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Blow up, growth, and decay of solutions for a class of coupled nonlinear viscoelastic Kirchhoff equations with distributed delay and variable exponents

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Abstract

In this work, we consider a quasilinear system of viscoelastic equations with dispersion, source, distributed delay, and variable exponents. Under a suitable hypothesis the blow-up and growth of solutions are proved, and by using an integral inequality due to Komornik the general decay result is obtained in the case of absence of the source term $f_1 = f_2 = 0$.

Mathematics Subject Classification: 35B40; 35L70; 93D20

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1 Introduction

Our understanding of real-world phenomena and our technology today are largely based on mathematical analysis for partial differential equations (PDEs) [1, 2, 4, 5]. This mathematical analysis helps us to visualize and understand different real-world problems [7, 8, 10, 11]. The mathematical analysis study of PDEs has also taught us to show a little modesty: we have discovered the impossibility of predicting certain phenomena governed by nonlinear PDEs in the medium term—think of the now famous butterfly effect: a small variation of the initial conditions can lead to very large variations in very long time. On the other hand, we have also learned to "hear the shape of a drum": it has been shown mathematically that the frequencies emitted by a drum during membrane vibration—a phenomenon described by a PDE—allow the drum shape to be perfectly reconstructed. One of the things to keep in mind about PDEs is that you usually do not want to get their solutions explicitly! What mathematics can do, on the other hand, is to say whether one or more solutions exist, and sometimes to very precisely describe certain properties of these solutions. However, the emergence of extremely powerful computers today makes it possible to obtain approximate solutions for partial derivative equations, even very complicated. This is what happens, for example, when you look at the weather forecast, or when we see the moving images of a simulation of airflow on the wing of airplane. The role of mathematicians is then to build approximation schemes and to demonstrate the relevance of the simulations by establishing a priori estimates on the made errors. When did EDP appear? They likely originated in the early days of rational mechanics in the seventeenth century, with figures like Newton and Leibniz playing crucial roles. As scientific disciplines, especially physics, advanced in energy functional, fluid mechanics equations, Navier–Stokes equations, where they contributed to the expansion of partial differential equations (PDEs).

To highlight a few key contributors, Euler's name stands out, as well as Navier and Stokes for fluid mechanics equations, Fourier for heat equations, Maxwell for electromagnetism equations, and Schrödinger, Heisenberg, and Einstein for quantum mechanics and the theory of relativity PDEs, respectively (see e.g. [1, 6, 9] and the references therein). Nevertheless, the systematic examination of partial differential equations (PDEs) is relatively recent, with mathematicians embarking on this endeavor only in the twentieth century. A significant leap occurred with Schwartz's formulation of the theory of distributions in the 1950s, and comparable progress emerged through Hörmander's work on pseudo-differential calculus in the early 1970s. Importantly, the study of PDEs remains highly active as we progress into the twenty-first century [12–16]. Mathematics serves as a potent tool in both scientific inquiry and engineering applications, enabling precise modeling, analysis, and solution exploration of complex mathematical systems fundamental to advancing our understanding of the natural world and optimizing technological innovations [17–19, 21–23]. This research not only influences applied sciences but also plays a crucial role in the ongoing evolution of mathematics itself, particularly in the domains of geometry and analysis. In this work, the following problem is addressed:

$$\begin{cases} |v_t|^\eta v_{tt} - M(\|\nabla v\|_2^2)\Delta v + \int_0^t h_1(t-r)\Delta v(r)\,dr - \Delta v_{tt} + \beta_1 |v_t(t)|^{m(y)-2} v_t(t) \\ \quad + \int_{\tau_1}^{\tau_2} \beta_2(r)|v_t(t-r)|^{m(y)-2} v_t(t-r)\,dr = f_1(v,w), \quad (y,t) \in \Omega \times (0,T), \\ |w_t|^\eta w_{tt} - M(\|\nabla w\|_2^2)\Delta w + \int_0^t h_2(t-r)\Delta w(r)\,dr \\ \quad - \Delta w_{tt} + \beta_3 |w_t(t)|^{s(y)-2} w_t(t) \\ \quad + \int_{\tau_1}^{\tau_2} \beta_4(r)|w_t(t-r)|^{s(y)-2} w_t(t-r)\,dr = f_2(v,w), \quad (y,t) \in \Omega \times (0,T), \\ v(y,t) = w(y,t) = 0, \quad (y,t) \in \partial\Omega \times (0,T), \\ v(y,0) = v_0(y), \quad v_t(y,0) = v_1(y), \quad y \in \Omega, \\ w(y,0) = w_0(y), \quad w_t(y,0) = w_1(y), \quad y \in \Omega, \\ v_t(y,-t) = f_0(y,t), \quad w_t(y,-t) = g_0(y,t) \quad \text{in } \Omega \times (0,\tau_2), \end{cases} \tag{1.1}$$

in which $\eta \geq 0$ for $N = 1,2$ and $0 < \eta \leq \frac{2}{N-2}$ for $N \geq 3$, and $h_i(.) : R^+ \to R^+$ $(i = 1,2)$ represents positive relaxation functions, which will be specified later. The term $-\Delta(.)tt$ denotes the dispersion term, and $M(\sigma)$ is a nonnegative locally Lipschitz function for $\gamma, \sigma \geq 0$ such that $M(\sigma) = \alpha 1 + \alpha_2 \sigma^\gamma$. Specifically, we choose $\alpha_1 = \alpha_2 = 1$, and

$$\begin{cases} f_1(v,w) = a_1|v+w|^{2(q(y)+1)}(v+w) + b_1|v|^{q(y)}.v.|w|^{q(y)+2}, \\ f_2(v,w) = a_1|v+w|^{2(q(y)+1)}(v+w) + b_1|w|^{q(y)}.w.|v|^{q(y)+2}. \end{cases} \tag{1.2}$$

In this context, we consider nonnegative constants $\tau_1 < \tau_2$ such that $\beta i : [\tau 1, \tau_2] \to \mathbb{R}$, where $i = 2,4$ represents the time delay in the distributive case. Furthermore, $q(.)$, $m(.)$, and $s(.)$ are variable exponents defined as measurable functions on $\overline{\Omega}$ in the following manner:

$$
\begin{aligned}
&1 \le q^- \le q(y) \le q^+ \le q^*,\\
&2 \le m^- \le m(y) \le m^+ \le m^*,\\
&2 \le s^- \le s(y) \le s^+ \le s^*,
\end{aligned}
\tag{1.3}
$$

where

$$
\begin{aligned}
&q^- = \inf_{y\in\overline{\Omega}} q(y), \qquad m^- = \inf_{y\in\overline{\Omega}} m(y), \qquad s^- = \inf_{y\in\overline{\Omega}} s(y),\\
&q^+ = \sup_{y\in\overline{\Omega}} q(y), \qquad m^+ = \sup_{y\in\overline{\Omega}} m(y), \qquad s^+ = \sup_{y\in\overline{\Omega}} s(y),
\end{aligned}
\tag{1.4}
$$

with

$$
\max\{m^+, s^+\} \le 2q^- + 1 \tag{1.5}
$$

and

$$
m^*, s^* = \frac{2(n-1)}{n-2} \quad \text{if } n \ge 3. \tag{1.6}
$$

This research is organized into distinct sections. In the following section, we present the hypotheses, concepts, and lemmas essential for our study. Section 2 is dedicated to proving the blow-up result, followed by the derivation of exponential growth of solutions. In Sect. 4, we establish the general decay when $f_1 = f_2 = 0$. The paper concludes with a comprehensive summary in the final section.

2 Fundamental theory

The importance of studying the blow-up of solutions in various systems lies in its ability to reveal critical thresholds, instabilities, and singularities that can significantly impact the behavior and evolution of dynamic processes [27–30]. Here, we will present some related theory and will define suitable assumptions for the proof of blow-up result.

(A1) Take a decreasing and differentiable function $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ in a manner that

$$
h_i(t) \ge 0, \qquad 1 - \int_0^\infty h_i(r)\,dr = l_i > 0, \quad i = 1, 2. \tag{2.1}
$$

(A2) One can find $\xi_1, \xi_2 > 0$ in a way that

$$
h_i'(t) \le -\xi_i h_i(t), \quad t \ge 0, i = 1, 2. \tag{2.2}
$$

(A3) $\beta_i : [\tau_1, \tau_2] \to \mathbb{R}$, $i = 2, 4$, are a bounded functions satisfying

$$
\begin{aligned}
&\delta \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right| dr < \beta_1, \quad \delta > 1,\\
&\delta \int_{\tau_1}^{\tau_2} \left|\beta_4(r)\right| dr < \beta_3, \quad \delta > 1.
\end{aligned}
\tag{2.3}
$$

Lemma 2.1 *There exists $F(v,w)$ in a manner that*

$$\begin{aligned} F(v,w) &= \frac{1}{2(q(y)+2)}\left[vf_1(v,w)+wf_2(v,w)\right] \\ &= \frac{1}{2(q(y)+2)}\left[a_1|v+w|^{2(q(y)+2)}+2b_1|vw|^{q(y)+2}\right] \geq 0, \end{aligned}$$

in which

$$\frac{\partial F}{\partial v}=f_1(v,w), \qquad \frac{\partial F}{\partial w}=f_2(v,w).$$

Here, consider $a_1=b_1=1$ for convenience.

Lemma 2.2 [26] *One can find $c_0>0$ and $c_1>0$ in a way that*

$$\begin{aligned} \frac{c_0}{2(q(y)+2)}\left(|v|^{2(q(y)+2)}+|w|^{2(q(y)+2)}\right) &\leq F(v,w) \\ &\leq \frac{c_1}{2(q(y)+2)}\left(|v|^{2(q(y)+2)}+|w|^{2(q(y)+2)}\right). \end{aligned} \tag{2.4}$$

Consider a measurable function $q:\Omega\rightarrow[1,\infty)$. We introduce the Lebesgue space with a variable exponent $q(.)$ as follows:

$$L^{q(.)}(\Omega)=\left\{v:\Omega\rightarrow\mathbb{R}; \text{measurable in } \Omega: \int_\Omega |v|^{q(.)}\,dy<\infty\right\},$$

with the norm defined by

$$\|v\|_{q(.)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{v}{\lambda}\right|^{q(y)}dy\leq 1\right\}.$$

Endowed with this norm, $L^{q(.)}(\Omega)$ forms a Banach space. Subsequently, we introduce the variable-exponent Sobolev space $W^{1,q(.)}(\Omega)$ as follows:

$$W^{1,q(.)}(\Omega)=\left\{v\in L^{q(.)}(\Omega); \nabla v \text{ exists and } |\nabla v|\in L^{q(.)}(\Omega)\right\},$$

with the norm given by

$$\|v\|_{1,q(.)}=\|v\|_{q(.)}+\|\nabla v\|_{q(.)},$$

$W^{1,q(.)}(\Omega)$ is a Banach space, and the closure of $C_0^\infty(\Omega)$ is given by $W_0^{1,q(.)}(\Omega)$.

