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Sequence spaces derived by q_λ operators in ℓ_p spaces and their geometric properties

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Abstract

In this paper, we establish a novel category of sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ by utlizing q-analogue Λ^q of Λ-matrix. Our investigation outlines several topological characteristics and inclusion results of these newly introduced sequence spaces, specifically identifying them as BK-spaces. Subsequently, we demonstrate that these novel sequence spaces are of nonabsolute type and establish their isometric isomorphism with ℓ_p and ℓ_∞. Moreover, we obtain the α-, β-, and γ-duals of these sequence spaces. We further characterize the class $(\ell_p^{q_\lambda}, X)$ of matrices, where X is any of the spaces ℓ_∞, c, or c_0. Lastly, our study delves into the exploration of specific geometric properties exhibited by the space $\ell_p^{q_\lambda}$.

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1 Introduction and preliminaries

Let ω denote the standard representation of the set that contains all complex sequences $x = (x_k)_{k=0}^\infty$. Given $z \in \omega$ and $X \subset \omega$, the expression $z^{-1} * X$ denotes the set

$$\{a \in \omega : a \cdot z = (a_k z_k)_{k=0}^\infty \in X\}.$$

Consider $A = (a_{nk})_{n,k=0}^\infty$ as an infinite matrix composed of complex entries. Let $x = (x_k)_{k=0}^\infty \in \omega$ and A_n be the nth row of the matrix A. As customary, we denote

$$(Ax)_n = \sum_{k=0}^\infty a_{nk} x_k$$

for all $n \in \mathbb{N}_0$ and $Ax = \{(Ax)_n\}_{n=0}^\infty$ provided that all the series $(Ax)_n$ converge. For $X \subset \omega$, the set

$$X_A = \{x \in \omega : Ax \in X\}$$

is referred to as the *matrix domain of A in X*. An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is categorized as a *triangle* if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for $n = 0, 1, \ldots$. The subsequent proposition represents a widely acknowledged result.

Proposition 1.1 ([20, 1.4.8] or [6, Remark 22(a)]) *Every triangle T has a unique inverse S, which is also a triangle, and $x = T(Sx) = S(Tx)$ for all $x \in \omega$.*

Throughout, we use the convention that every term with a negative subscript is equal to 0.

Finally, we write $\Sigma = (\Sigma_{nk})_{n,k=0}^{\infty}$ for the triangle with $\Sigma_{nk} = 1$ $(0 \leq k \leq n; n = 0, 1, \ldots)$.

Throughout this study, consider $(\lambda_k)_{k=0}^{\infty}$ as a strictly monotone increasing sequence of real numbers tending to infinity, where $\lambda_0 \geq 1$.

We now turn to certain basic definitions in q-theory.

Definition 1.2 The q-analogue $[\nu]_q$ $(q \in (0, 1))$ of a real number ν is defined by

$$[\nu]_q = \begin{cases} \frac{1-q^{\nu}}{1-q}, & \nu \in \mathbb{R}, \\ 0, & \nu = 0. \end{cases}$$

Here, $\mathbb{R}$ denotes the set of real numbers. Also, we denote $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Apparently, $[\nu]_q = \nu$ as $q \to 1^-$.

Definition 1.3 The notation $\binom{n}{\nu}_q$, for any two nonnegative integers n and ν, defined by

$$\binom{n}{\nu}_q = \begin{cases} \frac{[n]_q!}{[n-\nu]_q![\nu]_q!}, & n \geq \nu, \\ 0, & n < \nu, \end{cases}$$

is the natural q-analog of the binomial coefficient $\binom{n}{\nu}$. Here, $[\nu]_q! = \prod_{i=1}^{\nu}[i]_q$ is the natural q-analog of $\nu!$.

Now we define the matrix $\Lambda^q = (\lambda_{nk}^q)$ as follows:

$$\lambda_{nk}^q = \begin{cases} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} & 1 \leq k \leq n \\ 0 & k > n, \end{cases}$$

where $\lambda_{-1} = 0$.

This can also be elaborated as follows:

$$\Lambda^q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{[\lambda_1]_q} & \frac{q^{\lambda_0}[\lambda_1-\lambda_0]_q}{[\lambda_1]_q} & 0 & 0 & 0 & \cdots \\ \frac{1}{[\lambda_2]_q} & \frac{q^{\lambda_0}[\lambda_1-\lambda_0]_q}{[\lambda_2]_q} & \frac{q^{\lambda_1}[\lambda_2-\lambda_1]_q}{[\lambda_2]_q} & 0 & 0 & \cdots \\ \frac{1}{[\lambda_3]_q} & \frac{q^{\lambda_0}[\lambda_1-\lambda_0]_q}{[\lambda_3]_q} & \frac{q^{\lambda_1}[\lambda_2-\lambda_1]_q}{[\lambda_3]_q} & \frac{q^{\lambda_2}[\lambda_3-\lambda_2]_q}{[\lambda_3]_q} & 0 & \cdots \\ \frac{1}{[\lambda_4]_q} & \frac{q^{\lambda_0}[\lambda_1-\lambda_0]_q}{[\lambda_4]_q} & \frac{q^{\lambda_1}[\lambda_2-\lambda_1]_q}{[\lambda_4]_q} & \frac{q^{\lambda_2}[\lambda_3-\lambda_2]_q}{[\lambda_4]_q} & \frac{q^{\lambda_3}[\lambda_4-\lambda_3]_q}{[\lambda_4]_q} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is observed that

$$\sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}([\lambda_k - \lambda_{k-1}]_q)}{[\lambda_n]_q} = \frac{1}{[\lambda_n]_q} \sum_{k=0}^{n} \big([\lambda_k]_q - [\lambda_{k-1}]_q\big) = \frac{[\lambda_n]_q}{[\lambda_n]_q} = 1.$$

