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Bounds for novel extended beta and hypergeometric functions and related results

Rakesh K. Parmar[1] and Tibor K. Pogány[2,3*]

*Correspondence:
pogany.tibor@nik.uni-obuda.hu;
tibor.poganj@uniri.hr
[2]Institute of Applied Mathematics, John von Neumann Faculty of Informatics, Óbuda University, Bécsi út 96/b, 1034 Budapest, Hungary
[3]Faculty of Maritime Studies, University of Rijeka, Studentska 2, 51000 Rijeka, Croatia
Full list of author information is available at the end of the article

Abstract

We introduce a new unified extension of the integral form of Euler's beta function with a MacDonald function in the integrand and establish functional upper bounds for it. We use this definition to extend as well the Gaussian and Kummer's confluent hypergeometric functions, for which we provide bounding inequalities. Moreover, we use our extension of the beta function to define a new probability distribution, for which we establish raw moments and moment inequalities and, as by-products, Turán inequalities for the initially defined extended beta function.

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1 Introduction and preliminaries

In recent years, various extensions of the beta function (or Euler function of the first kind)

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \qquad \min\bigl\{\Re(a),\Re(b)\bigr\} > 0, \tag{1}$$

have been considered. Connections to the definition of beta function, extensions of a number of well-known higher transcendental functions, such as Gauss hypergeometric, Kummer confluent hypergeometric, Whittaker, Appell, Lauricella, Srivastava triple hypergeometric, Bessel, Struve, Bessel–Struve kernel, τ-hypergeometric, etc., have been investigated along with various potentially useful properties and certain connections with some well-known special functions, including applications in many diverse areas of mathematical, physical, engineering, and statistical sciences investigated by several authors in a set of publications; see, for instance, [1–8, 10, 12] and references therein.

We make the usual conventions: $\mathbb{Z}^-$, $\mathbb{R}_+$, and $\mathbb{C}$ denote the sets of negative integers, positive real numbers, and complex numbers, respectively; also, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.

In this note, we introduce the following unified approach to the class of generalized beta functions. Firstly, we recall the MacDonald function (or modified Bessel function of the second kind) of order μ [15, p. 251, Eq. (10.27.4)]:

$$K_\mu(z) = \frac{\pi}{2}\frac{I_{-\mu}(z) - I_\mu(z)}{\sin(\pi\mu)}, \qquad I_\mu(z) = \sum_{n\geq 0} \frac{(\frac{z}{2})^{2n+\mu}}{\Gamma(\mu+1+n)n!},$$

where I_μ is the modified Bessel function of the first kind; see [15, p. 249, Eq. (10.25.2)]. We point out that $I_\mu(x)$ is real when $\mu \in \mathbb{R}$ and $\arg(z) = 0$.

Definition 1 The extended beta function built with the MacDonald function reads

$$\mathrm{B}^\lambda_{p,q,\nu}(a,b) = \sqrt{\frac{2}{\pi}} \int_0^1 t^{a-1}(1-t)^{b-1}\sqrt{h_\theta(t)}K_{\nu+\frac{1}{2}}\big(h_\theta(t)\big)\,\mathrm{d}t, \tag{2}$$

where

$$h_\theta(t) = \frac{p}{t^\lambda} + \frac{q}{(1-t)^\lambda}, \quad \theta = (p,q,\lambda).$$

Here $\lambda > 0$, $\min\{\Re(p),\Re(q)\} > 0$, $\min\{a,b\} > \frac{\lambda}{2} > 0$, and $\nu \in \mathbb{R}$.

Bearing in mind that for a fixed ν [15, p. 255, Eq. (10.40.2)],

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left(1 + \frac{4\nu^2-1}{8z} + \mathcal{O}\big(z^{-2}\big)\right), \quad z \to \infty, \tag{3}$$

because of the parity with respect to the real order $\nu + \frac{1}{2}$, the asymptotic expansion is valid for all real ν, we have

$$K_{\nu+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left(1 + \frac{\nu(\nu+1)}{2z} + \mathcal{O}\big(z^{-2}\big)\right) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\big(1 + \mathcal{O}\big(z^{-1}\big)\big), \quad z \to \infty.$$

