.6** *If $\chi_\Im\in\mathcal{R}_{\sin}^m(\Im)$, then for a complex number $|\rho|\le\frac{4(2^m\Im(\nu))^2}{3(3^m\Im(\nu))}$, we have*

$$\left|d_3-\rho d_2^2\right| \le \frac{1}{3(3^m\Im(\nu))}, \tag{3.19}$$

*where $\Im(\nu)$ is given by (1.6). This inequality is sharp.*

**Corollary 3.7** [44]. *Let $\chi\in\mathcal{R}_{\sin}$. Then, for a complex number $\rho$,*

$$\left|d_3-\rho d_2^2\right| \le \frac{1}{3}\max\left\{1,\left|\frac{3\rho}{4}\right|\right\}.$$

**Theorem 3.8** *Let $\chi_\Im\in\mathcal{R}_{\sin}(\Im)$. Then, for a complex number $\rho$,*

$$\left|d_3-\rho d_2^2\right| \le \frac{1}{3\Im(\nu)}\max\left\{1,\frac{\rho(3\Im(\nu))}{(2\Im(\nu))^2}\right\},$$

*where $\Im(\nu)$ is given by (1.6). The result is sharp.*

*Proof* Using the same procedure as we adopted in Theorem 3.5, we get the required result. □

**Theorem 3.9** *Let* $\chi_{\Im} \in \mathcal{R}^m_{\sin}(\Im)$. *Then*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{3(3^m \Im(\nu))}, \tag{3.20}$$

*where* $\Im(\nu)$ *is given by* (1.6). *This inequality is sharp for the function*

$$\chi_3(z) = z + \frac{1}{3(3^m \Im(\nu))} z^3 + \cdots, \quad z \in E.$$

*Proof* From (3.9) and (3.10), we have

$$\begin{aligned}\left|d_3 - d_2^2\right| &= \frac{1}{6(3^m \Im(\nu))}\left|c_2 - \frac{1}{2}\left(1 + \frac{3(3^m \Im(\nu))}{(2^m \Im(\nu))^2}\right)c_1^2\right|, \\ &= \frac{1}{6(3^m \Im(\nu))}\left|c_2 - \phi c_1^2\right|,\end{aligned}$$

where $\phi = \frac{1}{2}(1 + \frac{3(3^m \Im(\nu))}{(2^m \Im(\nu))^2})$ since $\nu \geq 0$ and $0 < \phi < 1$. Now, by using (2.2) for $n = 2$ and $k = 1$, we obtain (3.20). □

**Theorem 3.10** *Let* $\chi_{\Im} \in \mathcal{R}_{\sin}(\Im)$. *Then*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{3\Im(\nu)},$$

*where* $\Im(\nu)$ *is given by* (1.6). *This inequality is sharp for*

$$\chi_3(z) = z + \frac{1}{3\Im(\nu)} z^3 + \cdots, \quad z \in E.$$

*Proof* Using the same procedure as we adopted in Theorem 3.9, we obtain the required result of Theorem 3.10. □

**Theorem 3.11** *Let* $\chi_{\Im} \in \mathcal{R}^m_{\sin}(\Im)$. *Then*

$$|d_2 d_3 - d_4| \leq \frac{1}{4(4^m \Im(\nu))}. \tag{3.21}$$

*This inequality is sharp for*

$$\chi_{4,\Im}(z) = z + \frac{1}{4(4^m \Im(\nu))} z^4 + \cdots, \quad z \in E,$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* From (3.9), (3.10), and (3.11), we have

$$|d_2 d_3 - d_4| = \left|\begin{array}{l}\left(\dfrac{4(4^m \Im(\nu)) + 5(2^m \Im(\nu))(3^m \Im(\nu))}{192(2^m \Im(\nu))(3^m \Im(\nu))(4^m \Im(\nu))}\right)c_1^3 \\ -\left(\dfrac{(4^m \Im(\nu)) + 3(2^m \Im(\nu))(3^m \Im(\nu))}{24(2^m \Im(\nu))(3^m \Im(\nu))(4^m \Im(\nu))}\right)c_1 c_2 + \left(\dfrac{1}{8(4^m 4^m \Im(\nu))}\right)c_3\end{array}\right|.$$

