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Global existence and attractivity for Riemann-Liouville fractional semilinear evolution equations involving weakly singular integral inequalities

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Abstract

In this paper, we obtain several results on the global existence, uniqueness and attractivity for fractional evolution equations involving the Riemann-Liouville type by exploiting some results on weakly singular integral inequalities in Banach spaces. Some boundedness conditions of the nonlinear term are considered to obtain the main results that generalize and improve some well-known works.

Keywords: Fractional evolution equation; Riemann-Liouville fractional derivative; Existence; Attractivity; Weakly singular integral inequality

1 Introduction

The aim of this paper is to present several results on the global existence, uniqueness and attractivity of the following fractional differential equation:

$$\begin{cases} {}_0^R D_t^\beta x(t) = Ax(t) + f(t,x(t)), & t \in (0,+\infty), \\ {}_0 I_t^{1-\beta} x(t)|_{t=0} = x_0, \end{cases} \tag{1.1}$$

where ${}_0^R D_t^\beta$ is the Riemann-Liouville fractional derivative with the order $\beta \in (0,1)$, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact C_0-semigroup $\{S(t)\}_{t\geq 0}$.

The attractivity of solutions plays a significant role in describing the properties of differential equations. Many researchers have investigated the attractivity of solutions of fractional differential equations. For instance, Furati and Tatar [5] proved that solutions of fractional differential equations with weighted initial data exist globally and decay as a power function. Kassim, Furati, and Tatar [10] studied the asymptotic behavior of solutions for a class of nonlinear fractional differential equations involving two Riemann-Liouville fractional derivatives of different orders. Gallegos and Duarte-Mermoud [6] studied the asymptotic behavior of solutions to Riemann-Liouville fractional systems. Zhou [26] studied the attractivity of solutions for fractional evolution equation with almost sectorial operators. Tuan, Czornik, Nieto and Niezabitowski [22] presented some results for existence of global solutions and attractivity for multidimensional fractional differential equations involving Riemann-Liouville fractional derivative. Sousa, Benchohra, and N'Guérékata [18] considered the attractivity of solutions of the fractional differential equation involving the ψ-Hilfer fractional derivative. For more references, we refer to [1, 15, 19, 20].

Since weakly singular integral inequalities are well-known tools for proving the existence, uniqueness, stability and attractivity of integral evolution equations and fractional differential equations, many scholars have begun to study weakly singular integral inequalities and have obtained several versions of weakly singular integral inequalities. See [3, 8, 9, 12, 13, 16, 21, 23, 28] for more details. Especially, Zhu [28–31] studied several results on the existence and attractivity for the following fractional differential equations with Riemann-Liouville fractional derivative in $\mathbb{R}$:

$$\begin{cases} {}_0^R D_t^\beta x(t) = f(t, x(t)), & t \in (0, +\infty), \\ \lim_{t\to 0^+} t^{1-\beta} x(t) = x_0. \end{cases} \tag{1.2}$$

Zhu presented some weakly singular integral inequalities to prove the main results under the following boundedness conditions

$$\left|f(t,x)\right| \le l(t)|x| + k(t), \tag{1.3}$$

$$\left|f(t,x)\right| \le l(t)|x|^\mu, \tag{1.4}$$

$$\left|f(t,x)\right| \le l(t)|x|^\mu + k(t), \tag{1.5}$$

$$\left|f(t,x)\right| \le l(t)\omega\left(t^{1-\beta}|x|\right), \tag{1.6}$$

where $\mu \in (0,1]$, $l, k \in C((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$ $(p > \frac{1}{\beta})$ and nonnegative nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_+)$ with $\lim_{t\to+\infty} \frac{t}{\omega(t)} = K(0 < K \le +\infty)$.

In this paper, by exploiting the Leray-Schauder alternative fixed point theorem and some weakly singular integral inequalities in Banach spaces, we first prove the existence of global mild solutions of problem (1.1). We also prove that there exists a unique mild solution of problem (1.1). Furthermore, we show that the mild solutions of problem (1.1) are globally attractive. Our results generalize and improve the results existing in literature. Finally, we provide several examples to illustrate the applicability of our results.

Below we will describe some of the new features. First, our problem is the natural generalization of many well-known works on the existence and global attractivity for fractional differential equations in finite-dimensional spaces. Second, some boundedness conditions of the nonlinear term are considered to obtain the main results that generalize and improve some well-know works. Instead of conditions (1.3)–(1.6), we deal with more general conditions in the Banach space:

$$\left\|f(t,x)\right\| \le l(t)\omega\left(t^{1-q}\|x\|\right), \tag{1.7}$$

$$\left\|f(t,x)\right\| \le l(t)\omega\left(t^{1-q}\|x\|\right) + k(t), \tag{1.8}$$

where $l, k \in C_{1-\beta}((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)(p > \frac{1}{\beta})$. Third, we obtain several useful nonlinear weakly singular integral inequalities in Banach spaces, which can also be used to control some problems. Fourth, problem (1.1) reduces the problems of first-order and Caputo fractional semilinear evolution equations and can be generalized to more complex forms, for instance, fractional impulsive evolution equations and fractional evolution inclusions.

The outline of this paper is as follows. In Sect. 2, we introduce some notations, definitions, and useful lemmas. In Sect. 3, we present several nonlinear weakly singular integral inequalities useful to prove the main results. In Sects. 4 and 5, we give some sufficient conditions on the global existence and attractivity of mild solutions of problem (1.1). In Sect. 6, some deduced results are given to illustrate our main results.

2 Preliminaries

In this section, we introduce some notations, definitions and lemmas which will be needed later.

The norm of a Banach space X will be denoted by $\|\cdot\|_X$. For an interval J, let $C(J,X)$ denote the Banach space of all continuous functions from J into X equipped with the norm $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$ and $L^p(J,X)(p>1)$ denote the Banach space of p-th integral functions from J into X equipped with the norm $\|x\|_{L^p} = (\int_J \|x(t)\|_X^p\,dt)^{\frac{1}{p}}$. Let $C_\beta(J,X) = \{x : y(t) = t^\beta x(t), y \in C(J,X)\}$ equipped with the norm $\|x\|_{C_\beta} = \sup\{t^\beta\|x(t)\|_X : t\in J\}$ and let $L^p_\beta(J,X) = \{x : y(t) = t^\beta x(t), y \in L^\beta(J,X)\}$ equipped with the norm $\|x\|_{L^p_\beta} = (\int_J t^\beta\|x(t)\|_X^p\,dt)^{\frac{1}{p}} : t\in J\}$. Obviously, the space $C_\beta(J,X)$, $L^p_\beta(J,X)$ is Banach spaces. For $a\geq 0$, let $C_0([a,+\infty),X) = \{x\in C([a,+\infty),X) : \lim_{t\to+\infty} x(t) = 0\}$. It is clear that $C_0([a,+\infty),X)$ is a Banach space equipped with the norm $\|x\|_0 = \sup_{a\leq t<+\infty}\|x(t)\|$.

Definition 2.1 ([4, 11, 14, 17]) The Riemann-Liouville fractional integral of order $\beta\in(0,1)$ for a function $f:\mathbb{R}_+\to X$ is defined by

$$ {}_0I_t^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s)\,ds, $$

where Γ is the gamma function.

Definition 2.2 ([4, 11, 14, 17]) The Riemann-Liouville fractional derivative of order $\beta\in(0,1)$ for a function $f:\mathbb{R}_+\to X$ is defined by

$$ {}_0^RD_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-s)^{-\beta} f(s)\,ds. $$

Definition 2.3 ([4, 11, 14, 17]) The Caputo fractional derivative of order $\beta\in(0,1)$ for a function $f:\mathbb{R}_+\to X$ is defined by

$$ {}_0^CD_t^\beta f(t) = {}_0^RD_t^\beta\big(f(t)-f(0)\big). $$

Lemma 2.4 ([2], Corollary 5.3) *Let u, ϕ, ψ and k be nonnegative continuous functions on $[a,b]$. Let ω be a continuous, nonnegative and nondecreasing function on $[0,+\infty)$, with $\omega(r)>0$ for $r>0$, and let $\Phi(t) = \max_{a\leq s\leq t}\phi(s)$ and $\Psi(t) = \max_{a\leq s\leq t}\psi(s)$. Assume that*

$$ u(t) \leq \phi(t) + \psi(t)\int_a^t k(s)\omega\big(u(s)\big)\,\mathrm{d}s, \quad \forall t\in[a,b]. \tag{2.1} $$

