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Generalizations of Levinson-type inequalities via new Green functions and Hermite interpolating polynomial

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Abstract
In this study, Levinson-type inequalities for the class of n-convex ($n \geq 3$) functions are generalized using new Green functions and the Hermite interpolating polynomial involving two types of data points. Some estimations for novel functionals are derived using f-divergence. Furthermore, different inequalities involving Shannon entropies are presented.

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1 Introduction and preliminaries

The theory of inequalities and convex functions have a strong connection. Convex functions are important to a number of fields of mathematics and play an essential part in the research of optimization problems and modern analysis. Numerous physicists and mathematicians have used higher-order convexity to exploit inequalities and solve problems requiring greater dimensions.

The divided difference is given in the following definition:

Divided Difference ([1, p.14]) For a function $h : [\hat{d}_1, \hat{d}_2] \to \mathbb{R}$, the nth order divided difference, at mutually exclusive points $u_0, \dots, u_n \in [\hat{d}_1, \hat{d}_2]$, is recursively defined by

$$[u_\sigma; h] = h(u_\sigma), \quad \sigma = 0, \dots, n,$$
$$[u_0, \dots, u_n; h] = \frac{[u_1, \dots, u_n; h] - [u_0, \dots, u_{n-1}; h]}{u_n - u_0}. \tag{1}$$

It is known that (1) is equivalent to

$$[u_0, \dots, u_n; h] = \sum_{\sigma=0}^{n} \frac{h(u_\sigma)}{l'(h_\sigma)}, \quad \text{where } l(u) = \prod_{e=0}^{n} (u - u_e).$$

In the following formulation (see [1, p. 15]), nth-order divided difference is used to define an n-convex function.

n-Convex function For $(n+1)$ different points $u_0,\dots,u_n \in [\hat{d}_1,\hat{d}_2]$, a function $f:[\hat{d}_1,\hat{d}_2]\to\mathbb{R}$ is said n-convex ($n\geq 0$) if and only if

$$[u_0,\dots,u_n;f]\geq 0$$

holds.

$$\text{If } [u_0,\dots,u_n;f]\leq 0, \quad \text{then } f \text{ is } n\text{-concave}.$$

The following criteria for an n-convex function is provided in [1, p. 16].

Theorem I *f is n-convex if and only if $f^{(n)}\geq 0$, given that $f^{(n)}$ exists.*

Levinson [2] extended Ky Fan's inequality for 3-convex functions as given:

Theorem A *Consider $f:\mathbb{I}_2=(0,2\gamma)\to\mathbb{R}$ with $\frac{d^3}{dz^3}f(z)\geq 0$. Consider $p_\sigma>0$ with $\sum_{\sigma=1}^{\hat{m}}p_\sigma=Q$ and $x_\sigma\in(0,\gamma)$. Then*

$$\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(x_\sigma)-f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma\right)\leq\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(2\gamma-x_\sigma)-f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma(2\gamma-x_\sigma)\right). \tag{2}$$

Popoviciu [3] noted that Levinson's inequality (2) has a significant role on $(0,2\gamma)$, while in [4], Bullen provided distinctive conformation of Popoviciu's results also gave converse of (2).

Theorem B (i) *Assume that $f:\mathrm{T}=[\hat{d}_1,\hat{d}_2]\to\mathbb{R}$ is a convex function of the third order and $x_\sigma,y_\sigma\in\mathrm{T}$ for $\sigma=1,2,\dots,\hat{m}$, $p_\sigma>0$ such that*

$$\min\{y_1\dots y_{\hat{m}}\}\geq\max\{x_1\dots x_{\hat{m}}\},\qquad y_1+x_1=\dots=y_{\hat{m}}+x_{\hat{m}} \tag{3}$$

then

$$\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(_\sigma)-f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma\right)\leq\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(y_\sigma)-f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma y_\sigma\right). \tag{4}$$

(ii) *For $p_\sigma>0$, f is 3-convex iff f is continuous and (3) and (4) hold.*

From (4), we have the following functional:

$$\mathbb{D}\big(f(\cdot)\big)=\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(y_\sigma)-f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma y_\sigma\right)-\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma f(x_\sigma)+f\left(\frac{1}{Q_{\hat{m}}}\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma\right). \tag{5}$$

In the next result, Pečarić [5] used weakening condition (3) to derive inequality (4).

Theorem C *Let* $f : \mathrm{T} \to \mathbb{R}$ *be such that* $f^3(t) \geq 0$ *and* $0 < p_\sigma$. *Let* $x_\sigma, y_\sigma \in \mathrm{T}$ *also be such that* $x_\sigma + y_\sigma = 2\breve{c}$, *for* $\sigma = 1, \dots, \hat{m}$, $x_\sigma + x_{\hat{m}-\sigma+1} \leq 2\breve{c}$ *and* $\frac{p_\sigma x_\sigma + p_{\hat{m}-\sigma+1} x_{\hat{m}-\sigma+1}}{p_\sigma + p_{\hat{m}-\sigma+1}} \leq \breve{c}$. *Then* (4) *is valid.*

In [6], Mercer stated that (4) is true for the symmetric distribution of the points, given in following theorem.

Theorem D *Let* f *be a* 3*-convex function, defined on* T, *and* p_σ *is such that* $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$. *Choose* x_σ, y_σ *such that* $\min\{y_1 \dots y_{\hat{m}}\} \geq \max\{x_1 \dots x_{\hat{m}}\}$ *and*

$$\sum_{\sigma=1}^{\hat{m}} p_\sigma \left(x_\sigma - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma \right)^2 = \sum_{\sigma=1}^{\hat{m}} p_\sigma \left(y_\sigma - \sum_{\sigma=1}^{\hat{m}} p_\sigma y_\sigma \right)^2. \tag{6}$$

Then (4) *holds.*

Let $\mathrm{T} = [\hat{d}_1, \hat{d}_2] \subset \mathbb{R}$, $\hat{d}_1 < \hat{d}_2$. In [7], Pečarić *et al.* proved Abel-Gontscharof-type identities by applying new type of Green functions:

$$f(\hat{u}) = f(\hat{d}_1) + (\hat{u} - \hat{d}_1) f'(\hat{d}_2) - \int_{\mathrm{T}} \hat{\mathbb{G}}_1(\hat{u}, \hat{z}) f''(\hat{z})\, d\hat{z}, \tag{7}$$

$$f(\hat{u}) = f(\hat{d}_2) - (\hat{d}_2 - \hat{u}) f'(\hat{d}_1) + \int_{\mathrm{T}} \hat{\mathbb{G}}_2(\hat{u}, \hat{z}) f''(\hat{z})\, dw, \tag{8}$$

$$f(\hat{u}) = f(\hat{d}_2) + (\hat{u} - \hat{d}_1) f'(\hat{d}_1) - (\hat{d}_2 - \hat{d}_1) f'(\hat{d}_2) + \int_{\mathrm{T}} \hat{\mathbb{G}}_3(\hat{u}, \hat{z}) f''(\hat{z})\, d\hat{z}, \tag{9}$$

$$f(\hat{u}) = f(\hat{d}_1) + (\hat{d}_2 - \hat{d}_1) f'(\hat{d}_1) - (\hat{d}_2 - \hat{u}) f'(\hat{d}_2) - \int_{\mathrm{T}} \hat{\mathbb{G}}_4(\hat{u}, \hat{z}) f''(\hat{z})\, d\hat{z}, \tag{10}$$

where $f : \mathrm{T} \to \mathbb{R}$ and for $\alpha = 1, \dots, 4$, $\hat{\mathbb{G}}_\alpha : \mathrm{T} \times \mathrm{T} \to \mathbb{R}$ are given as:

$$\hat{\mathbb{G}}_1(\hat{u}, \hat{z}) = \begin{cases} \hat{z} - \hat{d}_1, & \hat{d}_1 \leq \hat{z} \leq \hat{u}, \\ \hat{u} - \hat{d}_1, & \hat{u} \leq \hat{z} \leq \hat{d}_2. \end{cases} \tag{11}$$

$$\hat{\mathbb{G}}_2(\hat{u}, \hat{z}) = \begin{cases} \hat{u} - \hat{d}_2, & \hat{d}_1 \leq \hat{z} \leq \hat{u}, \\ \hat{z} - \hat{d}_2, & \hat{u} \leq \hat{z} \leq \hat{d}_2. \end{cases} \tag{12}$$

$$\hat{\mathbb{G}}_3(\hat{u}, \hat{z}) = \begin{cases} \hat{u} - \hat{d}_1, & \hat{d}_1 \leq \hat{z} \leq \hat{u}, \\ \hat{z} - \hat{d}_1, & \hat{u} \leq \hat{z} \leq \hat{d}_2. \end{cases} \tag{13}$$

$$\hat{\mathbb{G}}_4(\hat{u}, \hat{z}) = \begin{cases} \hat{z} - \hat{d}_2, & \hat{d}_1 \leq \hat{z} \leq \hat{u}, \\ \hat{u} - \hat{d}_2, & \hat{u} \leq \hat{z} \leq \hat{d}_2. \end{cases} \tag{14}$$

