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Continuity of the solutions sets for parametric set optimization problems

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Abstract

The current study focuses on exploring the stability of solution sets pertaining to set optimization problems, particularly with regard to the set order relation outlined by Karaman et al. 2018. Sufficient conditions are provided for the lower semicontinuity, upper semicontinuity, and compactness of *m*-minimal solution mappings in parametric set optimization, where the involved set-valued mapping is Lipschitz continuous.

Mathematics Subject Classification: 90C

Keywords: Set optimization; Stability; Semicontinuity; Locally Lipschitz continuity

1 Introduction

Over the past few decades, set optimization has garnered considerable attention from numerous researchers, primarily due to its widespread applications across various fields of applied mathematics, including vector optimization, interval optimization, game theory, mathematical economics, control system field, mathematical finance, and numerous others, see [2–11].

As we know, the stability of solutions holds great importance and serves as an intriguing subject in the field of set-valued optimization, see [12–20]. Xu and Li [13] introduced the notion of a lower-level mapping and established the semicontinuity of minimal solution mappings for a parametric set optimization problem. Han and Huang [7] delved into the convexity and semicontinuity of solution mappings in parametric set optimization problems, leveraging level mappings in their analysis. Karuna and Lalitha [14] primarily investigated the semicontinuity of approximate efficient solution mappings in the context of parametric set-valued optimization problems. Anh et al. [18] established the internal and external stability of the solutions sequence of perturbed problems, demonstrating their convergence towards a solution of the original problem. Li and Wei [16] conducted a thorough examination of the compactness and semicontinuity of E-minimal solution sets in parametric set optimization problems, utilizing improvement sets, and doing so under various pertinent conditions. Zhang and Huang [17] examined the semicontinuity and compactness of minimal solution mappings in parametric set optimization problems, taking into account the local Lipschitz continuity condition.

Lately, Karaman et al. [1] introduced a novel set order relations within the family of bounded sets, utilizing the Minkowski difference as a foundation. These new relations endow the family of bounded sets with a partial order, thus offering a fresh approach to the study of set optimization problems. Employing these novel set order relations, set optimization problems have been examined in [1, 20–22]. Nevertheless, to our knowledge, no research has been conducted on the continuity of solution maps for parametric set optimization problems using Karaman's set order relations so far. Therefore, it is of significant interest and importance to delve into the semicontinuity and compactness of solution maps for the parametric set optimization problem, leveraging Karaman's set order relations.

The rest of this paper is organized as follows. In Sect. 2, we introduce essential concepts, notations, and outcomes that are used throughout the paper. Section 3 delves into the lower semicontinuity, upper semicontinuity, and compactness of m-minimal solution mappings for parametric set optimization, specifically focusing on locally Lipschitz continuous set-valued mappings.

2 Preliminaries

Let $(X, \|\cdot\|_X)$, $(\Lambda, \|\cdot\|_\Lambda)$ and $(Y, \|\cdot\|_Y)$ be three real normed vector spaces. We denote the closed unit ball centered at origin in X (respectively, Y) by $\mathbb{B}_X$ (respectively, $\mathbb{B}_Y$), and an open ball centered at y with radius $r > 0$ in Y by $\mathbb{B}_Y(y, r)$. Let $K \subseteq Y$ be a convex, pointed and closed cone with nonempty interior. Additionally, we use $P(Y)$ to denote the family of nonempty proper subsets of Y, and $B(Y)$ to represent the family of nonempty proper bounded subsets of Y. Then we denote by $\mathrm{int} A$, $\mathrm{cl} A$ the topological interior, and the topological closure of $A \subseteq Y$, respectively.

For $A, B \in P(Y)$, the Minkowski (Pontryagin) difference of A and B, considered in [11], is given as

$$A \dot{-} B := \{y \in Y : y + B \subseteq A\}.$$

In [8], for any $A, B \in P(Y)$, the lower set less order relation $\preceq_K^l$ and the strict lower set less order relation $\ll_K^l$ on $P(Y)$ are defined by

$$A \preceq_K^l B \quad \Leftrightarrow \quad B \subseteq A + K;$$
$$A \ll_K^l B \quad \Leftrightarrow \quad B \subseteq A + \mathrm{int}K.$$

Lately, Karaman et al. [1] introduced the following order relations on $P(Y)$.

$$A \preceq^m B \quad \Leftrightarrow \quad (A \dot{-} B) \cap (-K) \neq \emptyset,$$

and

$$A \ll^m B \quad \Leftrightarrow \quad (A \dot{-} B) \cap (-\mathrm{int}K) \neq \emptyset.$$

Obviously, $\preceq^m$ is a partial order relation on $B(Y)$.

Lemma 2.1 [6] *Let A and B be nonempty subsets of Y. If $0 < \lambda < \delta$, B is convex, and $A + \delta B_Y \subseteq B + \lambda B_Y$, then $A \subseteq \mathrm{int}B$.*

For any nonempty subset S of X, let $F : X \to 2^Y$ be a set-valued mapping. We consider the constrained set optimization problem:

$$\text{(SOP)} \quad \min \quad F(x)$$
$$\text{s.t.} \quad x \in S.$$

Throughout the paper, it is assumed that $F(x) \neq \emptyset$ for all $x \in S$, and that $F(S) := \bigcup_{x\in S} F(x)$.

