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Ibragimov–Gadjiev operators preserving exponential functions

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Abstract

In this paper, a modification of general linear positive operators introduced by Ibragimov and Gadjiev in 1970 is constructed. It is shown that this modification preserves exponential mappings and also contains modified Bernstein-, Szász- and Baskakov-type operators as special cases. The convergence properties of corresponding operators on $[0,\infty)$ and in exponentially weighted spaces are investigated. Finally, the quantitative Voronovskaja theorem in terms of modulus of continuity for functions having exponential growth is examined.

Keywords: Ibragimov; Gadjiev operators; Modified Bernstein-type operators; Szász; Mirakjan operators; Modulus of continuity

1 Introduction

After the construction of a general sequence of positive operators by Ibragimov and Gadjiev [1] in 1970, some authors, inspired by this work, introduced various generalizations of these operators. This is because the results obtained for the generalized operators are also valid for the operators included in them. Thus, Aral and Acar [2] introduced a general class of Durrmeyer operators by modifying the Ibragimov–Gadjiev operators as

$$\begin{aligned} M_n(f;x) = (n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}\big(x,0,\alpha_n\psi_n(0)\big)\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \\ \times\int_0^\infty f(y)K_n^{(\nu)}\big(y,0,\alpha_n\psi_n(0)\big)\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\,dy, \end{aligned}$$

where

$$K_n^{(\nu)}\big(x,0,\alpha_n\psi_n(0)\big) = \frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}$$

$x,t\in\mathbb{R}^+$ and $-\infty<u<\infty$ is a sequence of functions of three variables x, t, u such that K_n is entire analytic function with respect to variable u for each $x,t\in\mathbb{R}^+$ and for each $n\in\mathbb{N}$. Bozma and Bars [3] defined a Kantorovich-type generalization on a variable bounded interval:

$$R_n(f;x) = (\beta_n + n + 2)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x)\frac{(-\alpha_n)^\nu}{\nu!}\int_{\frac{\nu+n+1}{\beta_n+n+2}}^{\frac{\nu+n+2}{\beta_n+n+2}} f(p)\,dp,$$

where $f : L_1[0,\frac{n+1}{n+2}] \to C[0,\frac{n+1}{n+2}]$. Herdem and Büyükyazıcı [4] constructed an extension in q-Calculus of these operators. The q-generalization of Ibragimov–Gadjiev operators have the following form:

$$L_n(f;q;x) = \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} f\left(\frac{[\nu]}{[n]^2\psi_n(0)}\right) D_{q,u}K_{n,\nu}^q(x,t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \frac{(-\alpha_n\psi_n(0))^\nu}{[\nu]!}$$

for $x \in \mathbb{R}^+$ and any function f defined on $\mathbb{R}^+$. Furthermore, Korovkin-type theorems for continuous and unbounded functions defined on $[0,\infty)$ were established, and some representation formulas using q-derivatives were given in [5]. Many other investigations about Ibragimov–Gadjiev operators may also be found in [6–11].

Now, we recall the original construction. Let $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ be the sequence of functions in $C[0,A]$ such that $\varphi_n(0) = 0$ and $\psi_n(t) > 0$ for all $t \in [0,A]$, $A > 0$. Let also $\{\alpha_n\}$ be a sequence of positive numbers having the following properties $\lim_{n\to\infty}\frac{\alpha_n}{n} = 1$, $\lim_{n\to\infty}\frac{1}{n^2\psi_n(0)} = 0$.

Assume that a sequence of functions of three variables $\{K_n(x,t,u)\}$ $(x,t \in [0,A], -\infty < u < \infty)$ satisfies the following conditions:

1^o Each function of this family is an entire analytic function with respect to u for fixed $x,t \in [0,A]$;

2^o $K_n(x,0,0) = 1$ for any $x \in [0,A]$ and for all $n \in \mathbb{N}$;

3^o $\{(-1)^\nu[\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)]_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\} \geq 0$ $(\nu, n \in \{1,2,\dots\};\ x \in [0,A])$;

4^o $-\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} = nx[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{n+m}(x,t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}](\nu, n \in \{1,2,\dots\}; x \in [0,A])$, where m is a number such that $m + n$ is zero or a naturel number.

Under these conditions, Ibragimov–Gadjiev operators are defined as

$$G_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\left[\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \tag{1}$$

for $x \in \mathbb{R}^+$ and any function f defined on $\mathbb{R}^+$.

We should mention that Ibragimov–Gadjiev operators contain, as a particular case, a series of operators. By choosing $K_n(x,t,u) = (1-\frac{ux}{1+t})^n, \alpha_n = n,\ \psi_n(0) = \frac{1}{n}$ the operators defined by (1) are transformed into Bernstein polynomials; for $\alpha_n = n$, $\psi_n(0) = \frac{1}{nb_n}$ $(\lim_{n\to\infty} b_n = \infty,\ \lim_{n\to\infty}\frac{b_n}{n} = 0)$, we also get Bernstein–Chlodowsky polynomials. For $K_n(x,t,u) = e^{-n(t+ux)}$, $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, we get Szász–Mirakyan operators. Moreover, if we choose $K_n(x,t,u) = K_n(t+ux)$, $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, then we obtain Baskakov operators.