In [8], the Hermite interpolating polynomial is defined as follows:

Let $\hat{d}_1, \hat{d}_2 \in \mathbb{R}$ with $\hat{d}_1 < \hat{d}_2$, and $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(2 \leq a)$ be the points. If $f \in C^n[\hat{d}_1, \hat{d}_2]$ and $\hat{\gamma}_H^{(i)}(s)$ exist, then the following Hermite conditions hold:

Hermite Conditions

$$\hat{\gamma}_H^{(i)}(\hat{g}_e) = f^{(i)}(\hat{g}_e); \quad 0 \leq i \leq k_e, 1 \leq e \leq a, \sum_{e=1}^{a} k_e + a = n.$$

Theorem H ([8]) *Let* $-\infty < \hat{d}_1 < \hat{d}_2 < \infty$ *and* $\hat{d}_1 < \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a \leq \hat{d}_2$ $(a \geq 2)$ *and* $f \in C^n([\hat{d}_1, \hat{d}_2])$. *Then*

$$f(\check{v}) = \hat{\gamma}_H(\check{v}) + R_H(f, \check{v}), \tag{15}$$

where

$$\hat{\gamma}_H(\check{v}) = \sum_{e=1}^{a} \sum_{i=0}^{k_e} H_{i_e}(\check{v}) f^{(i)}(\hat{g}_e)$$

is the Hermite interpolating polynomial, and H_{i_e} *are the polynomials for the Hermite basis defined as*

$$H_{i_e}(\check{v}) = \frac{1}{i!} \frac{\mathrm{T}(\check{v})}{(\check{v} - \hat{g}_e)^{k_e+1-i}} \sum_{k=0}^{k_e-i} \frac{1}{k!} \frac{d^k}{d\check{v}^k} \left(\frac{(\check{v} - \hat{g}_e)^{k_e+1}}{\mathrm{T}(\check{v})} \right) \bigg|_{\check{v}=\hat{g}_e} (\check{v} - \hat{g}_e)^k, \tag{16}$$

with

$$\mathrm{T}(\check{v}) = \Pi_{e=1}^{a} (\check{v} - \hat{g}_e)^{k_e+1},$$

and

$$R_H(f, \check{v}) = \int_{\mathrm{T}} f^{(n)}(s) \mathbb{G}_{H,n}(\check{v}, s)\, ds$$

is remainder, where $\mathbb{G}_{H,n}(\check{v}, s)$ *is given by*

$$\mathbb{G}_{H,n}(\check{v}, s) = \begin{cases} \sum_{e=1}^{a} \sum_{i=0}^{k_e} \frac{(\hat{g}_e - s)^{n-i-1}}{(n-i-1)!} H_{i_e}(\check{v}), & s \leq \check{v}; \\ -\sum_{e=r+1}^{a} \sum_{i=0}^{k_e} \frac{(\hat{g}_e - s)^{n-i-1}}{(n-i-1)!} H_{i_e}(\check{v}), & s \geq \check{v}, \end{cases} \tag{17}$$

for all $\hat{g}_r \leq s \leq \hat{g}_{r+1}$; $r = 0, 1, \ldots, a$, *with* $\hat{g}_0 = \hat{d}_1$ *and* $\hat{g}_{a+1} = \hat{d}_2$.

We observe that $0 \leq \mathbb{G}_{H,n-3}(\check{v}, s)$, and $\mathbb{G}_{H,n-3}$ represents derivative of order three with respect to the first variable.

The positivity of $\mathbb{G}_{H,n}(\check{v}, s)$ is described in [9] and [10], as follows:

Lemma 1 *The following statements are true for the Green function* $\mathbb{G}_{H,n}(\check{v}, s)$ *as given in* (17).

(i) $\frac{\mathbb{G}_{H,n}(\check{v},s)}{\mathrm{T}(\check{v})} > 0$ $\hat{g}_1 \leq \check{v} \leq \hat{g}_a$, $\hat{g}_1 \leq s \leq \hat{g}_a$;

(ii) $\mathbb{G}_{H,n}(\check{v}, s) \leq \frac{1}{(n-1)!(\hat{d}_2 - \hat{d}_1)} |\mathrm{T}(\check{v})|$;

(iii) $\int_{\mathrm{T}} \mathbb{G}_{H,n}(\check{v}, s)\, ds = \frac{\mathrm{T}(\check{v})}{n!}$.

In 2023, Rasheed *et al.* [11] defined a novel class of 3-convex Green functions and utilized them to state the following fruitful lemma.

Lemma 2 *Let f be defined on* T *such that f''' exists and* $\mathbb{G}_\alpha$ $(\alpha = 1,\dots,4)$ *are the two-point right focal problem-type Green functions given by* (22)–(25). *Then*

$$f(\hat{u}) = f(\hat{d}_1) + (\hat{u}-\hat{d}_1)f'(\hat{d}_2) + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2)f''(\hat{d}_1) - \frac{(\hat{u}-\hat{d}_1)^2}{2}f''(\hat{d}_2) + \int_{\mathrm{T}} G_1(\hat{u},\hat{z})f'''(\hat{z})\,d\hat{z}, \tag{18}$$

$$f(\hat{u}) = f(\hat{d}_2) - (\hat{d}_2-\hat{u})f'(\hat{d}_1) - f''(\hat{d}_1)\frac{(\hat{d}_2-\hat{u})^2}{2} + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2)f''(\hat{d}_2) - \int_{\mathrm{T}} G_2(\hat{u},\hat{z})f'''(\hat{z})\,d\hat{z}, \tag{19}$$

$$\begin{aligned} f(\hat{u}) &= f(\hat{d}_2) + (\hat{u}-\hat{d}_1)f'(\hat{d}_1) - (\hat{d}_2-\hat{d}_1)f'(\hat{d}_2) \\ &\quad - f''(\hat{d}_1)\left[\frac{(\hat{u}-\hat{d}_1)^2}{2} + (\hat{u}-\hat{d}_1)(\hat{d}_1-\hat{d}_2)\right] \\ &\quad + f''(\hat{d}_2)\left[\frac{(\hat{d}_2-\hat{d}_1)^2}{2} + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2)\right] - \int_{\mathrm{T}} G_3(\hat{u},\hat{z})f'''(\hat{z})\,d\hat{z}, \end{aligned} \tag{20}$$

$$\begin{aligned} f(\hat{u}) &= f(\hat{d}_1) + (\hat{d}_2-\hat{d}_1)f'(\hat{d}_1) - (\hat{d}_2-\hat{u})f'(\hat{d}_2) \\ &\quad + f''(\hat{d}_1)\left[(\hat{u}-\hat{d}_2)(\hat{u}-\hat{d}_1) + \frac{(\hat{d}_2-\hat{d}_1)^2}{2}\right] \\ &\quad - \left[(\hat{u}-\hat{d}_2)(\hat{d}_2-\hat{d}_1) + \frac{(\hat{u}-\hat{d}_2)^2}{2}\right]f''(\hat{d}_2) + \int_{\mathrm{T}} f'''(w)G_4(\hat{u},\hat{z})\,d\hat{z}, \end{aligned} \tag{21}$$

where $\mathbb{G}_\alpha : \mathrm{T}\times\mathrm{T} \to \mathbb{R}$, $\alpha \in \{1,2,3,4\}$ *given as*:

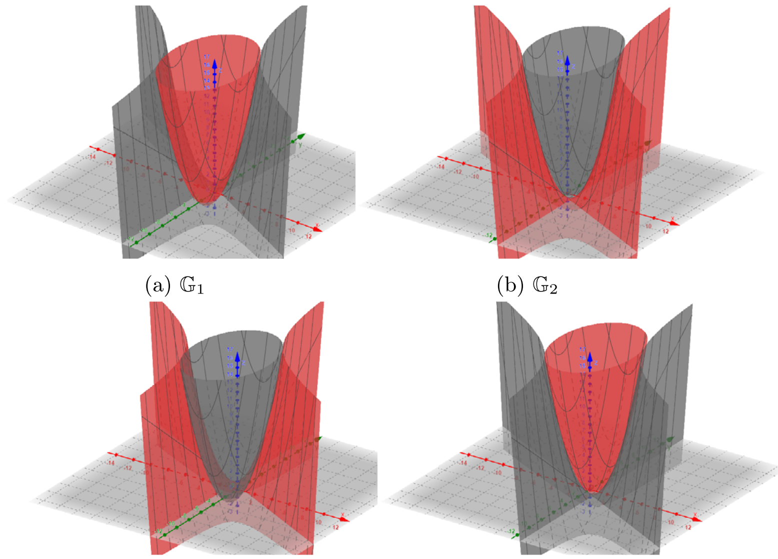
$$\mathbb{G}_1(\hat{u},\hat{z}) = \begin{cases} \frac{1}{2}(\hat{z}-\hat{d}_1)^2 + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2), & \hat{d}_1 \le \hat{z} \le \hat{u}, \\ (\hat{u}-\hat{d}_1)(\hat{z}-\hat{d}_2) + \frac{(\hat{u}-\hat{d}_1)^2}{2}, & \hat{u} \le \hat{z} \le \hat{d}_2. \end{cases} \tag{22}$$

$$\mathbb{G}_2(\hat{u},\hat{z}) = \begin{cases} (\hat{u}-\hat{d}_2)(\hat{z}-\hat{d}_1) + \frac{1}{2}(\hat{u}-\hat{d}_2)^2, & \hat{d}_1 \le \hat{z} \le \hat{u}, \\ \frac{(\hat{z}-\hat{d}_2)^2}{2} + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2), & \hat{u} \le \hat{z} \le \hat{d}_2. \end{cases} \tag{23}$$

$$\mathbb{G}_3(\hat{u},\hat{z}) = \begin{cases} (\hat{u}-\hat{d}_1)(\hat{z}-\hat{d}_2) + \frac{(\hat{u}-\hat{d}_1)^2}{2}, & \hat{d}_1 \le \hat{z} \le \hat{u}, \\ \frac{1}{2}(\hat{z}-\hat{d}_1)^2 + (\hat{u}-\hat{d}_1)(\hat{u}-\hat{d}_2), & \hat{u} \le \hat{z} \le \hat{d}_2. \end{cases} \tag{24}$$

$$\mathbb{G}_4(\hat{u},\hat{z}) = \begin{cases} \frac{(\hat{z}-\hat{d}_2)^2}{2} + (\hat{u}-\hat{d}_2)(\hat{u}-\hat{d}_1), & \hat{d}_1 \le \hat{z} \le \hat{u}, \\ (\hat{u}-\hat{d}_2)(\hat{z}-\hat{d}_1) + \frac{1}{2}(\hat{u}-\hat{d}_2)^2, & \hat{u} \le \hat{z} \le \hat{d}_2. \end{cases} \tag{25}$$

Remark 1 If integration by parts is applied to integral part of (7)–(10) by selecting $f''(\hat{z})$ as the first function and $\hat{\mathbb{G}}_\alpha(\hat{u},\hat{z})(\alpha = 1,2,3,4)$ as the second function, then (18)–(21) are derived. Graphical depection of $\hat{\mathbb{G}}_\alpha(\hat{u},\hat{z})(\alpha = 1,2,3,4)$ is given in figure (1).



(a) $\mathbb{G}_1$ (b) $\mathbb{G}_2$ (c) $\mathbb{G}_3$ (d) $\mathbb{G}_4$

Figure 1 Graph of $\mathbb{G}_\alpha(\alpha = 1,\ldots,4)$ for different ranges of $\hat{u}$ and $\hat{z}$

In [12], Adeel *et al.* established the Levinson inequality for the class of three convex functions using two Green's functions. They also provided valuable findings in information theory. For the class of higher-order convex functions, Adeel *et al.* [13] generalized Levinson-type inequalities using the Abel-Gontscharoff interpolation. Using Lidstone polynomials and Green functions in combination with Levinson-type inequalities, Adeel *et al.* [14] computed the Shannon entropy and calculated f-divergence.

In recent decades, several scholars have used the Hermite interpolation to modify the inequalities for higher-order convex functions. Butt *et al.* [15] generalized the Popoviciu inequality for higher-order convex functions using Hermite interpolation and also constructed some results relating to the Grüss- and Ostrowski-type inequalities. In another analysis, generalizations of Levinson-type inequalities are stated by Adeel *et al.* [16] via Hermite interpolating polynomial for n-convex functions and estimated bounds for the Shannon entropy and f-divergence. For n-convex functions, Adeel *et al.* [17] proved Levinson-type inequalities by applying Hermite interpolating polynomial and Green functions and also derived inequalities for the Shannon entropy and f-divergence. In [18], Mehmood *et al.* explored discrete and continuous cyclic refinements of Jensen's inequality and extended them from convex function to higher-order convex function through the use of various new Green functions by employing Hermite interpolating polynomial whose error term is approximated using Peano's kernel. In 2021, Ansari *et al.* [19] utilized Hermite's interpolation to derive a new generalization of an inequality for higher-order convex functions containing Csiszár divergence on time scales.

In [20], Adeel *et al.* employed Fink's identity to obtain new generalizations of Levinson-type inequalities for n-convex functions. Furthermore, they applied their results to evaluate different entropies. In [21], authors applied the Lidstone interpolating polynomial for $2n$-convex functions to derive different generalizations of Levinson-type inequalities. In [22], Bilal *et al.* generalized Shannon-type inequalities via diamond integrals. In [23], Bilal *et al.* defined Csiszár's f-divergence for diamond integrals and proved inequalities for different divergences.

2 Mian results

This section is divided in two subsections. First, we present results associated with Bullen-type inequalities associated with Green functions (22)–(25). Second, 3-convex Green functions (22)–(25) are used to generalize the Levinson-type inequalities for higher-order convex functions via Hermite interpolation.

2.1 Generalization of Bullen-type inequalities for higher-order convex function

We begin by defining the subsequent functional:

F: Suppose a function $f : \mathrm{T} = [\hat{d}_1, \hat{d}_2] \to \mathbb{R}$. Let $(p_1, \dots, p_{\hat{m}}) \in \mathbb{R}^{\hat{m}}$ and $(q_1, \dots, q_{\hat{\omega}}) \in \mathbb{R}^{\hat{\omega}}$ be such that $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$, $\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon = 1$ and $x_\sigma, y_\epsilon, \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma, \sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon \in \mathrm{T}$. Then

$$\mathbf{D}\big(f(\cdot)\big) = \sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon f(y_\epsilon) - f\left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right) - \sum_{\sigma=1}^{\hat{m}} p_\sigma f(x_\sigma) + f\left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right). \tag{26}$$

B: Let H_{i_e}, $\mathbb{G}_{\mathcal{H},n}$ be defined in (16) and (17) and $\mathbb{G}_\alpha(\cdot, \hat{z})$ for $\alpha = 1, \dots, 4$ be defined in (22)–(25).