We recall the concept of the optimal solutions for the problem (SOP) with regard to the set order relation $\preceq_K^l$.

Definition 2.1 [8] An element $\bar{x} \in S$ is said to be

(i) a K-l-minimal solution of (SOP) if $x \in S$ such that $F(x) \preceq_K^l F(\bar{x})$ imply $F(\bar{x}) \preceq_K^l F(x)$;

(ii) a weak K-l-minimal solution of (SOP) if $x \in S$ such that $F(x) \ll_K^l F(\bar{x})$ imply $F(\bar{x}) \ll_K^l F(x)$.

Let $E_l(S)$ and $W_l(S)$ represent the K-l-minimal solution set and the weak K-l-minimal solution set of (SOP), respectively. It is evident that $E_l(S) \subseteq W_l(S)$.

We now recall the notion of the minimal and weak minimal solutions of (SOP) with respect to the relation $\preceq^m$. To define solution concepts, we assume that $F(x) \in B(Y)$, for all $x \in S$.

Definition 2.2 [1] An element $\bar{x} \in S$ is said to be

(i) a m-minimal solution of (SOP) if there is no $x \in S$ such that $F(x) \preceq^m F(\bar{x})$ and $F(x) \neq F(\bar{x})$, that is, either $F(x) \npreceq^m F(\bar{x})$ or $F(x) = F(\bar{x})$, for any $x \in S$.

(ii) a weak m-minimal solution of (SOP) if there is no $x \in S$ such that $F(x) \ll^m F(\bar{x})$.

Let $E_m(S)$ and $W_m(S)$ represent the m-minimal solution set and the weak m-minimal solution set of (SOP), respectively.

Lemma 2.2 [21] *If K is a closed convex pointed cone with nonempty interior, then*

$$E_l(S) \subseteq W_l(S) \subseteq W_m(S).$$

A subset S of a topological space is said to be arcwise connected if, for every pair of points $x, y \in S$, there exists a continuous function $\varphi : [0,1] \to S$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Definition 2.3 [22] Let S be an arcwise connected subset of X. A set-valued mapping $F : X \to 2^Y$ is said to be strictly m-quasiconnected on S if for any $A \in P(Y)$ and for any $x, y \in S$ with $x \neq y$, $F(x) \preceq^m A$ and $F(y) \preceq^m A$, there exists a continuous path $\varphi : [0,1] \to S$ with $\varphi(0) = x$ and $\varphi(1) = y$ such that

$$F\big(\varphi(t)\big) \ll^m A, \quad \forall t \in (0,1).$$

Lemma 2.3 [22] *If S is an arcwise connected subset of X and $F : X \to 2^Y$ is strictly m-quasiconnected on S with nonempty values, then $E_m(S) = W_m(S)$.*

Definition 2.4 [2] Let $F : X \to 2^Y$ be a set-valued mapping. Then, F is said to be

(i) upper semicontinuous (u.s.c.) at $\bar{x}$ if, for any open neighborhood U of $F(\bar{x})$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for every $x \in N(\bar{x})$, $F(x) \subseteq U$.

(ii) lower semicontinuous (l.s.c.) at $\bar{x}$ if, for any open subset U of Y with $F(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $F(x) \cap U \neq \emptyset$ for all $x \in N(\bar{x})$.

We say that F is u.s.c. and l.s.c. on X if it is u.s.c. and l.s.c. at each point $x \in X$, respectively. We call that F is continuous on X if it is both u.s.c. and l.s.c. on X.

Lemma 2.4 [2] *A set-valued mapping $F : X \to 2^Y$ is l.s.c. at $\bar{x} \in X$ if and only if for any sequence $\{x_n\} \subseteq X$ with $x_n \to \bar{x}$ and for any $\bar{y} \in F(\bar{x})$, there exists $y_n \in F(x_n)$ such that $y_n \to \bar{y}$.*

Lemma 2.5 [2] *Let $F : X \to 2^Y$ be a set-valued mapping. For any given $\bar{x} \in X$, if $F(\bar{x})$ is compact, then F is u.s.c. at $\bar{x} \in X$ if and only if for any sequence $\{x_n\} \subseteq X$ with $x_n \to \bar{x}$ and for any $y_n \in F(x_n)$, there exist $\bar{y} \in F(\bar{x})$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \bar{y}$.*

Definition 2.5 [17] Let S be a nonempty subset of X. A set-valued mapping $F : X \to 2^Y$ is said to be locally Lipschitz continuous at $\bar{x} \in S$ if and only if there exist a constant $L > 0$ and a neighborhood $N(\bar{x})$ of $\bar{x}$ in X such that

$$F(x_1) \subseteq F(x_2) + L\|x_1 - x_2\|_X \mathbb{B}_Y, \quad \forall x_1, x_2 \in N(\bar{x}) \cap S.$$

We say that F is locally Lipschitz continuous on S if and only if F is locally Lipschitz continuous at every $\bar{x} \in S$.