In recent years, by defining the linear positive operators, which preserve the exponential functions by Aldaz and Render [12], several researchers introduced linear positive operators that reproduce the exponential functions by conveniently modifying the well-known operators. In [13], Acar et al. presented a modification of Szász-Mirakyan operators that reproduces the functions 1 and e^{2ax}, $a > 0$. They discussed approximation properties via a certain weighted modulus of continuity and a quantitative Voronovskaya-type theorem. In

[14], recovered a generalization of the Bernstein operators that reproduce the exponential functions e^{ax} and e^{2ax}, $a > 0$. The authors also showed that this way better approximates functions with modified operators than the classical one. After that, this method was studied and extended in numerous papers. We refer interested readers to [15–32].

In parallel with these developments, this paper aims to construct a new generalization of Ibragimov–Gadjiev operators, G_n^*, fixing the function e^{ax}, $a > 0$. Then, for these operators, we provide some approximation properties and present special cases as examples.

The rest of this work is organized as follows: In Sect. 2, the technique to construct the modified Ibragimov–Gadjiev operators is discussed. In Sect. 3, moments, central moments, and a recurrence formula are calculated. In Sect. 4, convergence properties on $[0,\infty)$ and in the light of weighted spaces are investigated. The rate of convergence using the exponential modulus of continuity is also examined. In Sect. 5, it is shown that the new operators, which will be constructed below, contain modified Bernstein-, Szász- and Baskakov-type operators that exist in the literature as a special case. Finally, in the last section, we summarize the main results and give some thoughts that can be applied to expand the scope of this study.

2 Construction of the operators

Let $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ be the sequence of functions in $C[0,A]$ such that $\varphi_n(0) = 0$ and $\psi_n(t) > 0$ for all $t \in [0,A]$, $A > 0$. Let also $\{\alpha_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \frac{\alpha_n}{n} = 1$, $\lim_{n\to\infty} \frac{1}{n^2\psi_n(0)} = 0$.

Assume that a sequence of functions of three variables $\{K_n(\lambda_n(x),t,u)\}$ ($x,t \in [0,A]$, $-\infty < u < \infty$ and $\lim_{n\to\infty} \lambda_n(x) = x$) satisfies the following conditions:

1^* Each function of this family is an entire analytic function with respect to u for fixed $x,t \in [0,A]$;

2^* $K_n(\lambda_n(x),0,0) = 1$ for any $x \in [0,A]$ and for all $n \in \mathbb{N}$;

3^* $\{(-1)^\nu [\frac{\partial^\nu}{\partial u^\nu} K_n(\lambda_n(x),t,u)]_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\} \geq 0$ ($\nu, n \in \{1,2,\ldots\}$; $x \in [0,A]$);

4^* $-\frac{\partial^\nu}{\partial u^\nu} K_n(\lambda_n(x),t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} = n\lambda_n(x)[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}} K_{n+m}(\lambda_n(x),t,u)|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}]$ ($\nu, n \in \{1,2,\ldots\}$; $x \in [0,A]$), where m is a number such that $m+n$ is zero or a natural number.

According to these conditions, modified Ibragimov–Gadjiev operators have the following form:

$$G_n^*(f;x) = \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n^2\psi_n(0)}} e^{ax} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\left[\frac{\partial^\nu}{\partial u^\nu} K_n\big(\lambda_n(x),t,u\big)\big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \tag{2}$$

for $x, a \in \mathbb{R}^+$ and any function f defined on $\mathbb{R}^+$. We also assume that G_n^* satisfies the following condition

$$\sum_{\nu=0}^{\infty} e^{-\frac{\mu\nu}{n^2\psi_n(0)}}\left[\frac{\partial^\nu}{\partial u^\nu} K_n\big(\lambda_n(x),t,u\big)\big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} = e^{\beta_n(\mu x)} \tag{3}$$

for all $n \in \mathbb{N}$ and $\mu \in \mathbb{Z}$. Here, $\beta_n(x)$ is a sequence such that $\lim_{n\to\infty} \beta_n(x) = x$.

It can be easily shown that the operators G_n^* preserve e^{ax}, i.e.,

$$G_n^*\big(e^{at};x\big) = e^{ax}.$$

We can use the Taylor expansion of $\{K_n(\lambda_n(x),t,u)\}$ due to condition (1^o). Setting $u=\varphi_n(t)$ and $u_1=\alpha_n\psi_n(t)$, we have

$$K_n\big(\lambda_n(x),t,\varphi_n(t)\big)=\sum_{\nu=0}^{\infty}\left[\frac{\partial^\nu}{\partial u^\nu}K_n\big(\lambda_n(x),t,u\big)\big|_{u=\alpha_n\psi_n(t)}\right]\frac{(\varphi_n(t)-\alpha_n\psi_n(t))^\nu}{\nu!}.$$

Since $\varphi_n(0)=0$ and $K_n(\lambda_n(x),0,0)=1$, by taking $t=0$ the above equation turns into

$$\sum_{\nu=0}^{\infty}\left[\frac{\partial^\nu}{\partial u^\nu}K_n\big(\lambda_n(x),t,u\big)\big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}=1. \tag{4}$$

Using Eq. (4), we get

$$\begin{aligned}
G_n^*\big(e^{at};x\big)&=\sum_{\nu=0}^{\infty}e^{-\frac{a\nu}{n^2\psi_n(0)}}e^{ax}e^{\frac{a\nu}{n^2\psi_n(0)}}\left[\frac{\partial^\nu}{\partial u^\nu}K_n\big(\lambda_n(x),t,u\big)\big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\\
&=e^{ax}\sum_{\nu=0}^{\infty}\left[\frac{\partial^\nu}{\partial u^\nu}K_n\big(\lambda_n(x),t,u\big)\big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\\
&=e^{ax}.
\end{aligned}$$

3 Auxilary results

In this part, we mention some obvious properties of the modified Ibragimov–Gadjiev operators.