For $v\in W_0^{1,q(.)}(\Omega)$, we give the equivalent norm

$$\|v\|_{1,q(.)}=\|\nabla v\|_{q(.)}.$$

$W_0^{-1,q'(.)}(\Omega)$ sign to the dual of $W_0^{1,q(.)}(\Omega)$ in which $\frac{1}{q(.)}+\frac{1}{q'(.)}=1$.

Also, we take the log-Hölder inequality

$$\left|q(y)-q(z)\right|\leq-\frac{A}{\log|y-z|} \tag{2.5}$$

for all$y,z\in\Omega$, with $|y-z|<\zeta$, where $0<\zeta<1$ and $A>0$.

Theorem 2.3 *Assume* (2.1)–(2.3) *hold. Then, for any* $(v_0, v_1, w_0, w_1, f_0, g_0) \in \mathcal{H}$, (1.1) *has a unique solution for some* $T > 0$:

$$\begin{aligned}
&v, w \in C\bigl([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\bigr),\\
&v_t \in C\bigl([0,T]; H_0^1(\Omega)\bigr) \cap L^{m(y)}\bigl(\Omega \times (0,T)\bigr) \cap \mathcal{H}_1,\\
&w_t \in C\bigl([0,T]; H_0^1(\Omega)\bigr) \cap L^{s(y)}\bigl(\Omega \times (0,T)\bigr) \cap \mathcal{H}_2,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{H}_1 = L^{m(y)}\bigl(\Omega \times (0,1) \times (\tau_1, \tau_2)\bigr),\\
&\mathcal{H}_2 = L^{s(y)}\bigl(\Omega \times (0,1) \times (\tau_1, \tau_2)\bigr),\\
&\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{H}_1 \times \mathcal{H}_2.
\end{aligned}$$

Proof We can prove the local existence result for (1.1) in suitable Sobolev spaces by exploiting the Faedo–Galerkin approximation method (see [3, 24]). □

Firstly, we take the following variables as mentioned in [25]:

$$\begin{aligned}
&x(y, \rho, r, t) = v_t(y, t - r\rho),\\
&z(y, \rho, r, t) = w_t(y, t - r\rho),
\end{aligned}$$

which verify

$$\begin{cases}
r x_t(y, \rho, r, t) + x_\rho(y, \rho, r, t) = 0,\\
x(y, 0, r, t) = v_t(y, t),
\end{cases} \tag{2.6}$$

and

$$\begin{cases}
r z_t(y, \rho, r, t) + z_\rho(y, \rho, r, t) = 0,\\
z(y, 0, r, t) = w_t(y, t).
\end{cases} \tag{2.7}$$

Then, problem (1.1) is equivalent to

$$\begin{cases}
|v_t|^\eta v_{tt} - M(\|\nabla v\|_2^2)\Delta v + \int_0^t h_1(t-r)\Delta v(r)\,dr - \Delta v_{tt} + \beta_1 |v_t(t)|^{m(y)-2} v_t(t)\\
\quad + \int_{\tau_1}^{\tau_2} \beta_2(r) |x(y,1,r,t)|^{m(y)-2} x(y,1,r,t)\,dr = f_1(v,w),\\
|w_t|^\eta w_{tt} - M(\|\nabla w\|_2^2)\Delta w + \int_0^t h_2(t-r)\Delta w(r)\,dr\\
\quad - \Delta w_{tt} + \beta_3 |w_t(t)|^{s(y)-2} w_t(t)\\
\quad + \int_{\tau_1}^{\tau_2} \beta_4(r) |z(y,1,r,t)|^{s(y)-2} z(y,1,r,t)\,dr = f_2(v,w),\\
r x_t(y, \rho, r, t) + x_\rho(y, \rho, r, t) = 0,\\
r z_t(y, \rho, r, t) + z_\rho(y, \rho, r, t) = 0.\\
v(y,0) = v_0(y), \qquad v_t(y,0) = v_1(y), \qquad w(y,0) = w_0(y),\\
w_t(y,0) = w_1(y), \quad \text{in } \Omega\\
x(y,\rho,r,0) = f_0(y,\rho r), \qquad z(y,\rho,r,0) = g_0(y,\rho r), \quad \text{in } \Omega \times (0,1) \times (0,\tau_2)\\
v(y,t) = w(y,t) = 0, \quad \text{in } \partial\Omega \times (0,T),
\end{cases} \tag{2.8}$$

where

$$(y,\rho,r,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,T).$$

In the upcoming step, the energy functional is introduced.

Lemma 2.4 *Let (2.1)–(2.3) be satisfied, and assume that (v,w,x,z) is a solution of (2.8), then $E(t)$ is nonincreasing, that is,*

$$\begin{aligned} E(t) ={}& \frac{1}{\eta+2}\bigl[\|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2}\bigr] + \frac{1}{2}\bigl[\|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\bigr] \\ &+ \frac{1}{2(\gamma+1)}\bigl[\|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)}\bigr] + \mathcal{W}(x,z) \\ &+ \frac{1}{2}\biggl[\biggl(1-\int_0^t h_1(r)\,dr\biggr)\|\nabla v\|_2^2 + \biggl(1-\int_0^t h_2(r)\,dr\biggr)\|\nabla w\|_2^2\biggr] \\ &+ \frac{1}{2}\bigl[(h_1 o\nabla v)(t) + (h_2 o\nabla w)(t)\bigr] - \int_\Omega F(v,w)\,dy \end{aligned} \tag{2.9}$$

fulfills

$$\begin{aligned} E'(t) \le{}& \frac{1}{2}\bigl[\bigl(h_1' o\nabla v\bigr)(t) + \bigl(h_2' o\nabla w\bigr)(t)\bigr] - \frac{1}{2}\bigl[h_1(t)\|\nabla v\|_2^2 + h_2(t)\|\nabla w\|_2^2\bigr] \\ &- C_0\biggl\{\int_\Omega \bigl|v_t(t)\bigr|^{m(y)}\,dy + \int_\Omega\int_{\tau_1}^{\tau_2} \bigl|\beta_2(r)\bigr|\bigl|x(y,1,r,t)\bigr|^{m(y)}\,dr\,dy \\ &+ \int_\Omega \bigl|w_t(t)\bigr|^{s(y)}\,dy + \int_\Omega\int_{\tau_1}^{\tau_2} \bigl|\beta_4(r)\bigr|\bigl|z(y,1,r,t)\bigr|^{s(y)}\,dr\,dy\biggr\} \\ \le{}& 0, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \mathcal{W}(x,z) ={}& \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} r\bigl|\beta_2(r)\bigr|\frac{(\delta m(y)-1)|x(y,\rho,r,t)|^{m(y)}}{m(y)}\,dr\,d\rho\,dy \\ &+ \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} r\bigl|\beta_4(r)\bigr|\frac{(\delta s(y)-1)|z(y,\rho,r,t)|^{s(y)}}{s(y)}\,dr\,d\rho\,dy. \end{aligned} \tag{2.11}$$

Proof By multiplying $(2.8)_1$, $(2.8)_2$ by v_t, w_t and integrating over Ω, we have

$$\begin{aligned} &\frac{d}{dt}\biggl\{\frac{1}{\eta+2}\|v_t\|_{\eta+2}^{\eta+2} + \frac{1}{\eta+2}\|w_t\|_{\eta+2}^{\eta+2} + \frac{1}{2}\|\nabla v_t\|_2^2 + \frac{1}{2}\|\nabla w_t\|_2^2 \\ &\quad+ \frac{1}{2(\gamma+1)}\bigl[\|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)}\bigr] \\ &\quad+ \frac{1}{2}\biggl(1-\int_0^t h_1(r)\,dr\biggr)\|\nabla v\|_2^2 + \frac{1}{2}\biggl(1-\int_0^t h_2(r)\,dr\biggr)\|\nabla w\|_2^2 \\ &\quad+ \frac{1}{2}(h_1 o\nabla v)(t) + \frac{1}{2}(h_2 o\nabla w)(t) - \int_\Omega F(v,w)\,dy\biggr\} \\ &= -\beta_1\int_\Omega \bigl|v_t(t)\bigr|^{m(y)}\,dy - \int_\Omega\int_{\tau_1}^{\tau_2} \beta_2(r)v_t\bigl|x(y,1,r,t)\bigr|^{m(y)-2}x(y,1,r,t)\,dr \end{aligned}$$

$$
\begin{aligned}
&- \beta_3 \int_\Omega \left|w_t(t)\right|^{s(y)}\,dy - \int_\Omega \int_{\tau_1}^{\tau_2} \beta_4(r) w_t \left|z(y,1,r,t)\right|^{s(y)-2} z(y,1,r,t)\,dr \\
&+ \frac{1}{2}\left(h_1' o \nabla v\right) - \frac{1}{2} h_1(t)\|\nabla v\|_2^2 + \frac{1}{2}\left(h_2' o \nabla w\right) - \frac{1}{2} h_2(t)\|\nabla w\|_2^2. \qquad (2.12)
\end{aligned}
$$

Now, multiplying $(2.8)_3$ by $(\frac{\delta m(y)-1}{m(y)}|x(y,1,r,t)|^{m(y)-1}|\beta_2(r)|)$, then integrating over $\Omega \times (0,1) \times (\tau_1, \tau_2)$, and applying $(2.6)_2$, the following is obtained:

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r\left|\beta_2(r)\right| \frac{(\delta m(y)-1)|x(y,\rho,r,t)|^{m(y)}}{m(y)}\,dr\,d\rho\,dr \\
&= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right|\left(\delta m(y)-1\right)|x|^{m(y)-1} x_\rho\,dr\,d\rho\,dy \\
&= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right| \frac{\delta m(y)-1}{m(y)} \frac{d}{d\rho}\left|x(y,\rho,r,t)\right|^{m(y)}\,dr\,d\rho\,dy \\
&= \int_\Omega \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right| \frac{\delta m(y)-1}{m(y)} \left(\left|x(y,0,r,t)\right|^{m(y)} - \left|x(y,1,r,t)\right|^{m(y)}\right)\,dr\,dy \\
&= \left(\int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right|\,dr\right) \int_\Omega \frac{\delta m(y)-1}{m(y)} \left|v_t(t)\right|^{m(y)}\,dy \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right| \frac{\delta m(y)-1}{m(y)} \left|x(y,1,r,t)\right|^{m(y)}\,dr\,dy, \qquad (2.13)
\end{aligned}
$$

and by the inequalities of Young, we have

$$
\begin{aligned}
&\int_\Omega v_t \left|x(y,1,r,t)\right|^{m(y)-2} x(y,1,r,t)\,dy \qquad (2.14) \\
&\quad \leq \int_\Omega \frac{1}{m(y)} \left|v_t(t)\right|^{m(y)}\,dy + \int_\Omega \frac{m(y)-1}{m(y)} \left|x(y,1,r,t)\right|^{m(y)}\,dy.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \beta_2(r) \int_\Omega v_t \left|x(y,1,r,t)\right|^{m(y)-2} x(y,1,r,t)\,dx\,ds \qquad (2.15) \\
&\quad \leq \left(\int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right|\,dr\right) \int_\Omega \frac{1}{m(y)} \left|v_t(t)\right|^{m(y)}\,dy \\
&\qquad + \int_\Omega \int_{\tau_1}^{\tau_2} \left|\beta_2(r)\right| \frac{m(y)-1}{m(r)} \left|x(y,1,r,t)\right|^{m(y)}\,ds\,dx.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r\left|\beta_4(r)\right| \frac{(\delta s(y)-1)|z(y,\rho,r,t)|^{s(y)}}{s(y)}\,dr\,d\rho\,dy \\
&= \left(\int_{\tau_1}^{\tau_2} \left|\beta_4(r)\right|\,dr\right) \int_\Omega \frac{\delta s(y)-1}{s(y)} \left|w_t(t)\right|^{s(y)}\,dy \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} \left|\beta_4(r)\right| \frac{\delta s(y)-1}{s(y)} \left|z(y,1,r,t)\right|^{s(y)}\,dr\,dy \qquad (2.16)
\end{aligned}
$$

and

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\beta_4(r)\int_\Omega w_t\big|z(y,1,r,t)\big|^{s(y)-2}z(y,1,r,t)\,dy\,dr\\&\quad\leq\left(\int_{\tau_1}^{\tau_2}\big|\beta_4(r)\big|\,dr\right)\int_\Omega\frac{1}{s(y)}\big|w_t(t)\big|^{s(y)}\,dy\\&\qquad+\int_\Omega\int_{\tau_1}^{\tau_2}\big|\beta_4(r)\big|\frac{s(y)-1}{s(y)}\big|z(y,1,r,t)\big|^{s(y)}\,ds\,dx.\end{aligned}\tag{2.17}$$

According to (2.12), (2.13), (2.15), (2.16), (2.17), we find (2.9) and

$$\begin{aligned}\frac{d}{dt}E(t)\leq&-\left(\beta_1-\delta\int_{\tau_1}^{\tau_2}\big|\beta_2(r)\big|\,dr\right)\int_\Omega\big|v_t(t)\big|^{m(y)}\,dy\\&-(\delta-1)\int_\Omega\int_{\tau_1}^{\tau_2}\big|\beta_2(r)\big|\big|x(y,1,r,t)\big|^{m(y)}\,dr\,dy\\&-\left(\beta_3-\delta\int_{\tau_1}^{\tau_2}\big|\beta_4(r)\big|\,dr\right)\int_\Omega\big|w_t(t)\big|^{s(y)}\,dy\\&-(\delta-1)\int_\Omega\int_{\tau_1}^{\tau_2}\big|\beta_4(r)\big|\big|z(y,1,r,t)\big|^{s(y)}\,dr\,dy\\&+\frac{1}{2}\big(h_1'o\nabla v\big)-\frac{1}{2}h_1(t)\|\nabla v\|_2^2+\frac{1}{2}\big(h_2'o\nabla w\big)-\frac{1}{2}h_2(t)\|\nabla w\|_2^2.\end{aligned}\tag{2.18}$$

Hence, by (2.3), we obtain (2.10), where

$$C_0=\min\left\{\left(\beta_1-\delta\int_{\tau_1}^{\tau_2}\big|\beta_2(s)\big|\,ds\right),\left(\beta_3-\delta\int_{\tau_1}^{\tau_2}\big|\beta_4(s)\big|\,ds\right),(\delta-1)\right\}>0,$$

and hence E is a decreasing function, which completes the proof. □

3 Blow-up

Here, we establish the blow-up result for the solution of (2.8). Initially, we introduce the functional as follows:

$$\begin{aligned}\mathbb{H}(t)=-E(t)=&-\frac{1}{\eta+2}\big[\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big]-\frac{1}{2}\big[\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big]\\&-\frac{1}{2(\gamma+1)}\big[\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big]-\mathcal{W}(x,z)\\&-\frac{1}{2}\left[\left(1-\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2+\left(1-\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right]\\&-\frac{1}{2}\big[(h_1o\nabla v)(t)+(h_2o\nabla w)(t)\big]+\int_\Omega F(v,w)\,dy.\end{aligned}\tag{3.1}$$

Theorem 3.1 *Assume that* (2.1)–(2.3) *hold and assume* $E(0)<0$, *then the solution of* (2.8) *blows up in finite time.*

Proof From (2.9), the following can be written:

$$E(t)\leq E(0)\leq 0.\tag{3.2}$$

Therefore

$$\begin{aligned}\mathbb{H}'(t) &= -E'(t)\\ &\geq C_0\left\{\int_\Omega \left|v_t(t)\right|^{m(y)}\,dy + \int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(r)\right|\left|x(y,1,r,t)\right|^{m(y)}\,dr\,dy\right.\\ &\quad\left.+\int_\Omega \left|w_t(t)\right|^{s(y)}\,dy + \int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_4(r)\right|\left|z(y,1,r,t)\right|^{s(y)}\,dr\,dy\right\}.\end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned}&\mathbb{H}'(t)\geq C_0\int_\Omega \left|v_t(t)\right|^{m(y)}\,dy\geq 0\\ &\mathbb{H}'(t)\geq C_0\int_\Omega \left|w_t(t)\right|^{s(y)}\,dy\geq 0\\ &\mathbb{H}'(t)\geq C_0\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(r)\right|\left|x(y,1,r,t)\right|^{m(y)}\,dr\,dy\geq 0\\ &\mathbb{H}'(t)\geq C_0\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_4(r)\right|\left|z(y,1,r,t)\right|^{s(y)}\,dr\,dy\geq 0.\end{aligned} \tag{3.4}$$

By (3.1) and (2.4), we have

$$\begin{aligned}0&\leq \mathbb{H}(0)\leq \mathbb{H}(t)\\ &\leq \int_\Omega F(v,w)\,dy\\ &\leq \int_\Omega \frac{c_1}{2(q(y)+2)}\left(|v|^{2(q(y)+2)}+|w|^{2(q(y)+2)}\right)dy\\ &\leq \frac{c_1}{2(q^-+2)}\left(\varrho(v)+\varrho(w)\right),\end{aligned} \tag{3.5}$$

in which

$$\varrho(v)=\varrho_{q(.)}(v)=\int_\Omega |v|^{2(q(y)+2)}\,dy.$$

Lemma 3.2 *Let* $\exists c>0$ *in a way that any solution of* (2.8) *fulfills*

$$\|v\|_{2(q^-+2)}^{2(q^-+2)}+\|w\|_{2(q^-+2)}^{2(q^-+2)}\leq c\left(\varrho(v)+\varrho(w)\right). \tag{3.6}$$

Proof Let

$$\Omega_1=\left\{y\in\Omega:\left|v(y,t)\right|\geq 1\right\},\qquad \Omega_2=\left\{y\in\Omega:\left|v(y,t)\right|<1\right\}, \tag{3.7}$$

we have

$$\begin{aligned}\varrho(v)&=\int_{\Omega_1}|v|^{2(q(y)+2)}\,dy+\int_{\Omega_2}|v|^{2(q(y)+2)}\,dy\\ &\geq \int_{\Omega_1}|v|^{2(q^-+2)}\,dy+c\left(\int_{\Omega_2}|v|^{2(q^-+2)}\,dy\right)^{\frac{2(q^++2)}{2(q^-+2)}},\end{aligned} \tag{3.8}$$

then

$$\varrho(v) \geq \int_{\Omega_1} |v|^{2(q^-+2)}\,dy$$
$$\left(\frac{\varrho(v)}{c}\right)^{\frac{2(q^-+2)}{2(q^++2)}} \geq \int_{\Omega_1} |v|^{2(q^-+2)}\,dy. \tag{3.9}$$

Hence, we get

$$\begin{aligned}\|v\|_{2(q^-+2)}^{2(q^-+2)} &\leq \varrho(v) + c\big(\varrho(v)\big)^{\frac{2(q^-+2)}{2(q^++2)}}\\ &\leq \big(\varrho(v)+\varrho(w)\big) + c\big(\varrho(v)+\varrho(w)\big)^{\frac{2(q^-+2)}{2(q^++2)}}\\ &\leq \big(\varrho(v)+\varrho(w)\big)\Big[1 + c\big(\varrho(v)+\varrho(w)\big)^{\frac{2(q^-+2)}{2(q^++2)}-1}\Big].\end{aligned} \tag{3.10}$$

According to (3.5), we have

$$\frac{\mathbb{H}(0)}{c} \leq \big(\varrho(v)+\varrho(w)\big).$$

Therefore,

$$\|v\|_{2(q^-+2)}^{2(q^-+2)} \leq \big(\varrho(v)+\varrho(w)\big)\Big[1 + c\big(\mathbb{H}(0)\big)^{\frac{2(q^-+2)}{2(q^++2)}-1}\Big].$$

Hence

$$\|v\|_{2(q^-+2)}^{2(q^-+2)} \leq c\big(\varrho(v)+\varrho(w)\big). \tag{3.11}$$