Definition 2.6 [22] Let S be a nonempty subset of X. A set-valued mapping $F : X \to 2^Y$ is said to be weak locally K-Lipschitz continuous at $\bar{x} \in S$ if and only if there exist a constant $L > 0$ and a neighborhood $N(\bar{x})$ of $\bar{x}$ in X such that

$$F(x_1) \cap \left\{ z + L\|x_1 - x_2\|_X \mathbb{B}_Y + K \right\} \neq \emptyset, \quad \forall x_1, x_2 \in N(\bar{x}) \cap S, \forall z \in F(x_2).$$

We say that F is weak locally K-Lipschitz continuous on S if and only if F is weak locally K-Lipschitz continuous at every $\bar{x} \in S$.

Remark 2.1 The following examples demonstrate that the concept of continuity differs from the concept of locally Lipschitz continuity for a set-valued mapping.

Example 2.1 [17] Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = \{x \in \mathbb{R} : x > 0\}$, and $K = \mathbb{R}^2_+$. Let $F : X \to 2^Y$ be a set-valued mapping defined as

$$F(x) = (\ln x, \ln x) + \mathbb{R}^2_+, \quad \forall x \in S.$$

Then it is easy to see that F is continuous on S, while it is not locally Lipschitz continous on S.

Example 2.2 [22] Let $S = X = \mathbb{R}$, and $Y = \mathbb{R}$. Consider $F : X \to 2^Y$ defined as

$$F(x) = \begin{cases} [-1, 1], & \text{if } x = 0, \\ \{0\}, & \text{if } x \neq 0. \end{cases}$$

Then one can easily see that F is locally Lipschitz at 0, but it is not l.s.c. at 0.

Example 2.3 [22] Let $S = X = \mathbb{R}$, and $Y = \mathbb{R}$. Consider $F : X \to 2^Y$ defined as

$$F(x) = \begin{cases} [-1,1], & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Then one can easily see that F is locally Lipschitz at 0, but it is not u.s.c. at 0.

Proposition 2.1 *We assume that $F : S \to 2^Y$ is a locally Lipschitz continuous set-valued mapping at $x_0 \in S$ and at $y_0 \in S$, respectively. Furthermore, $S \subseteq X$ is a nonempty, and $F(x_0)$ is closed. If, for any sequences $\{x_n\}$ and $\{y_n\}$ satisfying the conditions $x_n \to x_0$ and $y_n \to y_0$, it holds that $F(x_n) \preceq^m F(y_n)$ for sufficiently large n, then it follows that $F(x_0) \preceq^m F(y_0)$.*

Proof Since F is locally Lipschitz continuous at $x_0 \in S$, there exists a constant $L > 0$ such that, for n sufficiently large, we have

$$F(x_n) \subseteq F(x_0) + L\|x_n - x_0\|_X \mathbb{B}_Y. \tag{1}$$

Moreover, as F is locally Lipschitz continuous at $y_0 \in S$, there exists a constant $L_1 > 0$ such that, for n sufficiently large, we have

$$F(y_0) \subseteq F(y_n) + L_1\|y_n - y_0\|_X \mathbb{B}_Y. \tag{2}$$

It follows from (1), (2), and $F(x_n) \preceq^m F(y_n)$ for n sufficiently large that, there exists $k_n \in K$ such that

$$-k_n + F(y_0) \subseteq F(x_0) + L\|x_n - x_0\|_X \mathbb{B}_Y + L_1\|y_n - y_0\|_X \mathbb{B}_Y.$$

We conclude that $F(x_0) \preceq^m F(y_0)$. In other words, there exists a vector $k \in K$ such that $-k + F(y_0) \subseteq F(x_0)$. Suppose, on the contrary, that for any $k \in K$ there exists $u_0 \in F(y_0)$ such that $-k + u_0 \notin F(x_0)$. Since $F(x_0)$ is closed, there exists $\epsilon > 0$ such that $((-k + u_0) + \epsilon\mathbb{B}_Y) \cap F(x_0) = \emptyset$, and as a result

$$-k + u_0 \notin F(x_0) + \epsilon\mathbb{B}_Y. \tag{3}$$

Because $\mathbb{B}_Y$ is a closed unit ball and $x_n \to x_0$, $y_n \to y_0$, as $n \to \infty$, $L\|x_n - x_0\|_X \mathbb{B}_Y + L_1\|y_n - y_0\|_X \mathbb{B}_Y$ is reduced to the origin 0 as $n \to \infty$. Then, we get $-k_n + F(y_0) \subseteq F(x_0) + \epsilon\mathbb{B}_Y$ for n sufficiently large, which is a contradition to (3). Therefore, $-k + F(y_0) \subseteq F(x_0)$, and so $F(x_0) \preceq^m F(y_0)$. □

Proposition 2.2 *Suppose that $F : S \to 2^Y$ is locally Lipschitz continuous at $x_0 \in S$ and weak locally $-K$-Lipschitz continuous at $y_0 \in S$, respectively. Furthermore, $S \subseteq X$ is a nonempty, and $F(x_0)$ is compact. If for any sequences $\{x_n\}$ and $\{y_n\}$ satisfying $x_n \to x_0$, $y_n \to y_0$, such that $F(y_0) \ll^m F(x_0)$ and $F(y_n)$ are convex subsets of Y, then $F(y_n) \ll^m F(x_n)$ for n sufficiently large.*