Lemma 1 *Let $f(t)=e^{\theta at}$, $\theta\in\mathbb{Z}$. Then, for the operators defined by (2), we have*

$$G_n^*\big(e^{\theta at};x\big)=e^{ax+\beta_n((\theta-1)ax)}$$

here β_n is the same as (3).

We give the following lemma without proof since it similarly follows from the work by the authors dealing with moments for the q-Ibragimov–Gadjiev operators [4].

Lemma 2 *Let $e_i(t):=t^i$, $i=0,1,2$. Then the operators G_n^* satisfies*

$$\begin{aligned}
G_n^*(e_0;x)&=e^{ax+\beta_n(-ax)},\\
G_n^*(e_1;x)&=e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\lambda_n(x),\\
G_n^*(e_2;x)&=e^{(ax+e^{\beta_{n+2m}(-ax)}-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}\lambda_n^2(x)\\
&\quad+e^{(ax+\beta_{n+m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\lambda_n(x),\\
G_n^*(e_3;x)&=e^{(ax+\beta_{n+3m}(-ax)-\frac{3a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^3\frac{(n+m)(n+2m)}{n^2}\lambda_n^3(x)\\
&\quad+e^{(ax+\beta_{n+2m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{3(n+m)}{n}\frac{1}{n^2\psi_n(0)}\lambda_n^2(x)
\end{aligned}$$

$$
\begin{aligned}
&+ e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^2\lambda_n(x),\\
G_n^*(e_4;x) &= e^{(ax+\beta_{n+4m}(-ax)-\frac{4a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^4\frac{(n+m)(n+2m)(n+3m)}{n^3}\lambda_n^4(x)\\
&+ e^{(ax+\beta_{n+3m}(-ax)-\frac{3a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^3\frac{6(n+m)(n+2m)}{n^2}\frac{1}{n^2\psi_n(0)}\lambda_n^3(x)\\
&+ e^{(ax+\beta_{n+2m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{7(n+m)}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^2\lambda_n^2(x)\\
&+ e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^3\lambda_n(x).
\end{aligned}
$$

Lemma 3 *Let $\mu_{n,r}(x) = G_n^*((t-x)^r;x)$, $r = 0,1,2,4$. Then by considering above Lemma, we have*

$$
\begin{aligned}
\mu_{n,0}(x) &= e^{ax+\beta_n(-ax)},\\
\mu_{n,1}(x) &= \frac{\alpha_n}{n}\lambda_n(x)e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})} - xe^{ax+\beta_n(-ax)},\\
\mu_{n,2}(x) &= \left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}\lambda_n^2(x)e^{(ax+e^{\beta_{n+2m}(-ax)}-\frac{2a}{n^2\psi_n(0)})}\\
&- \frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\lambda_n(x)e^{(ax+\beta_{n+m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\\
&- 2x\frac{\alpha_n}{n}\lambda_n(x)e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})} + x^2e^{ax+\beta_n(-ax)},\\
\mu_{n,4}(x) &= e^{(ax+\beta_{n+4m}(-ax)-\frac{4a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^4\frac{(n+m)(n+2m)(n+3m)}{n^3}\lambda_n^4(x)\\
&+ e^{(ax+\beta_{n+3m}(-ax)-\frac{3a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^3\frac{6(n+m)(n+2m)}{n^2}\frac{1}{n^2\psi_n(0)}\lambda_n^3(x)\\
&+ e^{(ax+\beta_{n+2m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{7(n+m)}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^2\lambda_n^2(x)\\
&+ e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^3\lambda_n(x)\\
&- 4x\left\{e^{(ax+\beta_{n+3m}(-ax)-\frac{3a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^3\frac{(n+m)(n+2m)}{n^2}\lambda_n^3(x)\right.\\
&+ e^{(ax+\beta_{n+2m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{3(n+m)}{n}\frac{1}{n^2\psi_n(0)}\lambda_n^2(x)\\
&\left.+ e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\left(\frac{1}{n^2\psi_n(0)}\right)^2\lambda_n(x)\right\}\\
&+ 6x^2\left\{e^{(ax+e^{\beta_{n+2m}(-ax)}-\frac{2a}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}\lambda_n^2(x)\right.\\
&\left.+ e^{(ax+\beta_{n+m}(-ax)-\frac{2a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\lambda_n(x)\right\}\\
&- 4x^3e^{(ax+\beta_{n+m}(-ax)-\frac{a}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\lambda_n(x) + x^4.
\end{aligned}
$$

4 Main results

In this section, first, we analyze the uniform convergence of G_n^* on $[0,\infty)$ by means of the modulus of continuity. In 1970, Boyanov and Vaselinov [33] gave approximation properties of a function in an infinite interval.

Now, suppose that $C^*[0,\infty)$ denotes the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property $\lim_{x\to\infty} f(x)$ existing and finite, given with uniform norm $\|.\|_{C^*[0,\infty)}$.

Theorem A [33] *If the sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of linear positive operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt};x\right) = e^{-kx}, \quad k = 0,1,2,$$

uniformly in $[0,\infty)$ then for $f \in [0,\infty)$,

$$\lim_{n\to\infty} A_n(f;x) = f(x).$$

Later, Holhoş [34] expanded Theorem A to find the rate of uniform convergence.