Similarly, we find

$$\|w\|_{2(q^-+2)}^{2(q^-+2)} \leq c\big(\varrho(v)+\varrho(w)\big). \tag{3.12}$$

The adding of (3.11) and (3.12) gives us (3.6). □

Corollary 3.3

$$\int_{\Omega} |v|^{m(y)}\,dy \leq c\Big(\big(\varrho(v)+\varrho(w)\big)^{m^-/2(q^-+2)} + \big(\varrho(v)+\varrho(w)\big)^{m^+/2(q^-+2)}\Big),$$
$$\int_{\Omega} |w|^{s(y)}\,dy \leq c\Big(\big(\varrho(v)+\varrho(w)\big)^{s^-/2(q^-+2)} + \big(\varrho(v)+\varrho(w)\big)^{s^+/2(q^-+2)}\Big). \tag{3.13}$$

Proof From (1.5), we have

$$\begin{aligned}\int_{\Omega} |v|^{m(y)}\,dy &\leq \int_{\Omega_1} |v|^{m^+}\,dy + \int_{\Omega_2} |v|^{m^-}\,dy\\ &\leq c\left(\int_{\Omega_1} |v|^{2(q^-+2)}\,dy\right)^{\frac{m^+}{2(q^-+2)}} + c\left(\int_{\Omega_2} |v|^{2(q^-+2)}\,dy\right)^{\frac{m^-}{2(q^-+2)}}\\ &\leq c\big(\|v\|_{2(q^-+2)}^{m^+} + \|v\|_{2(q^-+2)}^{m^-}\big).\end{aligned} \tag{3.14}$$

According to Lemma 3.2, we find $(3.13)_1$. Similarly, we obtain $(3.13)_2$. □

Now, take

$$\mathcal{K}(t) = \mathbb{H}^{1-\alpha}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega \left[v|v_t|^\eta v_t + w|w_t|^\eta w_t\right] dy + \varepsilon \int_\Omega [\nabla v_t \nabla v + \nabla w_t \nabla w]\, dy, \tag{3.15}$$

in which $0 < \varepsilon$ will be considered later and take

$$0 < \alpha < \min\left\{\left(1 - \frac{1}{2(q^-+2)} - \frac{1}{\eta+2}\right), \frac{1+2\gamma}{4(\gamma+1)}, \frac{2q^- + 4 - m^-}{(2q^-+4)(m^+-1)}, \frac{2q^- + 4 - m^+}{(2q^-+4)(m^+-1)}, \frac{2q^- + 4 - r^+}{(2q^-+4)(s^+-1)}, \frac{2q^- + 4 - s^-}{(2q^-+4)(s^+-1)}\right\} < 1. \tag{3.16}$$

By multiplying $(2.8)_1$, $(2.8)_2$ by v, w and with the help of (4.4), the following is achieved:

$$\begin{aligned}
\mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\right) \\
&\quad + \underbrace{\varepsilon \int_\Omega \nabla v \int_0^t g(t-r)\nabla v(r)\, dr\, dy}_{J_1} + \underbrace{\varepsilon \int_\Omega \nabla w \int_0^t h(t-r)\nabla w(r)\, dr\, dy}_{J_2} \\
&\quad \underbrace{- \varepsilon\beta_1 \int_\Omega v v_t |v_t|^{m(y)-2}\, dy}_{J_3} \underbrace{- \varepsilon\beta_3 \int_\Omega w w_t |w_t|^{s(y)-2}\, dy}_{J_4} \\
&\quad \underbrace{- \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} \beta_2(r) v x(y,1,r,t)\left|x(y,1,r,t)\right|^{m(y)-2}\, dr\, dy}_{J_5} \\
&\quad \underbrace{- \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} \beta_4(r) w z(y,1,r,t)\left|z(y,1,r,t)\right|^{s(y)-2}\, dr\, dy}_{J_6} \\
&\quad - \varepsilon\left(\|\nabla v\|_2^2 + \|\nabla w\|_2^2\right) - \varepsilon\left(\|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)}\right) \\
&\quad + \underbrace{\varepsilon \int_\Omega \left(v f_1(v,w) + w f_2(v,w)\right) dy}_{J_7}.
\end{aligned}$$

By (2.1), we obtain

$$\begin{aligned}
\mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\right) \\
&\quad + \underbrace{\varepsilon \int_\Omega \nabla v \int_0^t g(t-r)\nabla v(r)\, dr\, dy}_{J_1} + \underbrace{\varepsilon \int_\Omega \nabla w \int_0^t h(t-r)\nabla w(r)\, dr\, dy}_{J_2} \\
&\quad \underbrace{- \varepsilon\beta_1 \int_\Omega v v_t |v_t|^{m(y)-2}\, dy}_{J_3} \underbrace{- \varepsilon\beta_3 \int_\Omega w w_t |w_t|^{s(y)-2}\, dy}_{J_4}
\end{aligned}$$

$$
\begin{aligned}
&\underbrace{-\varepsilon \int_{\Omega}\int_{\tau_1}^{\tau_2} \beta_2(r) v x(y,1,r,t)\left|x(y,1,r,t)\right|^{m(y)-2}\,dr\,dy}_{J_5} \\
&\underbrace{-\varepsilon \int_{\Omega}\int_{\tau_1}^{\tau_2} \beta_4(r) w z(y,1,r,t)\left|z(y,1,r,t)\right|^{s(y)-2}\,dr\,dy}_{J_6} \\
&-\varepsilon\left(\|\nabla v\|_2^2+\|\nabla w\|_2^2\right)-\varepsilon\left(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\right) \\
&+\underbrace{\varepsilon\left(2q^-+4\right)\int_{\Omega} F(v,w)\,dy}_{J_7}.
\end{aligned}
\tag{3.17}
$$

We have

$$
\begin{aligned}
J_1 &= \varepsilon \int_0^t h_1(t-r)\,dr \int_{\Omega} \nabla v.\left(\nabla v(r)-\nabla v(t)\right)dy\,dr+\varepsilon\int_0^t h_1(r)\,dr\|\nabla v\|_2^2 \\
&\geq \frac{\varepsilon}{2}\int_0^t h_1(r)\,dr\|\nabla v\|_2^2-\frac{\varepsilon}{2}(h_1 o\nabla v),
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
J_2 &= \varepsilon \int_0^t h_2(t-r)\,dr \int_{\Omega} \nabla w.\left(\nabla w(r)-\nabla w(t)\right)dy\,dr+\varepsilon\int_0^t h_2(r)\,dr\|\nabla w\|_2^2 \\
&\geq \frac{\varepsilon}{2}\int_0^t h_2(r)\,dr\|\nabla w\|_2^2-\frac{\varepsilon}{2}(h_2 o\nabla w).
\end{aligned}
\tag{3.19}
$$

From (4.5), we find

$$
\begin{aligned}
\mathcal{K}'(t) \geq{}& (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\left(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\right)+\varepsilon\left(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\right) \\
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right] \\
&-\frac{\varepsilon}{2}(h_1 o\nabla v)-\frac{\varepsilon}{2}(h_2 o\nabla w)-\varepsilon\left(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\right) \\
&+J_3+J_4+J_5+J_6+J_7.
\end{aligned}
\tag{3.20}
$$

Applying the inequality of Young, we have for $\delta_1, \delta_2 > 0$

$$
J_3 \leq \varepsilon\beta_1\left\{\frac{1}{m^-}\int_{\Omega}\delta_1^{m(y)}|v|^{m(y)}\,dy+\frac{m^+-1}{m^+}\int_{\Omega}\delta_1^{-\frac{m(y)}{m(y)-1}}|v_t|^{m(y)}\,dy\right\},
\tag{3.21}
$$

$$
J_4 \leq \varepsilon\beta_3\left\{\frac{1}{s^-}\int_{\Omega}\delta_2^{s(y)}|w|^{s(y)}\,dy+\frac{s^+-1}{s^+}\int_{\Omega}\delta_2^{-\frac{s(y)}{s(y)-1}}|w_t|^{s(y)}\,dy\right\},
\tag{3.22}
$$

and

$$
\begin{aligned}
J_5 \leq \varepsilon\Bigg\{&\frac{(\int_{\tau_1}^{\tau_2}|\beta_2(r)|\,dr)}{m^-}\int_{\Omega}\delta_1^{m(y)}|v|^{m(y)}\,dy \\
&+\frac{m^+-1}{m^-}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\beta_2(r)\right|\delta_1^{-\frac{m(y)}{m(y)-1}}\left|x(y,1,r,t)\right|^{m(y)}\,dr\,dy\Bigg\},
\end{aligned}
\tag{3.23}
$$

$$
J_6 \leq \varepsilon\Bigg\{\frac{(\int_{\tau_1}^{\tau_2}|\beta_4(r)|\,dr)}{s^-}\int_{\Omega}\delta_2^{s(y)}|w|^{s(y)}\,dy
$$

$$+\frac{s^+-1}{s^-}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_4(r)\right|\delta_2^{-\frac{s(y)}{s(y)-1}}\left|z(y,1,r,t)\right|^{s(y)}\,dr\,dy\Big\}. \tag{3.24}$$

Therefore, by setting δ_1, δ_2 so that

$$\delta_1^{-\frac{m(y)}{m(y)-1}}=\frac{C_0}{2}\kappa\mathbb{H}^{-\alpha}(t),\qquad \delta_2^{-\frac{s(y)}{s(y)-1}}=\frac{C_0}{2}\kappa\mathbb{H}^{-\alpha}(t), \tag{3.25}$$

putting in (3.20), the following is obtained:

$$\begin{aligned}\mathcal{K}'(t)\geq{}&\big[(1-\alpha)-\varepsilon\kappa(\widehat{m}+\widehat{s})\big]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\&-\varepsilon\bigg[\bigg(1-\frac{1}{2}\int_0^t h_1(r)\,dr\bigg)\|\nabla v\|_2^2+\bigg(1-\frac{1}{2}\int_0^t h_2(r)\,dr\bigg)\|\nabla w\|_2^2\bigg]\\&+\varepsilon\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)-\frac{\varepsilon}{2}(h_1o\nabla v)-\frac{\varepsilon}{2}(h_2o\nabla w)\\&-\varepsilon\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\bigg(\frac{C_0\kappa}{2}\bigg)^{1-m(y)}\mathbb{H}^{\alpha(m(y)-1)}(t)|v|^{m(y)}\,dy\\&-\varepsilon\frac{\beta_3(\delta+1)}{\delta s^-}\int_\Omega\bigg(\frac{C_0\kappa}{2}\bigg)^{1-s(y)}\mathbb{H}^{\alpha(s(y)-1)}(t)|w|^{s(y)}\,dy\\&-\varepsilon\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big)+J_7,\end{aligned} \tag{3.26}$$

in which $\widehat{m}=\frac{m^+-1}{m^-}$, $\widehat{s}=\frac{s^+-1}{s^-}$, by using (3.5) and (3.13), we have

$$\begin{aligned}&\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\bigg(\frac{C_0\kappa}{2}\bigg)^{1-m(y)}\mathbb{H}^{\alpha(m(y)-1)}(t)|v|^{m(y)}\,dy\\&\quad\leq\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\bigg(\frac{C_0\kappa}{2}\bigg)^{1-m^-}\mathbb{H}^{\alpha(m^+-1)}(t)|v|^{m(y)}\,dy\\&\quad=C_1\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega|v|^{m(y)}\,dy\\&\quad\leq C_2\Big\{\big(\varrho(v)+\varrho(w)\big)^{\frac{m^-}{2(q^-+2)}+\alpha(m^+-1)}\big(\varrho(v)+\varrho(w)\big)^{\frac{m^+}{2(q^-+2)}+\alpha(m^+-1)}\Big\}.\end{aligned} \tag{3.27}$$