Proof Since $F(y_0) \ll^m F(x_0)$, there exists $k_0 \in \mathrm{int}K$ such that $-k_0 + F(x_0) \subseteq F(y_0)$. Then, $-k_0 + F(x_0) \subseteq F(y_0) \subseteq \bigcup_{z\in F(y_0)} \mathbb{B}_Y(z,r)$, for any $r > 0$. Since $F(x_0)$ is compact, there is a finite set $\{z_1, z_2, \dots, z_n\} \subseteq F(y_0)$ such that

$$-k_0 + F(x_0) \subseteq \bigcup_{i=1}^{n} \mathbb{B}_Y(z_i, r).$$

Since $-k_0 + F(x_0)$ is compact, we notice that there exists $\epsilon > 0$ such that

$$-k_0 + F(x_0) + 3\epsilon\mathbb{B}_Y \subseteq \bigcup_{i=1}^{n} \mathbb{B}_Y(z_i, r). \tag{4}$$

Since F is locally Lipschitz at $x_0 \in S$, there exist a neighbourhood $N(x_0)$ of x_0 and a constant $L > 0$ such that

$$F(x) \subseteq F(x_0) + L\|x - x_0\|_X \mathbb{B}_Y, \quad \forall x \in N(x_0). \tag{5}$$

Since F is weak locally $-K$-Lipschitz continuous at $y_0 \in S$, there exist a neighborhood $N(y_0)$ of y_0 and a constant $L_1 > 0$ such that

$$F(y) \cap \left\{z + L_1\|y - y_0\|_X \mathbb{B}_Y - K\right\} \neq \emptyset, \quad \forall y \in N(y_0)\ \forall z \in F(y_0). \tag{6}$$

Since $x_n \to x_0$, $y_n \to y_0$, there are $n_1, n_2 \in N$ such that $x_n \in N(x_0)$ for any $n > n_1$ and $y_n \in N(y_0)$ for any $n > n_2$. Let $n_0 = \max\{n_1, n_1\}$, then, for any $n > n_0$, it follows from (5) that

$$2\epsilon\mathbb{B}_Y + F(x_n) \subseteq F(x_0) + L\|x_n - x_0\|_X \mathbb{B}_Y + 2\epsilon\mathbb{B}_Y. \tag{7}$$

Since $x_n \to x_0$ as $n \to \infty$ and $\mathbb{B}_Y$ is a closed unit ball, the radius of the closed ball $L\|x_n - x_0\|_X \mathbb{B}_Y$ will be close to zero as $n \to \infty$. Therefore, it follows from (4) and (7), that

$$\begin{aligned} -k_0 + 2\epsilon\mathbb{B}_Y + F(x_n) &\subseteq -k_0 + F(x_0) + 3\epsilon\mathbb{B}_Y \\ &\subseteq \bigcup_{i=1}^{n} \mathbb{B}_Y(z_i, r) = \{z_1, z_2, \dots, z_n\} + B_Y(0, r). \end{aligned} \tag{8}$$

For any $u \in -k_0 + 2\epsilon\mathbb{B}_Y + F(x_n)$, we make a conclusion from (8) that there is $i_0 \in \{1, 2, \dots, n\}$ and $b_1 \in B_Y(0, r)$ such that

$$u = z_{i_0} + b_1. \tag{9}$$

From (6), we get $F(y_n) \cap \{z + L_1\|y_n - y_0\|_X \mathbb{B}_Y - K\} \neq \emptyset$, $\forall y_n \in N(y_0)\ \forall z \in F(y_0)$. Since $y_n \to y_0$ as $n \to \infty$, the radius of the closed ball $L_1\|y_n - y_0\|_X \mathbb{B}_Y$ will be close to zero as $n \to \infty$. Therefore, $F(y_n) \cap \{z + \epsilon\mathbb{B}_Y - K\} \neq \emptyset$, $\forall y_n \in N(y_0)\ \forall z \in F(y_0)$. Then there exist $t_n \in F(y_n)$, $b_2 \in \mathbb{B}_Y$ and $k_1 \in K$ such that $t_n = z_{i_0} + \epsilon b_2 - k_1$, and so $z_{i_0} = t_n - \epsilon b_2 + k_1$. Combining it with (9), we obtain $u = z_{i_0} + b_1 = t_n - \epsilon b_2 + k_1 + b_1$, namely,

$$-k_1 + u = t_n - \epsilon b_2 + b_1 \in F(y_n) + \epsilon\mathbb{B}_Y + \mathbb{B}_Y(0, r). \tag{10}$$

Because $u \in -k_0 + 2\epsilon\mathbb{B}_Y + F(x_n)$, we get $-k_1 + u \in -k_0 - k_1 + 2\epsilon\mathbb{B}_Y + F(x_n)$. From this and (10), we have

$$-k_0 - k_1 + 2\epsilon\mathbb{B}_Y + F(x_n) \subseteq F(y_n) + \epsilon\mathbb{B}_Y + \mathbb{B}_Y(0,r) \subseteq F(y_n) + \epsilon\mathbb{B}_Y + r\mathbb{B}_Y. \tag{11}$$

Let $k_0 + k_1 = k_2$. Since $k_0 \in \text{int}K$, $k_1 \in K$, we have $k_2 \in \text{int}K$.

$$-k_2 + 2\epsilon\mathbb{B}_Y + F(x_n) \subseteq F(y_n) + \epsilon\mathbb{B}_Y + r\mathbb{B}_Y. \tag{12}$$

Let $r \to 0$, then $r + \epsilon \to \epsilon < 2\epsilon$, and hence, by Lemma 2.1, we obtain

$$-k_2 + F(x_n) \subseteq \text{int}F(y_n) \subseteq F(y_n).$$

This indicates that

$$F(y_n) \ll^m F(x_n),$$

for n sufficiently large. This finishes the proof. □

For any $\lambda \in \Lambda$, we consider the following constrained parametric set-valued optimization problem (for short, PSOP):

$$\begin{aligned} \text{(PSOP)} \quad & \min \quad F(x) \\ & \text{s.t.} \quad x \in S(\lambda), \end{aligned}$$

where $S : \Lambda \to 2^X$ and $F : X \to 2^Y$ are two set-valued mappings.