Theorem B [34] *Let $A_n : C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of positive linear operators satisfying that*

$$\left\|A_n(1;x) - 1\right\|_{[0,\infty)} = a_n,$$
$$\left\|A_n\left(e^{-t};x\right) - e^{-x}\right\|_{[0,\infty)} = b_n,$$
$$\left\|A_n\left(e^{-2t};x\right) - e^{-2x}\right\|_{[0,\infty)} = c_n.$$

Then, for every function of $f \in C^*[0,\infty)$,

$$\left\|A_n(f;x) - f(x)\right\|_{[0,\infty)} \le \|f\|_{[0,\infty)} a_n + (2 + a_n)\omega^*(f, \sqrt{a_n + 2b_n + c_n})$$

where

$$\omega^*(f,\delta) = \max_{\substack{|e^{-x}-e^{-t}|\le\delta \\ x,t>0}} \left|f(t) - f(x)\right|, \quad \delta > 0$$

is the modulus of continuity. Further,

$$\left|f(t) - f(x)\right| \le \left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\right)\omega^*(f,\delta).$$

Now, we are ready to prove our main theorem about uniform convergence of G_n^*.

Theorem 1 *For $f \in C^*[0,\infty)$, we have*

$$\left\|G_n^* f - f\right\|_{[0,\infty)} \le \|f\|_{[0,\infty)} a_n + (2 + a_n)\omega^*(f, \sqrt{a_n + 2b_n + c_n})$$

where

$$a_n = \left\| G_n^*(1;x) - 1 \right\|_{[0,\infty)}$$
$$b_n = \left\| G_n^*\left(e^{-t};x\right) - e^{-x} \right\|_{[0,\infty)}$$
$$c_n = \left\| G_n^*\left(e^{-2t};x\right) - e^{-2x} \right\|_{[0,\infty)}.$$

Moreover, a_n, b_n and c_n tend to zero as n goes to infinity so that G_n^* converges uniformly to f.

Proof Considering Lemma 1 and the definition of the operators G_n^*, we can write

$$G_n^*(1;x) = e^{ax+\beta_n(-ax)},$$
$$G_n\left(e^{-t};x\right) = \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n^2\psi_n(0)}} e^{ax} e^{-\frac{\nu}{n^2\psi_n(0)}} \left[\frac{\partial^\nu}{\partial u^\nu} K_n\left(\lambda(x),t,u\right)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$
$$= e^{ax} \sum_{\nu=0}^{\infty} \left(e^{-\frac{\nu(a+1)}{n^2\psi_n(0)}}\right) \left[\frac{\partial^\nu}{\partial u^\nu} K_n\left(\lambda(x),t,u\right)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$
$$= e^{ax} e^{\beta_n(-(a+1)x)}$$
$$G_n\left(e^{-2t};x\right) = \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n^2\psi_n(0)}} e^{ax} e^{-\frac{2\nu}{n^2\psi_n(0)}} \left[\frac{\partial^\nu}{\partial u^\nu} K_n\left(\lambda(x),t,u\right)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$
$$= e^{ax} \sum_{\nu=0}^{\infty} e^{-\frac{(a+2)\nu}{n^2\psi_n(0)}} \left[\frac{\partial^\nu}{\partial u^\nu} K_n\left(\lambda(x),t,u\right)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$
$$= e^{ax} e^{\beta_n(-(a+2)x)}.$$

Thus, the result follows immediately from (4) and Theorem B. □

Now, we examine the behavior of the operators G_n^* on some weighted spaces and then prove a quantitative Voronovskaja theorem in terms of modulus of continuity for functions having exponential growth.

Set $\varphi(x) = 1 + e^{2ax}$, $x \in \mathbb{R}^+$, and consider the following weighted spaces:

$$B_\varphi\left(\mathbb{R}^+\right) = \left\{f : \mathbb{R}^+ \to \mathbb{R} : \left|f(x)\right| \le M_f\varphi(x)\right\},$$
$$C_\varphi\left(\mathbb{R}^+\right) = C\left(\mathbb{R}^+\right) \cap B_\varphi\left(\mathbb{R}^+\right),$$
$$C_\varphi^k\left(\mathbb{R}^+\right) = \left\{f \in C\left(\mathbb{R}^+\right) : \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = k_f\right\},$$

where M_f and k_f are constants depending on f. All three spaces are normed spaces with the norm

$$\|f\|_\varphi = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

It is obvious that for any $f \in C_\varphi^k(\mathbb{R}^+)$, the inequality

$$\left\| G_n^*(f) \right\|_\varphi \le \|f\|_\varphi$$

holds, and we conclude that G_n^* maps $C_\varphi^k(\mathbb{R}^+)$ to $C_\varphi^k(\mathbb{R}^+)$ [25].

Theorem 2 *For each function $f \in C_\varphi^k(\mathbb{R}^+)$,*

$$\lim_{n\to\infty} \left\| G_n^*(f) - f \right\|_\varphi = 0.$$

Proof Using the general result established in [25], it is sufficient to verify that following three conditions

$$\lim_{n\to\infty} \left\| G_n^*\left(e^{\upsilon at}\right) - e^{\upsilon at} \right\|_\varphi = 0, \quad \upsilon = 0, 1, 2. \tag{5}$$

For $\upsilon = 0$, from Lemma 3, one has

$$\left\| G_n^*(1) - 1 \right\|_\varphi = \sup_{x\in\mathbb{R}^+} \frac{|e^{ax+\beta_n(-ax)} - 1|}{1 + e^{2ax}}.$$