Clearly, the matrix Λ^q is a triangle, and so it has a unique inverse (see Proposition 1.1) $\Omega^q = \{\Lambda^q\}^{-1} = (\lambda_{nk})^{-q}$ defined by

$$(\lambda_{nk})^{-q} = \begin{cases} \frac{[\lambda_n]_q}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}, & k < n, \\ \frac{[\lambda_n]_q}{q^{\lambda_{n-1}}[\lambda_n - \lambda_{n-1}]_q}, & k = n, \\ 0, & k > n. \end{cases}$$

Based on the matrix above, we define Λ^q-transform of a sequence $x \in \omega$. The Λ^q-transform of the sequence $x = (x_n)$ is a sequence $y = (y_n)$ given by

$$y_n = \big(\Lambda^q x\big)_n = \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} x_k, \quad n \in \mathbb{N}_0.$$

Also the terms with nonpositive subscripts like x_{-1}, x_0, etc. are considered to be naught. Then

$$\ell_p^{q_\lambda} = \big\{x = (x_n) \in \omega : \big(\Lambda^q x\big)_n \in \ell_p\big\}, \quad 1 \le p < \infty$$
$$\ell_\infty^{q_\lambda} = \big\{x = (x_n) \in \omega : \big(\Lambda^q x\big)_n \in \ell_\infty\big\}.$$

In other words, the above sequence spaces are as follows:

$$\ell_p^{q_\lambda} := \left\{u \in \omega : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k \right|^p < \infty \right\},$$
$$\ell_\infty^{q_\lambda} := \left\{u \in \omega : \sup_n \left| \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k \right| < \infty \right\}.$$

Equivalently, it can be easily seen that

$$\ell_p^{q_\lambda} := \ell_p\big(\Lambda^q\big) = (\ell_p)_{\Lambda^q} \quad \text{and} \quad \ell_\infty^{q_\lambda} := \ell_\infty\big(\Lambda^q\big) = (\ell_\infty)_{\Lambda^q}. \tag{1.1}$$

These newly defined spaces represent a generalization of numerous known sequence spaces documented in the literature, as evidenced below.

Remark 1.4 The sequence spaces introduced using the aforementioned matrix generalize numerous sequence spaces well-documented in the literature.

(1) When $\lambda_n = n + 1$ for all $n \in \mathbb{N}_0$, the sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are reduced to the sequence spaces previously defined in [21].

(2) When $q = 1$, the sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are reduced to the sequence spaces previously defined in [12].

(3) When $q = 1$ and $\lambda_n = n + 1$ for all $n \in \mathbb{N}_0$, the sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are reduced to the sequence spaces previously defined in [16].

The q-calculus has several applications in different areas, e.g., summability theory [13, 14], approximation theory [2, 15, 18], integral equations [10], etc. Most recently, the Cesàro q-difference sequence spaces have been studied in [22]. For more results related to this theory, readers are suggested to refer to the papers [1, 3–5, 7, 8]. While our primary focus remains on the spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$, we introduce the spaces $c_0^{q_\lambda}$ and c^{q_λ} to substantiate our findings.

$$c_0^{q_\lambda} := (c_0)_{\Lambda^q} = \left\{u \in \omega : \lim_n \left|\sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k\right|^p = 0\right\}, \tag{1.2}$$

$$c^{q_\lambda} := (c)_{\Lambda^q} = (c_0 \oplus e)_{\Lambda^q} = \left\{u \in \omega : \lim_n \left|\sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k\right|^p < \infty\right\}. \tag{1.3}$$

2 Topological results

Let us proceed to the primary findings of the paper.

Theorem 2.1 *The sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are categorized as BK-spaces with the norms*

$$\|u\|_{\ell_p^{q_\lambda}} = \left\|\left(\Lambda^q u\right)_n\right\|_p = \left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k\right|^p\right)^{1/p}$$

and

$$\|u\|_{\ell_\infty^{q_\lambda}} = \left\|\left(\Lambda^q u\right)_n\right\|_\infty = \sup_n \left|\sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k\right|.$$

Proof We have established that relation (1.1) holds true, where Λ^q is a triangle. Then, by using Theorem 4.3.12 of Wilansky [20] and the fact that ℓ_p and ℓ_∞ are BK-spaces, the result follows immediately. □

Proposition 2.2 *The sequence spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are of nonabsolute type, that is, $\|x\|_{\ell_p^{q_\lambda}} \neq \||x|\|_{\ell_p^{q_\lambda}}$ and $\|x\|_{\ell_\infty^{q_\lambda}} \neq \||x|\|_{\ell_\infty^{q_\lambda}}$, where $|x| = (|x_n|)$.*