Then

$$u(t) \le W^{-1}\left[W\big(\Phi(t)\big) + \Psi(t)\int_a^t k(s)\,\mathrm{d}s\right], \quad \forall t \in [a,T), \tag{2.2}$$

where $W(u) = \int_{u_0}^{u} \frac{1}{\omega(\tau)}\,d\tau$, $u_0, u > 0$, W^{-1} *is the inverse of* W *and*

$$T = \sup\left\{\tau \in [a,b] : W\big(\Phi(t)\big) + \Psi(t)\int_a^t k(s)\,\mathrm{d}s \in \mathrm{Dom}\big(W^{-1}\big), a \le t \le \tau\right\}.$$

Lemma 2.5 ([28]) *Let* $1 \le p < \infty$, φ *and* ϕ *be continuous and nonnegative functions on* $[0,\infty)$, *function* $l \in L^p_{Loc}([0,+\infty),\mathbb{R}_+)$, *and* u *be a continuous and nonnegative function with*

$$u(t) \le M(t) + \phi(t)\left(\int_0^t l^p(s)u^p(s)\,\mathrm{d}s\right)^{\frac{1}{p}}, \quad \forall t \in [0,\infty). \tag{2.3}$$

Then

$$u(t) \le M(t) + \phi(t)\left\{M(t)\exp\left(\int_0^t L(s)\,\mathrm{d}s\right)\right\}^{\frac{1}{p}}, \quad \forall t \in [0,\infty), \tag{2.4}$$

where $M(t) = \int_0^t 2^{p-1}l^p(s)\varphi^p(s)\,\mathrm{d}s$ *and* $L(t) = 2^{p-1}l^p(t)\phi^p(t)$.

Lemma 2.6 ([28]) *Let* $0 < \beta < 1$, $p > \frac{1}{\beta}$, $q = \frac{p}{p-1}$, $\rho \in L^p_{1-\beta}([0,1],\mathbb{R})$. *Then*

$$\left|\int_0^t \left(\frac{t}{t-s}\right)^{1-\beta}\rho(s)\,ds\right| \le \frac{2^{\frac{1}{q}}t^{\beta-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^t s^{p(1-\beta)}\big|\rho(s)\big|^p\,ds\right)^{\frac{1}{p}}, \quad \forall t \in [0,1]. \tag{2.5}$$

Lemma 2.7 ([29]) *Let* $0 < \beta < 1$, $p > \frac{1}{\beta}$, $q = \frac{p}{p-1}$, $\rho \in L^p_{1-\beta}([0,1],\mathbb{R})$. *Then*

$$\left|\int_0^t \left(\frac{t}{t-s}\right)^{1-\beta}\rho(s)\,\mathrm{d}s\right| \le \frac{2^{\frac{1}{q}}t^{\beta-1+\frac{1}{q}}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^t s^{p(1-\beta)}\big|\rho(s)\big|^p\,\mathrm{d}s\right)^{\frac{1}{p}}, \quad \forall t \in [0,1], \tag{2.6}$$

and if $0 < t_1 \le t_2 \le 1$, *then*

$$\begin{aligned}
&\left|\int_0^{t_2}\left(\frac{t_2}{t_2-s}\right)^{1-\beta}\rho(s)\,\mathrm{d}s - \int_0^{t_1}\left(\frac{t_1}{t_1-s}\right)^{1-\beta}\rho(s)\,\mathrm{d}s\right| \\
&\quad\le \frac{2^{\frac{1}{q}}(t_2-t_1)^{\beta-1+\frac{1}{q}}}{(qq-q+1)^{\frac{1}{q}}}\left(\int_{t_1}^{t_2} s^{p(1-\beta)}\big|\rho(s)\big|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\qquad + \left(\frac{(t_2-t_1)^{1+q(\beta-1)} + t_1^{1+q(\beta-1)} - t_2^{1+q(\beta-1)}}{qq-q+1}\right)^{\frac{1}{q}} \times \left(\int_0^{t_1} s^{p(1-\beta)}\big|\rho(s)\big|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Lemma 2.8 ([31]) *Let* $0 < \beta < 1$, $p > \frac{1}{\beta}$, $\rho \in L^p_{Loc,1-\beta}([1,+\infty),\mathbb{R})$ *and*

$$y(t) = \int_1^t (t-s)^{\beta-1}\rho(s)\,ds.$$

Then $y \in C([1,+\infty),\mathbb{R})$.

Lemma 2.9 ([31]) *Let $0 < \beta < 1$, $p > \frac{1}{\beta}$, $\rho \in L^p_{1-\beta}([0,1],\mathbb{R})$ and*

$$y(t) = \int_0^1 (t-s)^{\beta-1}\rho(s)\,ds.$$

Then $y \in C([1,+\infty),\mathbb{R})$.

3 Nonlinear weakly singular integral inequalities

In this section, we study some nonlinear weakly singular integral inequalities that will be useful to prove the main results.

Lemma 3.1 *Let $a,b \ge 0$, $1 > \alpha \ge \delta \ge 0$ and $0 < \beta < 1$, $p > \max\{\frac{1}{\beta}, \frac{1}{1-\alpha+\delta}\}$ and $q = \frac{p}{p-1}$. Let $f : (0,T) \times X \to X$ be a continuous function, and there exists a function $l \in C((0,T),\mathbb{R}_+) \cap L^p_{Loc,\alpha-\delta}([0,T),\mathbb{R}_+)$ and a nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_+)$ such that*

$$\left\| f(t,x)\right\| \le l(t)\omega\left(t^\alpha \|x\|\right), \quad \forall (t,x) \in (0,T) \times X.$$

Let $u \in C_\alpha([0,T),\mathbb{R}_+)$ with

$$\left\| u(t)\right\| \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}\left\| f\left(s,u(s)\right)\right\| \mathrm{d}s, \quad \forall t \in (0,T). \tag{3.1}$$

Then

$$\left\| u(t)\right\| \le t^{-\alpha}\left\{ W^{-1}\left[W\left(2^{p-1}a^p\right) + 2^{p-1}c^p t^{p\beta-1}\int_0^t s^{p(\alpha-\delta)} l^p(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{p}}, \quad \forall t \in (0,T_1), \tag{3.2}$$

where $c = \frac{b\Gamma^{\frac{1}{q}}(q(\beta-1)+1)\Gamma^{\frac{1}{q}}(q(\delta-\alpha)+1)}{\Gamma^{\frac{1}{q}}(q(\beta-1)+q(\delta-\alpha)+2)}$, $W(u) = \int_{u_0}^u \frac{1}{\omega^p(\tau^{1/p})}\,d\tau$, $u_0, u > 0$ and

$$\begin{aligned} T_1 = \sup\Big\{ &\tau \in (0,T) : W\left(2^{p-1}a^p\right) + 2^{p-1}c^p t^{p\beta-1} \\ &\times \int_0^t s^{p(\alpha-\delta)} l^p(s)\,\mathrm{d}s \in \mathrm{Dom}\left(W^{-1}\right), 0 < t \le \tau \Big\}. \end{aligned}$$

Proof For $t \in (0,T)$, let $v(t) = t^\alpha u(t)$. We get

$$\left\| v(t)\right\| \le a + bt^{\alpha-\delta}\int_0^t (t-s)^{\beta-1}\left\| f\left(s,s^{-\alpha}v(s)\right)\right\| \mathrm{d}s. \tag{3.3}$$

Using the Hölder inequality, we obtain

$$\begin{aligned} \left\| v(t)\right\| &\le a + bt^{\alpha-\delta}\int_0^t (t-s)^{\beta-1} l(s)\omega\left(\left\| v(s)\right\|\right)\mathrm{d}s \\ &= a + bt^{\alpha-\delta}\int_0^t (t-s)^{\beta-1}s^{\delta-\alpha}s^{\alpha-\delta} l(s)\omega\left(\left\| v(s)\right\|\right)\mathrm{d}sr \\ &\le a + bt^{\alpha-\delta}\left(\int_0^t (t-s)^{q(\beta-1)+1-1}s^{q(\delta-\alpha)}\,\mathrm{d}s\right)^{\frac{1}{q}} \times \left(\int_0^t s^{p(\alpha-\delta)} l^p(s)\omega^p\left(\left\| v(s)\right\|\right)\mathrm{d}s\right)^{\frac{1}{p}} \\ &\le a + ct^{q-\frac{1}{p}}\left(\int_0^t s^{p(\alpha-\delta)} l^p(s)\omega^p\left(\left\| v(s)\right\|\right)\mathrm{d}s\right)^{\frac{1}{p}}. \end{aligned} \tag{3.4}$$

Then

$$\left\|v(t)\right\|^{p} \leq 2^{p-1}a^{p} + 2^{p-1}c^{p}t^{p\beta-1}\int_{0}^{t} s^{p(\alpha-\delta)}l^{p}(s)\omega^{p}\left(\left\|v(s)\right\|\right)\mathrm{d}s. \tag{3.5}$$