By passing to limit condition, using equality (4), we have

$$\lim_{n\to\infty} \left\| G_n^*(1) - 1 \right\|_\varphi = 0.$$

We now prove for $\upsilon = 2$, similarly from Lemma 3 and equality (4), we get

$$\begin{aligned}
\left\| G_n^*\left(e^{2at}\right) - e^{2ax} \right\|_\varphi &= \sup_{x\in\mathbb{R}^+} \frac{|e^{ax+\beta_n(ax)} - e^{2ax}|}{1 + e^{2ax}} \\
&\le \frac{e^{ax}}{1 + e^{2ax}} \left| e^{\beta_n(ax)} - e^{ax} \right| \\
&\le \left| e^{\beta_n(ax)} - e^{ax} \right|,
\end{aligned}$$

which leads to

$$\lim_{n\to\infty} \left\| G_n^*\left(e^{2at}\right) - e^{2ax} \right\|_\varphi = 0.$$

Since $G_n^*(e^{at}; x) = e^{ax}$, condition (5) is implemented for $\upsilon = 1$. Hence, the proof is completed. □

Theorem 3 *Let $G_n^*: K \to C[0,\infty)$ be the sequence of linear positive operators preserving e^{ax}, $a > 0$. We suppose that for each constant $B > 0$ and fixed $x \in [0,\infty)$, G_n^* satisfy*

$$G_n^*\left((t-x)^2 e^{Bt}; x\right) \le C_a(B, x)\mu_{n,2}(x).$$

Additionally, if $f \in C^2[0,\infty) \cap K$ and $f'' \in Lip(c, B)$, $0 < c \le 1$, then for $x \in [0,\infty)$,

$$\left| G_n^*(f; x) - f(x) - f'(x)\mu_{n,1} - \frac{1}{2} f''(x)\mu_{n,2}(x) \right|$$

$$\leq \mu_{n,2}(x)\left(\frac{\sqrt{C_a(2B,x)}}{2}+\frac{C_a(B,x)}{2}+e^{2Bx}\right)\omega_1\left(f'',\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}},B\right).$$

Proof By considering the Taylor expansion of the function $f\in C^2[0,\infty)$ at $x\in[0,\infty)$, we obtain

$$f(t)=f(x)+f'(x)(t-x)+f''(x)\frac{(t-x)^2}{2}+h(t,x), \tag{6}$$

where

$$h(t,x)=\frac{(t-x)^2}{2}\big(f''(\xi)-f''(x)\big),\quad x<\xi<t.$$

Applying the operators G_n^* to equality (6), we have

$$\begin{aligned}\left|G_n^*(f;x)-f(x)-f'(x)\mu_{n,1}-\frac{1}{2}f''(x)\mu_{n,2}(x)\right| &= \left|G_n^*\big(h(t,x);x\big)\right|\\ &\leq G_n^*\big(\left|h(t,x)\right|;x\big).\end{aligned} \tag{7}$$

Additionally,

$$h(t,x)=\frac{(t-x)^2}{2}\big(f''(\xi)-f''(x)\big)\leq\frac{(t-x)^2}{2}\begin{cases}e^{Bx}\omega_1(f'',h,B), & |t-x|\leq h,\\ e^{Bx}\omega_1(f'',kh,B), & h\leq|t-x|\leq kh,\end{cases}$$

It was proved by Tachev et al. [35] that, for each $h>0$ and $k\in\mathbb{N}$,

$$\omega_1(f,kh,B)\leq ke^{B(k-1)h}\omega_1(f,h,B).$$

With the help of the above inequality, we obtain

$$\begin{aligned}\frac{(t-x)^2e^{Bx}}{2}\omega_1\big(f'',kh,B\big) &\leq \frac{(t-x)^2e^{Bx}}{2}ke^{B(k-1)h}\omega_1\big(f'',h,B\big)\\ &\leq \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)e^{Bx}e^{B|t-x|}\omega_1\big(f'',h,B\big)\\ &\leq \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\big(e^{Bt}+e^{2Bx}\big)\omega_1\big(f'',h,B\big).\end{aligned}$$

Thus,

$$\left|h(t,x)\right|\leq\frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\big(e^{Bt}+e^{2Bx}\big)\omega_1\big(f'',h,B\big).$$

Applying the operators G_n^* to both sides of the above inequality, we have

$$\begin{aligned}G_n^*\big(\left|h(t,x)\right|;x\big) &\leq \frac{1}{2}G_n^*\left(\left(\frac{|t-x|^3}{h}+|t-x|^2\right)\big(e^{Bt}+e^{2Bx}\big);x\right)\omega_1\big(f'',h,B\big)\\ &= \left(\frac{1}{2h}G_n^*\big(|t-x|^3e^{Bt};x\big)+\frac{1}{2}G_n^*\big(|t-x|^2e^{Bt};x\big)\right)\end{aligned}$$

$$+\frac{e^{2Bx}}{2h}G_n^*\big(|t-x|^3;x\big)+\frac{e^{2Bx}}{2}G_n^*\big(|t-x|^2;x\big)\omega_1\big(f'',h,B\big).$$

Using some computations, we get

$$\begin{aligned}
G_n^*\big(|t-x|^2e^{Bt};x\big) &= G_n^*\big(t^2e^{Bt};x\big)-2xG_n^*\big(te^{Bt};x\big)+x^2G_n^*\big(e^{Bt};x\big)\\
&= e^{(\beta_{n+2m}((B-a)x)+ax+\frac{(B-a)2}{n^2\psi_n(0)})}\left(\frac{\alpha_n}{n}\right)^2\frac{(n+m)}{n}\lambda^2(x)\\
&\quad + e^{(\beta_{n+m}((B-a)x)+ax+\frac{(B-a)}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\lambda(x)\\
&\quad - 2xe^{(\beta_{n+m}((B-a)x)+ax+\frac{(B-a)}{n^2\psi_n(0)})}\frac{\alpha_n}{n}\lambda(x)+x^2e^{ax+\beta_n((B-a)x)},
\end{aligned}$$