Specifying the values of parameters p, q, λ, and ν, we cover a whole spectrum of extended beta functions, which confirms that the newly extended beta function $\mathrm{B}^\lambda_{p,q,\nu}(a,b)$ is not artificially constructed. In fact, (2) is the so-called beta function transform and maps a suitable input function $\phi(t)$ into the multiparameter function [11]

$$\int_0^1 t^{a-1}(1-t)^{b-1}\phi(t)\,\mathrm{d}t.$$

In Definition 1, we have $\phi(t) = \sqrt{h_\theta(t)}K_{\nu+\frac{1}{2}}(h_\theta(t))$, $\sqrt{h_\theta(t)}$ being the necessarily implemented correcting factor function (up to the multiplicative constant $\sqrt{2/\pi}$); compare with relation (3). In turn, the constraint $\min\{a,b\} > \frac{\lambda}{2} > 0$ follows immediately by rewriting $\sqrt{h_\theta(t)}$ in (2) in a convenient form.

Setting the values of the parameters p, q, ν, λ in (2), we get several already known and frequently studied members of the family of beta functions. So, when $\lambda = 1$ and $q = p$, we arrive at the so-called (p,ν)-extended beta function introduced by Parmar et al. [16, p. 93, Eq. (13)]:

$$\mathrm{B}_{p,\nu}(a,b) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{a-\frac{3}{2}}(1-t)^{b-\frac{3}{2}}K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\mathrm{d}t,$$

where $\Re(p) \geq 0$, $\min\{\Re(a), \Re(b)\} > 0$, and $\sqrt{p}$ takes its principal value. This beta function is recently considered by Milovanović et al. [14] for establishing Gautschi–Pinelis-type upper bounds for the MacDonald and (p,ν)-extended beta functions.

For $\lambda = 1$ and $\nu = 0$, we arrive at the so-called (p,q)-extended beta function

$$\mathrm{B}_{p,q}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t.$$

The case where $\min\{\Re(a), \Re(b)\} > 0$ and $\min\{\Re(p), \Re(q)\} \geq 0$ was studied by Choi et al. [8]. In turn, if we put $q = p$ and $\nu = 0$ in (2), it reduces to the generalized extended beta function

$$\mathrm{B}^{\lambda}_{p,p,\frac{1}{2}}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{e}^{-p(t^{-\lambda}+(1-t)^{-\lambda})}\,\mathrm{d}t, \quad \lambda > 0, \Re(p) > 0,$$

which should be distinguished from the generalization of the beta function studied by Lee et al. [12, p. 189, Eq. (1.13)]:

$$\mathrm{B}(a,b;p;m) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{e}^{-pt^{-m}(1-t)^{-m}}\,\mathrm{d}t, \quad m > 0, \Re(p) > 0.$$

Another particular case occurs for $\lambda = 1$, $q = p$, and $\nu = 0$, where we get the p-extended beta function [1, p. 20, Eq. (1.7)]

$$\mathrm{B}_p(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t, \quad \Re(p) \geq 0; \min\{\Re(a), \Re(b)\} > 0.$$

Finally, if use (3) and set $\nu = 0$ and $p, q \searrow 0$, then $\mathrm{B}^{\lambda}_{p,q,\nu}(a,b) \to \mathrm{B}(a,b)$ gives Euler's integral (1).

In this paper, we investigate the extended beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(x,y)$ and define the associated hypergeometric and confluent (Kummer-type) hypergeometric functions. For each of these, we obtain an integral representation and derive functional upper bounds. Finally, we introduce a related probability distribution, which we exploit to present Turán-type inequalities for the newly defined extension of the already studied beta functions.

2 Novel extended hypergeometric functions

In this section, we extend the Gauss hypergeometric and confluent hypergeometric functions by making use of $\mathrm{B}^{\lambda}_{p,q,\nu}(x,y)$.

Definition 2 The power series of the extended hypergeometric function reads

$$F^{\lambda}_{p,q,\nu}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+n,c-b)}{\mathrm{B}(b,c-b)}\frac{z^n}{n!}, \tag{4}$$

provided that $p \geq 0$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\Re(c) > \Re(b) > 0$, $|z| < 1$.