Let $\mu(t) = \|v(t)\|^{p}$. Then

$$\mu(t) \leq 2^{p-1}a^{p} + 2^{p-1}c^{p}t^{p\beta-1}\int_{0}^{t} s^{p(\alpha-\delta)}l^{p}(s)\omega^{p}\left(\mu^{\frac{1}{p}}(s)\right)\mathrm{d}s. \tag{3.6}$$

Using Lemma 2.4, we obtain

$$\mu(t) \leq W^{-1}\left[W\left(2^{p-1}a^{p}\right) + 2^{p-1}c^{p}t^{p\beta-1}\int_{0}^{t} s^{p(\alpha-\delta)}l^{p}(s)\,\mathrm{d}s\right], \tag{3.7}$$

and

$$\left\|u(t)\right\| \leq t^{-\alpha}\left\{W^{-1}\left[W\left(2^{p-1}a^{p}\right) + 2^{p-1}c^{p}t^{p\beta-1}\int_{0}^{t} s^{p(\alpha-\delta)}l^{p}(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{p}}. \tag{3.8}$$

Thus, we complete the proof. □

The following conclusion is a consequence of Lemma 3.1 when $\alpha = 1-\beta$ and $\delta = 0$.

Corollary 3.2 *Let $a, b \geq 0$ and $0 < \beta < 1$, $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. Let $f : (0,T) \times X \to X$ be a continuous function, and there exists a function $l \in C((0,T),\mathbb{R}_{+})) \cap L^{p}_{Loc,1-\beta}([0,T),\mathbb{R}_{+})$ and a nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_{+})$ such that*

$$\left\|f(t,x)\right\| \leq l(t)\omega\left(t^{1-\beta}\|x\|\right), \quad \forall (t,x) \in (0,T) \times X.$$

Let $t^{1-\beta}u(t)$ be a continuous, nonnegative function on $[0,T)$ with

$$\left\|u(t)\right\| \leq at^{\beta-1} + b\int_{0}^{t}(t-s)^{\beta-1}\left\|f\left(s,u(s)\right)\right\|\mathrm{d}s, \quad \forall t \in (0,T). \tag{3.9}$$

Then

$$\left\|u(t)\right\| \leq t^{\beta-1}\left\{W^{-1}\left[W\left(2^{p-1}a^{p}\right) + 2^{p-1}c^{p}t^{p\beta-1} \times \int_{0}^{t} s^{p(1-\beta)}l^{p}(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{p}}, \quad \forall t \in (0,T_1), \tag{3.10}$$

where $c = \frac{b\Gamma^{\frac{2}{q}}(q(\beta-1)+1)}{\Gamma^{\frac{1}{q}}(2q(\beta-1)+2)}$, $W(u) = \int_{u_0}^{u}\frac{1}{\omega^{p}(\tau^{1/p})}\,d\tau$, $u_0, u > 0$ and

$$T_1 = \sup\left\{\tau \in (0,T) : W\left(2^{p-1}a^{p}\right) + 2^{p-1}c^{p}t^{p\beta-1} \times \int_{0}^{t} s^{p(1-\beta)}l^{p}(s)\,\mathrm{d}s \in \mathrm{Dom}\left(W^{-1}\right), 0 < t \leq \tau\right\}.$$

We can also obtain the following results.

Lemma 3.3 *Let $a, b \geq 0$, $1 > \alpha \geq \delta \geq 0$ and $0 < \beta < 1$, $p > \max\{\frac{1}{\beta}, \frac{1}{1-\alpha+\delta}\}$ and $q = \frac{p}{p-1}$. Let l be a nondecreasing continuous function on $(0, +\infty)$ with $l \in L^p_{Loc,-\delta}([0, +\infty), \mathbb{R}_+))$. Let $t^\alpha u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$\|u(t)\| \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) \|u(s)\| \,\mathrm{d}s, \quad \forall t \in (0, +\infty). \tag{3.11}$$

Then

$$\|u(t)\| \leq at^{-\alpha} + ct^{q-\alpha-\frac{1}{p}} \left\{ M(t) \exp\left(\int_0^t L(s) \,\mathrm{d}s \right) \right\}^{\frac{1}{p}}, \quad \forall t \in (0, +\infty), \tag{3.12}$$

where $M(t) = 2^{p-1} a^p \int_0^t s^{-p\delta} l^p(s) \,\mathrm{d}s$, $L(t) = 2^{p-1} c^p t^{pq-p\delta-1} l^p(t)$, and c is defined as in Lemma 3.1.

Proof Let $v(t) = t^\alpha u(t)$. Using (3.11) and the same procedure as in (3.4), we get

$$\|v(t)\| \leq a + bt^{\alpha-\delta} \int_0^t (t-s)^{\beta-1} l(s) s^{-\alpha} \|v(s)\| \,\mathrm{d}s, \tag{3.13}$$

and

$$\|v(t)\| \leq a + ct^{q-\frac{1}{p}} \left(\int_0^t s^{-p\delta} l^p(s) \|v(s)\|^p \,\mathrm{d}s \right)^{\frac{1}{p}}. \tag{3.14}$$

It follows from Lemma 2.5 that

$$\|v(t)\| \leq a + ct^{q-\frac{1}{p}} \left\{ M(t) \exp\left(\int_0^t L(s) \,\mathrm{d}s \right) \right\}^{\frac{1}{p}}, \quad \forall t \in [0, +\infty), \tag{3.15}$$

which completes the proof. □

Lemma 3.4 *Let $a, b \geq 0$, $1 > \alpha \geq \delta \geq 0$, $0 < \gamma < 1$ and $0 < \beta < 1$, $p > \max\{\frac{1}{\beta}, \frac{1}{1-\alpha+\delta}\}$ and $q = \frac{p}{p-1}$. Let l be a nonnegative nondecreasing continuous function on $(0, +\infty)$ with $l \in L^p_{Loc,(1-\gamma)\alpha-\delta}[0, +\infty)$. Let $t^\alpha u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$\|u(t)\| \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) \|u(s)\|^\gamma \,\mathrm{d}s, \quad \forall t \in (0, +\infty). \tag{3.16}$$

Then

$$\|u(t)\| \leq t^{-\alpha} \left[2^{(p-1)(1-\gamma)} a^{p(1-\gamma)} + (1-\gamma) 2^{p-1} c^p t^{p\beta-1} \times \int_0^t s^{p(1-\gamma)\alpha - p\delta} l^p(s) \,\mathrm{d}s \right]^{\frac{1}{p(1-\gamma)}}, \tag{3.17}$$

for all $t \in (0, +\infty)$, where c is defined as in Lemma 3.1.

Proof Let $v(t) = t^\alpha u(t)$. Using (3.16) and the same procedure as in (3.4), we get

$$\|v(t)\| \leq a + bt^{\alpha-\delta} \int_0^t (t-s)^{\beta-1} l(s) s^{-\gamma\alpha} \|v(s)\|^\gamma \,\mathrm{d}s, \tag{3.18}$$

and

$$\|v(t)\| \leq a + ct^{q-\frac{1}{p}}\left(\int_0^t s^{p(1-\gamma)\alpha-p\delta} l^p(s)\|v(s)\|^{p\gamma}\,\mathrm{d}s\right)^{\frac{1}{p}}. \tag{3.19}$$

Then from (3.19), we know

$$\|v(t)\|^p \leq 2^{p-1}a^p + 2^{p-1}c^p t^{p\beta-1}\int_0^t s^{p(1-\gamma)\alpha-p\delta} l^p(s)\|v(s)\|^{p\gamma}\,\mathrm{d}s. \tag{3.20}$$

Using Lemma 2.4, we get

$$\|v(t)\|^p \leq \left[2^{(p-1)(1-\gamma)}a^{p(1-\gamma)} + (1-\gamma)2^{p-1}c^p t^{p\beta-1}\int_0^t s^{p(1-\gamma)\alpha-p\delta} l^p(s)\,\mathrm{d}s\right]^{\frac{1}{1-\gamma}}. \tag{3.21}$$

Thus, we complete the proof. □

4 Global existence

In this section, we present the existence and uniqueness results for problem (1.1).