By (3.16), we find

$$\begin{aligned}&r=m^-+\alpha\big(2q^-+4\big)\big(m^+-1\big)\leq\big(2q^-+4\big),\\&r=m^++\alpha\big(2q^-+4\big)\big(m^+-1\big)\leq\big(2q^-+4\big),\end{aligned}$$

and by the inequality

$$x^\gamma\leq x+1\leq\bigg(1+\frac{1}{b}\bigg)(x+b),\quad\forall x\geq0,0<\gamma\leq1,b>0, \tag{3.28}$$

with $b=\frac{1}{\mathbb{H}(0)}$. Then we have

$$\big(\varrho(v)+\varrho(w)\big)^{\frac{m^-}{2(q^-+2)}+\alpha(m^+-1)}\leq\bigg(1+\frac{1}{\mathbb{H}(0)}\bigg)\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(0)\big)$$

$$\le C_3\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\big) \tag{3.29}$$

and

$$\big(\varrho(v)+\varrho(w)\big)^{\frac{m^+}{2(q^-+2)}+\alpha(m^+-1)} \le C_3\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\big), \tag{3.30}$$

where $C_3 = 1+\frac{1}{\mathbb{H}(0)}$. Substituting (3.29) and (3.30) into (3.27), we get

$$\begin{aligned}&\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-m(y)}\mathbb{H}^{\alpha(m(y)-1)}(t)|v|^{m(y)}\,dy\\&\quad\le C_4\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\big).\end{aligned} \tag{3.31}$$

Similarly, we find

$$\begin{aligned}&\frac{\beta_3(\delta+1)}{\delta s^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-s(y)}\mathbb{H}^{\alpha(s(y)-1)}(t)|w|^{s(y)}\,dy\\&\quad\le C_5\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\big),\end{aligned} \tag{3.32}$$

where $C_4 = C_4(\kappa) = C_3\frac{\beta_1(\delta+1)}{\delta m^-}(\frac{C_0\kappa}{2})^{1-m^-}$, $C_5 = C_5(\kappa) = C_3\frac{\beta_3(\delta+1)}{\delta s^-}(\frac{C_0\kappa}{2})^{1-s^-}$.
Combining (3.31), (3.32), and (3.26), and by (2.4), we obtain

$$\begin{aligned}\mathcal{K}'(t) &\ge \big[(1-\alpha)-\varepsilon\kappa(\widehat{m}+\widehat{s})\big]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\&\quad-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right]\\&\quad+\varepsilon\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)-\frac{\varepsilon}{2}(h_1o\nabla w)-\frac{\varepsilon}{2}(h_2o\nabla w)+J_7\\&\quad-\varepsilon(C_4+C_5)\big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\big)-\varepsilon\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big).\end{aligned} \tag{3.33}$$

Now, for $0<a<1$, from (3.1) and (2.4)

$$\begin{aligned}J_7 &= \varepsilon\big(2q^-+4\big)\int_\Omega F(v,w)\,dy\\&= \varepsilon a\big(2q^-+4\big)\int_\Omega F(v,w)\,dy\\&\quad+(1-a)\big(2q^-+4\big)\varepsilon\mathcal{W}(x,z)+\varepsilon(1-a)\big(2q^-+4\big)\mathbb{H}(t)\\&\quad+\frac{\varepsilon(1-a)(2q^-+4)}{\eta+2}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\&\quad+\varepsilon(1-a)\big(q^-+2\big)\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)\\&\quad+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t g(r)\,dr\right)\|\nabla v\|_2^2\\&\quad+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t h(r)\,dr\right)\|\nabla w\|_2^2\\&\quad+\varepsilon(1-a)\big(q^-+2\big)\big((h_1o\nabla v)+(h_2o\nabla w)\big)\end{aligned}$$

$$+\frac{\varepsilon(1-a)(q^-+2)}{(\gamma+1)}\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big). \tag{3.34}$$

Substituting (4.21) in (4.20) and applying (2.4), the following is obtained:

$$\begin{aligned}
\mathcal{K}'(t) \geq{}& \big\{(1-\alpha)-\varepsilon\kappa(\widehat{m}+\widehat{s})\big\}\mathbb{H}^{-\alpha}\mathbb{H}'(t)\\
&+\varepsilon\big\{(1-a)\big(q^-+2\big)+1\big\}\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)\\
&+\varepsilon\big\{(1-a)\big(2q^-+4\big)+1\big\}\mathcal{W}(x,z)\\
&+\varepsilon\left\{\frac{\varepsilon(1-a)(2q^-+4)}{\eta+2}+\frac{1}{\eta+1}\right\}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)\left(1-\int_0^t h_1(r)\,dr\right)-\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\right\}\|\nabla v\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)\left(1-\int_0^t h_2(r)\,dr\right)-\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\right\}\|\nabla w\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)-\frac{1}{2}\right\}(h_1 o\nabla v+h_2 o\nabla w)\\
&+\varepsilon\left\{\frac{(1-a)(q^-+2)}{\gamma+1}-1\right\}\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big)\\
&+\varepsilon\big\{c_0a-\big(C_4(\kappa)+C_5(\kappa)\big)\big\}\big(\varrho(v)+\varrho(w)\big)\\
&+\varepsilon\big\{(1-a)\big(2q^-+4\big)-\big(C_4(\kappa)+C_5(\kappa)\big)\big\}\mathbb{H}(t).
\end{aligned} \tag{3.35}$$

Here, choose $0<a$ in a manner that

$$\big(q^-+2\big)(1-a)>1+\gamma.$$

Further, we have

$$\begin{aligned}
\lambda_1 &:= \big(q^-+2\big)(1-a)-1>0\\
\lambda_2 &:= \big(q^-+2\big)(1-a)-\frac{1}{2}>0\\
\lambda_3 &:= \frac{(q^-+2)(1-a)}{\gamma+1}-1>0,
\end{aligned}$$

and suppose

$$\max\left\{\int_0^\infty h_1(r)\,dr,\int_0^\infty h_2(r)\,dr\right\}<\frac{(q^-+2)(1-a)-1}{((q^-+2)(1-a)-\frac{1}{2})}=\frac{2\lambda_1}{2\lambda_1+1}, \tag{3.36}$$

which gives

$$\begin{aligned}
\lambda_4 &= \left\{\big((q^-+2)(1-a)-1\big)-\int_0^t h_1(r)\,dr\left(\big(q^-+2\big)(1-a)-\frac{1}{2}\right)\right\}>0,\\
\lambda_5 &= \left\{\big((q^-+2)(1-a)-1\big)-\int_0^t h_2(r)\,dr\left(\big(q^-+2\big)(1-a)-\frac{1}{2}\right)\right\}>0.
\end{aligned}$$

After that, select κ large enough that

$$\lambda_6 = ac_0 - \big(C_4(\kappa) + C_5(\kappa)\big) > 0,$$
$$\lambda_7 = 2\big(q^- + 2\big)(1-a) - \big(C_4(\kappa) + C_5(\kappa)\big) > 0.$$

In the last stage, take κ, a, and we pick ε in a way that

$$\lambda_8 = (1-\alpha) - \varepsilon\kappa(\widehat{m} + \widehat{s}) > 0,$$

and

$$\begin{aligned}\mathcal{K}(0) &= \mathbb{H}^{1-\alpha}(0) + \frac{\varepsilon}{\eta+1}\int_\Omega \big[v_0|v_1|^\eta v_1 + w_0|w_1|^\eta w_1\big]\,dy \\ &\quad + \varepsilon\int_\Omega [\nabla v_1\nabla v_0 + \nabla w_1 \nabla w_0]\,dy > 0. \end{aligned}\tag{3.37}$$

Thus, for some $\mu > 0$, (3.35) implies

$$\begin{aligned}\mathcal{K}'(t) \ge \mu\big\{&\mathbb{H}(t) + \|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)} \\ &+ \|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + (h_1 o\nabla v) + (h_2 o \nabla w) \\ &+ \varrho(v) + \varrho(w) + \mathcal{W}(x,z)\big\}\end{aligned}\tag{3.38}$$

and

$$\mathcal{K}(t) \ge \mathcal{K}(0) > 0, \quad t > 0.\tag{3.39}$$

In the coming step, applying the inequalities of Holder and Young, we get

$$\begin{aligned}\left|\int_\Omega \big(v|v_t|^\eta v_t + w|w_t|^\eta w_t\big)\,dy\right|^{\frac{1}{1-\alpha}} &\le C\Big[\|v\|_{2(q^-+2)}^{\frac{\theta}{1-\alpha}} + \|v_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}} \\ &\quad + \|w\|_{2(q^-+2)}^{\frac{\theta}{1-\alpha}} + \|w_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}}\Big],\end{aligned}\tag{3.40}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

Select $\mu = (\eta+2)(1-\alpha)$ to obtain the following:

$$\frac{\theta}{1-\alpha} = \frac{\eta+2}{(1-\alpha)(\eta+2)-1} \le 2\big(q^- + 2\big).$$