We denote the solution mappings $E_m : \Lambda \to 2^X$, $W_m : \Lambda \to 2^X$ for (PSOP) as follows: $E_m(\lambda) = E_m(S(\lambda))$, $W_m(\lambda) = W_m(S(\lambda))$.

Remark 2.2 Clearly, for any $\lambda \in \Lambda$, $E_m(\lambda) \subseteq W_m(\lambda)$.

Definition 2.7 Let $S : \Lambda \to 2^X$ and $F : X \to 2^Y$ are two set-valued mappings with nonempty values. The level mapping $L : \Lambda \times X \to 2^X$ is defined by

$$L(\lambda, x) = \{y \in S(\lambda) : F(y) \preceq^m F(x)\}, \quad (\lambda, x) \in \Lambda \times X.$$

Remark 2.3 Clearly, for any $\lambda \in \Lambda$, $x \in S(\lambda)$, $E_m(L(\lambda, x)) \subseteq E_m(\lambda) = E_m(S(\lambda))$.

Proposition 2.3 *Let $S : \Lambda \to 2^X$ be a nonempty closed-valued. $F : S \to 2^Y$ is locally Lipschitz continuous and closed-valued on $S(\lambda)$, then $L(\lambda, x)$ is closed for all $(\lambda, x) \in \Lambda \times X$.*

Proof For any $(\lambda, x) \in \Lambda \times X$, let $\{y_n\} \subseteq L(\lambda, x)$ be a sequence with $y_n \to y_0$, then $y_0 \in S(\lambda)$ and $F(y_n) \preceq^m F(x)$. Since F is locally Lipschitz continuous and closed-valued on $S(\lambda)$, it follows from Proposition 2.1 that $F(y_0) \preceq^m F(x)$ and as a result $y_0 \in L(\lambda, x)$. □

From Theorem 5.1 in [10], Proposition 2.3, and Lemma 2.2, we can get the following Lemma.

Lemma 2.6 *Let S be a nonempty and compact subset of X and $F : S \to 2^Y$ is locally Lipschitz continuous and closed-valued on S, then (SOP) has a m-minimal solution.*

Lemma 2.7 [22] *If $S(\lambda)$ is an arcwise connected subset of X and $F : X \to 2^Y$ is strictly m-quasiconnected on $S(\lambda)$ with nonempty values and $x_0 \in E_m(\lambda)$, then $L(\lambda, x_0) = x_0$.*

3 Stability of solutions of (PSOP)

In this section, we undertake an analysis of the continuity properties of solution mappings for (PSOP). Initially, we establish the upper semi-continuity and compactness of the weak m-minimal solution mapping associated with (PSOP).

Theorem 3.1 *Let $\lambda_0 \in \Lambda$. Suppose that*

(i) S is continuous at λ_0, and $S(\lambda_0)$ is compact;

(ii) F is locally Lipschitz continuous and weak locally $-K$-Lipschitz continuous on $S(\lambda_0)$ with nonempty and convex compact values.

Then, $W_m(\lambda)$ is u.s.c. at λ_0 and $W_m(\lambda_0)$ is compact.

Proof Since $E_m(S) \subseteq W_m(S)$. It follows from Lemma 2.6 that $W_m(\lambda_0) \neq \emptyset$. Now, we claim that $W_m(\lambda)$ is u.s.c. at λ_0. By the contrary, we suppose that $W_m(\lambda)$ is not u.s.c. at λ_0, then there is at least a neighborhood U with $W_m(\lambda_0) \subseteq U$, a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$, and $x_n \in W_m(\lambda_n)$ such that

$$x_n \notin U, \quad \forall n \in N. \tag{13}$$

Since S is u.s.c. at λ_0 and $S(\lambda_0)$ is compact, by Lemma 2.5, there exists $x_0 \in S(\lambda_0)$ such that $x_n \to x_0$. (Of course, if necessary, we can extract a subsequence).

We now assert that $x_0 \in W_m(\lambda_0)$. If $x_0 \notin W_m(\lambda_0)$, then there is $y_0 \in S(\lambda_0)$ such that $F(y_0) \ll^m F(x_0)$. It follows from the l.s.c. of S at λ_0 and Lemma 2.4, there exists a sequence $\{y_n\}$ with $y_n \in S(\lambda_n)$ such that $y_n \to y_0$. Since F is locally Lipschitz continuous and weak locally $-K$-Lipschitz continuous on $S(\lambda_0)$ with nonempty and convex compact values. By Proposition 2.2 we get $F(y_n) \ll^m F(x_n)$ for large enough n, which is a contradiction to $x_n \in W_m(\lambda_n)$, therefore $x_0 \in W_m(\lambda_0)$. Based on the assumption that $x_n \to x_0$, it follows that $x_n \in U$ for n sufficiently large. However, this contradicts (13). Consequently, $W_m(\lambda)$ is u.s.c. at λ_0.