For all $p \geq 0$ and $\Re(c) > \Re(b) > 0$, the extended confluent hypergeometric function

$$\Phi^{\lambda}_{p,q,\nu}(b;c;z) = \sum_{n=0}^{\infty} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+n,c-b)}{\mathrm{B}(b,c-b)}\frac{z^n}{n!}. \tag{5}$$

To distinguish the previously defined functions, we call them $\mathrm{B}^{\lambda}_{p,q,\nu}$-extended hypergeometric and $\mathrm{B}^{\lambda}_{p,q,\nu}$-extended confluent hypergeometric functions, respectively.

Firstly, we derive integral expressions for the extended Gauss hypergeometric and confluent hypergeometric function.

Theorem 1 *We have the following integral representations for the* $\mathrm{B}^{\lambda}_{p,q,\nu}$*–extended hypergeometric function*:

$$F^{\lambda}_{p,q,\nu}(a,b;c;z) = \frac{\sqrt{2/\pi}}{\mathrm{B}(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \sqrt{h_\theta(t)} K_{\nu+\frac{1}{2}}\big(h_\theta(t)\big)\,\mathrm{d}t \tag{6}$$

for all $p > 0$ *and* $|\arg(1-z)| < \pi$ *or for* $p = 0$ *and* $\Re(c) > \Re(b) > 0$.

Moreover, for all $p > 0$ *or for* $p = 0$ *and* $\Re(c) > \Re(b) > 0$, *we have*

$$\Phi^{\lambda}_{p,q,\nu}(b;c;z) = \frac{\sqrt{2/\pi}}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\mathrm{e}^{zt} \sqrt{h_\theta(t)} K_{\nu+\frac{1}{2}}\big(h_\theta(t)\big)\,\mathrm{d}t. \tag{7}$$

Proof Substituting definition (2) of $\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)$ into (4), we have

$$\begin{aligned} F^{\lambda}_{p,q,\nu}(a,b;c;z) &= \sqrt{\frac{2}{\pi}} \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-\frac{\lambda}{2}-1}(1-t)^{c-b-\frac{\lambda}{2}-1} \\ &\quad \cdot \sqrt{p(1-t)^\lambda + qt^\lambda} K_{\nu+\frac{1}{2}}\left(\frac{p}{t^\lambda} + \frac{q}{(1-t)^\lambda}\right) \sum_{n=0}^\infty (a)_n \frac{(zt)^n}{n!}\,\mathrm{d}t. \end{aligned}$$

Employing the binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^\infty (a)_n \frac{(zt)^n}{n!}$$

in the integrand, which obviously holds since $|zt| < |t| < 1$, we obtain the stated integral in (6).

A similar argument can be used to establish the integral representation of the extended confluent hypergeometric function in (7). □

Setting $z = 1$ in (6) and using the definition (2), we readily obtain a related summation result.

Corollary 1.1 *For all* $p > 0$ *or for* $p = 0$ *and* $\Re(c-a-b) > 0$, *we have*

$$F^{\lambda}_{p,q,\nu}(a,b;c;1) = \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b,c-a-b)}{\mathrm{B}(b,c-b)}.$$

3 Bound for extended beta and consequences

In this section, we expose our first main result on a functional upper bound for $\mathrm{B}^{\lambda}_{p,q,\nu}$ with some related consequences.

Theorem 2 *Let $p,q>0$, $\lambda\in(0,1)\cup(1,\infty)$, and $\nu\in\mathbb{R}$. Then for all $2\min\{a,b\}>\lambda>0$, we have*

$$\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)\le\frac{\sqrt{2pq}K_{\nu+\frac12}((p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}})^{\lambda+1})}{\sqrt{\pi}(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}})^{\frac{\lambda-1}{2}}}\mathrm{B}\left(a-\frac{\lambda}{2},b-\frac{\lambda}{2}\right)=:R^{\lambda}_{p,q,\nu}(a,b). \tag{8}$$

Moreover, for $\lambda=1$, $p\neq q$, and $2\min\{a,b\}>1$,

$$R^{1}_{p,q,\nu}(a,b)=\begin{cases}\lim_{\lambda\nearrow1}R^{\lambda}_{p,q,\nu}(a,b)=\frac{\sqrt{2pq}}{\sqrt{\pi}\min\{p,q\}}K_{\nu+\frac12}((\sqrt{p}+\sqrt{q})^2)\mathrm{B}(a-\frac12,b-\frac12),\\ \lim_{\lambda\searrow1}R^{\lambda}_{p,q,\nu}(a,b)=\frac{\sqrt{2\max\{p,q\}}}{\sqrt{\pi}}K_{\nu+\frac12}((\sqrt{p}+\sqrt{q})^2)\mathrm{B}(a-\frac12,b-\frac12),\end{cases} \tag{9}$$

whilst

$$R^{1}_{p,p,\nu}(a,b)=\sqrt{\frac{2p}{\pi}}K_{\nu+\frac12}(4p)\mathrm{B}\left(a-\frac12,b-\frac12\right). \tag{10}$$

