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Upper bounds for the spectral radius of matrices having the Perron–Frobenius property

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Abstract

A new upper bound for the spectral radius of matrices having the Perron–Frobenius property is given by considering the position of positive entries. Some examples involving the largest zero of polynomials and the spectral radius of the iterative matrix for the Perron–Frobenius splitting are given to show the superiority of the theoretical result.

Keywords: Perron–Frobenius property; Perron–Frobenius splitting; Polynomials; Spectral radius

1 Introduction

Matrices having the Perron–Frobenius property [7] arise in many different fields of science and engineering, such as steady state behavior of Markov chains, population growth models, and Web search engines [2, 9, 11, 12].

Definition 1 [7] If $A \in \mathbb{R}^{n\times n}$, then A possesses the Perron–Frobenius (P–F) property if the spectral radius

$$\rho(A) := \max_{\lambda\in\sigma(A)} \{|\lambda|\}$$

is a positive eigenvalue of A and $\rho(A)$ possesses a corresponding nonnegative eigenvector, where $\sigma(A)$ is the spectrum of matrix A.

The well-known classes of nonnegative matrices, eventually positive matrices, and eventually nonnegative matrices [1, 3] are all included in matrices having the P–F property. There are various problems on matrices having the P–F property, for details, see [8, 10, 13, 14]. One of such problems is to bound its dominant eigenvalue $\rho(A)$, and the first result stated as below is due to Noutsos in 2006 [7].

Theorem 1 [7, *Theorem* 2.5] *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be a matrix having the P–F property. Then*

$$\rho(A) \le Bnd_{DN}(A) := \max_{i\in N} R_i(A^\top), \tag{1}$$

where $N = \{1,2,\dots,n\}$ *and* $R_i(A^\top) = \sum_{j\in N} a_{ij}$.

The upper bound (1) due to Noutsos for matrices having the P–F property was improved by He, Liu, and Lv in 2023 by using the positive part of $R_i(A)$ [4].

Theorem 2 [4, *Theorems* 2, 3 *and* 5] *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be a matrix having the P–F property. Then*

$$\rho(A) \le Bnd_{HLL_1}(A) := \max_{i\in N}\{a_{ii} + r_i^+(A)\} \tag{2}$$

and

$$\begin{aligned}\rho(A) &\le Bnd_{HLL_2}(A)\\ &:= \frac{1}{2}\max_{i,j\in N,\ j\ne i}\Big(a_{ii} + a_{jj} + \big((a_{ii} - a_{jj})^2 + 4r_i^+(A)r_j^+(A)\big)^{\frac{1}{2}}\Big).\end{aligned} \tag{3}$$

Furthermore, $Bnd_{HLL_2}(A) \le Bnd_{HLL_1}(A)$*, where* $r_i^+(A) := \sum_{\substack{j\in N,\ j\ne i\\ a_{ij}>0}} a_{ij}$.

Besides bounds (2) and (3), He et al. also provided an S-type upper bound for the spectral radius in [4], but this bound needs more computations. In this paper, we present a new upper bound for the spectral radius for matrices having the P–F property, and show that the new bound is sharper than bounds (2) and (3). Some numerical examples are also given to show the superiority of the new bound.

2 Main results

In this section we give a new upper bound for the spectral radius of matrices having the P–F property.

Theorem 3 *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be a matrix having the Perron–Frobenius property, and* $r_i^+(A) > 0$ *for any* $i \in N$*. Then*

$$\rho(A) \le Bnd(A), \tag{4}$$

where $Bnd(A) := \frac{1}{2}\max_{\substack{i,j\in N, j\ne i\\ a_{ij}>0}}(a_{ii} + a_{jj} + ((a_{ii} - a_{jj})^2 + 4r_i^+(A)r_j^+(A))^{\frac{1}{2}})$.