Consequently, by the application of (3.5), (3.16), and (3.28), we have

$$\|v\|_{2(q^-+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} \le d\big(\|v\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{H}(t)\big)$$
$$\|w\|_{2(q^-+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} \le d\big(\|w\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{H}(t)\big), \quad \forall t \ge 0.$$

Therefore, we have

$$\left|\int_\Omega \big(v|v_t|^\eta v_t + w|w_t|^\eta w_t\big)\,dy\right|^{\frac{1}{1-\alpha}}$$

$$\leq c\Big\{\varrho(v) + \varrho(w) + \|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \mathbb{H}(t)\Big\}. \tag{3.41}$$

In the same way, we have

$$\left|\int_\Omega (\nabla v \nabla v_t + \nabla w \nabla w_t)\, dy\right|^{\frac{1}{1-\alpha}} \leq C\Big[\|\nabla v\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla v_t\|_2^{\frac{\mu}{1-\alpha}} + \|\nabla w\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla w_t\|_2^{\frac{\mu}{1-\alpha}}\Big],$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

For the next step, assume $\theta = 2(\gamma+1)(1-\alpha)$ to get

$$\frac{\mu}{1-\alpha} = \frac{2(\gamma+1)}{2(1-\alpha)(\gamma+1)-1} \leq 2$$

$$\left|\int_\Omega (\nabla v \nabla v_t + \nabla w \nabla w_t)\, dy\right|^{\frac{1}{1-\alpha}} \leq c\Big\{\|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)} + \|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\Big\}. \tag{3.42}$$

Thus, by (3.41) and (3.42),

$$\begin{aligned}
\mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \bigg(\mathbb{H}^{1-\alpha}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega \big(v|v_t|^\eta v_t + w|w_t|^\eta w_t\big)\, dy \\
&\quad + \varepsilon \int_\Omega (\nabla v_t \nabla v + \nabla w_t \nabla w)\, dy\bigg)^{\frac{1}{1-\alpha}} \\
&\leq c\bigg(\mathbb{H}(t) + \left|\int_\Omega \big(v|v_t|^\eta v_t + w|w_t|^\eta w_t\big)\, dy\right|^{\frac{1}{1-\alpha}} + \|\nabla v\|_2^{\frac{2}{1-\alpha}} + \|\nabla w\|_2^{\frac{2}{1-\alpha}} \\
&\quad + \|\nabla v_t\|_2^{\frac{2}{1-\alpha}} + \|\nabla w_t\|_2^{\frac{2}{1-\alpha}}\bigg) \\
&\leq c\big(\mathbb{H}(t) + \|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)} + \|\nabla v_t\|_2^2 \\
&\quad + \|\nabla w_t\|_2^2 + (h_1 o \nabla v) + (h_2 o \nabla w) + \varrho(v) + \varrho(w)\big) \\
&\leq c\Big\{\mathbb{H}(t) + \|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)} \\
&\quad + \|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + (h_1 o \nabla v) + (h_2 o \nabla w) \\
&\quad + \varrho(v) + \varrho(w) + \mathcal{W}(x,z)\Big\}.
\end{aligned} \tag{3.43}$$

Now, (3.38) and (3.43) imply

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.44}$$

in which $0 < \lambda$, this relies only on β and c.

Further simplification of (4.31) leads to

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda \frac{\alpha}{(1-\alpha)} t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Thus, it completes the proof. □

4 Growth of solution

Here, the exponential growth of solution of problem (2.8) will be established.

For this, the functional is defined as follows:

$$\begin{aligned}
\mathbb{H}(t) &= -E(t) \\
&= -\frac{1}{\eta+2}\left[\|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2}\right] - \frac{1}{2}\left[\|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\right] \\
&\quad - \frac{1}{2(\gamma+1)}\left[\|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)}\right] - \mathcal{W}(x,z) \\
&\quad - \frac{1}{2}\left[\left(1-\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2 + \left(1-\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right] \\
&\quad - \frac{1}{2}\left[(h_1 o \nabla v)(t) + (h_2 o \nabla w)(t)\right] + \int_\Omega F(v,w)\,dy.
\end{aligned} \tag{4.1}$$

Theorem 4.1 *Assume that* (2.1)–(2.3) *are satisfied, and suppose* $E(0) < 0$, *then*

$$2\left(q^- + 2\right) > \frac{\eta+2}{\eta+1}. \tag{4.2}$$

Then the solution of problem (2.8) *grows exponentially.*

Proof To prove the required result, (2.9) implies

$$E(t) \leq E(0) \leq 0 \tag{4.3}$$

with the help of (3.3) and (3.4).

Now, take the following:

$$\begin{aligned}
\mathcal{R}(t) &= \mathbb{H}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega \left[v|v_t|^\eta v_t + w|w_t|^\eta w_t\right] dy \\
&\quad + \varepsilon\int_\Omega [\nabla v_t \nabla v + \nabla w_t \nabla w]\,dy,
\end{aligned} \tag{4.4}$$

in which $\varepsilon > 0$ will be chosen in a later stage.

From $(2.8)_1$, $(2.8)_2$, and (4.4), we have

$$\begin{aligned}
\mathcal{R}'(t) &= \mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2\right) \\
&\quad + \underbrace{\varepsilon\int_\Omega \nabla v \int_0^t g(t-r)\nabla v(r)\,dr\,dy}_{I_1} + \underbrace{\varepsilon\int_\Omega \nabla w \int_0^t h(t-r)\nabla w(r)\,dr\,dy}_{I_2}
\end{aligned}$$

$$
\begin{aligned}
&\underbrace{-\varepsilon\beta_1\int_\Omega vv_t|v_t|^{m(y)-2}\,dy}_{I_3}\underbrace{-\varepsilon\beta_3\int_\Omega ww_t|w_t|^{s(y)-2}\,dy}_{I_4}\\
&\underbrace{-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\beta_2(r)vx(y,1,r,t)\big|x(y,1,r,t)\big|^{m(y)-2}\,dr\,dy}_{I_5}\\
&\underbrace{-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\beta_4(r)wz(y,1,r,t)\big|z(y,1,r,t)\big|^{s(y)-2}\,dr\,dy}_{I_6}\\
&-\varepsilon\big(\|\nabla v\|_2^2+\|\nabla w\|_2^2\big)-\varepsilon\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big)\\
&\underbrace{+\varepsilon\int_\Omega\big(vf_1(v,w)+wf_2(v,w)\big)\,dy}_{I_7}.
\end{aligned}
$$

By (2.1), we obtain

$$
\begin{aligned}
\mathcal{R}'(t)\geq{}&\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)+\varepsilon\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)\\
&\underbrace{+\varepsilon\int_\Omega\nabla v\int_0^t g(t-r)\nabla v(r)\,dr\,dy}_{I_1}\underbrace{+\varepsilon\int_\Omega\nabla w\int_0^t h(t-r)\nabla w(r)\,dr\,dy}_{I_2}\\
&\underbrace{-\varepsilon\beta_1\int_\Omega vv_t|v_t|^{m(y)-2}\,dy}_{I_3}\underbrace{-\varepsilon\beta_3\int_\Omega ww_t|w_t|^{s(y)-2}\,dy}_{I_4}\\
&\underbrace{-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\beta_2(r)vx(y,1,r,t)\big|x(y,1,r,t)\big|^{m(y)-2}\,dr\,dy}_{I_5}\\
&\underbrace{-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\beta_4(r)wz(y,1,r,t)\big|z(y,1,r,t)\big|^{s(y)-2}\,dr\,dy}_{I_6}\\
&-\varepsilon\big(\|\nabla v\|_2^2+\|\nabla w\|_2^2\big)-\varepsilon\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big)\\
&\underbrace{+\varepsilon\big(2q^-+4\big)\int_\Omega F(v,w)\,dy}_{I_7}.
\end{aligned}
\tag{4.5}
$$

Similar to J_1, J_2 in (3.21) and (3.22), we estimate I_1, I_2:

$$
I_1=J_1\geq\frac{\varepsilon}{2}\int_0^t h_1(r)\,dr\|\nabla v\|_2^2-\frac{\varepsilon}{2}(h_1o\nabla v),
\tag{4.6}
$$

$$
I_2=J_2\geq\frac{\varepsilon}{2}\int_0^t h_2(r)\,dr\|\nabla w\|_2^2-\frac{\varepsilon}{2}(h_2o\nabla w).
\tag{4.7}
$$

From (4.5), we find

$$
\mathcal{K}'(t)\geq\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)+\varepsilon\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)
$$

$$
\begin{aligned}
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right]\\
&-\frac{\varepsilon}{2}(h_1 o\nabla v)-\frac{\varepsilon}{2}(h_2 o\nabla w)-\varepsilon\left(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\right)\\
&+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
\tag{4.8}
$$

Similar to J_3, J_4, J_5, and J_6 in (3.20)–(3.24), we estimate I_i, $i=3,\dots,6$. By Young's inequality, we find for $\delta_1,\delta_2>0$

$$
I_3 \le \varepsilon\beta_1\left\{\frac{1}{m^-}\int_\Omega \delta_1^{m(y)}|v|^{m(y)}\,dy+\frac{m^+-1}{m^+}\int_\Omega \delta_1^{-\frac{m(y)}{m(y)-1}}|v_t|^{m(y)}\,dy\right\},
\tag{4.9}
$$

$$
I_4 \le \varepsilon\beta_3\left\{\frac{1}{s^-}\int_\Omega \delta_2^{s(y)}|w|^{s(y)}\,dy+\frac{s^+-1}{s^+}\int_\Omega \delta_2^{-\frac{s(y)}{s(y)-1}}|w_t|^{s(y)}\,dy\right\},
\tag{4.10}
$$

and

$$
\begin{aligned}
I_5 \le \varepsilon\Bigg\{&\frac{(\int_{\tau_1}^{\tau_2}|\beta_2(r)|\,dr)}{m^-}\int_\Omega \delta_1^{m(y)}|v|^{m(y)}\,dy\\
&+\frac{m^+-1}{m^-}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(r)\right|\delta_1^{-\frac{m(y)}{m(y)-1}}\left|x(y,1,r,t)\right|^{m(y)}\,dr\,dy\Bigg\},
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
I_6 \le \varepsilon\Bigg\{&\frac{(\int_{\tau_1}^{\tau_2}|\beta_4(r)|\,dr)}{s^-}\int_\Omega \delta_2^{s(y)}|w|^{s(y)}\,dy\\
&+\frac{s^+-1}{s^-}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_4(r)\right|\delta_2^{-\frac{s(y)}{s(y)-1}}\left|z(y,1,r,t)\right|^{s(y)}\,dr\,dy\Bigg\}.
\end{aligned}
\tag{4.12}
$$