Proof Consider the sequence $w = (1, -1, 0, \dots)$. Consequently,

$$\left(\Lambda^q w\right)_n = \left(1, \frac{1}{[\lambda_1]_q} - \frac{q^{\lambda_0}[\lambda_1 - \lambda_0]}{[\lambda_1]_q}, \dots\right)$$

and

$$\left(\Lambda^q |w|\right)_n = \left(1, \frac{1}{[\lambda_1]_q} + \frac{q^{\lambda_0}[\lambda_1 - \lambda_0]}{[\lambda_1]_q}, \dots\right).$$

This clearly indicates that $\|x\|_{\ell_p^{q_\lambda}} \neq \||x|\|_{\ell_p^{q_\lambda}}$. □

Theorem 2.3 *If $1 \leq r < s < \infty$, then $\ell_r^{q_\lambda} \subset \ell_s^{q_\lambda}$, and the inclusion is proper.*

Proof It is known that under the given condition $\ell_r \subset \ell_s$ and by using relation (1.1), we obtain that $\ell_r^{q_\lambda} \subset \ell_s^{q_\lambda}$. The properness of the inclusion follows from the following example.

Example 2.4 For $1 \leq r < s$, let us consider that $x = (x_n) \in \ell_s \setminus \ell_r$ and

$$y_n = \frac{x_n[\lambda_n]_q - x_{n-1}[\lambda_{n-1}]_q}{q^{\lambda_{n-1}}[\lambda_n - \lambda_{n-1}]_q}.$$

Then

$$\begin{aligned}
(\Lambda^q y)_n &= \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} y_k = \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} \frac{x_k[\lambda_k]_q - x_{k-1}[\lambda_{k-1}]_q}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q} \\
&= \frac{1}{[\lambda_n]_q} \sum_{k=0}^{n} \left(x_k[\lambda_k]_q - x_{k-1}[\lambda_{k-1}]_q \right) = x_n.
\end{aligned}$$

Hence, $\Lambda^q y = x \in \ell_s \setminus \ell_r$. Equivalently, $y \in \ell_s^{q_\lambda} \setminus \ell_r^{q_\lambda}$. □

Theorem 2.5 *$\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are isometrically isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof The mapping $K : \ell_p^{q_\lambda} \to \ell_p$ defined by the relation

$$w \to v = K(w) = \Lambda^q w$$

is clearly linear and injective. To demonstrate its surjectivity, we consider a sequence $v = (v_n) \in \ell_p$. Then we have that

$$w_k = \frac{1}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q} \sum_{j=k-1}^{k} (-1)^{k-j} [\lambda_j]_q v_j.$$

For $1 \leq p < \infty$, we obtain that

$$\begin{aligned}
\|w\|_{\ell_p^{q_\lambda}} &= \left(\sum_{i=1}^{\infty} \left| \sum_{k=0}^{i} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_i]_q} w_k \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} \left| \sum_{k=0}^{i} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_i]_q} \left(\frac{1}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q} \sum_{j=k-1}^{k} (-1)^{k-j} [\lambda_j]_q v_j \right) \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} |v_i|^p \right)^{\frac{1}{p}} = \|v\|_{l_p} < \infty.
\end{aligned}$$

For the case $p = \infty$, a similar approach enables us to establish the theorem. □

Theorem 2.6 *The inclusion $\ell_p^{q_\lambda} \subset c_0^{q_\lambda} \subset c^{q_\lambda} \subset \ell_\infty^{q_\lambda}$ is strict.*

Proof In what follows, we prove that $\ell_p^{q_\lambda} \subset c_0^{q_\lambda}$ and $c^{q_\lambda} \subset \ell_\infty^{q_\lambda}$. Let us suppose that $w = (w_n) \in \ell_p^{q_\lambda}$. From relation (1.1), we have that $(w_n) \in \ell_p^{q_\lambda}$. It is known that $\ell_p \subset c_0$, and from this it follows that $\Lambda^q(w) \in c_0$, respectively $w \in c_0^{q_\lambda}$. To prove that inclusion is strict, we consider the following example. □

Example 2.7 Let $w = (w_k)$, where

$$w_k = \frac{1}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{[\lambda_j]_q}{(j+1)^{\frac{1}{p}}}.$$

Then

$$\left(\Lambda^q w\right)_n = \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]}{[\lambda_n]} w_k = \frac{1}{(n+1)^{\frac{1}{p}}},$$

which implies that $w \in c_0^{q_\lambda}$ but not in $\ell_p^{q_\lambda}$. This proves the first part. For the second inclusion, we know that $c \subset \ell_\infty$, from which it yields that $c^{q_\lambda} \subset \ell_\infty^{q_\lambda}$. To prove the strict inclusion, we consider the following sequence:

$$w_k = \frac{1}{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q} \sum_{j=k-1}^{k} (-1)^{k-j} [\lambda_j]_q (-1)^j.$$

Then

$$\left(\Lambda^q w\right)_n = \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]}{[\lambda_n]} w_k = (-1)^n,$$

and we obtain that $w \in \ell_\infty^{q_\lambda} \setminus c^{q_\lambda}$.