Proof Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ be an entrywise nonnegative nonzero eigenvector of matrix A corresponding to $\rho(A)$, that is,

$$A\mathbf{x} = \rho(A)\mathbf{x}. \tag{5}$$

Let

$$x_{i_0}x_{j_0} = \max_{a_{ij}>0,\ i\ne j} x_i x_j.$$

From (5), we have that, for any $i \in N$,

$$\rho(A)x_i = \sum_{k\in N} a_{ik}x_k \tag{6}$$

and

$$\begin{aligned}\big(\rho(A) - a_{ii}\big)x_ix_i &= \sum_{j\in N, j\neq i} a_{ij}x_jx_i \\ &= \sum_{\substack{j\in N,\ j\neq i\\ a_{ij}>0}} a_{ij}x_jx_i + \sum_{\substack{j\in N,\ j\neq i\\ a_{ij}\leq 0}} a_{ij}x_jx_i \\ &\leq \sum_{\substack{j\in N,\ j\neq i\\ a_{ij}>0}} a_{ij}x_jx_i \\ &\leq \sum_{\substack{j\in N,\ j\neq i\\ a_{ij}>0}} a_{ij}x_{i_0}x_{j_0} \\ &\leq r_i^+(A)x_{i_0}x_{j_0}.\end{aligned} \tag{7}$$

Case I. If $x_{i_0}x_{j_0} > 0$, then $x_{i_0} > 0$ and $x_{j_0} > 0$. By (7), we have

$$\big(\rho(A) - a_{i_0i_0}\big)x_{i_0}x_{i_0} \leq r_{i_0}^+(A)x_{i_0}x_{j_0} \tag{8}$$

and

$$\big(\rho(A) - a_{j_0j_0}\big)x_{j_0}x_{j_0} \leq r_{j_0}^+(A)x_{i_0}x_{j_0}. \tag{9}$$

Case I-a. If $\rho(A) - a_{i_0i_0} > 0$ and $\rho(A) - a_{j_0j_0} > 0$, then multiplying (8) by (9) gives

$$\big(\rho(A) - a_{i_0i_0}\big)\big(\rho(A) - a_{j_0j_0}\big)x_{i_0}^2x_{j_0}^2 \leq r_{i_0}^+(A)r_{j_0}^+(A)x_{i_0}^2x_{j_0}^2,$$

and hence

$$\big(\rho(A) - a_{i_0i_0}\big)\big(\rho(A) - a_{j_0j_0}\big) \leq r_{i_0}^+(A)r_{j_0}^+(A). \tag{10}$$

Solving $\rho(A)$ in (10) gives

$$\begin{aligned}\rho(A) &\leq \frac{1}{2}\Big(a_{i_0i_0} + a_{j_0j_0} + \big((a_{i_0i_0} - a_{j_0j_0})^2 + 4r_{i_0}^+(A)r_{j_0}^+(A)\big)^{\frac{1}{2}}\Big) \\ &\leq \frac{1}{2}\max_{\substack{i,j\in N,\ j\neq i\\ a_{ij}>0}}\Big(a_{ii} + a_{jj} + \big((a_{ii} - a_{jj})^2 + 4r_i^+(A)r_j^+(A)\big)^{\frac{1}{2}}\Big),\end{aligned}$$

i.e., inequality (4) holds.

Case I-b. If $\rho(A) - a_{i_0i_0} \leq 0$ or $\rho(A) - a_{j_0j_0} \leq 0$, then $\rho(A) \leq \max\{a_{i_0i_0}, a_{j_0j_0}\}$. Without loss of generality, suppose that $a_{i_0i_0} \geq a_{j_0j_0}$, then

$$\rho(A) \leq \frac{1}{2}(a_{i_0i_0} + a_{j_0j_0} + a_{i_0i_0} - a_{j_0j_0})$$

$$\begin{aligned}
&\leq \frac{1}{2}\left(a_{i_0i_0} + a_{j_0j_0} + \left((a_{i_0i_0} - a_{j_0j_0})^2 + 4r_{i_0}^+(A)r_{j_0}^+(A)\right)^{\frac{1}{2}}\right) \\
&\leq \frac{1}{2}\max_{\substack{i,j\in N,\ j\neq i\\ a_{ij}>0}}\left(a_{ii} + a_{jj} + \left((a_{ii} - a_{jj})^2 + 4r_i^+(A)r_j^+(A)\right)^{\frac{1}{2}}\right),
\end{aligned}$$

i.e., inequality (4) holds.