By using the applications of Lemma 2.3, and after some simple calculations, we get

$$|d_2d_3 - d_4| \leq \frac{1}{4(4^m\Im(\nu))}.$$

This completes the proof. □

**Corollary 3.12** [44]. *Let* $\chi \in \mathcal{R}_{\sin}$ $(m = 0$ *and* $\nu = 0)$. *Then*

$$|d_2d_3 - d_4| \leq \frac{1}{4}.$$

*This inequality is sharp for*

$$\chi_4(z) = z + \frac{1}{4}z^4 + \cdots .$$

**Theorem 3.13** *Let* $\chi_\Im \in \mathcal{R}_{\sin}(\Im)$. *Then*

$$|d_2d_3 - d_4| \leq \frac{1}{4\Im(\nu)},$$

*where* $\Im(\nu)$ *is given by* (1.6). *This inequality is sharp for*

$$\chi_{4,\Im}(z) = z + \frac{1}{4\Im(\nu)}z^4 + \cdots, \quad z \in E.$$

*Proof* Using the same procedure of Theorem 3.11, we obtain the result of Theorem 3.13. □

*Remark* 3.14 For $\nu = 0$, in Theorem 3.13, we get known result proved in [44].

**Theorem 3.15** *Let* $\chi_\Im \in \mathcal{R}^m_{\sin}(\Im)$. *Then*

$$\left|d_2d_4 - d_3^2\right| \leq \frac{1}{9(3^m\Im(\nu))^2}. \tag{3.22}$$

*This inequality is sharp for*

$$\chi_{3,\Im}(z) = z + \frac{1}{3(3^m\Im(\nu))}z^3 + \cdots, \quad z \in E,$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* From (3.9), (3.10), and (3.11), we have

$$\left|d_2d_4 - d_3^2\right| = \left|C(m,\nu)c_1c_3 - B(m,\nu)c_1^2c_2 - D(m,\nu)c_2^2 - A(m,\nu)c_1^4\right|,$$

where

$$A(m,\nu) = \frac{5}{(2^m\Im(\nu))(4^m\Im(\nu))}, \qquad D(m,\nu) = \frac{1}{32(3^m\Im(\nu))^2}$$

$$C(m,\nu) = \frac{1}{32(2^m\Im(\nu))(4^m\Im(\nu))},$$

$$B(m,\nu)=\frac{9(3^m\Im(\nu))^2-8(2^m\Im(\nu))(4^m\Im(\nu))}{288(2^m\Im(\nu))(4^m\Im(\nu))(3^m\Im(\nu))^2}.$$

Using Lemma 2.3, we obtain

$$\left|d_2d_4-d_3^2\right|=\left|\begin{array}{l} T_1(m,\nu)c_1^4+T_2(m,\nu)c_1^2\big(4-c_1^2\big)x-\big(3^m\Im(\nu)\big)T_3(m,\nu)c_1^2x \\ -\frac{(4-c_1^2)c_1^2x^2}{128(2^m\Im(\nu))(4^m\Im(\nu))}-\frac{(4-c_1^2)^2x^2}{144(3^m\Im(\nu))^2}+\frac{c(4-c_1^2)(1-|x|^2)}{144(2^m\Im(\nu))(4^m\Im(\nu))}z \end{array}\right|,$$

where

$$\begin{aligned} T_1(m,\nu)&=\frac{1}{128(2^m\Im(\nu))(4^m\Im(\nu))}-T_3(q)\\ &\quad+\left(\frac{15(3^m\Im(\nu))^2-16(2^m\Im(\nu))(4^m\Im(\nu))}{2304(2^m\Im(\nu))(4^m\Im(\nu))(3^m\Im(\nu))^2}\right)-\frac{1}{144(3^m\Im(\nu))^2},\\ T_2(m,\nu)&=\left(\big(3^m\Im(\nu)\big)^2-\big(2^m\Im(\nu)\big)\big(4^m\Im(\nu)\big)\right),\\ T_3(m,\nu)&=\frac{(9(3^m\Im(\nu))^2-8(2^m\Im(\nu))(4^m\Im(\nu)))}{576(2^m\Im(\nu))(4^m\Im(\nu))(3^m\Im(\nu))^2}. \end{aligned}$$