Theorem 2.8 *The inclusion $\ell_\infty \subset \ell_\infty^{q_\lambda}$ holds true, the inclusion being strict.*

Proof Let us consider that $w = (w_n) \in \ell_\infty$. Then

$$\begin{aligned} \|w\|_{\ell_\infty^{q_\lambda}} = \left\| \left(\Lambda^q w\right)_n \right\|_\infty &= \sup_n \left| \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} w_k \right| \\ &\le \sup_n \|w\|_{\ell_\infty} \sup_n \left| \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} \right| \le \|w\|_{\ell_\infty}. \end{aligned}$$

To prove that inclusion is strict, we consider the following sequence. □

Example 2.9

$$u_k = \begin{cases} 1; & 0 \le k \le n-1 \\ \frac{[\lambda_{k-1}]_q}{[\lambda_k]_q - [\lambda_{k-1}]_q}; & k = n \\ 0; & k > n. \end{cases}$$

Then

$$[\lambda_1]_q \le [\lambda_2]_q \le \cdots \le [\lambda_n]_q \le \cdots,$$

from which it follows that $0 < \frac{[\lambda_{n-1}]_q}{[\lambda_n]_q} \le 1$ and $\lim_n \frac{[\lambda_{n-1}]_q}{[\lambda_n]_q} = 1$, respectively, $\sup_n \frac{[\lambda_{n-1}]_q}{[\lambda_n]_q} = 1$.

Under these conditions we have that $(u_k) \in \ell_\infty^{q_\lambda}$ since

$$\sup_n \left| \sum_{k=0}^{n} \frac{q^{\lambda_{k-1}}[\lambda_k - \lambda_{k-1}]_q}{[\lambda_n]_q} u_k \right| = \sup_n \frac{2[\lambda_{n-1}]_q}{[\lambda_n]_q} = 2.$$

On the other side,

$$\sup_n \frac{[\lambda_{n-1}]_q}{[\lambda_n]_q - [\lambda_{n-1}]_q} = \infty,$$

so $(u_k) \notin \ell_\infty$.

3 α-, β- and γ-duals

In this section, we formulate the α-, β-, and γ-duals associated with the spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$. The following lemmas are essential for substantiating the findings that we wish to achieve.

Throughout our discussion, $\mathcal{N}$ represents the family comprising all finite subsets of the set $\mathbb{N}_0$.

It is presumed that $U = (u_{nk})$ represents an infinite matrix over the set of complex numbers.

Lemma 3.1 [9, 19] *Each of the subsequent assertions holds true*:

(i) $U \in (\ell_\infty, \ell_1)$ *iff*

$$\sup_{K \in \mathcal{N}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} u_{nk} \right| < \infty. \tag{3.1}$$

(ii) $U \in (\ell_\infty, c)$ *iff*

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} u_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{3.2}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |u_{nk}| = \sum_{k=0}^{\infty} \left| \lim_{n\to\infty} u_{nk} \right|. \tag{3.3}$$

(iii) $U \in (\ell_\infty, \ell_\infty)$ *iff*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |u_{nk}| < \infty.$$

(iv) *Let* $1 < p < \infty$. *Then* $U \in (\ell_p, \ell_\infty)$ *iff*

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |u_{nk}|^{p'} < \infty. \tag{3.4}$$

(v) *Let* $1 < p < \infty$. *Then* $U \in (\ell_p, c)$ *iff* (3.2) *and* (3.4) *hold.*

(vi) *Let* $1 < p < \infty$. *Then* $U \in (\ell_p, \ell_1)$ *iff*

$$\sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{n \in N} u_{nk} \right|^{p'} < \infty, \quad (1 < p < \infty). \tag{3.5}$$

Theorem 3.2 *Define the sets S_1 and S_2 by*

$$S_1 := \left\{ s = (s_k) \in \omega : \sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{n \in K} b_{nk} \right|^{p'} < \infty \right\},$$

$$S_2 := \left\{ s = (s_k) \in \omega : \sup_{K \in \mathcal{N}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} b_{nk} \right| < \infty \right\},$$

where the matrix $B = (b_{nk})$ is defined by

$$b_{nk} = \begin{cases} \frac{1}{q^{\lambda_{n-1}}} [\lambda_n - \lambda_{n-1}]_q \sum_{k=n-1}^{n} (-1)^{n-k} [\lambda_k]_q v_k s_n, & 0 \le k \le n, \\ 0, & k > n. \end{cases} \tag{3.6}$$

Then

1. $[\ell_p^{q_\lambda}]^\alpha = S_1$.
2. $[\ell_\infty^{q_\lambda}]^\alpha = S_2$.