Definition 4.1 A function $x \in C_{1-\beta}((0,T],X)$ is called a mild solution of problem (1.1) if it satisfies the following fractional integral equation

$$x(t) = t^{\beta-1}S_\beta(t)x_0 + \int_0^t (t-s)^{\beta-1}S_\beta(t-s)f\big(s,x(s)\big)\,ds, \quad \forall t \in [0,T],$$

where

$$S_\beta(t) = \beta\int_0^\infty \theta\xi_\beta(\theta)S\big(t^\beta\theta\big)\,d\theta, \qquad \xi_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\varpi_\beta\big(\theta^{-\frac{1}{\beta}}\big),$$

$$\varpi_\beta(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\beta-1}\frac{\Gamma(n\beta+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

$$\xi_\beta(\theta) \geq 0, \quad \theta \in (0,\infty), \int_0^\infty \xi_\beta(\theta)\,d\theta = 1.$$

Lemma 4.2 ([16, 27]) *If the C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded, then the operator $S_\beta(t)$ has the following properties:*

(i)

$$\|S_\beta(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|, \quad \forall x \in X, t \geq 0,$$

where $\sup_{t\in[0,\infty)}\|T(t)\| \leq M < \infty$;

(ii) $S_\beta(t)(t \geq 0)$ is strongly continuous;

(iii) $S_\beta(t)(t > 0)$ is compact if $S(t)(t > 0)$ is compact.

Theorem 4.3 *Let $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. Suppose $f : (0,T] \times X \to X$ is a continuous function, and there exists a function $l \in C_{1-\beta}((0,T],\mathbb{R}_+) \cap L^p_{1-\beta}([0,T],\mathbb{R}_+)$ and a nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_+)$ such that*

$$\|f(t,x)\| \leq l(t)\omega\big(t^{1-\beta}\|x\|\big), \quad \forall (t,x) \in (0,T] \times X.$$

Then problem (1.1) has at least one mild solution in $C_{1-\beta}((0,T],X)$ provided that

$$W\big(2^{p-1}\|x_0\|^p\big)+2^{p-1}c^p t^{p\beta-1}\int_0^t s^{p(1-\beta)}l^p(s)\,\mathrm{d}s\in \mathrm{Dom}\big(W^{-1}\big),\quad \forall t\in(0,T],$$

where $c=\frac{M\Gamma^{\frac{2}{q}}(q(\beta-1)+1)}{\Gamma(q)\Gamma^{\frac{1}{q}}(2q(\beta-1)+2)}$, $W(u)=\int_{u_0}^{u}\frac{1}{\omega^p(\tau^{1/p})}\,d\tau$, $u_0,u>0$.

Proof Define the operator $G:C_{1-\beta}((0,T],X)\to C_{1-\beta}((0,T],X)$ by

$$Gx(t)=t^{\beta-1}S_\beta(t)x_0+\int_0^t(t-s)^{\beta-1}S_\beta(t-s)f\big(s,x(s)\big)\,ds. \tag{4.1}$$

Step 1. We will prove that G is compact. To see this, let $\Omega\in C_{1-\beta}((0,T],X)$ be bounded and $\|x\|_{1-\beta}\le R$ for each $x\in\Omega$ with some $R>0$. We will show that $t^{1-\beta}G(\Omega)$ is uniformly bounded and equicontinuous on $[0,T]$. First, we prove that $t^{1-\beta}G(\Omega)$ is uniformly bounded. For $x\in\Omega$, we have

$$\begin{aligned}
\big\|t^{1-\beta}Gx(t)\big\| &\le \frac{M}{\Gamma(q)}\|x_0\|+\frac{Mt^{1-\beta}}{\Gamma(q)}\bigg\|\int_0^t(t-s)^{\beta-1}f\big(s,x(s)\big)\,\mathrm{d}s\bigg\|\\
&\le \frac{M}{\Gamma(q)}\|x_0\|+\frac{M}{\Gamma(q)}\bigg(\int_0^t\Big(\frac{1}{t-s}+\frac{1}{s}\Big)^{q(1-\beta)}\mathrm{d}s\bigg)^{\frac{1}{q}}\\
&\quad\times\bigg(\int_0^t s^{p(1-\beta)}l^p(s)\omega^p\big(s^{1-\beta}\big\|x(s)\big\|\big)\,\mathrm{d}s\bigg)^{\frac{1}{p}}\\
&\le \frac{M}{\Gamma(q)}\|x_0\|+\frac{2^{\frac{1}{q}M}\omega(R)t^{\beta-1+\frac{1}{q}}}{\Gamma(q)(q(\beta-1)+1)^{\frac{1}{q}}}\bigg(\int_0^t s^{p(1-\beta)}l^p(s)\,\mathrm{d}s\bigg)^{\frac{1}{p}}\\
&\le \frac{M}{\Gamma(q)}\|x_0\|+\frac{2^{\frac{1}{q}M}\omega(R)T^{\beta-1+\frac{1}{q}}}{\Gamma(q)(q(\beta-1)+1)^{\frac{1}{q}}}\bigg(\int_0^T s^{p(1-\beta)}l^p(s)\,\mathrm{d}s\bigg)^{\frac{1}{p}}.
\end{aligned}$$

This proves that the set $t^{1-\beta}G(\Omega)$ is uniformly bounded. Second, we prove that $t^{1-\beta}G(\Omega)$ is an equicontinuous family. For any $x\in\Omega$, let $0\le t_1<t_2\le T$, we get

$$\begin{aligned}
&\big\|t_2^{1-\beta}Gx(t_2)-t_1^{1-\beta}Gx(t_1)\big\|\\
&=\frac{M}{\Gamma(q)}\bigg\|\int_0^{t_2}\Big(\frac{t_2}{t_2-s}\Big)^{1-\beta}f\big(s,x(s)\big)\,\mathrm{d}s-\int_0^{t_1}\Big(\frac{t_1}{t_1-s}\Big)^{1-\beta}f\big(s,x(s)\big)\,\mathrm{d}s\bigg\|\\
&\le\frac{M}{\Gamma(q)}\bigg\|\int_0^{t_1}\Big(\Big(\frac{t_2}{t_2-s}\Big)^{1-\beta}-\Big(\frac{t_1}{t_1-s}\Big)^{1-\beta}\Big)f\big(s,x(s)\big)\,\mathrm{d}s\bigg\|\\
&\quad+\frac{M}{\Gamma(q)}\bigg\|\int_{t_1}^{t_2}\Big(\frac{t_2}{t_2-s}\Big)^{1-\beta}f\big(s,x(s)\big)\,\mathrm{d}s\bigg\|.
\end{aligned}\tag{4.2}$$

Since $\|f(t,x(t))\|\le l(t)\omega(t^{1-\beta}\|x(t)\|)\le l(t)\omega(R)$ and $l\in C_{1-\beta}((0,T],\mathbb{R}_+)\cap L^p_{1-\beta}([0,T],\mathbb{R}_+)$. By Lemma 2.7, we know that the right-hand side of (4.2) tends to zero as $t_2\to t_1$. Therefore, $t^{1-\beta}G(\Omega)$ is an equicontinuous family. From Lemma 4.2, it follows that $t^{1-\beta}G(\Omega)$ is relatively compact for each $t\in[0,T]$.

Step 2. We now show that G is continuous. Let $x_n \to x$ in $C_{1-\beta}((0,T],X)$. Then there exists $r > 0$ such that $\|x_n\|_{1-\beta} \le r$ and $\|x\|_{1-\beta} \le r$. For every $s \in (0,T]$, we have

$$f\bigl(s,x_n(s)\bigr) \to f\bigl(s,x(s)\bigr) \quad \text{as } n \to +\infty$$

and

$$\left(\frac{t}{t-s}\right)^{1-\beta} \bigl\|f\bigl(s,x_n(s)\bigr) - f\bigl(s,x(s)\bigr)\bigr\| \le 2\omega(r)\left(\frac{t}{t-s}\right)^{1-\beta} l(s). \tag{4.3}$$

Since $l \in C_{1-\beta}((0,T],\mathbb{R}) \cap L^p_{1-\beta}([0,T],\mathbb{R})$, using (2.6) in Lemma 2.7, we know that the function

$$s \to 2\omega(r)\left(\frac{t}{t-s}\right)^{1-\beta} l(s)$$

is integrable for $s \in (0,t)$. Then we deduce that

$$\left\| \int_0^t \left(\frac{t}{t-s}\right)^{1-\beta} \bigl[f\bigl(s,x_n(s)\bigr) - f\bigl(s,x(s)\bigr)\bigr]\,\mathrm{d}s \right\| \to 0 \quad \text{as } n \to +\infty.$$

Therefore, $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ pointwise on $[0, T]$ as $n \to +\infty$. With the fact that G is compact, we get that $G : C_{1-\beta}((0,T],X) \to C_{1-\beta}((0,T],X)$ is continuous.