Impelled from identity (5), the following results are constructed:

Theorem 1 *Assume that* **F** *and* **B** *hold. Consider the points* $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(a \geq 2)$ *and* $f \in C^n[\hat{d}_1, \hat{d}_2]$. *Then, for* $\alpha = 1, 4$,

$$\begin{aligned}\mathbf{D}\big(f(\cdot)\big) &= \frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right] \\ &\quad \times \big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) + \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z} \\ &\quad + \int_{\mathrm{T}}\int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z}, s) f^{(n)}(s)\, ds\, d\hat{z}\end{aligned} \tag{27}$$

and for $\alpha = 2, 3$,

$$\begin{aligned}\mathbf{D}\big(f(\cdot)\big) &= \frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right] \\ &\quad \times \big(2f^{(2)}(\hat{d}_2) - f^{(2)}(\hat{d}_1)\big) - \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z} \\ &\quad - \int_{\mathrm{T}}\int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z}, s) f^{(n)}(s)\, ds\, d\hat{z}\end{aligned} \tag{28}$$

where

$$\mathbf{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) = \sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon \mathbb{G}_\alpha(y_\epsilon,\hat{z}) - \mathbb{G}_\alpha\left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon,\hat{z}\right) - \sum_{\sigma=1}^{\hat{m}} p_\sigma \mathbb{G}_\alpha(x_\sigma,\hat{z}) + \mathbb{G}_\alpha\left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma,\hat{z}\right). \tag{29}$$

Proof Let $\alpha = 1,4$, then applying (26) to the identities (18) and (21) and using linearity of $\mathbf{D}(f(\cdot))$, we have

$$\mathbf{D}\big(f(\cdot)\big) = \frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right] \times \big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) + \int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(.,\hat{z})\big) f^{(3)}(\hat{z})\,d\hat{z}. \tag{30}$$

From Theorem H, $f^{(3)}(\hat{z})$ becomes

$$f^{(3)}(\hat{z}) = \sum_{e=1}^{a}\sum_{i=0}^{k_e} H_{i_e}(\hat{z}) f^{(i+3)}(\hat{g}_e) + \int_{\mathrm{T}} \mathbb{G}_{\mathcal{H},n-3}(\hat{z},s) f^{(n)}(s)\,ds. \tag{31}$$

Using (31) in (30), we get (27).

Following the same steps, we get (28) for $\alpha = 2,3$. □

Under the condition defined by (6), the generalized form of the Bullen-type inequality (for positive weights) is presented.

Corollary 1 *Assume* **B**. *Let* $(p_1,\dots,p_{\hat{m}}) \in \mathbb{R}^{\hat{m}}$ *be such that* $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$ *and* x_σ, $y_{\hat{\omega}}$ *satisfy* (6) *and* $\max\{x_1 \dots x_{\hat{m}}\} \le \min\{y_1 \dots y_{\hat{m}}\}$ *and* $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(a \ge 2)$ *be the points. If* $f \in C^n[\hat{d}_1,\hat{d}_2]$, *then* (27) *and* (28) *hold.*

For n-convex functions, the following form of identities (27) and (28) are given.

Theorem 2 *Assume all the suppositions of Theorem* 1 *for the n-convex function* f.

If

$$\int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\,d\hat{z} \ge 0, \quad s \in \mathrm{T}, \tag{32}$$

then for $\alpha = 1,4$,

$$\mathbf{D}\big(f(\cdot)\big) \ge \frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right] \times \big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) + \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) H_{i_e}(\hat{z})\,d\hat{z} \tag{33}$$

and for $\alpha = 2, 3$,

$$\mathbf{D}\big(f(\cdot)\big) \leq \frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon y_\epsilon\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right] \times \big(2f^{(2)}(\hat{d}_2) - f^{(2)}(\hat{d}_1)\big) - \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbf{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z}. \tag{34}$$

Proof As f is n-convex ($n \geq 3$), then by Theorem I, we have

$$f^{(n)}(\hat{z}) \geq 0, \quad \forall\, \hat{z} \in \mathrm{T}.$$

Thus, using (32) in (27) and (28), we get (33) and (34), respectively. □

Remark 2

(i) According to Theorem 2, if the inequality in (32) is reversed, then inequalities in (33) and (34) hold conversely.

(ii) If f is n-concave, then inequalities (33) and (34) hold in the opposite direction.

Remark 3 $\mathbf{D}(\cdot)$ is reduced in $\mathbb{D}(\cdot)$ if $\hat{\omega} = \hat{m}$, $p_{\hat{m}} = q_{\hat{\omega}}$ and weights are positive. Then for $\alpha = 1, 4$, (27), (32) and (33) become

$$\mathbb{D}\big(f(\cdot)\big) = \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z} + \int_{\mathrm{T}}\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z},s) f^{(n)}(s)\, ds\, d\hat{z}, \tag{35}$$

$$\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\, d\hat{z} \geq 0, \quad s \in \mathrm{T} \tag{36}$$

and

$$\mathbb{D}\big(f(\cdot)\big) \geq \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z}, \tag{37}$$

respectively.

For $\alpha = 2, 3$, (28), (32) and (34) become

$$\mathbb{D}\big(f(\cdot)\big) = -\sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e) \int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z} - \int_{\mathrm{T}}\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z},s) f^{(n)}(s)\, ds\, d\hat{z}, \tag{38}$$

$$\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) \mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\, d\hat{z}, \geq 0 \quad s \in \mathrm{T} \tag{39}$$

and

$$\mathbb{D}\big(f(\cdot)\big) \leq -\sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}, \tag{40}$$

respectively.

Theorem 3 *Assume* **B**. *Let* $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(a \geq 2)$ *be the points and* $f \in C^n[\hat{d}_1, \hat{d}_2]$. *Choose positive weights* $(p_1, \ldots, p_{\hat{m}})$ *such that* $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$. *Then*

(i) *For every* $e = 2, \ldots, a$, *if* k_e *is odd, then* (37) *and* (40) *hold.*

(ii) *Let* (37) *and* (40) *be fulfilled and the function*

$$F(\hat{z}) = \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)H_{i_e}(\hat{z}). \tag{41}$$

If $F(\hat{z}) \geq 0$, *then* (37) *and* (40) *become*

$$\mathbb{D}\big(f(\cdot)\big) \geq 0 \tag{42}$$

and

$$\mathbb{D}\big(f(\cdot)\big) \leq 0, \tag{43}$$

respectively.

Proof

(i) Since the weights are positive and the Green functions $\mathbb{G}_\alpha(\cdot,\hat{z})$ are 3-convex, $\mathbb{D}(\mathbb{G}_\alpha(\cdot,\hat{z})) \geq 0$, for fixed α. Additionally, for each $e = 2, \ldots, a$, k_e is odd this implies $\mathrm{T}(\cdot) \geq 0$ and by part (i) of Lemma 1

$$\mathbb{G}_{\mathcal{H},n-3}(\cdot, s) \geq 0,$$

hence (32) holds. Thus, applying Theorem 2 for the n-convex function f, we obtain (37) and (40).

(ii) Using (41) in (37) and (40) gives (42) and (43), respectively. □

Theorem 4 *Let positive real numbers* $p_1, \ldots, p_{\hat{m}}$ *be such that* $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$. *Let also* $x_\sigma, y_\sigma \in \mathrm{T}$ *be such that* $x_\sigma + y_\sigma = 2\check{c}$, *for* $\sigma = 1, \ldots, \hat{m}$, $x_\sigma + x_{\hat{m}-\sigma+1} \leq 2\check{c}$ *and* $\frac{p_\sigma x_\sigma + p_{\hat{m}-\sigma+1}x_{\hat{m}-\sigma+1}}{p_\sigma + p_{\hat{m}-\sigma+1}} \leq \check{c}$. *Then for an* n*-convex function* f, *we have the following*

(i) *For each* $e = 2, \ldots, a$, *if* k_e *is odd, then* (37) *and* (40) *hold.*

(ii) *Let* (37) *and* (40) *hold and*

$$F(\hat{z}) = \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)H_{i_e}(\hat{z}) \tag{44}$$

be 3*-convex. Then* (37) *and* (40) *become*

$$\mathbb{D}\big(f(\cdot)\big) \geq 0 \tag{45}$$

and

$$\mathbb{D}\big(f(\cdot)\big) \leq 0, \tag{46}$$

respectively.