Next, we claim that $W_m(\lambda_0)$ is compact. Given that $W_m(\lambda_0) \subseteq S(\lambda_0)$ and $S(\lambda_0)$ is compact, it suffices to demonstrate that $W_m(\lambda_0)$ is closed. Let $\{z_n\} \subseteq W_m(\lambda_0)$ be a sequence with $z_n \to z_0$. Assuming $z_0 \notin W_m(\lambda_0)$, there exists an element $z^* \in S(\lambda_0)$ satisfying $F(z^*) \ll^m F(z_0)$. By employing the same reasoning as previously demonstrated, we can conclude that for large enough n, $F(z^*) \ll^m F(z_n)$ holds. However, this contradicts $\{z_n\} \subseteq W_m(\lambda_0)$, therefore $z_0 \in W_m(\lambda_0)$. We can deduce that $W_m(\lambda_0)$ is also compact. □

Now, we present an example to demonstrate Theorem 3.1.

Example 3.1 Let $X = Y = \mathbb{R}^2$, $\Lambda = [0, 1]$, and $K = \mathbb{R}^2_+$. Assume that $S(\lambda) = [0, \lambda] \times [0, \lambda]$ for all $\lambda \in \Lambda$. Define a set-valued mapping $F : X \to 2^Y$ as $F(x, y) = (x^2y^2, x^2y^2) + \mathbb{B}_Y$ for all $(x, y) \in \mathbb{R}^2$. Let $\lambda_0 = 1$. It is straightforward to verify that all the conditions of Theorem 3.1 are satisfied. Through a simple computation, we can determine that $W_m(\lambda) = [0, \lambda] \times \{0\} \cup \{0\} \times [0, \lambda]$ for all $\lambda \in \Lambda$. Evidently, we observe that $W_m(\lambda)$ is u.s.c. at 1.

Theorem 3.2 *Let $\lambda_0 \in \Lambda$. Suppose that*

(*i*) *S is continuous at λ_0, and $S(\lambda_0)$ is compact*;

(*ii*) *F is locally Lipschitz continuous and weak locally –K-Lipschitz continuous on $S(\lambda_0)$ with nonempty and convex compact values*;

(*iii*) *F is strictly m-quasiconnected on $S(\lambda_0)$.*

Then, $E_m(\lambda)$ is u.s.c. at λ_0 and $E_m(\lambda_0)$ is compact.

Proof It follows from Lemma 2.6 that $E_m(\lambda_0) \neq \emptyset$. Now, we assert that $E_m(\lambda)$ is u.s.c. at λ_0. By the contrary, we suppose that $E_m(\lambda)$ is not u.s.c. at λ_0, then there exists at least a neighborhood U such that $E_m(\lambda_0) \subseteq U$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$E_m(\lambda_n) \nsubseteq U. \tag{14}$$

Since F is strictly m-quasiconnected on $S(\lambda_0)$, based on Lemma 2.3, we deduce that $E_m(\lambda_0) = W_m(\lambda_0) \subseteq U$. Leveraging Theorem 3.1, we can infer that $W_m(\lambda)$ is u.s.c. at λ_0. Consequently, there exists a neighborhood V of λ_0 such that $W_m(\lambda) \subseteq U$, for all $\lambda \in V$. Because $\lambda_n \to \lambda_0$, it follows that $\lambda_n \in V$ for n large enough. Therefore, $E_m(\lambda_n) \subseteq W_m(\lambda_n) \subseteq U$ for n large enough. However, this conclusion contradicts (14). Consequently, $E_m(\lambda)$ is u.s.c. at λ_0. Since $E_m(\lambda_0) = W_m(\lambda_0)$, as a result, by Theorem 3.1 $E_m(\lambda_0)$ is compact. □

In the theorem that follows, we explore the lower semicontinuity of the m-minimal solution for the (PSOP) problem.

Lemma 3.1 *Let $\lambda_0 \in \Lambda$. Assume that*

(*i*) *S is continuous at λ_0, S is compact-valued on Λ and $S(\lambda_0)$ is an arcwise connected subset of X*;

(*ii*) *F is locally Lipschitz continuous on $S(\lambda_0)$ with nonempty and closed values*;

(*iii*) *F is strictly m-quasiconnected on $S(\lambda_0)$.*

Then, $L(\lambda, x)$ is u.s.c. at $\lambda_0 \times S(\lambda_0)$.