Step 3. We shall prove that the set $\Lambda = \{x \in C_{1-\beta}((0,T],X) : x = \lambda Gx \text{ for some } 0 < \lambda < 1\}$ is bounded. Indeed, for $x \in \Lambda$, one has

$$\begin{aligned} \bigl\|x(t)\bigr\| &\le \frac{M}{\Gamma(q)} t^{\beta-1}\|x_0\| + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{\beta-1} \bigl\|f\bigl(s,x(s)\bigr)\bigr\|\,\mathrm{d}s \\ &\le \frac{M}{\Gamma(q)} t^{\beta-1}\|x_0\| + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{\beta-1} l(s)\omega\bigl(s^{1-\beta}\bigl\|x(s)\bigr\|\bigr)\,\mathrm{d}s. \end{aligned}$$

Using Corollary 3.2, we obtain

$$\begin{aligned} \bigl\|x(t)\bigr\| \le \frac{M}{\Gamma(q)} t^{\beta-1} \left\{ W^{-1}\left[W\bigl(2^{p-1}\|x_0\|^p\bigr) + 2^{p-1}c^p t^{p\beta-1} \int_0^t s^{p(1-\beta)} l^p(s)\,\mathrm{d}s \right] \right\}^{\frac{1}{p}}, \\ \forall t \in (0,T], \end{aligned} \tag{4.4}$$

and

$$\|x\|_{1-\beta} \le \frac{M}{\Gamma(q)} \left\{ W^{-1}\left[W\bigl(2^{p-1}\|x_0\|^p\bigr) + 2^{p-1}c^p T^{p\beta-1} \int_0^T s^{p(1-\beta)} l^p(s)\,\mathrm{d}s \right] \right\}^{\frac{1}{p}}. \tag{4.5}$$

Then the set Λ is bounded.

Finally, by applying the fixed point theorem in Theorem 6.5.4 in [7], the operator G has a fixed point $x \in C_{1-\beta}((0,T],X)$, which is the mild solution of problem (1.1). □

Now we investigate the existence of global mild solutions of problem (1.1).

Theorem 4.4 *Let $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. Suppose $f : (0,+\infty) \times X \to X$ is a continuous function, and there exists a nonnegative function $l \in C((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$ and a nonnegative nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_+)$ with $\lim_{t\to+\infty} \frac{t}{\omega(t)} = K(0 < K \leq +\infty)$ such that*

$$\left\|f(t,x)\right\| \leq l(t)\omega\left(t^{1-\beta}\|x\|\right), \quad \forall (t,x) \in (0,+\infty) \times X.$$

Then problem (1.1) has at least one global mild solution in $C_{1-\beta}((0,+\infty),X)$.

Proof Letting $\mu(t) = \omega^p(t^{\frac{1}{p}})$, we know

$$\lim_{t\to+\infty} \frac{t}{\mu(t)} = \lim_{t\to+\infty} \frac{t}{\omega^p(t^{\frac{1}{p}})} = K^p. \tag{4.6}$$

Since $\int_{u_0}^{+\infty} \frac{1}{\tau}\, d\tau$ is divergent ($u_0 > 0$), from (4.6), we get that $\int_{u_0}^{+\infty} \frac{1}{\mu(\tau)}\, d\tau$ is also divergent. Since $W(u) = \int_{u_0}^{u} \frac{1}{\mu(\tau)}\, d\tau = \int_{u_0}^{u} \frac{1}{\omega^p(\tau^{1/p})}\, d\tau$, then we get $[0,+\infty) \in \mathrm{Dom}(W^{-1})$ and

$$W\left(2^{p-1}\|x_0\|^p\right) + 2^{p-1}c^p t^{p\beta-1} \int_0^t s^{p(1-\beta)} l^p(s)\, \mathrm{d}s \in \mathrm{Dom}\left(W^{-1}\right), \quad \forall t \in [0,+\infty),$$

where c is defined as in Theorem 4.3.

For any $T > 0$, from Theorem 4.3, we know that problem (1.1) has at least one mild solution in $C_{1-\beta}((0,T],X)$. Since T can be chosen arbitrarily large, then problem (1.1) has at least one global mild solution in $C_{1-\beta}((0,+\infty),X)$. Thus, we complete the proof of Theorem 4.4. □

From Theorem 4.4, we can immediately obtain the following conclusion.

Corollary 4.5 *Let $0 < \gamma \leq 1$, $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. Suppose $f : (0,+\infty) \times X \to X$ is a continuous function, and there exist nonnegative functions $l,k \in C_{1-\beta}((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$ such that*

$$\left\|f(t,x)\right\| \leq l(t)\|x\|^\gamma + k(t), \quad \forall (t,x) \in (0,+\infty) \times X.$$

Then problem (1.1) has at least one mild solution in $C_{1-\beta}((0,+\infty),X)$.

Proof Since

$$\left\|f(t,x)\right\| \leq t^{\gamma(\beta-1)}l(t)\left(t^{1-\beta}\|x\|\right)^\gamma + k(t) \leq \left(t^{\gamma(\beta-1)}l(t) + k(t)\right)\left(\left(t^{1-\beta}\|x\|\right)^\gamma + 1\right), \tag{4.7}$$

then we know

$$l_1(t) := t^{1-\beta}\left(t^{\gamma(\beta-1)}l(t) + k(t)\right) = t^{(1-\gamma)(1-\beta)}l(t) + t^{1-\beta}k(t), \quad l_1 \in L^p_{Loc}\left([0,+\infty),\mathbb{R}_+\right),$$

and if $0 < \gamma < 1$, then

$$\lim_{t\to+\infty} \frac{t}{t^\gamma + 1} = +\infty; \tag{4.8}$$

if $\gamma = 1$, then

$$\lim_{t\to+\infty} \frac{t}{t^\gamma + 1} = 1. \tag{4.9}$$

Applying Theorem 4.4, we know that problem (1.1) has at least one mild solution in $C_{1-\beta}(0,+\infty)$. Thus, the proof is complete. □

Theorem 4.6 *Let $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. If $f : (0,+\infty) \times X \to \mathbb{R}$ is a continuous function with and $f(\cdot,0) \in L^p_{1-\beta,Loc}[0,+\infty)$, and there exists a function $l \in C_{1-\beta}((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$ such that*

$$\|f(t,x) - f(t,y)\| \le l(t)\|x-y\|, \quad \forall x,y \in \mathbb{R}, t \in (0,+\infty).$$

Then problem (1.1) has a unique mild solution on $(0,+\infty)$.

Proof We know

$$\|f(t,x)\| \le \|f(t,x) - f(t,0)\| + \|f(t,0)\| \le l(t)\|x\| + \|f(t,0)\|. \tag{4.10}$$

Since $f(\cdot,0) \in L^p_{Loc,1-\beta}([0,+\infty),X)$ and $l \in C_{1-\beta}((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$, applying Corollary 4.5, we know that problem (1.1) has at least one mild solution in $C_{1-\beta}((0,+\infty),X)$. We suppose that x_1, x_2 are two global mild solutions of problem (1.1). Then

$$\begin{aligned}\|x_1(t) - x_2(t)\| &= \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\beta-1}\big(f(s,x_1(s)) - f(s,x_2(s))\big)\,\mathrm{d}s \right\| \\ &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\beta-1} l(s)\|x_1(s) - x_2(s)\|\,\mathrm{d}s.\end{aligned}$$

Using Theorem 3.3, we can get $x_1(t) = x_2(t)$. Thus, the proof is complete. □

5 Global attractivity

Definition 5.1 The mild solution $x \in C_{1-\beta}((0,+\infty),X)$ of problem (1.2) is said to be globally attractive if $\lim_{t\to+\infty} x(t) = 0$.

The main result in the section reads as follows.

Theorem 5.2 *Let $0 < \beta < \gamma < 1$, $0 < \mu \le 1$, $p > \frac{1}{\beta}, l$, $k \in C_{1-\beta}((0,+\infty),\mathbb{R}_+) \cap L^p_{Loc,1-\beta}([0,+\infty),\mathbb{R}_+)$ be such that there exists a constant $K > 0$ such that*

$$t^\gamma l(t) \le K, \qquad t^\gamma k(t) \le K, \quad \forall t \in [1,+\infty).$$

Suppose $f : (0,+\infty) \times X \to X$ is a continuous function and

$$\|f(t,x)\| \le l(t)\|x\|^\mu + k(t), \quad \forall (t,x) \in (0,+\infty) \times X. \tag{5.1}$$

Then problem (1.2) has at least one globally attractive mild solution.

For convenience, we first obtain several lemmas under the assumptions in Theorem 5.2, which will be useful in the proof of the main theorem.