Case I-c. If $\rho(A) - a_{i_0i_0} \leq 0$ and $\rho(A) - a_{j_0j_0} \leq 0$, then $\rho(A) \leq \min\{a_{i_0i_0}, a_{j_0j_0}\}$, and thus $\rho(A) \leq \min\{a_{i_0i_0}, a_{j_0j_0}\} \leq \max\{a_{i_0i_0}, a_{j_0j_0}\}$. From Case I-b, inequality (4) also holds.

Case II. If $x_{i_0}x_{j_0} = 0$, then from $\mathbf{x} \neq 0$, there exists one index $k \in N$ such that $x_k \neq 0$. For this index k, by the assumption that $r_i^+(A) > 0$ for any $i \in N$, we have $r_k^+(A) > 0$, and hence there is an index $k_0 \in N$ and $k_0 \neq k$ such that $a_{kk_0} > 0$. Furthermore, by (7) it follows that

$$\left(\rho(A) - a_{kk}\right)x_k x_k \leq \sum_{\substack{j\in N,\ j\neq k\\ a_{kj}>0}} a_{kj}x_j x_k \leq r_k^+(A)x_{i_0}x_{j_0} = 0,$$

and hence $\rho(A) \leq a_{kk}$. Similarly to Case I-b and Case I-c, we have

$$\begin{aligned}
\rho(A) &\leq a_{kk} \\
&= \frac{1}{2}(a_{kk} + a_{k_0k_0} + a_{kk} - a_{k_0k_0}) \\
&\leq \frac{1}{2}\left(a_{kk} + a_{k_0k_0} + \left((a_{kk} - a_{k_0k_0})^2 + 4r_k^+(A)r_{k_0}^+(A)\right)^{\frac{1}{2}}\right) \\
&\leq \frac{1}{2}\max_{\substack{i,j\in N,\ j\neq i\\ a_{ij}>0}}\left(a_{ii} + a_{jj} + \left((a_{ii} - a_{jj})^2 + 4r_i^+(A)r_j^+(A)\right)^{\frac{1}{2}}\right),
\end{aligned}$$

i.e., inequality (4) also holds. The conclusion follows from Case I and Case II. □

Remark here that the difference of bounds (3) and (4) is the restriction under the *max*, from which it holds obviously that $Bnd(A) \leq Bnd_{HLL_2} \leq Bnd_{HLL_1}$, and the bound $Bnd(A)$ need less computations than Bnd_{HLL_2} in general.

For a nonnegative matrix A, it holds that $r_i(A) = r_i^+(A)$ for any $i \in N$. Hence, we can obtain the following bound for the spectral radius for nonnegative matrices because a nonnegative matrix is a matrix having the P–F property.

Corollary 1 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and $r_i(A) > 0$ for any $i \in N$. Then*

$$\rho(A) \leq Bnd_N(A), \tag{11}$$

where $Bnd_N(A) := \frac{1}{2}\max_{\substack{i,j\in N, j\neq i\\ a_{ij}\neq 0}}(a_{ii} + a_{jj} + ((a_{ii} - a_{jj})^2 + 4r_i(A)r_j(A))^{\frac{1}{2}})$.

Remark here that the upper bound (11) for the spectral radius of nonnegative matrices is exactly the bound provided by Kolotilina [5].

Table 1 Upper bounds for $\rho(A)$

$\rho(A)$	Bound (1)	Bound (2)	Bound (3)	Our bound (4)
3.4292	4.4800	4.4100	4.3253	4.0754

Example 1 Consider the matrix

$$A = \begin{bmatrix} -0.2 & 2 & 0.00 & 1.28 & -0.02 \\ 1 & 0.62 & 0.06 & 0.85 & 1.00 \\ -0.04 & 0.06 & 2.93 & -0.7 & 1.3 \\ 1.4 & 0.85 & -0.70 & 1.06 & 1.1 \\ 0 & 0.95 & 1.0 & 1.2 & 0.1 \end{bmatrix}$$

with the spectral radius $\rho(A) = 3.4292$ and the corresponding eigenvector $\mathbf{x} = (0.3107, 0.3707, 0.6992, 0.3082, 0.4269)^\top$. Bound (1) in Theorem 1, bounds (2) and (3) in Theorem 2, and our bound (4) in Theorem 3 are listed in Table 1. From Table 1 it can be seen that our bound (4) is sharper than bounds (2) and (3), and sharper than bound (1) in some cases.