Proof We start with the defining integral (2) for $p\neq q$. Obviously,

$$\begin{aligned}\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)&\le\sqrt{\frac{2}{\pi}}\max_{0\le t\le1}\sqrt{p(1-t)^{\lambda}+qt^{\lambda}}\sup_{0<t<1}K_{\nu+\frac12}\left(\frac{p}{t^{\lambda}}+\frac{q}{(1-t)^{\lambda}}\right)\\&\quad\cdot\int_0^1 t^{a-\frac{\lambda}{2}-1}(1-t)^{b-\frac{\lambda}{2}-1}\,\mathrm{d}t\\&\le\sqrt{\frac{2}{\pi}}\sqrt{\max_{0\le t\le1}\{p(1-t)^{\lambda}+qt^{\lambda}\}}K_{\nu+\frac12}\left(\inf_{0<t<1}\left\{\frac{p}{t^{\lambda}}+\frac{q}{(1-t)^{\lambda}}\right\}\right)\\&\quad\cdot\mathrm{B}\left(a-\frac{\lambda}{2},b-\frac{\lambda}{2}\right),\end{aligned} \tag{11}$$

where both estimated functions in the integrand are positive in the declared range of parameters, and the MacDonald function $K_\mu(z)$ is decreasing and continuous for $z>0$. For

$$h_1(t)=p(1-t)^{\lambda}+qt^{\lambda},$$

we have $h_1'(t)=\lambda[-p(1-t)^{\lambda-1}+qt^{\lambda-1}]$, and the stationary point becomes

$$t_0=\frac{1}{1+(\frac{q}{p})^{\frac{1}{\lambda-1}}}\in(0,1).$$

Furthermore,

$$h_1''(t)=\lambda(\lambda-1)\big[p(1-t)^{\lambda-2}+qt^{\lambda-2}\big]<0,\quad \lambda\in(0,1),$$

and therefore t_0 is the abscissa of maximum for $h_1(t)$. Routine calculations lead to

$$\sqrt{h_1(t_0)}=\frac{\sqrt{pq}}{(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}})^{\frac{\lambda-1}{2}}}, \tag{12}$$

which ensures (8), and we infer $R^{1}_{p,q,\nu}(a,b)$ by the limit asn $\lambda \nearrow 1$. However, letting $p < q$, say, the denominator in (12) behaves as

$$\left[1 + \left(\frac{p}{q}\right)^{\frac{1}{1-\lambda}}\right]^{\frac{\lambda-1}{2}} \underset{\lambda\nearrow 1}{\longrightarrow} 1,$$

which confirms the first part of the upper bound formula in (9).

In turn, for $\lambda > 1$, $h_1''(t) > 0$ for all $t \in [0,1]$, i.e., this function is convex, and, consequently,

$$\max_{0\le t\le 1} h_1(t) = \max\big\{h_1(0), h_1(1)\big\} = \max\{p,q\},$$

which, together with (11), completes the proof for different p and q. In the case $p = q$, we straightforwardly get the behavior of $h_1(t)$.

Now denote the argument function of the MacDonald function by

$$h_\theta(t) = pt^{-\lambda} + q(1-t)^{-\lambda}.$$

The stationary point t_1 is the solution of $h_\theta'(t) = -\lambda[pt^{-\lambda-1} - q(1-t)^{-\lambda-1}] = 0$ with respect to t, that is,

$$t_1 = \frac{1}{1 + (\frac{q}{p})^{\frac{1}{\lambda+1}}} \in (0,1).$$

Since $h_\theta''(t) = \lambda(\lambda+1)[pt^{-\lambda-2} + q(1-t)^{-\lambda-2}] > 0$ for all $t \in (0,1)$, the value $h_\theta(t_1)$ is the global minimum of the considered function, where

$$h_\theta(t_1) = \min_{0<t<1} h_\theta(t) = \big(p^{\frac{1}{\lambda+1}} + p^{\frac{1}{\lambda+1}}\big)^{\lambda+1} \underset{\lambda\to 1}{\longrightarrow} (\sqrt{p} + \sqrt{q})^2,$$

which proves (9) and also the claim (10) by taking $p = q$ in the previous relation. □