Proof For $s = (s_k) \in \omega$, consider the following equality:

$$s_k w_k = \frac{1}{q^{\lambda_{k-1}}} [\lambda_k - \lambda_{k-1}]_q \sum_{j=k-1}^{k} (-1)^{k-j} [\lambda_j]_q v_j s_k = (Bv)_k$$

for each $k \in \mathbb{N}$. Note that $v = (v_k)$ represents the Λ^q-transform of the sequence $w = (w_k)$, and the matrix $B = (b_{nk})$ is defined analogously to (3.6). It is evident that $sw = (s_k w_k) \in \ell_1$ whenever $w \in \ell_p^{q_\lambda}$ iff $Bv \in \ell_1$ whenever $v \in \ell_p$. This implies that $s = (s_k) \in [\ell_p^{q_\lambda}]^\alpha$ iff $B \in (\ell_p, \ell_1)$. By utilizing Lemma 3.1 (vi), we can conclude that

$$\left[\ell_p^{q_\lambda}\right]^\alpha = S_1.$$

The determination of the α-dual of the space $\ell_\infty^{q_\lambda}$ is established similarly by employing Lemma 3.1 (i). To avoid redundancy in the statements, the proof is omitted. □

Theorem 3.3 *Define the matrix $B' = (b'_{nk})$ by*

$$b'_{nk} = [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}} [\lambda_j - \lambda_{j-1}]_q} s_j$$

for all $n, k \in \mathbb{N}$. Then each of the subsequent statements holds true:

(i) $s = (s_k) \in [\ell_p^{q_\lambda}]^\beta$ *iff* $B' = (b'_{nk}) \in (\ell_p, c)$ *and*

$$\left\{ \frac{[\lambda_m]_q s_m}{q^{\lambda_{m-1}} [\lambda_m - \lambda_{m-1}]_q} \right\} \in \ell_\infty. \tag{3.7}$$

(ii) $s = (s_k) \in [\ell_\infty^{q_\lambda}]^\beta$ *iff* $B' = (b'_{nk}) \in (\ell_\infty, c)$ *and*

$$\left\{ \frac{[\lambda_m]_q s_m}{q^{\lambda_{m-1}} [\lambda_m - \lambda_{m-1}]_q} \right\} \in c_0. \tag{3.8}$$

Proof (i) Suppose that $s = (s_k) \in [\ell_p^{q_\lambda}]^\beta$. This implies that the series $\sum_{k=0}^\infty s_k w_k \in c$ for all $w = (w_k) \in \ell_p^{q_\lambda}$. Employing Abel's mth partial sum of the series $\sum_{k=0}^\infty s_k w_k$, we derive the following equality:

$$\begin{aligned}\sum_{k=0}^m s_k w_k &= \sum_{k=0}^m s_k \left[\frac{1}{q^{\lambda_{k-1}}} [\lambda_k - \lambda_{k-1}]_q \sum_{j=k-1}^k (-1)^{k-j} [\lambda_j]_q v_j \right] \\ &= \sum_{k=0}^{m-1} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}} [\lambda_j - \lambda_{j-1}]_q} s_j v_k + \frac{[\lambda_m]_q}{q^{\lambda_{m-1}} [\lambda_m - \lambda_{m-1}]_q} s_m v_m\end{aligned} \tag{3.9}$$

$\forall m \in \mathbb{N}$. Given the fact that $\ell_p^{q_\lambda} \cong \ell_p$, we take the limit as $m \to \infty$ in (3.9). As per the assumption, the series $\sum_{k=0}^\infty s_k w_k \in c$. This directly leads to the fact that

$$\sum_{k=0}^\infty [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}} [\lambda_j - \lambda_{j-1}]_q} s_j v_k \in c,$$

and the term $[\lambda_m]_q s_m / q^{\lambda_{m-1}} [\lambda_m - \lambda_{m-1}]_q$ in the right-hand side of (3.9) is expected to tend to zero as $m \to \infty$. Furthermore, given that $\ell_p \subset c_0$, which is established with $\{[\lambda_m]_q s_m / q^{\lambda_{m-1}} [\lambda_m - \lambda_{m-1}]_q\} \in \ell_\infty$, we consequently deduce that

$$\sum_{k=0}^\infty s_k w_k = \sum_{k=0}^\infty [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}} [\lambda_j - \lambda_{j-1}]_q} s_j v_k = \left(B' v\right)_k. \tag{3.10}$$

Thus, $B' = (b'_{nk}) \in (\ell_p, c)$. Alternatively, the matrix B' satisfies part (v) of Lemma 3.1, hence establishing the necessity of these conditions.

Conversely, assume that $B' = (b'_{nk}) \in (\ell_p, c)$ and condition (3.7) remains valid. By employing (3.9), we derive once again relation (3.10). Consequently, as $B' = (b'_{nk}) \in (\ell_p, c)$, it follows that the series $\sum_{k=0}^\infty s_k w_k \in c$ for all $w = (w_k) \in \ell_p^{q_\lambda}$. Thus, $s = (s_k) \in [\ell_p^{q_\lambda}]^\beta$, affirming that the conditions are sufficient.