Therefore, by setting δ_1, δ_2 so that

$$
\delta_1^{-\frac{m(y)}{m(y)-1}}=\frac{C_0}{2}\kappa,\qquad \delta_2^{-\frac{s(y)}{s(y)-1}}=\frac{C_0}{2}\kappa,
\tag{4.13}
$$

substituting in (4.8), the following is achieved:

$$
\begin{aligned}
\mathcal{R}'(t)\ge{}&\left[1-\varepsilon\kappa(\widehat{m}+\widehat{s})\right]\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\left(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\right)\\
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right]\\
&+\varepsilon\left(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\right)-\frac{\varepsilon}{2}(h_1 o\nabla v)-\frac{\varepsilon}{2}(h_2 o\nabla w)\\
&-\varepsilon\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-m(y)}|v|^{m(y)}\,dy\\
&-\varepsilon\frac{\beta_3(\delta+1)}{\delta s^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-s(y)}|w|^{s(y)}\,dy\\
&-\varepsilon\left(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\right)+I_7,
\end{aligned}
\tag{4.14}
$$

where $\widehat{m}=\frac{m^+-1}{m^-}$, $\widehat{s}=\frac{s^+-1}{s^-}$. By using (3.5) and (3.13), we have

$$
\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-m(y)}|v|^{m(y)}\,dy\le\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega\left(\frac{C_0\kappa}{2}\right)^{1-m^-}|v|^{m(y)}\,dy
$$

$$
\begin{aligned}
&= C_8 \int_\Omega |v|^{m(y)}\,dy \\
&\leq C_9\Big\{\big(\varrho(v)+\varrho(w)\big)^{\frac{m^-}{2(q^-+2)}} \\
&\quad + \big(\varrho(v)+\varrho(w)\big)^{\frac{m^+}{2(q^-+2)}}\Big\}. \qquad (4.15)
\end{aligned}
$$

By (1.5), we find

$$
r = m^- \leq \big(2q^- + 4\big), \qquad r = m^+ \leq \big(2q^- + 4\big),
$$

and by (3.28) with $b = \frac{1}{\mathbb{H}(0)}$. Then we have

$$
\begin{aligned}
\big(\varrho(v)+\varrho(w)\big)^{\frac{m^-}{2(q^-+2)}} &\leq \left(1+\frac{1}{\mathbb{H}(0)}\right)\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(0)\Big) \\
&\leq C_{10}\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\Big) \qquad (4.16)
\end{aligned}
$$

and

$$
\big(\varrho(v)+\varrho(w)\big)^{\frac{m^+}{2(q^-+2)}} \leq C_{10}\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\Big), \qquad (4.17)
$$

where $C_{10} = 1 + \frac{1}{\mathbb{H}(0)}$. Substituting (4.16) and (4.17) into (4.15), we get

$$
\frac{\beta_1(\delta+1)}{\delta m^-}\int_\Omega \left(\frac{C_0\kappa}{2}\right)^{1-m(y)} |v|^{m(y)}\,dy \leq C_{11}\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\Big). \qquad (4.18)
$$

Similarly, we find

$$
\frac{\beta_3(\delta+1)}{\delta s^-}\int_\Omega \left(\frac{C_0\kappa}{2}\right)^{1-s(y)} |w|^{s(y)}\,dy \leq C_{12}\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\Big), \qquad (4.19)
$$

where $C_{11} = C_{11}(\kappa) = C_9\frac{\beta_1(\delta+1)}{\delta m^-}(\frac{C_0\kappa}{2})^{1-m^-}$, $C_{12} = C_{12}(\kappa) = C_9\frac{\beta_3(\delta+1)}{\delta s^-}(\frac{C_0\kappa}{2})^{1-s^-}$.

Combining (4.18), (4.19), and (4.14), we have

$$
\begin{aligned}
\mathcal{R}'(t) \geq{}& \big[1-\varepsilon\kappa(\widehat{m}+\widehat{s})\big]\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big) \\
&- \varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2 + \left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\|\nabla w\|_2^2\right] \\
&+ \varepsilon\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big) - \frac{\varepsilon}{2}(h_1 o\nabla v) - \frac{\varepsilon}{2}(h_2 o\nabla w) + I_7 \qquad (4.20) \\
&- \varepsilon(C_{11}+C_{12})\Big(\big(\varrho(v)+\varrho(w)\big)+\mathbb{H}(t)\Big) - \varepsilon\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\big).
\end{aligned}
$$

Now, for $0 < a < 1$, from (4.1) and (2.4)

$$
\begin{aligned}
J_7 &= \varepsilon\big(2q^-+4\big)\int_\Omega F(v,w)\,dy \\
&= \varepsilon a\big(2q^-+4\big)\int_\Omega F(v,w)\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+(1-a)\big(2q^{-}+4\big)\varepsilon\mathcal{W}(x,z)+\varepsilon(1-a)\big(2q^{-}+4\big)\mathbb{H}(t)\\
&+\frac{\varepsilon(1-a)(2q^{-}+4)}{\eta+2}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\
&+\varepsilon(1-a)\big(q^{-}+2\big)\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)\\
&+\varepsilon(1-a)\big(q^{-}+2\big)\left(1-\int_0^t g(r)\,dr\right)\|\nabla v\|_2^2\\
&+\varepsilon(1-a)\big(q^{-}+2\big)\left(1-\int_0^t h(r)\,dr\right)\|\nabla w\|_2^2\\
&+\varepsilon(1-a)\big(q^{-}+2\big)\big((h_1 o\nabla v)+(h_2 o\nabla w)\big)\\
&+\frac{\varepsilon(1-a)(q^{-}+2)}{(\gamma+1)}\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big).
\end{aligned}
\tag{4.21}
$$

Substituting (4.21) in (4.20) and applying (2.4), we have

$$
\begin{aligned}
\mathcal{R}'(t)\geq{}&\big\{1-\varepsilon\kappa(\widehat{m}+\widehat{s})\big\}\mathbb{H}'(t)\\
&+\varepsilon\big\{(1-a)\big(q^{-}+2\big)+1\big\}\big(\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2\big)\\
&+\varepsilon\big\{(1-a)\big(2q^{-}+4\big)+1\big\}\mathcal{W}(x,z)\\
&+\varepsilon\left\{\frac{\varepsilon(1-a)(2q^{-}+4)}{\eta+2}+\frac{1}{\eta+1}\right\}\big(\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}\big)\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)\left(1-\int_0^t h_1(r)\,dr\right)-\left(1-\frac{1}{2}\int_0^t h_1(r)\,dr\right)\right\}\|\nabla v\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)\left(1-\int_0^t h_2(r)\,dr\right)-\left(1-\frac{1}{2}\int_0^t h_2(r)\,dr\right)\right\}\|\nabla w\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)-\frac{1}{2}\right\}(h_1 o\nabla v+h_2 o\nabla w)\\
&+\varepsilon\left\{\frac{(1-a)(q^{-}+2)}{\gamma+1}-1\right\}\big(\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\big)\\
&+\varepsilon\big\{c_0 a-\big(C_{11}(\kappa)+C_{12}(\kappa)\big)\big\}\big(\varrho(v)+\varrho(w)\big)\\
&+\varepsilon\big\{(1-a)\big(2q^{-}+4\big)-\big(C_{11}(\kappa)+C_{12}(\kappa)\big)\big\}\mathbb{H}(t).
\end{aligned}
\tag{4.22}
$$

Here, assume that $0<a$ is small in a manner that

$$
\big(q^{-}+2\big)(1-a)>1+\gamma,
$$

we have

$$
\begin{aligned}
&\lambda_1:=\big(q^{-}+2\big)(1-a)-1>0,\\
&\lambda_2:=\big(q^{-}+2\big)(1-a)-\frac{1}{2}>0,\\
&\lambda_3:=\frac{(q^{-}+2)(1-a)}{\gamma+1}-1>0,
\end{aligned}
$$

and we assume

$$\max\left\{\int_0^\infty h_1(r)\,dr, \int_0^\infty h_2(r)\,dr\right\} < \frac{(q^- + 2)(1-a) - 1}{((q^- + 2)(1-a) - \frac{1}{2})} = \frac{2\lambda_1}{2\lambda_1 + 1}, \tag{4.23}$$

which gives

$$\lambda_4 = \left\{\left((q^- + 2)(1-a) - 1\right) - \int_0^t h_1(r)\,dr\left(\left(q^- + 2\right)(1-a) - \frac{1}{2}\right)\right\} > 0,$$
$$\lambda_5 = \left\{\left((q^- + 2)(1-a) - 1\right) - \int_0^t h_2(r)\,dr\left(\left(q^- + 2\right)(1-a) - \frac{1}{2}\right)\right\} > 0.$$

After this, select κ large in a way that

$$\lambda_6 = ac_0 - \left(C_{11}(\kappa) + C_{12}(\kappa)\right) > 0,$$
$$\lambda_7 = 2\left(q^- + 2\right)(1-a) - \left(C_{11}(\kappa) + C_{12}(\kappa)\right) > 0.$$

At the last stage, fix κ, a and pick ε small such that

$$\lambda_8 = (1-\alpha) - \varepsilon\kappa(\widehat{m} + \widehat{s}) > 0$$

and

$$\mathcal{R}(0) = \mathbb{H}(0) + \frac{\varepsilon}{\eta + 1}\int_\Omega \left[v_0|v_1|^\eta v_1 + w_0|w_1|^\eta w_1\right] dy,$$
$$+ \varepsilon \int_\Omega [\nabla v_1 \nabla v_0 + \nabla w_1 \nabla w_0]\,dy > 0, \tag{4.24}$$

and from (4.4)

$$\mathcal{R}(t) \le \frac{c_1}{2(q^- + 2)}\left[\|v\|_{2(q^+ + 2)}^{2(q^+ + 2)} + \|w\|_{2(q^+ + 2)}^{2(q^+ + 2)}\right]. \tag{4.25}$$