At the remaining part of this section, we derive functional bounds for the real-argument $\mathrm{B}^{\lambda}_{p,q,\nu}$-extended hypergeometric function $F^{\lambda}_{p,q,\nu}$ and the $\mathrm{B}^{\lambda}_{p,q,\nu}$-extended confluent hypergeometric $\Phi^{\lambda}_{p,q,\nu}$ in terms of Euler's beta function. To obtain these bounds, we apply the results of Theorems 1 and 2.

Theorem 3 *For all* $p, q > 0$, $\lambda \in (0,1) \cup (1,\infty)$, *and* $\nu \in \mathbb{R}$ *or for* $p = 0$ *and* $\Re(c) > \Re(b) > 0$, *we have*

$$\big|F^{\lambda}_{p,q,\nu}(a,b;c;z)\big| \le \frac{g(z)}{\mathrm{B}(b,c-b)} R^{\lambda}_{p,q,\nu}(b,c-b), \tag{13}$$

provided that $2\min\{b, c-b\} > \lambda$, $\Re(z) > 0$. *Here*

$$g(z) = \begin{cases} (1-|z|)^{-a}, & a > 0,\\ 1, & a = 0,\\ 2^{-a}, & a < 0. \end{cases}$$

Moreover,

$$\left|\Phi_{p,q,\nu}^{\lambda}(b;c;z)\right| \le \frac{\mathrm{e}^{|z|}}{\mathrm{B}(b,c-b)} R_{p,q,\nu}^{\lambda}(b,c-b), \quad \Re(z) > 0. \tag{14}$$

In both upper bounds, we have

$$R_{p,q,\nu}^{\lambda}(b,c-b) = \frac{\sqrt{2pq}K_{\nu+\frac{1}{2}}((p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}})^{\lambda+1})}{\sqrt{\pi}(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}})^{\frac{\lambda-1}{2}}} \mathrm{B}\left(b-\frac{\lambda}{2}, c-b-\frac{\lambda}{2}\right),$$

whilst for $\lambda = 1$, $p \neq q$, *and* $2\min\{b, c-b\} > \lambda$,

$$R_{p,q,\nu}^{1}(b,c-b) = \begin{cases} \lim_{\lambda\nearrow 1} R_{p,q,\nu}^{\lambda}(b,c-b) = \frac{\sqrt{2pq}}{\sqrt{\pi}\min\{p,q\}} K_{\nu+\frac{1}{2}}((\sqrt{p}+\sqrt{q})^2) \\ \qquad \cdot \mathrm{B}(b-\frac{1}{2}, c-b-\frac{1}{2}), \\ \lim_{\lambda\searrow 1} R_{p,q,\nu}^{\lambda}(b,c-b) = \frac{\sqrt{2\max\{p,q\}}}{\sqrt{\pi}} K_{\nu+\frac{1}{2}}((\sqrt{p}+\sqrt{q})^2) \\ \qquad \cdot \mathrm{B}(b-\frac{1}{2}, c-b-\frac{1}{2}). \end{cases}$$

Finally,

$$R_{p,p,\nu}^{1}(b,c-b) = \sqrt{\frac{2p}{\pi}} K_{\nu+\frac{1}{2}}(4p) \mathrm{B}\left(b-\frac{1}{2}, c-b-\frac{1}{2}\right).$$

Proof Consider the bound upon the function $g(z) = |1-zt|^{-a}$, which occurs in the integrand of the integral representation (6) of the extended hypergeometric function:

$$|1-zt|^{-a} \le g(z) = \begin{cases} (1-|z|)^{-a}, & a > 0, \\ 1, & a = 0, \\ 2^{-a}, & a < 0. \end{cases}$$

The bounds are an immediate consequence of the triangle inequality and the fact that we integrate with respect to $t \in (0,1)$. Hence, again by the triangle inequality, by Theorems 1 and 2 we have the estimate

$$\left|F_{p,q,\nu}^{\lambda}(a,b;c;z)\right| \le \sqrt{\frac{2}{\pi}} \frac{g(z)}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-\frac{\lambda}{2}-1}(1-t)^{c-b-\frac{\lambda}{2}-1} \cdot \sqrt{p(1-t)^{\lambda}+qt^{\lambda}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t^{\lambda}} + \frac{q}{(1-t)^{\lambda}}\right) \mathrm{d}t.$$