(ii) This can be readily derived following a similar method employed in proving part (i) by utilizing $\{[\lambda_m]_q s_m / q^{m-1} [\lambda_m - \lambda_{m-1}]_q\} \in c_0$ in place of ℓ_∞. □

Theorem 3.4 *Each of the following assertions holds true*:

(i) $s = (s_k) \in [\ell_p^{q_\lambda}]^\gamma$ *iff* $B' = (b'_{nk}) \in (\ell_p, \ell_\infty)$ *and condition* (3.7) *holds.*

(ii) $s = (s_k) \in [\ell_\infty^{q_\lambda}]^\gamma$ *iff* $B' = (b'_{nk}) \in (\ell_\infty, \ell_\infty)$ *and condition* (3.8) *holds.*

Proof This is derived by employing a similar methodology as used in proving parts (i) and (ii) of Theorem 3.3. The distinction lies in utilizing part (iv) of Lemma 3.1 in place of part (v) of Lemma 3.1 to establish the first result, and employing part (iii) of Lemma 3.1 in place of part (v) of Lemma 3.1 to establish the second result. The details of the proof are omitted to avoid redundancy in the statements. □

4 Matrix transformations

Here, some class $(\ell_p^{q_\lambda}, X)$ of matrix transformations is characterized, where X represents any of the spaces ℓ_∞, c, or c_0. Define the matrix $\tilde{A}(q) = (\tilde{a}^q_{nk})_{n,k \in \mathbb{N}_0}$ via an infinite matrix

$A = (a_{nk})_{n,k\in\mathbb{N}_0}$ by

$$\tilde{a}^q_{nk} = [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj}.$$

Theorem 4.1 *$A \in (\ell_p^{q_\lambda}, \ell_\infty)$ iff*

$$\sup_{n\in\mathbb{N}_0} \left| \sum_{k=0}^{\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k \right|^{p'} < \infty, \tag{4.1}$$

$$\left(\frac{[\lambda_m]_q}{q^{\lambda_{m-1}}[\lambda_m - \lambda_{m-1}]_q} a_{nm} \right)_{m\in\mathbb{N}_0} \in \ell_\infty. \tag{4.2}$$

Proof We consider the established fact that $\ell_p^{q_\lambda}$ is linearly norm-isomorphic to ℓ_p.

Suppose $A \in (\ell_p^{q_\lambda}, \ell_\infty)$. Consequently, Ax exists and is contained in the space ℓ_∞ for all $x \in \ell_p^{q_\lambda}$. This implies that $A_n \in \{\ell_p^{q_\lambda}\}^\beta$ for each $n \in \mathbb{N}_0$. Therefore, conditions (4.1) and (4.2) are necessary.

Conversely, assume that conditions (4.1) and (4.2) hold and take any $x \in \ell_p^{q_\lambda}$. Then $A_n \in \{\ell_p^{q_\lambda}\}^\beta$, confirming the existence of Ax. Consequently, we deduce that

$$\begin{aligned}
\sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \left[\frac{1}{q^{\lambda_{k-1}}} [\lambda_k - \lambda_{k-1}]_q \sum_{j=k-1}^{k} (-1)^{k-j} [\lambda_j]_q y_j \right] \\
&= \sum_{k=0}^{m-1} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k + \frac{[\lambda_m]_q}{q^{\lambda_{m-1}}[\lambda_m - \lambda_{m-1}]_q} a_{nm} y_m
\end{aligned} \tag{4.3}$$

$\forall m \in \mathbb{N}$. Taking the limit as $m \to \infty$ in (4.3) and applying condition (4.2), we derive that

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k = \left(\tilde{A}(q)y\right)_n \tag{4.4}$$

for all $n \in \mathbb{N}_0$. Employing (4.1) and applying Holder's inequality, we observe that

$$\begin{aligned}
\frac{\|Ax\|_{\ell_\infty}}{\|y\|_{\ell_p}} &= \sup_{n\in\mathbb{N}_0} \frac{1}{\|y\|_{\ell_p}} \left| \sum_{k=0}^{\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k \right| \\
&\leq \sup_{n\in\mathbb{N}_0} \frac{1}{\|y\|_{\ell_p}} \left\{ \left| \sum_{k=0}^{\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k \right|^{p'} \right\}^{1/p'} \left(\sum_{k=0}^{\infty} |y_k|^p \right)^{1/p} \\
&= \sup_{n\in\mathbb{N}_0} \left\{ \left| \sum_{k=0}^{\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} y_k \right|^{p'} \right\}^{1/p'} < \infty.
\end{aligned}$$

Consequently, we conclude that $A \in (\ell_p^{q_\lambda}, \ell_\infty)$. Hence, conditions (4.1) and (4.2) are sufficient. □

Theorem 4.2 *$A \in (\ell_p^{q_\lambda}, c)$ iff each of assertions (4.1) and (4.2) holds true and*

$$\exists \alpha_k \in \mathbb{C} \text{ such that } \lim_{n\to\infty} [\lambda_k]_q \sum_{j=k}^{k+1} (-1)^{j-k} \frac{1}{q^{\lambda_{j-1}}[\lambda_j - \lambda_{j-1}]_q} a_{nj} = \alpha_k. \tag{4.5}$$

Proof Let $A \in (\ell_p^{q_\lambda}, c)$. Hence, Ax exists and is in the space c for all $x \in \ell_p^{q_\lambda}$. As $c \subset \ell_\infty$, it follows from Theorem 4.1 that conditions (4.1) and (4.2) are necessary.

Consider $e^{(k)}$ as a sequence with 1 in the kth position and 0 elsewhere. Observing (4.4) with $x = \Omega^q e^{(k)}$, we notice that

$$Ax = A\big(\Omega^q e^{(k)}\big) = \tilde{A}(q)\big(\Lambda^q\big(\Omega^q e^{(\nu)}\big)\big) = \tilde{A}(q)e^{(k)} = \big(\tilde{A}(q)\big)_n.$$

By assumption, $(\tilde{A}(q))_n \in c$. This validates the necessity of condition (4.5).