For sufficiently large n, it is obvious that

$$G_n^*\big(|t-x|^2e^{Bt};x\big)\le C_a(B,x)\mu_{n,2}(x). \tag{8}$$

Making use of the Cauchy–Schwarz inequality, we have the following inequalities

$$\begin{aligned}
G_n^*\big(|t-x|^3e^{Bt};x\big) &\le \sqrt{G_n^*\big(|t-x|^2e^{2Bt};x\big)}\sqrt{G_n^*\big(|t-x|^4;x\big)}\\
&\le \sqrt{C_a(2B,x)\mu_{n,2}(x)}\sqrt{\mu_{n,4}(x)}, \qquad (9)\\
G_n^*\big(|t-x|^3;x\big) &\le \sqrt{G_n^*\big(|t-x|^4;x\big)}\sqrt{G_n^*\big(|t-x|^2;x\big)}\\
&\le \sqrt{\mu_{n,4}(x)}\sqrt{\mu_{n,2}(x)}. \qquad (10)
\end{aligned}$$

Thus, using inequalities (8), (9), and (10) in (7), we obtain

$$\begin{aligned}
&\left|G_n^*(f;x)-f(x)-f'(x)\mu_{n,1}-\frac{1}{2}f''(x)\mu_{n,2}(x)\right|\\
&\quad\le\bigg(\frac{1}{2h}\sqrt{C_a(2B,x)\mu_{n,2}(x)}\sqrt{\mu_{n,4}(x)}+\frac{1}{2}C_a(B,x)\mu_{n,2}(x)\\
&\qquad+\frac{e^{2Bx}}{2h}\sqrt{\mu_{n,4}(x)}\sqrt{\mu_{n,2}(x)}+\frac{e^{2Bx}}{2}\mu_{n,2}(x)\bigg)\omega_1\big(f'',h,B\big).
\end{aligned}$$

Finally, when $h=\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}$ is chosen and substituted in the above equality, we get

$$\begin{aligned}
&\left|G_n^*(f;x)-f(x)-f'(x)\mu_{n,1}-\frac{1}{2}f''(x)\mu_{n,2}(x)\right|\\
&\quad\le\mu_{n,2}(x)\left(\frac{\sqrt{C_a(2B,x)}}{2}+\frac{C_a(B,x)}{2}+e^{2Bx}\right)\omega_1\left(f'',\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}},B\right).
\end{aligned}$$

Note that, for fixed $x\in[0,\infty)$, $\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}\to 0$ as $n\to\infty$, guarantees the convergence of Theorem 2. □

5 An application of modified Ibragimov–Gadjiev operators

Just like classical Ibragimov–Gadjiev operators, modified Ibragimov–Gadjiev operators also contain some modified operators preserving exponential functions under appropriate selection of $K_n(\lambda_n(x),t,u),\alpha_n$ and $\psi_n(0)$.

If property 4^* is applied ν-times to the $K_n(\lambda_n(x),t,u)$, the operators defined by (1) can be reduced to the form

$$G_n^*(f;x)=\sum_{\nu=0}^{\infty}e^{-\frac{a\nu}{n^2\psi_n(0)}}e^{ax}f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{n(n+m)\dots(n+(\nu-1)m)}{\nu!}$$
$$\times\left(\lambda_n(x)\alpha_n\psi_n(0)\right)^{\nu}K_{n+\nu m}\left(\lambda_n(x),0,\alpha_n\psi_n(0)\right). \tag{11}$$

1. In case

$$K_n\left(\lambda_n(x),t,u\right)=\left(1-\frac{u\lambda_n(x)}{1+t}\right)^n,$$

the operator (11) turns into the form

$$G_n^*(f;x)=\sum_{\nu=0}^{\infty}e^{-\frac{a\nu}{n^2\psi_n(0)}}e^{ax}f\left(\frac{\nu}{n^2\psi_n(0)}\right)\binom{n}{\nu}$$
$$\times\left(\lambda_n(x)\alpha_n\psi_n(0)\right)^{\nu}\left(1-\alpha_n\psi_n(0)\lambda_n(x)\right)^{n-\nu}. \tag{12}$$

Conditions (1^*)–(4^*) are fulfilled, and $m=-1$. For $\alpha_n=n$, $\psi_n(0)=\frac{1}{n}$, we have modified Bernstein operators

$$G_n^*(f;x)=\sum_{\nu=0}^{\infty}e^{-\frac{a\nu}{n}}e^{ax}f\left(\frac{\nu}{n^2\psi_n(0)}\right)\binom{n}{\nu}\left(\lambda_n(x)\right)^{\nu}\left(1-\lambda_n(x)\right)^{n-\nu},$$

where

$$\lambda_n(x)=\frac{e^{-ax/n}-1}{e^{a/n}-1}$$

defined by Aral et al. [14]. Use of L'Hospital's rule gives

$$\lim_{n\to\infty}\lambda_n(x)=x$$

as claimed.