Let $|z|=1$, $|x|=t$, $t\in[0;1]$, $|c_1|=c\in[0;2]$. Then, using the triangle inequality, we get

$$\begin{aligned} &\left|d_2d_4-d_3^2\right|\\ &\quad\le\left|T_1(m,\nu)\right|c^4+\left|T_2(m,\nu)\right|c^2\big(4-c^2\big)t+\big(3^m\Im(\nu)\big)T_3(m,\nu)c_1^2t\\ &\qquad+\frac{(4-c_1^2)c_1^2t^2}{128(2^m\Im(\nu))(4^m\Im(\nu))}+\frac{(4-c_1^2)^2t^2}{144(3^m\Im(\nu))^2}+\frac{c(4-c^2)(1-|t|^2)}{144(2^m\Im(\nu))(4^m\Im(\nu))}\\ &\quad=H(c,t). \end{aligned}$$

Taking the derivative of $H(c,t)$ w.r.t., $t$ we get

$$\begin{aligned} \frac{\partial H(c,t)}{\partial t}&=\left|T_2(m,\nu)\right|c^2\big(4-c^2\big)+\big(3^m\Im(\nu)\big)\left|T_3(m,\nu)\right|c_1^2\\ &\quad+\frac{(4-c_1^2)^2t}{72(3^m\Im(\nu))^2}\\ &>0, \end{aligned}$$

which shows that $H(c,t)$ increases on $[0,1]$ with respect to $t$. That is, $H(c,t)$ has a maximum value at $t=1$, which is

$$\begin{aligned} \max H(c,t)=H(c,1)&\\ &=\left|T_1(m,\nu)\right|c^4+\left|T_2(m,\nu)\right|c^2\big(4-c^2\big)+\big(3^m\Im(\nu)\big)T_3(m,\nu)c^2\\ &\quad+\frac{(4-c^2)c^2}{128(2^m\Im(\nu))(4^m\Im(\nu))}+\frac{(4-c^2)^2}{144(3^m\Im(\nu))^2}\\ &=G(c). \end{aligned}$$

Differentiation gives

$$G'(c)=4\left|T_1(m,\nu)\right|c^3+\left|T_2(m,\nu)\right|8c+2\big(3^m\Im(\nu)\big)\left|T_3(m,\nu)\right|c$$

$$
\begin{aligned}
&+\frac{8c-2c^3-2c^3}{128(2^m\Im(\nu))(4^m\Im(\nu))}-\frac{4c(4-c^2)}{144(3^m\Im(\nu))^2}\\
&=\biggl(\bigl|T_2(m,\nu)\bigr|8+2\bigl(3^m\Im(\nu)\bigr)\bigl|T_3(m,\nu)\bigr|\\
&\quad-\frac{16}{144(3^m\Im(\nu))^2}+\frac{8}{128(2^m\Im(\nu))(4^m\Im(\nu))}\biggr)c\\
&\quad+\biggl(\bigl|T_1(m,\nu)\bigr|-\frac{1}{128(2^m\Im(\nu))(4^m\Im(\nu))}+\frac{1}{144(3^m\Im(\nu))^2}\biggr)c^3.
\end{aligned}
$$

If $G'(c)=0$, then the root is $c=0$ and

$$
c^2=\frac{-(|T_2(m,\nu)|8+2(3^m\Im(\nu))|T_3(m,\nu)|-\frac{16}{144(3^m\Im(\nu))^2}+\frac{8}{128(2^m\Im(\nu))(4^m\Im(\nu))})}{4(|T_1(m,\nu)|-\frac{1}{128(2^m\Im(\nu))(4^m\Im(\nu))}+\frac{1}{144(3^m\Im(\nu))^2})}.
$$