Conversely, assuming that each of conditions (4.1), (4.2), and (4.5) is satisfied and considering $x \in \ell_p^{q_\lambda}$, it follows that $A_n \in \{\ell_p^{q_\lambda}\}^\beta$ for all $n \in \mathbb{N}_0$. Consequently, Ax exists for all $x \in \ell_p^{q_\lambda}$. This leads to the derivation of equality (4.4). Considering conditions (4.1) and (4.5), we ascertain that the matrix $\tilde{A}(q)$ fulfills conditions (3.2) and (3.4). This implies that $Ax = \tilde{A}y \in c$ in view of (4.4). Therefore, we conclude that $A \in (\ell_p^{q_\lambda}, c)$. □

The following result is presented without proof by substituting the space c with the space c_0 in the preceding theorem.

Theorem 4.3 *Let $A = (a_{nk})$ be an infinite matrix over the complex field $\mathbb{C}$. Then $A \in (\ell_p^{q_\lambda}, c_0)$ iff each of conditions (4.1) and (4.2) holds true, and condition (4.5) also holds true with $\alpha_k = 0$ for all $k \in \mathbb{N}_0$.*

5 Some geometric properties of the spaces $\ell_p^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$

In this section, we illustrate geometric structures, namely the approximation property, Dunford–Pettis property, Hahn–Banach extension property, and rotundity, of the spaces $\ell_p^{q_\lambda}$ $(1 \le p < \infty)$ and $\ell_\infty^{q_\lambda}$. At this juncture, it is anticipated that readers familiarize themselves with the fundamental concepts and definitions encompassing the approximation property [11, Definition 3.4.26], weak compactness of a linear operator [11, Definition 3.5.1], complete continuous or Dunford–Pettis property [11, Definition 3.5.15], and Hahn–Banch extension theorem [11, Art 1.9.6, p. 75]. Given that these definitions are accessible in Megginson's work [11] or other standard literature pertaining to the geometry of Banach spaces, detailed exposition is omitted herein.

.

Theorem 5.1 [11, *Theorem* 3.4.27] *The space ℓ_p $(1 \le p < \infty)$ possesses the approximation property.*

Lemma 5.2 [11, *Exercise* 3.50, *p.* 339] *Suppose that X and Y are two normed spaces and $C : X \to Y$ is continuous. Then, if Q is weakly compact in X, then $C(Q)$ is weakly compact in Y.*

We add that the space ℓ_∞ exhibits Hahn–Banach extension property. This can be evidenced from the following theorem.

Theorem 5.3 [17] *Let $C_0 : A \to \ell_\infty$ be a bounded linear operator, where A is a linear subspace of a Banach space X. Then the operator C_0 can be extended to a bounded linear operator $C : X \to \ell_\infty$ such that $\|C_0\| = \|C\|$.*

Let

$$S_\lambda = \{s \in \lambda : \|s\| = 1\}.$$

Definition 5.4 [11, Definition 5.1.1] A normed space λ is called rotund (or strictly convex) if given any $s_1, s_2 \in S_\lambda$ $(s_1 \neq s_2)$ and $0 < \alpha < 1$,

$$\|\alpha s_1 + (1-\alpha)s_2\| < 1.$$

Proposition 5.5 [11, *Proposition* 5.1.2] *A normed space λ is rotund iff for any $s_1, s_2 \in S_\lambda$ $(s_1 \neq s_2)$*

$$\left\| \frac{s_1 + s_2}{2} \right\| < 1.$$

Proposition 5.6 [11, *Proposition* 5.1.9] *Any normed space that is isometrically isomorphic to a rotund space is also rotund.*

The primary result of this section is presented as follows.

Theorem 5.7 *Let $1 \le p < \infty$. Then the space $\ell_p^{q_\lambda}$ has approximation property.*

Proof For any Banach space X, consider a compact linear operator $C : X \to \ell_p^{q_\lambda}$. Consequently, given a bounded sequence $s = (s_n) \in X$, the sequence (Cs_n) contains a convergent subsequence (Cs_{n_k}) in $\ell_p^{q_\lambda}$. This implies that

$$\|Cs_{n_u} - Cs_{n_v}\|^p_{\ell_p^{q_\lambda}} = \left\|C(s_{n_u} - s_{n_v})\right\|^p_{\ell_p^{q_\lambda}} = \left\|\left(\Lambda^q C\right)(s_{n_u} - s_{n_v})\right\|^p_{\ell_p} \to 0$$

as $u, v \to \infty$. Hence, the operator $\Lambda^q C : X \to \ell_p$ is both well-defined and compact. Consequently, our focus shifts to the space ℓ_p, known for possessing the approximation property. As a consequence, $\exists$ a sequence $\{T_n\}$ consisting of finite rank bounded linear operators from X to ℓ_p such that

$$\left\|\Lambda^q C - T_n\right\| \to 0$$

as $n \to \infty$. In light of this observation, we come to understand that the sequence $(\Lambda^q T_n)$ comprising bounded linear operators from X to $\ell_p^{q_\lambda}$ fulfills the criteria for a sequence of finite rank. Furthermore,

$$\begin{aligned} \left\|C - \Omega^q T_n\right\| &= \sup_{\|s\|=1} \left\|\left(C - \Omega^q T_n\right)s\right\|^p_{\ell_p^{q_\lambda}} \\ &= \sup_{\|s\|=1} \left\|Cs - \left(\Omega^q T_n\right)s\right\|^p_{\ell_p^{q_\lambda}} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|s\|=1} \left\| \Lambda^q Cs - T_n s \right\|_{\ell_p}^p \\
&= \sup_{\|s\|=1} \left\| \left(\Lambda^q C - T_n \right) s \right\|_{\ell_p}^p \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, the proof is completed. □