Proof Assume to the contrary that there exists $x_0 \in S(\lambda_0)$ such that $L(\lambda, x)$ is not u.s.c. at (λ_0, x_0). Then, there exist a neighborhood U with $L(\lambda_0, x_0) \subseteq U$ and a sequence $\{(\lambda_n, x_n)\}$ with $(\lambda_n, x_n) \to (\lambda_0, x_0)$ such that $L(\lambda_n, x_n) \nsubseteq U$. Therefore, there exists a sequence $\{y_n\}$ with $y_n \in L(\lambda_n, x_n)$ and

$$y_n \notin U, \quad \forall n \in \mathbb{N}. \tag{15}$$

Since S is u.s.c. and compact-valued at λ_0, by Lemma 2.5, there exist $y_0 \in S(\lambda_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$. Without loss of generality, let $y_n \to y_0$. It follows from $y_n \in L(\lambda_n, x_n)$ that

$$F(y_n) \leq^m F(x_n), \quad \forall (\lambda_n, x_n) \in \Lambda \times X.$$

By Proposition 2.1 we have $F(y_0) \leq^m F(x_0)$. and so $y_0 \in L(\lambda_0, x_0)$. Therefore, from $y_n \to y_0$, we have $y_n \in U$ for large enough n, which contradicts (15). Therefore, $L(\lambda)$ is u.s.c. at $\lambda_0 \times S(\lambda_0)$. □

Theorem 3.3 *Let $\lambda_0 \in \Lambda$. Assume that*

(i) S is continuous at λ_0, S is compact-valued on Λ and $S(\lambda_0)$ is an arcwise connected subset of X;

(ii) F is locally Lipschitz continuous on $S(\lambda_0)$ with nonempty and closed values;

(iii) F is strictly m-quasiconnected on $S(\lambda_0)$.

Then, $E_m(\lambda)$ is l.s.c. at λ_0.

Proof We conclude from Lemma 2.6 that $E_m(\lambda_0) \neq \emptyset$. Now, we assert that $E_m(\lambda)$ is l.s.c. at λ_0. By the contrary, we suppose that $E_m(\lambda)$ is not l.s.c. at λ_0, then there exists a point $y \in E_m(\lambda_0)$, along with a neighborhood U of 0 in X. Additionally, there is a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$(y + U) \cap E_m(\lambda_n) = \emptyset, \quad \forall n \in N. \tag{16}$$

It follows from $y \in E_m(\lambda_0)$ that $y \in S(\lambda_0)$. Because S is l.s.c. at λ_0, according to Lemma 2.4, there exists a sequence $\{y_n\}$ with $y_n \in S(\lambda_n)$ such that $y_n \to y$. By Lemma 2.7, we have that $L(\lambda_0, y) = \{y\}$. Then $L(\lambda_0, y) \subseteq y + U$. By Lemma 3.1 we know that $L(\cdot,\cdot)$ u.s.c. at (λ_0, y). Then for large enough n,

$$L(\lambda_n, y_n) \subseteq y + U. \tag{17}$$

It is clear that $y_n \in L(\lambda_n, y_n)$ and so $L(\lambda_n, y_n) \neq \emptyset$. By Proposition 2.3 we know that $L(\lambda_n, y_n)$ is closed. Noting that $S(\lambda_n)$ is compact and $L(\lambda_n, y_n) \subseteq S(\lambda_n)$, we get that $L(\lambda_n, y_n)$ is compact. It follows from Lemma 2.6 that $E_m(L(\lambda_n, y_n)) \neq \emptyset$. Let $z_n \in E_m(L(\lambda_n, y_n))$, it follows from (17) that

$$z_n \in E_m\big(L(\lambda_n, y_n)\big) \subseteq L(\lambda_n, y_n) \subseteq y + U, \tag{18}$$

for large enough n. By Remark 2.2, we get

$$z_n \in E_m\big(L(\lambda_n, y_n)\big) \subseteq E_m(\lambda_n), \tag{19}$$

for large enough n. It follows from (18) and (19) that $z_n \in (y + U) \cap E_m(\lambda_n)$ for large enough n, which contradicts (16). Therefore, $E_m(\lambda)$ is l.s.c. at λ_0. □

Lastly, we demonstrate the lower semicontinuity of the weak m-minimal solution for the (PSOP) problem, as outlined below:

Theorem 3.4 *Let $\lambda_0 \in \Lambda$. Assume that*

(i) S is continuous at λ_0, S is compact-valued on Λ and $S(\lambda_0)$ is an arcwise connected subset of X;

(ii) F is locally Lipschitz continuous on $S(\lambda_0)$ with nonempty and closed values;

(iii) F is strictly m-quasiconnected on $S(\lambda_0)$.

Then, $W_m(\lambda)$ is l.s.c. at λ_0.

Proof Since $E_m(S) \subseteq W_m(S)$. It follows from Lemma 2.6 that $W_m(\lambda_0) \neq \emptyset$. Now, we assert that $W_m(\lambda)$ is l.s.c. at λ_0. By Lemma 2.3, it follows that $W_m(\lambda_0) = E_m(\lambda_0)$. For any

$y_0 \in W_m(\lambda_0)$, we get $y_0 \in E_m(\lambda_0)$. By Theorem 3.3, we know $E_m(\lambda)$ is l.s.c. at λ_0. For any neighborhood U of y_0, there exists a neighborhood V of λ_0 such that

$$U \cap E_m(\lambda) \neq \emptyset, \quad \forall \lambda \in V.$$

Using Remark 2.1, we have $E_m(\lambda) \subseteq W_m(\lambda)$.

$$U \cap W_m(\lambda) \neq \emptyset, \quad \forall \lambda \in V.$$

This shows that $W_m(\lambda)$ is l.s.c. at λ_0. □

Now, we present an example to exemplify Theorem 3.4.