By arguments similar to those in [14, p. 1436, Proposition 2] we see that for $p, q > 0$ and $t \in (0,1)$, the MacDonald function of the second kind $K_{\nu+\frac{1}{2}}$ is positive, which means that

$$\left|F_{p,q,\nu}^{\lambda}(a,b;c;z)\right| \le \frac{g(z)}{\mathrm{B}(b,c-b)} \mathrm{B}_{p,q,\nu}^{\lambda}(b,c-b) \le \frac{g(z)}{\mathrm{B}(b,c-b)} R_{p,q,\nu}^{\lambda}(b,c-b),$$

where the derivation of $R_{p,q,\nu}^{\lambda}(b,c-b)$ is described in the proof of Theorem 2; compare also relations (8)–(10). The bound (13) is proved.

Mimicking this prooof, we easily establish the functional upper bound (14) upon the Kummer-type confluent hypergeometric function $\Phi^{\lambda}_{p,q,\nu}$ for the quoted range of parameters. □

Remark 1 Having in mind that

$$\int_0^1 \frac{\mathrm{d}t}{(1-zt)^a} = \frac{1-(1-z)^{1-a}}{(1-a)z}; \qquad \int_0^1 \mathrm{e}^{zt}\,\mathrm{d}t = \frac{\mathrm{e}^z-1}{z},$$

following the proof of Theorem 3, we could refine the functional upper bounds in the previous theorem.

4 The extended beta distribution

As a probabilistic application of $\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)$, let ξ be a random variable (r.v.) on a standard probability space $(\Omega,\mathfrak{F},\mathsf{P})$ distributed according to the so-called $\mathrm{B}^{\lambda}_{p,q,\nu}$-extended beta distribution, which we introduce by employing $\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)$. The defining probability density function (PDF) reads

$$f_\Theta(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{x^{a-1}(1-t)^{b-1}}{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)}\sqrt{h_\theta(x)}K_{\nu+\frac12}(h_\theta(x)), & 0<t<1, \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

The range of the parameter vector $\Theta = (a,b,p,q,\lambda) \in \mathbb{R}^5_+$ is $\min\{a,b\} > \frac{\lambda}{2} > 0$ and $\nu \in \mathbb{R}$, as it originates from the MacDonald function $K_{\nu+\frac12}$ in (2), and we recall that it is even in parameter.

The related cumulative distribution function associated with the PDF (15) can be expressed as

$$F_\Theta(x) = \begin{cases} 0, & x \le 0, \\ \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b;x)}{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)}, & 0<x\le 1, \\ 1, & x>1,\end{cases}$$

where for all $\Theta > 0$ and $\nu \in \mathbb{R}$,

$$\mathrm{B}^{\lambda}_{p,q,\nu}(a,b;x) = \sqrt{\frac{2}{\pi}}\int_0^x t^{a-1}(1-t)^{b-1}\sqrt{h_\theta(t)}K_{\nu+\frac12}\big(h_\theta(t)\big)\,\mathrm{d}t,$$

which turns out to be the extended *incomplete* extended beta function. Finally, we write $\xi \sim \mathrm{B}^{\lambda}_{p,q,\nu}$ or $\xi \sim f_\Theta(t)$, where Θ stands for the parameter vector (a,b,p,q,λ). Our first result concerns the rth raw moment of $\xi \sim \mathrm{B}^{\lambda}_{p,q,\nu}$:

$$m_r = \mathsf{E}\xi^r = \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+r,b)}{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)}, \quad a+r > \frac{\lambda}{2} > 0.$$

Particular frequently used cases are the mean and variance:

$$m_1 = \mathsf{E}\xi = \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+1,b)}{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)},$$
$$\operatorname{Var}\xi = \mathsf{E}\xi^2 - (\mathsf{E}\xi)^2 = \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)\mathrm{B}^{\lambda}_{p,q,\nu}(a+2,b) - [\mathrm{B}^{\lambda}_{p,q,\nu}(a+1,b)]^2}{\{\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)\}^2}.$$