2. In (12), for $\alpha_n=n$, $\psi_n(0)=\frac{1}{nb_n}$ ($\lim b_n=\infty$, $\lim\frac{b_n}{n}=0$), the operator (11) becomes modified Bernstein–Chlodowsky operators

$$G_n^*(f;x)=\sum_{\nu=0}^{\infty}e^{\frac{-a\nu b_n}{n}}e^{ax}f\left(\frac{\nu b_n}{n}\right)\binom{n}{\nu}\left(\frac{\lambda_n(x)}{b_n}\right)^{\nu}\left(1-\frac{\lambda_n(x)}{b_n}\right)^{n-\nu}$$

with

$$\lambda_n(x)=b_n\frac{e^{ax/n}-1}{e^{ab_n/n}-1},\qquad\lim_{n\to\infty}\lambda_n(x)=x$$

defined by Özsaraç et al. [26].

3. By choosing

$$K_n\big(\lambda_n(x), t, u\big) = e^{-n(t+u\lambda_n(x))},$$

conditions (1*)–(4*) are fulfilled, and $m = 0$. The operator (11) becomes

$$G_n^*(f;x) = \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n^2\psi_n(0)}} e^{ax} f\left(\frac{\nu}{n^2\psi_n(0)}\right) \frac{(n\lambda_n(x))^\nu}{\nu!} \big(\alpha_n\psi_n(0)\big)^\nu e^{-n(\alpha_n\psi_n(0)\lambda_n(x))}. \tag{13}$$

If we choose $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, we get modified Szász–Mirakjan operators

$$G_n^*(f;x) = e^{-n\lambda_n(x)} \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n}} e^{ax} f\left(\frac{\nu}{n}\right) \frac{(n\lambda_n(x))^\nu}{\nu!},$$

where

$$\lambda_n(x) = \frac{-ax}{n(e^{-a/n} - 1)}$$

introduced by Acu et al. [15].

Besides, with the choice of $\lambda_n(x)$ in (13) as

$$\lambda_n(x) = \frac{ax}{n(e^{a/n} - 1)}$$

another variant of Szász–Mirakjan operators is obtained, which was presented by Goyal [24]. As can be seen easily, using L'Hospital's rule, the limit of both $\lambda_n(x)$ yields

$$\lim_{n\to\infty} \lambda_n(x) = x.$$

It must be noted that the new variants of Szász–Mirakjan operators obtained by different selection of $\lambda_n(x)$ differ in terms of the functions they preserve as well as their structural features.

4. In addition, if we choose

$$K_n\big(\lambda_n(x), t, u\big) = \big(1 + t + u\lambda_n(x)\big)^{-n},$$

all conditions are fulfilled and $m = 1$. Thus, the operator defined by (11) turns into

$$\begin{aligned} G_n(f;x) = {} & \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n^2\psi_n(0)}} e^{ax} f\left(\frac{\nu}{n^2\psi_n(0)}\right) \binom{n+\nu-1}{\nu} \\ & \times \big(\lambda_n(x)\alpha_n\psi_n(0)\big)^\nu K_{n+\nu}\big(\lambda_n(x), 0, \alpha_n\psi_n(0)\big). \end{aligned}$$

Choosing $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, we get Baskakov operators

$$G_n(f;x) = \sum_{\nu=0}^{\infty} e^{-\frac{a\nu}{n}} e^{ax} f\left(\frac{\nu}{n}\right) \binom{n+\nu-1}{\nu} \frac{\lambda_n^\nu(x)}{(1+\lambda_n(x))^{n+\nu}},$$

where

$$\lambda_n(x) = \frac{e^{-\frac{ax}{n}} - 1}{1 - e^{\frac{a}{n}}}, \qquad \lim_{n\to\infty} \lambda_n(x) = x$$

defined by Özsaraç and Acar [25].

By choosing the appropriate sequences of $K_n(\lambda_n(x), t, u)$, α_n and $\psi_n(0)$, one can obtain other new operators, and we leave it to readers.

6 Conclusions

Through this work, a new generalization of Ibragimov–Gadjiev operators, which fixes the function $e^{ax}, a > 0$, has been constructed. Then, for these operators, some approximation properties have been provided, and it has been shown that the newly defined operators contain modified Bernstein-, Szász-, and Baskakov-type operators, which were studied by several authors, as special cases. The relationship between these operators obtained by different choices of $\lambda_n(x)$ has also been revealed.

It is worth noting to readers that one can obtain new operators by taking different sequences of $K_n(\lambda_n(x), t, u)$, α_n and $\psi_n(0)$. Moreover, the other approximation properties not covered in this study may also be investigated.