Proof Proof is similar to Theorem 3. □

2.2 Levinson-type inequality for *n*-convex ($n \geq 3$) functions

In this section, results are given for the generalization of Levinson-type inequality using new green functions $\mathbb{G}_\alpha(\alpha = 1,\dots,4)$ and interpolating Hermite polynomial. For this, first, we have

$\boldsymbol{\mathcal{H}}$: Consider $x_1,\dots,x_{\hat{m}} \in (0,\gamma)$ and $f : \mathbb{I}_2 = [0,2\gamma] \to \mathbb{R}$. Choose $(p_1,\dots,p_{\hat{m}}) \in \mathbb{R}^{\hat{m}}$ and $(q_1,\dots,q_{\hat{\omega}}) \in \mathbb{R}^{\hat{\omega}}$ be such that $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$ and $\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon = 1$. Also, let x_σ, $\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)$ and $\sum_{\sigma=1}^{\hat{m}} p_\sigma \in \mathbb{I}_2$. Then

$$\begin{aligned}\check{\mathbf{D}}\big(f(\cdot)\big) = {}&\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon f(2\gamma - x_\epsilon) - f\left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)\right) - \sum_{\sigma=1}^{\hat{m}} p_\sigma f(x_\sigma)\\ &+ f\left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right).\end{aligned} \tag{47}$$

For the next results, we construct the following identities:

Theorem 5 *Assume* $\boldsymbol{\mathcal{H}}$ *and* **B**. *Let* $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(a \geq 2)$ *be the points and* $f \in C^n[\hat{d}_1,\hat{d}_2]$. *Then for* $0 \leq \hat{d}_1 < \hat{d}_2 \leq 2\gamma$ *and* $\alpha = 1,4$, *we have*

$$\begin{aligned}\check{\mathbf{D}}\big(f(\cdot)\big) = {}&\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right]\\ &\times \frac{1}{2}\big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) + \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \check{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\\ &+ \int_{\mathrm{T}}\int_{\mathrm{T}} \check{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)f^{(n)}(s)\,ds\,d\hat{z}\end{aligned} \tag{48}$$

and for $\alpha = 2,3$,

$$\begin{aligned}\check{\mathbf{D}}\big(f(\cdot)\big) = {}&\frac{1}{2}\left[\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)^2 - \left(\sum_{\epsilon=1}^{\hat{\omega}} q_\epsilon(2\gamma - x_\epsilon)\right)^2 - \sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma^2 + \left(\sum_{\sigma=1}^{\hat{m}} p_\sigma x_\sigma\right)^2\right]\\ &\times \big(2f^{(2)}(\hat{d}_2) - f^{(2)}(\hat{d}_1)\big) - \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \check{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\\ &- \int_{\mathrm{T}}\int_{\mathrm{T}} \check{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)f^{(n)}(s)\,ds\,d\hat{z},\end{aligned} \tag{49}$$

where $\breve{\mathbf{D}}(f(\cdot))$ *is defined in* (47) *and*

$$\breve{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)=\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon\mathbb{G}_k(2\gamma-x_\epsilon,\hat{z})-\mathbb{G}_k\left(\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon(2\gamma-x_\epsilon,\hat{z})\right)-\sum_{\sigma=1}^{\hat{m}}p_\sigma\mathbb{G}_k(x_\sigma,\hat{z})+\mathbb{G}_k\left(\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma,\hat{z}\right). \tag{50}$$

Proof Replace T, $\mathbf{D}(\cdot)$ and y_ϵ with $\mathbb{I}_2$, $\breve{\mathbf{D}}(\cdot)$ and $(2\gamma-x_\epsilon)$ in Theorem 1, respectively, to get the required result. □

For n-convex functions, we give the following form of identity (48).

Theorem 6 *Consider f is an n-convex function and all the conditions of Theorem 2 hold. If*

$$\int_{\mathrm{T}}\breve{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\,d\hat{z}\geq 0,\quad s\in\mathbb{I}_2, \tag{51}$$

then for $\alpha=1,4$,

$$\begin{aligned}\breve{\mathbf{D}}\big(f(\cdot)\big)\geq{}&\left[\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon(2\gamma-x_\epsilon)^2-\left(\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon(2\gamma-x_\epsilon)\right)^2-\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma^2+\left(\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma\right)^2\right]\\&\times\frac{1}{2}\big(2f^{(2)}(\hat{d}_1)-f^{(2)}(\hat{d}_2)\big)+\sum_{e=1}^{a}\sum_{i=0}^{k_e}f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}}\breve{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,t)\big)H_{i_e}(\hat{z})\,d\hat{z}\end{aligned} \tag{52}$$

and for $\alpha=2,3$,

$$\begin{aligned}\breve{\mathbf{D}}\big(f(\cdot)\big)\leq{}&\left[\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon(2\gamma-x_\epsilon)^2-\left(\sum_{\epsilon=1}^{\hat{\omega}}q_\epsilon(2\gamma-x_\epsilon)\right)^2-\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma^2+\left(\sum_{\sigma=1}^{\hat{m}}p_\sigma x_\sigma\right)^2\right]\\&\times\frac{1}{2}\big(2f^{(2)}(\hat{d}_2)-f^{(2)}(\hat{d}_1)\big)-\sum_{e=1}^{a}\sum_{i=0}^{k_e}f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}}\breve{\mathbf{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\end{aligned} \tag{53}$$

where $0\leq\hat{d}_1<\hat{d}_2\leq 2\gamma$.

Proof As a consequences of conditions mentioned in the statement, the proof is similar to Theorem 2. □

Remark 4 $\breve{\mathbf{D}}(\cdot)$ is reduced in $\breve{\mathbb{D}}(\cdot)$ if $\hat{\omega}=\hat{m}$, $p_{\hat{m}}=q_{\hat{\omega}}$ and weights are positive. Then for $\alpha=1,4$, (48), (51) and (52) become

$$\begin{aligned}\breve{\mathbb{D}}\big(f(\cdot)\big)={}&\sum_{e=1}^{a}\sum_{i=0}^{k_e}f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}}\breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\\&+\int_{\mathrm{T}}\int_{\mathrm{T}}\breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)f^{(n)}(s)\,ds\,d\hat{z},\end{aligned} \tag{54}$$

$$\int_{\mathrm{T}} \breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\,d\hat{z} \geq 0, \quad s \in \mathbb{I}_2 \tag{55}$$

and

$$\breve{\mathbb{D}}\big(f(\cdot)\big) \geq \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}. \tag{56}$$

For $\alpha = 2,3$, (49), (51), and (53) become

$$\begin{aligned}\breve{\mathbb{D}}\big(f(\cdot)\big) = {} & -\sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,t)\big)H_{i_e}(\hat{z})\,d\hat{z} \\ & -\int_{\mathrm{T}}\int_{\mathrm{T}} \breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)f^{(n)}(s)\,ds\,d\hat{z},\end{aligned} \tag{57}$$

$$\int_{\mathrm{T}} \breve{\mathbb{D}}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)\mathbb{G}_{\mathcal{H},n-3}(\hat{z},s)\,d\hat{z} \geq 0, \quad s \in \mathbb{I}_2 \tag{58}$$

and

$$\breve{\mathbb{D}}\big(f(\cdot)\big) \leq -\sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}} \mathbb{D}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}, \tag{59}$$

respectively.