Lemma 5.3 *Under the assumptions in Theorem 5.2, problem (1.2) has at least one mild solution $x_1 \in C_{1-\beta}((0,T],X)$ provided that $T > 1$ and $M_2 = \frac{KT^{\beta-\gamma}\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)} < 1$.*

Proof From (5.1), we have

$$\|f(t,x)\| \le t^{\mu(\beta-1)}l(t)\big(t^{1-\beta}\|x\|\big)^{\mu} + k(t) \le \big(t^{\mu(\beta-1)}l(t) + k(t)\big)\big(\big(t^{1-\beta}\|x\|\big)^{\mu} + 1\big). \tag{5.2}$$

Let $\omega(t) = t^{\mu} + 1$ and $W(u) = \int_{u_0}^{u} \frac{1}{\omega^p(t^{1/p})}\,dt = \int_{u_0}^{u} \frac{1}{(t^{\mu/p}+1)^p}\,dt$, where $u_0, u > 0$, then we get $[0,+\infty) \subset \mathrm{Dom}(W^{-1})$. Using Theorem 4.3, we know that problem (1.2) has at least one mild solution $x_1 \in C_{1-\beta}((0,T],X)$ that satisfies the following integral equation

$$x_1(t) = t^{\beta-1}S_\beta(t)x_0 + \int_0^t (t-s)^{\beta-1}S_\beta(t-s)f\big(s,x_1(s)\big)\,ds, \quad \forall t \in (0,T]. \tag{5.3}$$

□

Now let us define the operator $F : C_0([T,+\infty),X) \to C_0([T,+\infty),X)$ by the following formula

$$\begin{aligned}(Fx)(t) = {}& t^{\beta-1}S_\beta(t)x_0 + \int_0^T (t-s)^{\beta-1}S_\beta(t-s)f\big(s,x_1(s)\big)\,ds \\ &+ \int_T^t (t-s)^{\beta-1}S_\beta(t-s)f\big(s,x_1(s)\big)\,ds,\end{aligned} \tag{5.4}$$

where $x_1 \in C_{1-\beta}((0,T],X)$ is a mild solution of problem (1.2) given in Lemma 5.3, and T is as in Lemma 5.3. For convenience, we denote $R_1 = \|x_1\|_{1-\beta} = \sup_{0<t\le T} t^{1-\beta}\|x_1(t)\|$.

Let $R > 1$ be sufficiently larger such that

$$M_1 + M_2\big(R^{\mu} + 1\big) \le R, \tag{5.5}$$

where M_2 is as defined in Lemma 5.3, and M_1 is defined in the following Lemma 5.4. Define a set U as follows

$$U = \Big\{x \in C_0\big([T,+\infty),X\big) : \|x\|_0 = \sup_{T\le t<+\infty} \big\|x(t)\big\| \le R\Big\}. \tag{5.6}$$

It is easy to see that U is a non-empty, closed, convex and bounded subset of $C_0([T,+\infty), X)$.

Lemma 5.4 *Under the assumptions in Theorem 5.2, F maps U into U.*

Proof For any $x \in U$, we have

$$\begin{aligned}\left\|(Fx)(t)\right\| &\le t^{\beta-1}\|x_0\| + \frac{M}{\Gamma(\beta)}\int_0^T (t-s)^{\beta-1}\left\|f\big(s,x_1(s)\big)\right\|ds \\ &\quad + \frac{M}{\Gamma(\beta)}\int_T^t (t-s)^{\beta-1}\left\|f\big(s,x(s)\big)\right\|ds \\ &\le T^{\beta-1}\|x_0\| + \frac{M}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\big(l(s)\left\|x_1(s)\right\|^{\mu} + k(s)\big)\,ds \\ &\quad + \frac{M}{\Gamma(\beta)}\int_T^t (t-s)^{\beta-1}\big(l(s)\left\|x(s)\right\|^{\mu} + k(s)\big)\,ds \\ &\le T^{\beta-1}\|x_0\| + \frac{M}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\big[R_1^{\mu}s^{\mu(\beta-1)}l(s) + k(s)\big]\,ds \\ &\quad + \frac{M}{\Gamma(\beta)}\int_T^t (t-s)^{\beta-1}K\big(R^{\mu}+1\big)s^{-\gamma}\,ds.\end{aligned} \tag{5.7}$$

Using Lemma 2.6, we get

$$\begin{aligned}&T^{\beta-1}\|x_0\| + \frac{M}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\big[R_1^{\mu}s^{\mu(\beta-1)}l(s) + k(s)\big]\,ds \\ &\quad = T^{\beta-1}\|x_0\| + \frac{MT^{\beta-1}R_1^{\mu}}{\Gamma(\beta)}\int_0^T \left(\frac{T}{T-s}\right)^{1-\beta} s^{\mu(\beta-1)}l(s)\,ds \\ &\qquad + \frac{MT^{\beta-1}}{\Gamma(\beta)}\int_0^T \left(\frac{T}{T-s}\right)^{1-\beta} k(s)\,ds \\ &\quad \le T^{\beta-1}\|x_0\| + \frac{2MT^{2\beta-1-\frac{1}{p}}R_1^{\mu}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^T s^{p(1-\mu)(1-\beta)}l^p(s)\,ds\right)^{\frac{1}{p}} \\ &\qquad + \frac{2MT^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^T s^{p(1-\beta)}k^p(s)\,ds\right)^{\frac{1}{p}} \\ &\quad = M_1.\end{aligned} \tag{5.8}$$

Since $\beta < \lambda$, we get

$$\begin{aligned}\frac{MK(R^{\mu}+1)}{\Gamma(\beta)}\int_T^t (t-s)^{\beta-1}s^{-\gamma}\,ds &\le \frac{MK(R^{\mu}+1)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{-\gamma}\,ds \\ &= \frac{MK(R^{\mu}+1)\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)}t^{\beta-\gamma} \\ &\le \frac{MK(R^{\mu}+1)\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)}T^{\beta-\gamma} \\ &= M_2\big(R^{\mu}+1\big).\end{aligned} \tag{5.9}$$

Using (5.5), (5.7), (5.8), and (5.9), we get

$$\left\|(Fx)(t)\right\| \le M_1 + M_2\big(R^{\mu}+1\big) \le R. \tag{5.10}$$

Thus, $\|Fx\|_0 \le R$ for any $x \in U$.

We now prove that Fx is a continuous function on $[T,+\infty)$. Since

$$t^{1-\beta}\left\|f\big(t,x_1(t)\big)\right\| \le t^{1-\beta}l(t)\left\|x_1(t)\right\|^{\mu} + t^{1-\beta}k(t) \le R_1^{\mu}t^{(1-\mu)(1-\beta)}l(t) + t^{1-\beta}k(t),$$

then we have $f(\cdot,x_1(\cdot)) \in L^p_{1-\beta}[0,T],X)$. Using Lemma 2.9, we get that $\int_0^T(\cdot - s)^{\beta-1}f(s, x_1(s))\,ds$ is continuous on $[T,+\infty)$. Since

$$\left\|f\big(t,x(t)\big)\right\| \le l(t)\left\|x(t)\right\|^{\mu} + k(t) \le R^{\mu}l(t) + k(t),$$

where $x \in U$, then $f(\cdot,x(\cdot))$ is continuous on $[T,+\infty)$ and $f(\cdot,x(\cdot)) \in L^p_{Loc,1-\beta}([T,+\infty),X)$. Using Lemma 2.8, we get that $\int_T^{\cdot}(\cdot - s)^{\beta-1}f(s,x(s))\,ds$ is continuous on $[T,+\infty)$. Therefore, Fx is a continuous function on $[T,+\infty)$ when $x \in U$.