Example 3.2 Let $X = \mathbb{R}$, $\Lambda = [0,1]$, $Y = \mathbb{R}^2$, and $K = \mathbb{R}^2_+$. Assume that $S(\lambda) = [1,2]$ for all $\lambda \in \Lambda$. A set-valued mapping $F : X \to 2^Y$ can be defined as follows:

$$F(x) = [1,x] \times [1,x], \quad \forall x \in S.$$

Let $\lambda_0 = 0$, then it is straightforward to verify that all the conditions stated in Theorem 3.4 are met. Through a straightforward computation, we deduce that $W_m(\lambda) = [1,2]$ for all $\lambda \in \Lambda$. Evidently, we observe that $W_m(\lambda)$ is l.s.c. at 0.

Remark 3.1 According to Remark 2.1, it is evident that the semicontinuity exhibited by objective set-valued mappings in [7, 12–16, 21] is distinct from the local Lipschitz continuity employed for objective set-valued mappings in Theorems 3.1 to 3.4.

Remark 3.2 The objective set-valued mappings with nonempty and closed values used in Theorems 3.3-3.4 are weaker than the objective set-valued mappings with nonempty and compact values used in [7, 12–16, 21]. That $S(\lambda_0)$ is an arcwise connected subset, which is also used in Theorems 3.3–3.4 is weaker than $S(\lambda_0)$, a convex subset appearing in [7, 12–16, 21].

Remark 3.3 By Lemma 2.2, we can know that $E_l(S) \subseteq W_l(S) \subseteq W_m(S)$. Hence, we have studied Theorems 3.1 and Theorems 3.4 for a larger set than any in [7, 12–16, 21].

Author contributions

The main theorems are proved by Yang and Li. Li and Xu drafted the manuscript, read and approved the final manuscript.

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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References

1. Karaman, E., Soyertem, M., Güvenç, I.A., Tozkan, D., Küçük, M., Küçük, Y.: Partial order relations on family of sets and scalarizations for set optimization. Positivity **22**(3), 783–802 (2018)
2. Khan, A.A., Tammer, C., Zălinescu, C.: Set-Valued Optimization. Springer, Heidelberg (2015)
3. Dontchev, A.L., Zolezzi, T.: In: Well-Posed Optimization Problems. Lecture Notes in Mathematics, vol. 1543. Springer, Berlin (1993).
4. Durea, M.: Scalarization for pointwise well-posedness vectorial problems. Math. Methods Oper. Res. **66**, 409–418 (2007)
5. Gupta, M., Srivastava, M.: Approximate solutions and Levitin-Polyak well-posedness for set optimization using weak efficiency. J. Optim. Theory Appl. **186**, 191–208 (2020)
6. Han, Y.: Painlevé-Kuratowski convergences of the solution sets for set optimization problems with cone quasiconnectedness. Optimization **71**, 2185–2208 (2022)
7. Han, Y., Huang, N.J.: Well-posedness and stability of solutions for set optimization problems. Optimization **66**, 17–33 (2017)
8. Jahn, J., Ha, T.X.D.: New order relations in set optimization. J. Optim. Theory Appl. **148**, 209–236 (2011)
9. Tanino, T.: Stability and sensitivity analysis in convex vector optimization. SIAM J. Control Optim. **26**, 524–536 (1988)
10. Khushboo, C.S., Lalitha, C.S.: A unified minimal solution in set optimization. J. Glob. Optim. **74**, 195–211 (2019)
11. Pallaschke, D., Urbanski, R.: Pairs of Compact Convex Sets. Kluwer Academic, Dordrecht (2002)
12. Xu, Y.D., Li, S.J.: Continuity of the solution set mappings to a parametric set optimization problem. Optim. Lett. **8**, 2315–2327 (2014)
13. Xu, Y.D., Li, S.J.: On the solution continuity of parametric set optimization problems. Math. Methods Oper. Res. **84**, 223–237 (2016)
14. Karuna, Lalitha, C.S.: Continuity of approximate weak efficient solution set map in parametric set optimization. J. Nonlinear Convex Anal. **19**, 1247–1262 (2018)
15. Khoshkhabar-amiranloo, S.: Stability of minimal solutions to parametric set optimization problems. Appl. Anal. **97**, 2510–2522 (2018)
16. Li, T.Y., Wei, Y.H.: The stability of the minimal solution sets for set optimization problems via improvement sets. J. Inequal. Appl., **75** (2023). https://doi.org/10.1186/s13660-023-02982-6
17. Zhang, C.L., Huang, N.: Well-posedness and stability in set optimization with applications. Positivity **25**, 1153–1173 (2021)
18. Anh, L.Q., Duy, T.Q., Hien, D.V.: Convergence of solutions to set optimization problems with the set less order relation. J. Optim. Theory Appl. **185**, 416–432 (2020)
19. Karuna, Lalitha, C.S.: External and internal stability in set optimization. Optimization **68**, 833–852 (2019)
20. Preechasilp, P., Wangkeeree, R.: A note on semicontinuity of the solution mapping for parametric set optimization problems. Optim. Lett. **13**, 1085–1094 (2019)
21. Khushboo, Lalitha, C.S.: Scalarizations for a set optimization problem using generalized oriented distance function. Positivity **23**, 1195–1213 (2019)
22. Ansari, Q.H., Hussain, N., Sharma, P.K.: Convergence of the solution sets for set optimization problems. J. Nonlinear Var. Anal. **6**, 165–183 (2022)

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