Theorem 5.8 *The space $\ell_1^{q_\lambda}$ possesses the D-P property.*

Proof Let C be a weakly compact operator from the Banach space $\ell_1(\Lambda^q)$ to a space X. Consequently, $C\Omega^q$ denotes a bounded linear operator from ℓ_1 to X. We wish to demonstrate the complete continuity of C.

Consider B, a bounded set in ℓ_1. Thus, $\Omega^q B$ is a bounded set in $\ell_1^{q_\lambda}$. Given the weak compactness of C, it therefore follows that the set

$$
C\left(\Omega^q B\right) = \left(\Omega^q\right) B
$$

is relatively weakly compact in X. Hence, we ascertain that the operator $C\Omega^q$ is a weakly compact operator from ℓ_1 to X. As the space ℓ_1 possesses the D-P property, it implies the complete continuity of the operator $C\Omega^q$.

Consider Q as a weakly compact subset of $\ell_1^{q_\lambda}$. By Lemma 5.2, it follows that $\Lambda^q Q$ constitutes a weakly compact subset of ℓ_1. Given the complete continuity of $C\Omega^q$, it follows that $C\Omega^q(\Lambda^q Q) = C(Q)$ represents a compact set in Y. This confirms the desired conclusion that C is completely continuous. □

Next, we wish to demonstrate that the space $\ell_\infty^{q_\lambda}$ possesses the Hahn–Banach extension property.

Theorem 5.9 *Suppose that ν represents a linear subspace of a Banach space X, and let $C_0 \in B(\nu, \ell_\infty^{q_\lambda})$. Then the operator C_0 can be extended to a bounded linear operator $C \in B(X, \ell_\infty^{q_\lambda})$, while preserving the norm, i.e., $\|C_0\| = \|C\|$.*

Proof Suppose $C_0 \in B(\nu, \ell_\infty^{q_\lambda})$. Consequently, $\Lambda^q C_0 \in B(\nu, \ell_\infty)$. As per Theorem 5.3, given that ℓ_∞ possesses the Hahn–Banach extension property, we can extend the operator $\Lambda^q C_0$ to an operator $T \in B(X, \ell_\infty)$ while preserving the norm $\|\Lambda^q C_0\| = \|T\|$. Choose the operator $C = \Omega^q T$. It is evident that $C \in B(X, \ell_\infty^{q_\lambda})$. Furthermore, for any $s \in \nu$, we observe that

$$
Cs = \left(\Omega^q T\right)s = \Omega^q(Ts) = \Omega^q\left(\left(\Lambda^q C_0\right)s\right) = C_0 s.
$$

Furthermore

$$
\|C\| = \left\|\Omega^q T\right\| = \left\|\Omega^q\left(\Lambda^q C_0\right)\right\| = \|C_0\|,
$$

as desired. □

Theorem 5.10 *Let $1 < p < \infty$. Then the space $\ell_p^{q_\lambda}$ is rotund.*

Proof This immediately follows from Proposition 5.6 and the fact that ℓ_p is a rotund space for $1 < p < \infty$. □

Theorem 5.11 *The spaces $\ell_1^{q_\lambda}$ and $\ell_\infty^{q_\lambda}$ are not rotund.*

Proof Choose any two sequences $s_1, s_2 \in \ell_1^{q_\lambda}$ given by

$$s_1 = \left(1, -\frac{1}{q}, 0, 0, \dots\right) \quad \text{and} \quad s_2 = \left(0, \frac{1+q}{q}, -\frac{1}{q^2}, 0, \dots\right).$$

Then $\Lambda^q s_1 = (1, 0, 0, 0, \dots)$ and $\Lambda^q s_2 = (0, 1, 0, 0, \dots)$. It follows that $\|s_1\|_{\ell_1^{q_\lambda}} = 1$ and $\|s_2\|_{\ell_1^{q_\lambda}} = 1$. That is, $s_1, s_2 \in S_{\ell_1^{q_\lambda}}$.

Let $s = \frac{s_1+s_2}{2} = \frac{1}{2}(1, 1, -\frac{1}{q^2}, 0, 0, \dots)$. Then $\Lambda^q s = (1, 1, 0, 0, \dots)$. Thus,

$$\|s\|_{\ell_1^{q_\lambda}} = \left\|\Lambda^q s\right\|_{\ell_1} = 1.$$

Hence, we see that

$$\|s\|_{\ell_1^{q_\lambda}} \not< 1.$$

Therefore, the space $\ell_1^{q_\lambda}$ is not rotund. In the similar way, nonrotundness of the space $\ell_\infty^{q_\lambda}$ can be established. □

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