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References

1. Ibragimov, I.I., Gadjiev, A.D.: On a sequence of linear positive operators. Sov. Math. Dokl. **11**, 1092–1095 (1970)
2. Aral, A., Acar, T.: On approximation properties of generalized Durrmeyer operators, modern mathematical methods and high performance computing in science and technology. In: Springer Proc Math. Stat., vol. 171, pp. 1–15. Springer, Singapore (2016)
3. Bozma, G., Bars, E.: Approximation with a Kantorovich type Ibragimov–Gadjiev operator. Euroasia J. Math. Eng. Nat. Med. Sci. **9**(20), 74–83 (2022)
4. Herdem, S., Buyukyazıcı, İ.: Ibragimov–Gadjiev operators based on q-integers. Adv. Differ. Equ. **2008**, 304 (2018)
5. Herdem, S., Buyukyazıcı, İ.: Weighted approximation by q-ibragimov-Gadjiev operators. Math. Commun. **25**(2), 201–212 (2020)
6. Gadjiev, A.D., Ispir, N.: On a sequence of linear positive operators in weighted space. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **11**, 45–56 (1999)
7. Bozma, G., Bars, E.: On the approximation with an Ibragimov–Gadjiev type operator. Euroasia J. Math. Eng. Nat. Med. Sci. **9**(20), 74–83 (2022)
8. Gonul, N., Coskun, E.: Approximation with modified Gadjiev–Ibragimov operators in $C[0, A]$. J. Comput. Anal. Appl. **15**(1), 868–879 (2013)
9. Bilgin, N.G., Coskun, N.: Comparison result of some Gadjiev–Ibragimov type operators. Karaelmas J. Sci. Eng. **8**(1), 188–196 (2018)
10. Bilgin, N.G., Ozgur, N.: Approximation by two dimensional Gadjiev–Ibragimov type operators. Ikon. J. Math. **1**(1), 1–10 (2019)
11. Deniz, E., Aral, A.: Convergence properties of Ibragimov–Gadjiev–Durrmeyer operators. Creative Math. Inform. **24**(1), 17–26 (2015)
12. Aldaz, J.M., Render, H.: Optimality of generalized Bernstein operators. J. Approx. Theory **162**, 1407–1416 (2010)
13. Acar, T., Aral, A., Gonska, H.: On Szász–Mirakyan operators preserving e^{2ax}, $a > 0$. Mediterr. J. Math. **14**(6), 1–14 (2017)

14. Aral, A., C'ardenas-Morales, D., Garrancho, P.: Bernstein-type operators that reproduce exponential functions. J. Math. Inequal. **12**(3), 861–872 (2018)
15. Acu, A.M., Tachev, G.: Yet another new variant of Szász–Mirakyan operator. Symmetry **13**(11), Article ID 2018 (2021)
16. Aral, A., Inoan, D., Raşa, I.: Approximation properties of Szász-Mirakyan operators preserving exponential functions. Positivity **23**, 233–246 (2019)
17. Bodur, M., Yılmaz Ö, G., Aral, A.: Approximation by Baskakov–Szász-Stancu operators preserving exponential functions. Constr. Math. Anal. **1**(1), 1–8 (2018)
18. Deo, N., Pratap, R.: Approximation by integral form of Jain and Pethe operators. Proc. Natl. Acad. Sci. India Sect. A Phys. Sci. **92**, 31–38 (2022)
19. Deo, N., Pratap, R.: α-Bernstein–Kantorovich operators. Afr. Math. **31**, 609–618 (2020)
20. Gupta, V.: A new type of exponential operator. Filomat **37**(14), 4629–4638 (2023)
21. Gupta, V., Milovanović, G.V.: A solution to exponential operators. Results Math. **77**, 207 (2022)
22. Gupta, V., Tachev, G.: On approximation properties of Phillips operators preserving exponential functions. Mediterr. J. Math. **14**(4), Article ID 177 (2017)
23. Kanat, K., Sofyalioglu, M.: On Stancu type Szász–Mirakyan–Durrmeyer operators preserving e^{2ax}, $a > 0$. Gazi Univ. J. Sci. **34**(1), 196–209 (2021)
24. Goyal, M.: Reconstruction of Szasz–Mirakyan operators preserving exponential type functions. Filomat **37**(2), 427–434 (2023)
25. Özsaraç, F., Acar, T.: Reconstruction of Baskakov operators preserving some exponential functions. Math. Meth. Appl. Sci., spl. **42**, 5124–5132 (2019)
26. Özsaraç, F., Aral, A., Karslı, H.: On Bernstein–Chlodovsky Type Operators Preserving Exponential Functions. Springer Proceedings in Mathematics & Statistics, vol. 306. Springer, Singapore (2018)
27. Pratap, R.: The family of λ-Bernstein–Durrmeyer operators based on certain parameters. Math. Found. Comput. **6**(3), 546–557 (2023)
28. Sofyalıoğlu Aksoy, M.: New modification of the post Widder operators preserving exponential functions. Arab J. Basic Appl. Sci. **31**(1), 93–103 (2024)
29. Sofyalıoğlu, M., Kanat, K.: Approximation properties of the post-Widder operators preserving $e^{2ax}, a > 0$. Math. Methods Appl. Sci. **43**(7), 4272–4285 (2020)
30. Sofyalıoğlu, M., Kanat, K.: Approximation properties of generalized Baskakov–Schurer–Szasz–Stancu operators preserving $e^{-2ax}, a > 0$. J. Inequal. Appl. **1**, 112 (2019)
31. Usta, F., Mursaleen, M., Çakır, İ.: Approximation properties of Bernstein–Stancu operators preserving e^{-2x}. Filomat **37**(5), 1523–1534 (2023)
32. Yılmaz, Ö.G., Gupta, V., Aral, A.: On Baskakov operators preserving the exponential function. J. Numer. Anal. Approx. Theory **46**(2), 150–161 (2017)
33. Boyanov, B.D., Vaselinov, V.M.: A note on the approximation of functions in an infinite interval by linear positive operators. Bull. Math. Soc. Sci. Math. Roum. **14**(62), 9–13 (1970)
34. Holhoş, A.: The rate of approximation of functions in an infinite interval by positive linear operators. Stud. Univ. Babeş–Bolyai, Math. **55**(2), 133–142 (2010)
35. Tachev, G., Gupta, V., Aral, A.: Voronovskaja's theorem for functions with exponential growth. Georgian Math. J. **27**, 459–468 (2018). https://doi.org/10.1515/gmj-2018-0041

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