Theorem 7 *Assume* **B**. *Let an n-convex function $f : \mathbb{I}_2 \to \mathbb{R}$ and $(p_1,\dots,p_{\hat{m}}) \in \mathbb{R}^+$ be such that $\sum_{\sigma=1}^{\hat{m}} p_\sigma = 1$. Let also $f \in C^n([0,2\gamma])$ and $\hat{d}_1 = \hat{g}_1 < \hat{g}_2 < \cdots < \hat{g}_a = \hat{d}_2$ $(a \geq 2)$ be the points. Then*

(i) *If k_e is odd for each $e = 2,\dots,a$, then (56) and (59) hold.*

(ii) *Consider the function*

$$F(\hat{z}) = \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)H_{i_e}(\hat{z}) \tag{60}$$

is nonnegative, and (56) and (59) also hold. Then (56) and (59) become

$$\breve{\mathbb{D}}\big(f(\cdot)\big) \geq 0 \tag{61}$$

and

$$\breve{\mathbb{D}}\big(f(\cdot)\big) \leq 0, \tag{62}$$

respectively, where $0 \leq \hat{d}_1 < \hat{d}_2 \leq 2\gamma$.

Proof Proof is same as of Theorem 3. □

3 Applications to information theory

Information theory is a branch of science that deals with data quantification, storage, and transfer. In 1948, Claude Shannon [24] gave the idea of information theory and described entropy as the fundamental unit of information. In other words, it is also possible to determine the information using the probability density function. Divergence measure is an idea in probability theory that helps solve certain problems because divergence measure is used to calculate the distance between the two probability distributions. Moreover, divergence measures are used to solve many problems in probability theory. Information and divergence measures are extremely valuable and essential in many fields, including Sensor networks [25], finance [26], economics [27], and approximation of probability distributions [28].

Levinson- type inequalities are essential for generalizing inequalities for divergence between probability distributions. The key conclusions from Sect. 2 are linked to information theory in this part, using the Shannon entropy and f-divergence.

3.1 Csiszár divergence

Csiszár [29, 30] presented the subsequent definition.

Definition 1 If $f:\mathbb{R}_+\to\mathbb{R}_+$ is a convex function, choose $\tilde{\mathbf{v}},\tilde{\mathbf{l}}\in\mathbb{R}_+^{\hat{m}}$ such that $\sum_{\sigma=1}^{\hat{m}}\nu_\sigma=1$ and $\sum_{\sigma=1}^{\hat{m}}l_\sigma=1$. Then the Csiszár f-divergence is defines as follows:

$$\mathbb{I}_f(\tilde{\mathbf{v}},\tilde{\mathbf{l}}):=\sum_{\sigma=1}^{\hat{m}}l_\sigma f\left(\frac{\nu_\sigma}{l_\sigma}\right). \tag{63}$$

In [31], Horv́ath *et al.* generalized (63) as follows:

Definition 2 If $f:\mathbb{I}\to\mathbb{R}$ is such that $\mathbb{I}\subset\mathbb{R}$, choose $\tilde{\mathbf{v}}=(\nu_1,\dots,\nu_{\hat{m}})\in\mathbb{R}^{\hat{m}}$ and $\tilde{\mathbf{l}}=(l_1,\dots,l_{\hat{m}})\in(0,\infty)^{\hat{m}}$ such that

$$\frac{\nu_\sigma}{l_\sigma}\in\mathbb{I},\quad \sigma=1,\dots,\hat{m}.$$

Then

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{v}},\tilde{\mathbf{l}}):=\sum_{\sigma=1}^{\hat{m}}l_\sigma f\left(\frac{\nu_\sigma}{l_\sigma}\right). \tag{64}$$

Theorem 8 *Assume that $f\in C^n[\hat{d}_1,\hat{d}_2]$ is an n-convex function and $\breve{\mathbf{r}}=(\breve{r}_1,\dots,\breve{r}_{\hat{m}})$, $\breve{\mathbf{k}}=(\breve{k}_1,\dots,\breve{k}_{\hat{m}})\in(0,\infty)^{\hat{m}}$ and $\breve{\mathbf{w}}=(\breve{w}_1,\dots,\breve{w_\omega})$, $\breve{\mathbf{t}}=(\breve{t}_1,\dots,\breve{t\omega})\in(0,\infty)^{\hat{\omega}}$ are such that*

$$\frac{\breve{r}_\sigma}{\breve{k}_\sigma}\in\mathbb{I},\quad \sigma=1,\dots,\hat{m},$$

and

$$\frac{\breve{w}_\epsilon}{\breve{t}_\epsilon}\in\mathbb{I},\quad \epsilon=1,\dots,\hat{\omega}.$$

If k_e is odd for each $e = 2, \dots, a$, then for $\alpha = 1, 4$,

$$\begin{aligned}\mathbb{J}_{\text{cis}}\big(f(\cdot)\big) \geq \frac{1}{2}\Bigg[&\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\sum_{\epsilon=1}^{\hat{\omega}}\frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \Bigg(\sum_{\epsilon=1}^{\hat{\omega}}\frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon}\Bigg)^2 - \frac{1}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\sum_{\sigma=1}^{\hat{m}}\frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} \\ &+ \Bigg(\sum_{\sigma=1}^{\hat{m}}\frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\Bigg)^2\Bigg]\big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) \\ &+ \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}}\mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\end{aligned} \tag{65}$$

and for $\alpha = 2, 3$,

$$\begin{aligned}\mathbb{J}_{\text{cis}}\big(f(\cdot)\big) \leq \frac{1}{2}\Bigg[&\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\sum_{\epsilon=1}^{\hat{\omega}}\frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \Bigg(\sum_{\epsilon=1}^{\hat{\omega}}\frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon}\Bigg)^2 - \frac{1}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\sum_{\sigma=1}^{\hat{m}}\frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} \\ &+ \Bigg(\sum_{\sigma=1}^{\hat{m}}\frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\Bigg)^2\Bigg]\big(2f^{(2)}(\hat{d}_1) - f^{(2)}(\hat{d}_2)\big) \\ &- \sum_{e=1}^{a}\sum_{i=0}^{k_e} f^{(i+3)}(\hat{g}_e)\int_{\mathrm{T}}\mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z},\end{aligned} \tag{66}$$

where

$$\begin{aligned}\mathbb{J}_{\text{cis}}\big(\mathbf{f}(\cdot)\big) = {}&\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon}\hat{\mathbb{I}}_{\mathbf{f}}(\breve{\mathbf{w}},\breve{\mathbf{t}}) - \mathbf{f}\Bigg(\sum_{\epsilon=1}^{\hat{\omega}}\frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon}\Bigg) - \frac{1}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\hat{\mathbb{I}}_{\mathbf{f}}(\breve{\mathbf{r}},\breve{\mathbf{k}}) \\ &+ \mathbf{f}\Bigg(\sum_{\sigma=1}^{\hat{m}}\frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\Bigg)\end{aligned} \tag{67}$$

and

$$\begin{aligned}\mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) = {}&\sum_{\epsilon=1}^{\hat{\omega}}\frac{\breve{t}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon}\mathbb{G}_k\Bigg(\frac{\breve{w}_\epsilon}{\breve{t}_\epsilon},\hat{z}\Bigg) - \mathbb{G}_k\Bigg(\sum_{\epsilon=1}^{\hat{\omega}}\frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}}\breve{t}_\epsilon},\hat{z}\Bigg) \\ &- \sum_{\sigma=1}^{\hat{m}}\frac{\breve{k}_\sigma}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma}\mathbb{G}_k\Bigg(\frac{\breve{r}_\sigma}{\breve{k}_\sigma},\hat{z}\Bigg) + \mathbb{G}_k\Bigg(\sum_{\sigma=1}^{\hat{m}}\frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}}\breve{k}_\sigma},\hat{z}\Bigg).\end{aligned} \tag{68}$$

Proof Since the weights are positive and the Green functions $\mathbb{G}_\alpha(\cdot,\hat{z})$ given in (22)–(25) are 3-convex, therefore,

$$\mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big) \geq 0$$

for fixed $\alpha = 1, 2, 3, 4$.