Again, taking the derivative of $G'(c)$, we have

$$
\begin{aligned}
G''(0)&=\bigl|T_2(m,\nu)\bigr|8+2\bigl(3^m\Im(\nu)\bigr)\bigl|T_3(m,\nu)\bigr|\\
&\quad-\frac{16}{144(3^m\Im(\nu))^2}+\frac{8}{128(2^m\Im(\nu))(4^m\Im(\nu))}\\
&\quad+3\biggl(\bigl|T_1(m,\nu)\bigr|-\frac{1}{128(2^m\Im(\nu))(4^m\Im(\nu))}+\frac{1}{144(3^m\Im(\nu))^2}\biggr)c^2.
\end{aligned}
$$

We see that

$$
G''(0)<0,
$$

so the function $G(c)$ can attain the maximum value at $c=0$; which is

$$
\bigl|d_2d_4-d_3^2\bigr|\leq\frac{1}{9(3^m\Im(\nu))^2}.
$$

This completes the result. □

**Theorem 3.16** *Let* $\chi_\Im\in\mathcal{R}_{\sin}(\Im)$. *Then*

$$
\bigl|d_2d_4-d_3^2\bigr|\leq\frac{1}{9(\Im(\nu))^2},
$$

*where* $\Im(\nu)$ *is given by* (1.6). *This inequality is sharp for*

$$
\chi_{3,\Im}(z)=z+\frac{1}{3\Im(\nu)}z^3+\cdots,\quad z\in E.
$$

*Proof* Using the same procedure as we adopted in Theorem 3.15, we obtain the result of Theorem 3.16. □

**Theorem 3.17** *Let* $\chi_\Im\in\mathcal{R}^m_{\sin}(\Im)$. *Then*

$$
\bigl|H_{3,1}(\chi)\bigr|\leq\frac{1}{27(3^m\Im(\nu))(3^m\Im(\nu))^2}+\frac{1}{16(4^m\Im(\nu))(4^m\Im(\nu))}
$$

$$+\frac{1}{15(5^m\Im(\nu))(3^m\Im(\nu))},$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* From (1.4), it is easy to see that

$$H_{3,1}(\chi)=d_3\big(d_2d_4-d_3^2\big)+d_4(d_2d_3-d_4)+d_5\big(d_3-d_2^2\big),$$

where $d_1=1$. This implies that

$$\big|H_{3,1}(\chi)\big|\leq|d_3|\big|d_2d_4-d_3^2\big|+|d_4||d_2d_3-d_4|+|d_5|\big|d_3-d_2^2\big|.$$

By using (3.1), (3.2), (3.3), (3.4), (3.19), (3.21), and (3.22), we have

$$\begin{aligned}\big|H_{3,1}(\chi)\big|&\leq\frac{1}{27(3^m\Im(\nu))(3^m\Im(\nu))^2}+\frac{1}{16(4^m\Im(\nu))(4^m\Im(\nu))}\\&\quad+\frac{1}{15(5^m\Im(\nu))(3^m\Im(\nu))},\end{aligned}$$

which is our required result. □

**Theorem 3.18** *Let* $\chi_\Im\in\mathcal{R}_{\sin}(\Im)$. *Then*

$$\begin{aligned}\big|H_{3,1}(\chi)\big|&\leq\frac{1}{27(\Im(\nu))^3}+\frac{1}{16(\Im(\nu))^2}\\&\quad+\frac{1}{15(\Im(\nu))^2},\end{aligned}$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* Using the same procedure as we adopted in Theorem 3.17, we obtain the result of Theorem 3.18. □

*Remark* 3.19 For $m=1$, $\nu=0$, in Theorem 3.18, we get known result proved in [44].