Theorem 4 *For all* $\Theta = (a,b,p,q,\lambda) \in \mathbb{R}^5_+$ *with* $\min\{a,b\} > \frac{\lambda}{2} > 0$, *we have the Turán inequality*

$$\mathrm{B}^{\lambda}_{p,q,\nu}(a+s,b)\mathrm{B}^{\lambda}_{p,q,\nu}(a+s+2r,b) \geq \left[\mathrm{B}^{\lambda}_{p,q,\nu}(a+s+r,b)\right]^2, \quad s,r > 0. \tag{16}$$

Moreover,

$$\mathrm{B}^{\lambda}_{p,q,\nu}(a+2s,b)\mathrm{B}^{\lambda}_{p,q,\nu}(a+2r,b) \geq \left[\mathrm{B}^{\lambda}_{p,q,\nu}(a+s+r,b)\right]^2, \quad s \geq r \geq 0.$$

Proof The inequality for the raw moments $m_r, r > 0$, of nonnegative random variables [13, p. 28, Eqs. (1.4.6)],

$$m^2_{s+r} \leq m_s m_{s+2r}, \quad s,r > 0,$$

proves the first stated inequality.

Next, following again Lukacs [13, p. 393, a)], for all $0 \leq r \leq s$, the slightly different moment inequality $m^2_{s+r} \leq m_{2s} \cdot m_{2r}$ is implied by the familiar CBS inequality with a simple rescaling of the integrand in m_{s+r}. The rest is obvious. □

Remark 2 The immediate consequence of the nonnegativity of the variance is the following contiguous Turán inequality:

$$\left[\mathrm{B}^{\lambda}_{p,q,\nu}(a+1,b)\right]^2 \leq \mathrm{B}^{\lambda}_{p,q,\nu}(a,b)\mathrm{B}^{\lambda}_{p,q,\nu}(a+2,b)$$

with respect to the variable a of the extended beta function. Obviously, for $r = 1$ and $s = 0$, relation (16) implies the same result.

To establish a bound of other kind, we use the integral moment inequality [9, p. 143, Theorem 192]

$$\mathfrak{M}_r(h,p) < \mathfrak{M}_s(h,p), \quad 0 < r < s, \tag{17}$$

where

$$\mathfrak{M}_r(h,p) = \int_\alpha^\beta h^r(t)p(t)\,\mathrm{d}t$$

for a suitable integrable nonnegative input function h. The integration interval (α,β) is either finite or infinite, and the nonnegative weight function p satisfies $\int_\alpha^\beta p(t)\,\mathrm{d}t = 1$. In our case the abbreviation $\mathfrak{M}_s(x^s, f_\Theta) = (m_s)^{1/s}$ is used for the r.v. ξ with distribution $\mathrm{B}^{\lambda}_{a,b,\nu}$

and $(\alpha,\beta) = (0,1)$. Inserting m_s into the moment inequality (17), we obtain the following Lyapunov-type inequality for the integral mean of the $\mathrm{B}^{\lambda}_{p,q,\nu}$ distribution.

Theorem 5 *Let a r.v. ξ be $\mathrm{B}^{\lambda}_{p,q,\nu}$-distributed. Then, for all $\Theta = (a,b,p,q,\lambda) \in \mathbb{R}^5_+$, $\nu \in \mathbb{R}$, and $\min\{a,b\} + r > \frac{\lambda}{2} > 0$, we have*

$$\frac{[\mathrm{B}^{\lambda}_{p,q,\nu}(a+s,b)]^{\frac{1}{s}}}{[\mathrm{B}^{\lambda}_{p,q,\nu}(a+r,b)]^{\frac{1}{r}}} > \left[\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)\right]^{\frac{1}{s}-\frac{1}{r}}, \qquad s > r > 0.$$

Finally, we point out that the $\mathrm{B}^{\lambda}_{p,q,\nu}$ distribution and the properties of the here defined incomplete extended beta function deserve a precise study to be addressed in the future research.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University-A Central University, Pondicherry, 605014, Puducherry, India. [2]Institute of Applied Mathematics, John von Neumann Faculty of Informatics, Óbuda University, Bécsi út 96/b, 1034 Budapest, Hungary. [3]Faculty of Maritime Studies, University of Rijeka, Studentska 2, 51000 Rijeka, Croatia.

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