Since k_e is odd for each $e = 2, \dots, a$, so we have $\mathrm{T}(\cdot) \geq 0$ and by part (i) of Lemma 1

$$\mathbb{G}_{\mathcal{H},n-3}(\cdot, s) \geq 0$$

hence $\mathbb{G}_{\mathcal{H},n}$ is 3-convex, therefore (32) holds. Thus, using $p_\sigma = \frac{\breve{k}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}$, $x_\sigma = \frac{\breve{r}_\sigma}{\breve{k}_\sigma}$, $q_\epsilon = \frac{\breve{t}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}$, $y_\epsilon = \frac{\breve{w}_\epsilon}{\breve{t}_\epsilon}$ in Theorem 2, (33) and (34) become (65) and (66), respectively. □

3.2 Shannon entropy

Definition 3 (see [31]) For positive probability distribution $\tilde{\mathbf{l}} = (l_1, \dots, l_{\hat{m}})$, the Shannon entropy is given by

$$\mathbb{S} := -\sum_{\sigma=1}^{\hat{m}} l_\sigma \log(l_\sigma). \tag{69}$$

In order to avoid many notions, we define the following functional:

$\mathbb{Q}$: Let $\breve{\mathbf{r}} = (\breve{r}_1, \dots, \breve{r}_{\hat{m}})$, $\breve{\mathbf{k}} = (\breve{k}_1, \dots, \breve{k}_{\hat{m}}) \in (0,\infty)^{\hat{m}}$ and $\breve{\mathbf{w}} = (\breve{w}_1, \dots, \breve{w}\hat{\omega})$, $\breve{\mathbf{t}} = (\breve{t}_1, \dots, \breve{t}\hat{\omega}) \in (0,\infty)^{\hat{\omega}}$.

We denote b as a base of log function.

Corollary 2 *Assume* $\mathbb{Q}$.

(i) *If* $n = 3, 5, \dots$ *and* $b > 1$, *then for* $\alpha = 1, 4$,

$$\begin{aligned}\mathbb{J}_s(\cdot) \geq{}& \left(\frac{\hat{d}_1^2 - 2\hat{d}_2^2}{(\hat{d}_1\hat{d}_2)^2}\right)\left[\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\sum_{\epsilon=1}^{\hat{\omega}} \frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \left(\sum_{\epsilon=1}^{\hat{\omega}} \frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\right)^2\right.\\ &\left.- \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\sum_{\sigma=1}^{\hat{m}} \frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} + \left(\sum_{\sigma=1}^{\hat{m}} \frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\right)^2\right]\\ &+ \sum_{e=1}^{a}\sum_{i=0}^{k_e} \frac{(-1)^{i+2}(i+2)!}{(c_j)^{i+3}}\int_{\mathrm{T}} \mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z}\end{aligned} \tag{70}$$

and for $\alpha = 2, 3$,

$$\begin{aligned}\mathbb{J}_s(\cdot) \leq{}& \left(\frac{\hat{d}_2^2 - 2\hat{d}_1^2}{(\hat{d}_1\hat{d}_2)^2}\right)\left[\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\sum_{\epsilon=1}^{\hat{\omega}} \frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \left(\sum_{\epsilon=1}^{\hat{\omega}} \frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\right)^2\right.\\ &\left.- \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\sum_{\sigma=1}^{\hat{m}} \frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} + \left(\sum_{\sigma=1}^{\hat{m}} \frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\right)^2\right]\\ &- \sum_{e=1}^{a}\sum_{i=0}^{k_e} \frac{(-1)^{i+2}(i+2)!}{(c_j)^{i+3}}\int_{\mathrm{T}} \mathbb{J}\big(\mathbb{G}_\alpha(\cdot,\hat{z})\big)H_{i_e}(\hat{z})\,d\hat{z},\end{aligned} \tag{71}$$

where $\mathbb{J}(\mathbb{G}_\alpha(\cdot,\hat{z}))$ *is defined in* (68), *and* $\mathbb{J}_s(\cdot)$ *is given by*

$$\begin{aligned}\mathbb{J}_s(\cdot) ={}& \sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon \log(\breve{w}_\epsilon) + \tilde{\mathbb{S}} - \log\left(\sum_{\epsilon=1}^{\hat{\omega}} \breve{w}_\epsilon\right) - \sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma \log(\breve{r}_\sigma) + \mathbb{S}\\ &+ \log\left(\sum_{\sigma=1}^{\hat{m}} \breve{r}_\sigma\right).\end{aligned} \tag{72}$$

(ii) *If* k_e *is odd and* $1 > b$ *or* $n = 4, 6, \dots$, *then inequalities in* (70) *and* (71) *are conversed.*

Proof

(i) Since $f(x) = \log(x)$ is an n-convex function for $n = 3, 5, \ldots$, and $b > 1$, putting $f(x) = \log(x)$ in Theorem 8 gives (70) and (71), where $\mathbb{S}$ is given in (69) and

$$\tilde{\mathbb{S}} = -\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon \log(\breve{t}_\epsilon).$$

(ii) As k_e is odd and $f(x) = \log(x)$ is n-concave for $n = 4, 6, \ldots$, then by Remark 2(ii), inequality (33) is reversed. Hence, using $f(x) = \log(x)$ and $p_\sigma = \frac{\breve{k}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}$, $x_\sigma = \frac{\breve{r}_\sigma}{\breve{k}_\sigma}$, $q_\epsilon = \frac{\breve{t}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}$, $y_\epsilon = \frac{\breve{w}_\epsilon}{\breve{t}_\epsilon}$ in reversed inequality (33) and (34), we obtain (70) and (71) in the reverse direction. □

Corollary 3 *Assume $\mathbb{Q}$ with odd values of k_e.*

(i) *If $b > 1$ and $n =$ even ($n \geq 4$), then for $\alpha = 1, 4$,*

$$\begin{aligned} \mathbf{J}_s(\cdot) \geq{}& \left(\frac{\hat{d}_1^2 - 2\hat{d}_2^2}{(\hat{d}_1\hat{d}_2)^2}\right)\left[\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon} \sum_{\epsilon=1}^{\hat{\omega}} \frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \left(\sum_{\epsilon=1}^{\hat{\omega}} \frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\right)^2 \right. \\ & \left. - \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma} \sum_{\sigma=1}^{\hat{m}} \frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} + \left(\sum_{\sigma=1}^{\hat{m}} \frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\right)^2\right] \\ & + \sum_{e=1}^{a} \sum_{i=0}^{k_e} \frac{(-1)^{i+2}(i+1)!}{(c_j)^{i+2}} \int_{\mathrm{T}} \mathbb{J}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z}, \end{aligned} \tag{73}$$

and for $\alpha = 2, 3$,

$$\begin{aligned} \mathbf{J}_s(\cdot) \leq{}& \left(\frac{\hat{d}_2^2 - 2\hat{d}_1^2}{(\hat{d}_1\hat{d}_2)^2}\right)\left[\frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon} \sum_{\epsilon=1}^{\hat{\omega}} \frac{(\breve{w}_\epsilon)^2}{\breve{t}_\epsilon} - \left(\sum_{\epsilon=1}^{\hat{\omega}} \frac{\breve{w}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\right)^2 \right. \\ & \left. - \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma} \sum_{\sigma=1}^{\hat{m}} \frac{(\breve{r}_\sigma)^2}{\breve{k}_\sigma} + \left(\sum_{\sigma=1}^{\hat{m}} \frac{\breve{r}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\right)^2\right] \\ & - \sum_{e=1}^{a} \sum_{i=0}^{k_e} \frac{(-1)^{i+2}(i+1)!}{(c_j)^{i+2}} \int_{\mathrm{T}} \mathbb{J}\big(\mathbb{G}_\alpha(\cdot, \hat{z})\big) H_{i_e}(\hat{z})\, d\hat{z}, \end{aligned} \tag{74}$$

where

$$\begin{aligned} \mathbf{J}_s(\cdot) ={}& \frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}\left(\tilde{S} + \sum_{\epsilon=1}^{\hat{\omega}} \breve{w}_\epsilon \log(\breve{t}_\epsilon)\right) - \frac{1}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon} \log\left(\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon\right) \\ & - \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}\left(S + \sum_{\sigma=1}^{\hat{m}} \breve{r}_\sigma \log(\breve{k}_\sigma)\right) + \frac{1}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma} \log\left(\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma\right). \end{aligned}$$