Now let us prove that $(Fx)(t) \to 0$ as $t \to +\infty$. For any $x \in U$, we have

$$\begin{aligned}\left|(Fx)(t)\right| \le t^{\beta-1}\|x_0\| &+ \frac{M}{\Gamma(\beta)}\int_0^T (t-s)^{\beta-1}\left\|f\big(s,x_1(s)\big)\right\|ds \\ &+ \frac{M}{\Gamma(\beta)}\int_T^t (t-s)^{\beta-1}\left\|f\big(s,x(s)\big)\right\|ds.\end{aligned} \tag{5.11}$$

Using Lemma 2.6, we have

$$\begin{aligned}&\int_0^T (t-s)^{\beta-1}\left\|f\big(s,x_1(s)\big)\right\|ds \\ &\quad\le t^{\beta-1}\int_0^T\left(\frac{t}{t-s}\right)^{1-\beta}\left\|f\big(s,x_1(s)\big)\right\|ds \\ &\quad\le t^{\beta-1}\int_0^T\left(\frac{T}{T-s}\right)^{1-\beta}\left\|f\big(s,x_1(s)\big)\right\|ds \\ &\quad\le R_1^{\mu}t^{\beta-1}\int_0^T\left(\frac{T}{T-s}\right)^{1-\beta}s^{\mu(\beta-1)}l(s)\,ds + t^{\beta-1}\int_0^T\left(\frac{T}{T-s}\right)^{1-\beta}k(s)\,ds \\ &\quad\le \frac{2T^{\beta-\frac{1}{p}}R_1^{\mu}t^{\beta-1}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^T s^{p(1-\mu)(1-\beta)}l^p(s)\,ds\right)^{\frac{1}{p}} \\ &\qquad+ \frac{2T^{\beta-\frac{1}{p}}t^{\beta-1}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^T s^{p(1-\beta)}k^p(s)\,ds\right)^{\frac{1}{p}},\end{aligned} \tag{5.12}$$

then we get that

$$\int_0^T (t-s)^{\beta-1}\left\|f\big(s,x_1(s)\big)\right\|ds \to 0, \quad \text{as } t \to +\infty. \tag{5.13}$$

Moreover, we know

$$\begin{aligned}\int_T^t (t-s)^{\beta-1}\left\|f\bigl(s,x(s)\bigr)\right\| ds &\le \int_T^t (t-s)^{\beta-1} l(s)\left\|x(s)\right\|^{\mu} ds + \int_T^t (t-s)^{\beta-1} k(s)\, ds \\ &\le KR^{\mu}\int_T^t (t-s)^{\beta-1} s^{-\gamma}\, ds + K\int_T^t (t-s)^{\beta-1} s^{-\gamma}\, ds \\ &\le K\bigl(R^{\mu}+1\bigr)\int_0^t (t-s)^{\beta-1} s^{-\gamma}\, ds \\ &= \frac{K(R^{\mu}+1)\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)} t^{\beta-\gamma}.\end{aligned} \tag{5.14}$$

Since $0<\beta<\gamma<1$, we get that

$$\int_T^t (t-s)^{\beta-1}\left\|f\bigl(s,x(s)\bigr)\right\| ds \to 0 \quad \text{as } t\to+\infty. \tag{5.15}$$

Using (5.13) and (5.15), we get $(Fx)(t)\to 0$ as $t\to+\infty$.

Thus, F maps U into U. The proof is complete. □

Lemma 5.5 *Under the assumptions in Theorem 5.2, $F: U\to U$ is completely continuous.*

Proof For any $T_1>T>1$ and $x\in U$, let $T\le t_1<t_2\le T_1$, then we get

$$\begin{aligned}&\left\|(Fx)(t_2)-(Fx)(t_1)\right\| \\ &\quad\le \left\|t_2^{1-\beta}(Fx)(t_2)-t_2^{1-\beta}(Fx)(t_1)\right\| \\ &\quad\le \left\|t_2^{1-\beta}(Fx)(t_2)-t_1^{1-\beta}(Fx)(t_1)\right\| + \left\|t_1^{1-\beta}(Fx)(t_1)-t_2^{1-\beta}(Fx)(t_1)\right\| \\ &\quad\le \frac{1}{\Gamma(\beta)}\left\|\int_0^T\left(\frac{t_2}{t_2-s}\right)^{1-\beta} f\bigl(s,x_1(s)\bigr)\, ds - \int_0^T\left(\frac{t_1}{t_1-s}\right)^{1-\beta} f\bigl(s,x_1(s)\bigr)\, ds\right\| \\ &\qquad + \frac{1}{\Gamma(\beta)}\left\|\int_T^{t_2}\left(\frac{t_2}{t_2-s}\right)^{1-\beta} f\bigl(s,x(s)\bigr)\, ds - \int_T^{t_1}\left(\frac{t_1}{t_1-s}\right)^{1-\beta} f\bigl(s,x(s)\bigr)\, ds\right\| \\ &\qquad + R\left|t_1^{1-\beta}-t_2^{1-\beta}\right|.\end{aligned} \tag{5.16}$$

Using Lemmas 2.8 and 2.9, we can obtain that FU is equicontinuous on $[T,T_1]$. From the inequality (5.10), we know that $(Fx)(t)$ is relatively compact for any $t\in[T,+\infty)$ and $x\in U$. Using the proof of Lemma 5.4, we can get that $\lim_{t\to+\infty}|(Fx)(t)|=0$ is uniformly for $x\in U$. Therefore, we get that the set FU is relatively compact.

We now show that F is continuous, that is $x_n\to x$ implies $Fx_n\to Fx$. Since $x_n(t)\to x(t)$, then $f(t,x_n(t))\to f(t,x(t))$ for $t\in[T,+\infty)$. Therefore, we have

$$(t-s)^{\beta-1} f\bigl(s,x_n(s)\bigr)\to (t-s)^{\beta-1} f\bigl(s,x(s)\bigr),\quad \forall s\in[T,t). \tag{5.17}$$

Since $l,k\in C([T,+\infty),\mathbb{R}_+)$ and

$$(t-s)^{\beta-1}\left\|f\bigl(s,x_n(s)\bigr)\right\| \le (t-s)^{\beta-1}\bigl(l(s)\left\|x_n(s)\right\|^{\mu}+k(s)\bigr) \le (t-s)^{\beta-1}\bigl(R^{\mu}l(s)+k(s)\bigr), \tag{5.18}$$

then we have $(t-\cdot)^{\beta-1}f(\cdot,x_n(\cdot))\in L^1([T,t],X)$. From (5.17) and (5.18), using the Lebesgue dominated convergence theorem, we have

$$\left\|(Fx_n)(t)-(Fx)(t)\right\| = \frac{1}{\Gamma(\beta)}\left\|\int_T^t (t-s)^{\beta-1}\bigl[f\bigl(s,x_n(s)\bigr)-f\bigl(s,x(s)\bigr)\bigr]\,ds\right\| \to 0$$

as $n\to+\infty$. Therefore, $(Fx_n)(t)\to(Fx)(t)$ pointwise on $[T,+\infty)$ as $n\to+\infty$. With the fact that F is compact, then $\|Fx_n-Fx\|_0\to 0$ as $n\to+\infty$, which implies F is continuous.

Therefore, $F:U\to U$ is completely continuous. □

Lemma 5.6 *Under the assumptions in Theorem 5.2, the following integral equation*

$$\begin{aligned} x(t) = {} & t^{\beta-1}S_\beta(t)x_0 + \int_0^T (t-s)^{\beta-1}S_\beta(t-s)f\bigl(s,x_1(s)\bigr)\,ds \\ & + \int_T^t (t-s)^{\beta-1}S_\beta(t-s)f\bigl(s,x(s)\bigr)\,ds \end{aligned} \tag{5.19}$$

has at least one mild solution in $C_0([T,+\infty),X)$, where $x_1\in C_{1-\beta}((0,T],X)$ is the mild solution of problem (5.3), and T is as in Lemma 5.3.

Proof Using Lemma 5.4, Lemma 5.5, and Theorem 4.3, we have that the integral equation (3.21) has at least one mild solution $x_2\in C_0([T,+\infty),X)$. □

Now we give the proof of Theorem 5.2.

Proof of Theorem 5.2 We denote

$$x(t) = \begin{cases} x_1(t) & t\in(0,T], \\ x_2(t) & t\in[T,+\infty), \end{cases}$$

where $x_1\in C_{1-\beta}((0,T],X)$ is a mild solution of problem (5.3), and $x_2\in C_0([T,+\infty),X)$ is a mild solution of the integral equation (5.19). From (5.3) and (5.19), we know that x is continuous on $(0,+\infty)$, and we have that x is the mild solution of the following integral equation

$$\begin{aligned} x(t) = {} & t^{\beta-1}S_\beta(t)x_0 + \int_0^T (t-s)^{\beta-1}S_\beta(t-s)f\bigl(s,x(s)\bigr)\,ds \\ & + \int_T^t (t-s)^{\beta-1}S_\beta(t-s)f\bigl(s,x(s)\bigr)\,ds. \end{aligned} \tag{5.20}$$

From Theorem 4.3, we know that x is also a global mild solution of problem (1.2) and

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x_2(t) = 0.$$

Thus, the mild solution x of problem (1.2) is globally attractive. □

The following conclusion is a consequence of Theorem 5.2.

Theorem 5.7 *Under the assumptions in Theorem 5.2, problem (1.2) has at least one mild solution $x \in C_{1-\beta}((0,+\infty),X)$ and*

$$\|x(t)\| = t^{\beta-1}\|x_0\| + o\left(t^{\beta-\gamma_1}\right) \quad as\ t \to +\infty, \tag{5.21}$$

where $\beta < \gamma_1 < \gamma < 1$.