Example 2 Consider the matrix

$$A = \begin{bmatrix} 0.8310 & 0.2305 & -0.3332 & -0.8384 \\ -0.1629 & 0.5589 & 0.4768 & -0.2525 \\ 5.0414 & 1.6179 & -4.6812 & -0.0670 \\ 1.5611 & -6.0991 & 1.3457 & 7.0798 \end{bmatrix}$$

and the Perron–Frobenius splitting [7, 8] of A with $A = M - N$, where

$$M = \begin{bmatrix} 0.5967 & -0.4283 & 0.3641 & -0.6811 \\ 0.5439 & 0.6875 & 0.6861 & 0.4109 \\ 4.7261 & 1.2240 & -5.2960 & 0.5392 \\ 2.1495 & -6.7270 & 1.0426 & 6.6704 \end{bmatrix},$$

and the iterative matrix

$$M^{-1}N = \begin{bmatrix} 0.1346 & -0.3462 & 0.3846 & 0.3846 \\ 0.3846 & 0.3462 & -0.1538 & 0.3077 \\ 0.3077 & -0.1154 & 0.3846 & 0.3077 \\ 0.3846 & 0.3846 & -0.3846 & 0.0769 \end{bmatrix}$$

has the P–F property with the spectral radius $\rho(M^{-1}N) = 0.5392$ and the corresponding eigenvector $\mathbf{x} = (0.4460, 0.4497, 0.7663, 0.1077)^\top$. Bounds (1), (2), (3), and (4) for $\rho(M^{-1}N)$ are given in Table 2. From Table 2 it can be seen that bounds (1), (2), and (3) are larger than 1. However our bound (4) is less than 1, which implies that we can conclude that the Perron–Frobenius splitting for this case is convergent by our bound.

Table 2 Upper bounds for $\rho(M^{-1}N)$

$\rho(A)$	Bound (1)	Bound (2)	Bound (3)	Our bound (4)
0.5392	1.2115	1.0385	1.0184	0.9778

Table 3 Upper bounds for $\rho(A)$

$\rho(A)$	Bound (1)	Bound (2)	Bound (3)	Melman's bound	Our bound (4)
1.1453	1.8000	1.5000	1.4491	1.3282	1.2845

Example 3 Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1.1 \\ 1 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & -0.1 \\ 0 & 0 & 1 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & -0.1 \end{pmatrix},$$

which is the companion matrix of the polynomial

$$p(z) = z^5 + 0.1z^4 - 0.4z^3 + 0.1z^2 - 0.5z - 1.1.$$

Matrix A has the P–F property with the dominant eigenvalue $\rho(A) = 1.1453$ and the corresponding eigenvector $\mathbf{x} = (0.3872, 0.5141, 0.4137, 0.5020, 0.4032)^\top$, where $\rho(A) = 1.1453$ is also the largest zero of the polynomial $p(z)$. Besides bounds (1), (2), (3), and (4), we give the upper bound proposed by Melman (see Theorem 3.1 of [6]) for zeros of the polynomial, see Table 3. From Table 3 it can be seen that our bound (4) is sharper than bounds (2) and (3), and sharper than bound (1) and Melman's bound in Theorem 3.1 of [6] in some cases.

3 Conclusions

In this paper we propose a new upper bound for the spectral radius by considering the position of the positive entries for a given matrix having the Perron–Frobenius property. We conjecture that by this technique the S-type upper bound for the spectral radius in [4] can be improved further.

Author contributions

K.M. Li (first author): conceptualization, methodology, writing-original draft; C.Q. Li (Second author, corresponding author): conceptualization, writing-original draft, writing review and editing.

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Availability of supporting data

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Ethical approval

Not applicable

Competing interests
The authors declare no competing interests.

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