(ii) *If $n =$ odd ($n \geq 3$) or $b < 1$, then inequality in (73) is inverted.*

Proof

(i) Since $f(x) = -x\log(x)$ is n-convex for $n = 4, 6, \ldots$, and $b > 1$, substituting $f(x) = -x\log(x)$ in Theorem 8 gives (73) and (74), where

$$\tilde{S} = -\sum_{\epsilon=1}^{\hat{\omega}} \breve{w}_\epsilon \log(\breve{w}_\epsilon)$$

and

$$S = -\sum_{\sigma=1}^{\hat{m}} \breve{r}_\sigma \log(\breve{r}_\sigma).$$

(ii) Since $f(x) = -x\log(x)$ is n-concave ($n = 3, 5, \ldots$), then (33) holds in the reverse direction by Remark 2(ii). Thus, using $p_\sigma = \frac{\breve{k}_\sigma}{\sum_{\sigma=1}^{\hat{m}} \breve{k}_\sigma}$, $x_\sigma = \frac{\breve{r}_\sigma}{\breve{k}_\sigma}$, $q_\epsilon = \frac{\breve{t}_\epsilon}{\sum_{\epsilon=1}^{\hat{\omega}} \breve{t}_\epsilon}$ and $y_\epsilon = \frac{\breve{w}_\epsilon}{\breve{t}_\epsilon}$ in reversed inequality (33) and (34), we have (73) and (74) in the reverse direction.

□

4 Conclusion

The purpose of this study is to generalize Levinson-type inequalities (with real weights) for two distinct types of data points that use convex functions of higher order. Newly defined 3-convex Green functions and Hermite interpolating polynomial are utilized for the class of n-convex ($n \geq 3$) functions. We are able to find applications to information theory, as well as the bounds for obtained entropies and divergences. Moreover, other interpolations, e.g., Lidstone interpolation, Taylor's polynomial, and Montgomery identity, are also useful for exploring the related results.

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Author contributions

AR initiated the work and made calculations. KAK supervised and validated the draft. JP dealt with the formal analysis and investigation. GP included the applications to information theory. All the authors read and approved the final manuscript.

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Data availability

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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References

1. Pečarić, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York (1992)

2. Levinson, N.: Generalization of an inequality of Kay Fan. J. Math. Anal. Appl. **6**, 133–134 (1969)
3. Popoviciu, T.: Sur une inegalite de N. Levinson Mathematica (Cluj) **6**, 301–306 (1969)
4. Bullen, P.S.: An inequality of N. Levinson. Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika, 109–112 (1973)
5. Pečarić, J.: On an inequality on N. Levinson. Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika, 71–74 (1980)
6. Mercer, A.M.: 94.33 Short proofs of Jensen's and Levinson's inequalities. Math. Gaz. **94**(531), 492–495 (2010)
7. Mehmood, N., Agarwal, R.P., Butt, S.I., Pečarić, J.: New generalizations of Popoviciu type inequalities via new Green's functions and Montgomery identity. J. Inequal. Appl. **2017**, 108 (2017)
8. Agarwal, R.P., Wong, P.J.Y.: Error Inequalities in Polynomial Interpolation and Their Applications. Kluwer Academic, Dordrecht (1993)
9. Beesack, P.: On the Green's function of an *N*-point boundary value problem. Pac. J. Math. **12**(3), 801–812 (1962)
10. Levin, A.Y.: Some problems bearing on the oscillation of solution of liear differential equations. Sov. Math. Dokl. **4**, 121–124 (1963)
11. Rasheed, A., Khan, K.A., Pečarić, J., Pečarić, Đ.: Generalization of the Levinson inequality via new Green functions with applications to information theory. J. Inequal. Appl. **2023**(1), 124 (2023)
12. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Generalization of the Levinson inequality with applications to information theory. J. Inequal. Appl. **2019**(1), 230 (2019)
13. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Levinson type inequalities for higher order convex functions via Abel-Gontscharoff interpolation. Adv. Differ. Equ. **2019**(1), 430 (2019)
14. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Estimation of f-divergence and Shannon entropy by Levinson type inequalities via new Green's functions and Lidstone polynomial. Adv. Differ. Equ. **2020**(1), 27 (2020)
15. Butt, S.I., Khan, K.A., Pečarić, J.: Popoviciu type inequalities via Hermite's polynomial. Math. Inequal. Appl. **19**(4), 1309–1318 (2016)
16. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Estimation of f-divergence and Shannon entropy by using Levinson type inequalities for higher order convex functions via Hermite interpolating polynomial. J. Inequal. Appl. **2020**(1), 137 (2020)
17. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Entropy results for Levinson-type inequalities via Green functions and Hermite interpolating polynomial. Aequ. Math. **96**(1), 1–16 (2022)
18. Mehmood, N., Butt, S.I., Pečarić, Đ., Pečarić, J.: New bounds for Shannon, relative and Mandelbrot entropies via Hermite interpolating polynomial. Demonstr. Math. **51**(1), 112–130 (2018)
19. Ansari, I., Khan, K.A., Nosheen, A., Pečarić, Đ., Pečarić, J.: New entropic bounds on time scales via Hermite interpolating polynomial. J. Inequal. Appl. **2021**(1), 195 (2021)
20. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Estimation of f-divergence and Shannon entropy by Bullen type inequalities via Fink's identity. Filomat **36**(2), 527–538 (2022)
21. Adeel, M., Khan, K.A., Pečarić, Đ., Pečarić, J.: Estimation of f-divergence and Shannon entropy by Levinson type inequalities via Lidstone interpolating polynomial. Trans. A. Razmadze Math. **175**(1), 1–11 (2021)
22. Bilal, M., Khan, K.A., Nosheen, A., Pečarić, J.: Generalizations of Shannon type inequalities via diamond integrals on time scales. J. Inequal. Appl. **2023**(1), 24 (2023)
23. Bilal, M., Khan, K.A., Nosheen, A., Pečarić, J.: Some inequalities related to Csiszár divergence via diamond integral on time scales. J. Inequal. Appl. **2023**(1), 55 (2023)
24. Shannon, C.E.: A mathematical theory of communication. Bell Syst. Tech. J. **27**(3), 379–423 (1948)
25. Pardo, L.: Statistical Inference Based on Divergence Measures. Chapman & Hall, London (2018)
26. Sen, A., Sen, M.A., Foster, J.E., Amartya, S., Foster, J.E.: On Economic Inequality. Oxford University Press, London (1997)
27. Theil, H.: Economics and Information Theory. North-Holland, Amsterdam (1967)
28. Chow, C.K.C.N., Liu, C.: Approximating discrete probability distributions with dependence trees. IEEE Trans. Inf. Theory **14**(3), 462–467 (1968)
29. Csiszár, I.: Information measures: a critical survey. In: Tans. 7th Prague Conf. On Info. Th., Statist. Decis. Funct., Random Process and 8th European Meeting of Statist., B, pp. 73–86. Academia, Prague (1978)
30. Csiszár, I.: Information-type measures of difference of probability distributions and indirect observations. Studia Sci. Math. Hung. **2**, 299–318 (1967)
31. Horváth, L., Pečarić, Đ., Pečarić, J.: Estimations of f-and Rényi divergences by using a cyclic refinement of the Jensen's inequality. Bull. Malays. Math. Sci. Soc. **42**, 933–946 (2019)

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