Proof From Theorem 5.2, we get that $x \in C_{1-\beta}((0,+\infty),X)$ is a globally attractive mild solution of problem (1.2). Since $0 < \beta < \gamma_1 < \gamma < 1$, for $t > T$, then we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\|x(t) - t^{\beta-1}x_0\|}{t^{\beta-\gamma_1}} \\
&\quad \le \lim_{t\to+\infty} \frac{\int_0^t (t-s)^{\beta-1}\|f(s,x(s))\|\,ds}{\Gamma(\beta)t^{\beta-\gamma_1}} \\
&\quad = \lim_{t\to+\infty} \frac{\int_0^T (t-s)^{\beta-1}\|f(s,x_1(s))\|\,ds}{\Gamma(\beta)t^{\beta-\gamma_1}} + \lim_{t\to+\infty} \frac{\int_T^t (t-s)^{\beta-1}\|f(s,x_2(s))\|\,ds}{\Gamma(\beta)t^{\beta-\gamma_1}}.
\end{aligned} \tag{5.22}$$

Using (5.12) and (5.14), we get

$$\lim_{t\to+\infty} \frac{\|x(t) - t^{\beta-1}x_0\|}{t^{\beta-\gamma_1}} = 0.$$

Thus, $x(t) = t^{\beta-1}x_0 + o(t^{\beta-\gamma_1})$ as $t \to +\infty$. □

Remark 5.8 In fact, from (5.12) and (5.14), we get that the mild solution x of problem (1.2) satisfies

$$\|x(t)\| \le t^{\beta-1}\|x_0\| + K_1 t^{\beta-1} + K_2 t^{\beta-\gamma}, \quad \forall t \in [T,+\infty), \tag{5.23}$$

where K_1 and K_2 are nonnegative constants.

Theorem 5.9 *Let $0 < \mu \le 1$, $0 < \beta < 1$, $\gamma > \beta$, $p > 1$, $\beta > \frac{1}{p} > 2\beta - 1$, $l \in C_{(1-\mu)(1-\beta)}((0,+\infty), \mathbb{R}_+) \cap L^p_{Loc,(1-\mu)(1-\beta)}([0,+\infty),\mathbb{R}_+)$. Suppose that there exists a constant $K > 0$ such that*

$$t^{\gamma} l(t) \le K, \quad \forall t \in [1,+\infty), \tag{5.24}$$

and $f : (0,+\infty) \times X \to X$ is a continuous function with

$$\|f(t,x)\| \le l(t)\|x\|^{\mu}, \quad \forall (t,x) \in (0,+\infty) \times X.$$

Then problem (1.2) is global attractive.

6 Deduced results

In this section, we derive some deduced results for the following first-order and Caputo fractional semilinear evolution equations

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t)), & t \in (0,+\infty), \\ x(0) = x_0. \end{cases} \tag{6.1}$$

$$\begin{cases} {}_0^C D_t^\beta x(t) = Ax(t) + f(t, x(t)), & t \in (0, +\infty), \\ x(0) = x_0. \end{cases} \tag{6.2}$$

Definition 6.1 A function $x \in C([0,T], X)$ is called a mild solution of problem (6.1) if it satisfies the following fractional integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f\bigl(s, x(s)\bigr)\,ds, \quad \forall t \in [0,T].$$

Definition 6.2 A function $x \in C([0,T], X)$ is called a mild solution of problem (6.2) if it satisfies the following fractional integral equation

$$x(t) = t^{\beta-1}S'_\beta(t)x_0 + \int_0^t (t-s)^{\beta-1}S_\beta(t-s)f\bigl(s, x(s)\bigr)\,ds, \quad \forall t \in [0,T],$$

where

$$S'_\beta(t) = \int_0^\infty \xi_\beta(\theta)S\bigl(t^\beta\theta\bigr)\,d\theta.$$

Theorem 6.3 *Suppose $f : (0,T] \times X \to X$ is a continuous function, and there exists a function $l \in C([0,T], \mathbb{R}_+)$ and a nondecreasing function $\omega \in C([0,+\infty), \mathbb{R}_+)$ such that*

$$\bigl\|f(t,x)\bigr\| \le l(t)\omega\bigl(\|x\|\bigr), \quad \forall (t,x) \in (0,T] \times X.$$

Then problem (6.1) has at least one mild solution in $C([0,T], X)$ provided that

$$W\bigl(M\|x_0\|\bigr) + M\int_0^t l(s)\,\mathrm{d}s \in \mathrm{Dom}\bigl(W^{-1}\bigr), \quad \forall t \in (0,T],$$

where $W(u) = \int_{u_0}^u \frac{1}{\omega(\tau)}\,d\tau$, $u_0, u > 0$.

Proof Define the operator $G_1 : C([0,T], X) \to C([0,T], X)$ by

$$G_1x(t) = S(t)x_0 + \int_0^t S(t-s)f\bigl(s, x(s)\bigr)\,ds, \quad \forall t \in [0,T].$$

Similar to the proof of Theorem 4.3, we only prove that the set $\Lambda_1 = \{x \in C([0,T], X) : x = \lambda G_1 x$ for some $0 < \lambda < 1\}$ is bounded. Indeed, for $x \in \Lambda_1$ one has

$$\begin{aligned} \bigl\|x(t)\bigr\| &\le M\|x_0\| + M\int_0^t \bigl\|f\bigl(s, x(s)\bigr)\bigr\|\,\mathrm{d}s \\ &\le M\|x_0\| + M\int_0^t l(s)\omega\bigl(\bigl\|x(s)\bigr\|\bigr)\,\mathrm{d}s \end{aligned}$$

Using Lemma 2.4, we obtain

$$\bigl\|x(t)\bigr\| \le W^{-1}\biggl[W\bigl(M\|x_0\|\bigr) + M\int_0^t l(s)\,\mathrm{d}s\biggr],$$

which shows that the set Λ_1 is bounded. The proof is complete. □

Theorem 6.4 *Let $p > \frac{1}{\beta}$ and $q = \frac{p}{p-1}$. Suppose $f : (0,T] \times X \to X$ is a continuous function, and there exists a function $l \in C((0,T],\mathbb{R}_+) \cap L^p([0,T],\mathbb{R}_+)$ and a nondecreasing function $\omega \in C([0,+\infty),\mathbb{R}_+)$ such that*

$$\left\|f(t,x)\right\| \leq l(t)\omega\left(\|x\|\right), \quad \forall (t,x) \in (0,T] \times X.$$

Then problem (6.2) has at least one mild solution in $C([0,T],X)$ provided that

$$W\left(2^{p-1}\left(\frac{M}{\Gamma(q)}\right)^p\right) + 2^{p-1}c^p t^{p\beta-1}\int_0^t s^p l^p(s)\,\mathrm{d}s \in \mathrm{Dom}\left(W^{-1}\right), \quad \forall t \in (0,T],$$

where $c = \frac{M\Gamma^{\frac{1}{q}}(q(\beta-1)+1)\Gamma^{\frac{1}{q}}(q+1)}{\Gamma(q)\Gamma^{\frac{1}{q}}(q(\beta-1)+q+2)}$, $W(u) = \int_{u_0}^u \frac{1}{\omega^p(\tau^{1/p})}\,d\tau$, $u_0, u > 0$.

Proof Define the operator $G_2 : C([0,T],X) \to C([0,T],X)$ by

$$G_2x(t) = S'_\beta(t)x_0 + \int_0^t (t-s)^{\beta-1}S_\beta(t-s)f\left(s,x(s)\right)ds, \quad \forall t \in [0,T]. \tag{6.3}$$

Similar to the proof of Theorem 4.3, we only prove that the set $\Lambda_2 = \{x \in C([0,T],X) : x = \lambda G_2x$ for some $0 < \lambda < 1\}$ is bounded. Indeed, for $x \in \Lambda_2$ one has

$$\begin{aligned}\left\|x(t)\right\| &\leq \frac{M}{\Gamma(q)}\|x_0\| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{\beta-1}\left\|f\left(s,x(s)\right)\right\|\,\mathrm{d}s \\ &\leq \frac{M}{\Gamma(q)}\|x_0\| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{\beta-1}l(s)\omega\left(\left\|x(s)\right\|\right)\mathrm{d}s.\end{aligned}$$

By Lemma 3.1 for $\alpha = \delta = 0$, we obtain

$$\left\|x(t)\right\| \leq \left\{W^{-1}\left[W\left(2^{p-1}\left(\frac{M}{\Gamma(q)}\right)^p\right) + 2^{p-1}c^p t^{p\beta-1}\int_0^t s^p l^p(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{p}}, \quad \forall t \in (0,T_1),$$

which shows that the set Λ_2 is bounded. The proof is complete. □

Remark 6.5 Theorems 6.3 and 6.4 generalize and improve the results on the existence of mild solutions in [24, 25, 27].

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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