## 4 Bound of $|H_{4,1}(\chi)|$ for the functions class $R^m_{\sin}(\Im)$ and $R_{\sin}(\Im)$

First of all we can deduce the form of $H_{4,1}(\chi)$ from (1.3) in the following way:

$$\begin{aligned}H_{4,1}(\chi)&=d_7\big(H_{3,1}(\chi)\big)-2d_5d_6(d_2d_3-d_4)-2d_4d_6\big(d_2d_4-d_3^2\big)\\&\quad-d_6^2\big(d_3-d_2^2\big)+d_5^2\big(d_2d_4+2d_3^2\big)+d_5^2\big(d_2d_4-d_3^2\big)\\&\quad-d_5^3+d_4^4-3d_3d_4^2d_5.\end{aligned}\tag{4.1}$$

We need the following simple result for the function class $\mathcal{R}^m_{\sin}(\Im)$, that is, if $\chi\in\mathcal{R}^m_{\sin}(\Im)$ of the form (1.8), then

$$\begin{aligned}\big|d_2d_4+2d_3^2\big|&\leq\big|d_2d_4-d_3^2\big|+3|d_3|^2\\&\leq\frac{4}{9(3^m\Im(\nu))^2}.\end{aligned}\tag{4.2}$$

Now we move towards the forth-order Hankel determinant.

**Theorem 4.1** *Let* $\chi_\varphi \in \mathcal{R}^m_{\sin}(\Im)$. *Then*

$$\begin{aligned}
\left|H_{4,1}(\chi)\right| \le{}& \frac{1}{7(7^m\Im(\nu))}\Bigg(\frac{1}{27(3^m\Im(\nu))(3^m\Im(\nu))^2}\\
&+\frac{1}{16(4^m\Im(\nu))(4^m\Im(\nu))}+\frac{1}{15(5^m\Im(\nu))(3^m\Im(\nu))}\Bigg)\\
&+\frac{1}{108(4^m\Im(\nu))(6^m\Im(\nu))(3^m\Im(\nu))^2}\\
&+\frac{1}{120(5^m\Im(\nu))(6^m\Im(\nu))(4^m\Im(\nu))}\\
&+\frac{1}{108(3^m\Im(\nu))(6^m\Im(\nu))^2}+\frac{1}{225(3^m\Im(\nu))^2(5^m\Im(\nu))^2}\\
&+\frac{1}{54(3^m\Im(\nu))^2(5^m\Im(\nu))^2}\\
&-\frac{1}{125(5^m\Im(\nu))^3}+\frac{1}{256(4^m\Im(\nu))^4}\\
&-\frac{1}{80(3^m\Im(\nu))(4^m\Im(\nu))^2(5^m\Im(\nu))}.
\end{aligned}$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* Taking modulus on both sides of (4.1) and then applying the triangle inequality, we obtain

$$\begin{aligned}
\left|H_{4,1}(\chi)\right| \le{}& |d_7|\left|H_{3,1}(\chi)\right| + 2|d_4||d_6|\left|d_2d_4-d_3^2\right| + 2|d_5||d_6||d_2d_3-d_4|\\
&+|d_6|^2\left|d_3-d_2^2\right| + |d_5|^2\left|d_2d_4-d_3^2\right| + |d_5|^2\left|d_2d_4+2d_3^2\right|\\
&+|d_5|^3+|d_4|^4+3|d_3||d_4|^2|d_5|.
\end{aligned}$$

Now, by using (3.1), (3.2),(3.3),(3.4),(3.17), (3.20), (3.21), (3.22), and (4.2), we get the required result. □

**Theorem 4.2** *Let* $\chi_\varphi \in \mathcal{R}_{\sin}(\Im)$. *Then*

$$\begin{aligned}
\left|H_{4,1}(\chi)\right| \le{}& \frac{1}{7\Im(\nu)}\left(\frac{1}{27(\Im(\nu))^3}+\frac{1}{16(\Im(\nu))^2}+\frac{1}{15(\Im(\nu))^2}\right)\\
&+\frac{1}{108(\Im(\nu))^4}+\frac{1}{120(\Im(\nu))^3}\\
&+\frac{1}{108(\Im(\nu))^3}+\frac{1}{225(\Im(\nu))^4}+\frac{1}{54(\Im(\nu))^4}\\
&-\frac{1}{125(\Im(\nu))^3}+\frac{1}{256(\Im(\nu))^4}-\frac{1}{80(\Im(\nu))^4},
\end{aligned}$$

*where* $\Im(\nu)$ *is given by* (1.6).

*Proof* By using a similar method as we adopted in the above theorem, we get the required result. □

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No datasets were generated or analysed during the current study.

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**Competing interests**
The authors declare no competing interests.

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