Thus, for some $\mu_1 > 0$, (4.22) implies

$$\begin{aligned}\mathcal{R}'(t) \ge \mu_1\Big\{&\mathbb{H}(t) + \|v_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \|\nabla v\|_2^{2(\gamma+1)} + \|\nabla w\|_2^{2(\gamma+1)} \\ &+ \|\nabla v_t\|_2^2 + \|\nabla w_t\|_2^2 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + (h_1 o \nabla v) + (h_2 o \nabla w) \\ &+ \varrho(v) + \varrho(w) + \mathcal{W}(x,z)\Big\}\end{aligned} \tag{4.26}$$

and

$$\mathcal{R}(t) \ge \mathcal{R}(0) > 0, \quad t > 0. \tag{4.27}$$

Further, applying the inequalities of Holder and Young, we get

$$\begin{aligned}\left|\int_\Omega \left(v|v_t|^\eta v_t + w|w_t|^\eta w_t\right) dy\right| \le C\Big[&\|v\|_{2(q^-+2)}^\theta + \|v_t\|_{\eta+2}^\mu \\ &+ \|w\|_{2(q^-+2)}^\theta + \|w_t\|_{\eta+2}^\mu\Big],\end{aligned} \tag{4.28}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. Next, assume $\mu=(\eta+2)$ to reach

$$\theta=\frac{(\eta+2)}{(\eta+1)}\leq 2\big(q^{-}+2\big).$$

Subsequently, by using (4.2) and (3.28), we obtain

$$\begin{aligned}
&\|v\|_{2(q^{-}+2)}^{\frac{\eta+2}{(\eta+1)}}\leq K\big(\|v\|_{2(q^{-}+2)}^{2(q^{-}+2)}+\mathbb{H}(t)\big)\\
&\|w\|_{2(q^{-}+2)}^{\frac{\eta+2}{(\eta+1)}}\leq K\big(\|w\|_{2(q^{-}+2)}^{2(q^{-}+2)}+\mathbb{H}(t)\big),\quad \forall t\geq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\int_{\Omega}\big(v|v_t|^{\eta}v_t+w|w_t|^{\eta}w_t\big)\,dy\right|\\
&\quad\leq c\big\{\big(\varrho(v)+\varrho(v)\big)+\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}+\mathbb{H}(t)\big\}.
\end{aligned}\tag{4.29}$$

Hence

$$\begin{aligned}
\mathcal{R}(t)&=\left(\mathbb{H}(t)+\frac{\varepsilon}{\eta+1}\int_{\Omega}\big(v|v_t|^{\eta}v_t+w|w_t|^{\eta}w_t\big)\,dy\right.\\
&\quad\left.+\varepsilon\int_{\Omega}(\nabla v_t\nabla v+\nabla w_t\nabla w)\,dy\right)\\
&\leq c\big(\mathbb{H}(t)+\|v_t\|_{\eta+2}^{\eta+2}+\|w_t\|_{\eta+2}^{\eta+2}+\|\nabla v\|_2^2+\|\nabla w\|_2^2\\
&\quad+\|\nabla v_t\|_2^2+\|\nabla w_t\|_2^2+\|\nabla v\|_2^{2(\gamma+1)}+\|\nabla w\|_2^{2(\gamma+1)}\\
&\quad+(h_1 o\nabla v)+(h_2 o\nabla w)+\mathcal{W}(x,z)+\big(\varrho(v)+\varrho(v)\big)\big).
\end{aligned}\tag{4.30}$$

From (4.26) and (4.30), we have

$$\mathcal{R}'(t)\geq\lambda_1\mathcal{R}(t),\tag{4.31}$$

where $\lambda_1>0$, this relies on μ_1 and c only. Further, (4.31) implies

$$\mathcal{R}(t)\geq\mathcal{R}(0)e^{(\lambda_1 t)},\quad \forall t>0.\tag{4.32}$$

From (4.4) and (4.25), we get

$$\mathcal{R}(t)\leq c\big(\|v\|_{2(q^{+}+2)}^{2(q^{+}+2)}+\|w\|_{2(q^{+}+2)}^{2(q^{+}+2)}\big).\tag{4.33}$$

Then (4.32) and (4.33) imply

$$\|v\|_{2(q^{+}+2)}^{2(q^{+}+2)}+\|w\|_{2(q^{+}+2)}^{2(q^{+}+2)}\geq Ce^{(\lambda_1 t)},\quad \forall t>0.$$

Therefore, we deduce that the solution experiences exponential growth in the $L^{2(p^{+}+2)}$ norm. This concludes the proof. □

5 General decay

In this section, we state and prove the general decay of system (2.8) in the case $f_1 = f_2 = 0$. For this goal, problem (2.8) can be written as

$$\begin{cases} |v_t|^{\eta} v_{tt} - M(\|\nabla v\|_2^2)\Delta v + \int_0^t h_1(t-r)\Delta v(r)\,dr - \Delta v_{tt} + \beta_1 |v_t(t)|^{m(y)-2} v_t(t) \\ \quad + \int_{\tau_1}^{\tau_2} |\beta_2(r)||x(y,1,r,t)|^{m(y)-2} x(y,1,r,t)\,dr = 0, \\ r x_t(y,\rho,r,t) + x_\rho(y,\rho,r,t) = 0, \\ v(y,0) = v_0(y), \qquad v_t(y,0) = v_1(y), \quad \text{in } \Omega \\ x(y,\rho,r,0) = f_0(y,\rho r), \quad \text{in } \Omega \times (0,1) \times (0,\tau_2) \\ v(y,t) = 0, \quad \text{in } \partial\Omega \times (0,T), \end{cases} \tag{5.1}$$

where

$$(y,\rho,r,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,T).$$

Here, we introduce the modified energy functional $\mathcal{E}$ of (5.1) as follows:

$$\begin{aligned} \mathcal{E}(t) = {} & \frac{1}{\eta+2}\|v_t\|_{\eta+2}^{\eta+2} + \frac{1}{2}\|\nabla v_t\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla v\|_2^{2(\gamma+1)} + \mathcal{F}(x) \\ & + \frac{1}{2}\left(1 - \int_0^t h_1(r)\,dr\right)\|\nabla v\|_2^2 + \frac{1}{2}(h_1 o \nabla v)(t). \end{aligned} \tag{5.2}$$

Similar to Lemma 2.4, the energy functional fulfills for assumption (2.3)

$$\begin{aligned} \mathcal{E}'(t) \le {} & -C_0 \left\{ \int_\Omega |v_t(t)|^{m(y)}\,dy + \int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(r)| |x(y,1,r,t)|^{m(y)}\,dr\,dy \right\} \\ & + \frac{1}{2}(h_1' o \nabla v)(t) - \frac{1}{2} h_1(t)\|\nabla v\|_2^2 \le 0, \end{aligned} \tag{5.3}$$

where

$$\mathcal{F}(z) := \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r|\beta_2(r)| \frac{(\delta m(y)-1)|x(y,\rho,r,t)|^{m(y)}}{m(y)}\,dr\,d\rho\,dy. \tag{5.4}$$

Remark 5.1 In this case $f_1 = f_2 = 0$. Condition (2.3) remains true for ($\delta = 1$), i.e., it can be replaced by

$$\int_{\tau_1}^{\tau_2} |\beta_2(r)|\,dr < \beta_1. \tag{5.5}$$

Also, relation (5.3) becomes of the form

$$\mathcal{E}'(t) \le -C_0 \int_\Omega |v_t(t)|^{m(y)}\,dy + \frac{1}{2}(h_1' o \nabla v)(t) - \frac{1}{2} h_1(t)\|\nabla v\|_2^2 \le 0. \tag{5.6}$$

Lemma 5.2 (*Komornik*, [20]) *Assume a nonincreasing function* $E : \mathbb{R}_+ \to \mathbb{R}_+$, *and suppose that* $\exists \sigma, \omega > 0$ *in a manner that*

$$\int_r^\infty E^{1+\sigma}(t)\,dt \le \frac{1}{\Omega} E^\sigma(0) E(r) = cE(r), \quad \forall r > 0. \tag{5.7}$$

Then we have $\forall t \geq 0$

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{\frac{1}{\sigma}}, & \text{if } \sigma > 0, \\ E(t) \leq cE(0)e^{-\omega t}, & \text{if } \sigma = 0. \end{cases} \tag{5.8}$$

Theorem 5.3 *Assume that* (1.3), (2.1)–(2.3), *and* (2.5) *hold. Then* $\exists c, \lambda > 0$ *such that the solution of* (5.1) *fulfills*

$$\begin{cases} \mathcal{E}(t) \leq c\mathcal{E}(0)/(1+t)^{\frac{2}{m^+-2}}, & \text{if } m^+ > 2, \\ \mathcal{E}(t) \leq c\mathcal{E}^{-\lambda t}, & \text{if } m(y) = 2. \end{cases} \tag{5.9}$$

Proof Multiplying $(5.1)_1$ by $v\mathcal{E}^p(t)$ for $p > 0$ to be specified later and integrating the result over $\Omega \times (s, T), s < T$, we have

$$\begin{aligned} &\int_r^T \mathcal{E}^p(t) \int_\Omega \biggl\{ v|v_t|^\eta v_{tt} - M\bigl(\|\nabla v\|_2^2\bigr) v\Delta v + \int_0^t h_1(t-r) v\Delta v(r)\,dr \\ &-v\Delta v_{tt} + \beta_1 v v_t \bigl|v_t(t)\bigr|^{m(y)-2} \\ &+ \int_{\tau_1}^{\tau_2} \bigl|\beta_2(r)\bigr| v x(y,1,r,t) \bigl|x(y,1,r,t)\bigr|^{m(y)-2}\,dr \biggr\}\,dy\,dt = 0, \end{aligned} \tag{5.10}$$

which implies that

$$\begin{aligned} &\int_r^T \mathcal{E}^p(t) \int_\Omega \biggl\{ \frac{d}{dt}\frac{1}{\eta+1}\bigl(v|v_t|^\eta v_t\bigr) - \frac{1}{\eta+1}|v_t|^{\eta+2} + \frac{d}{dt}(\nabla v \nabla v_t) - |\nabla v_t|^2 \\ &+ M\bigl(\|\nabla v\|_2^2\bigr)|\nabla v|^2 - \int_0^t h_1(t-r)\nabla v \nabla v(r)\,dr + \beta_1 v v_t \bigl|v_t(t)\bigr|^{m(y)-2} \\ &+ \int_{\tau_1}^{\tau_2} \bigl|\beta_2(r)\bigr| v x(y,1,r,t) \bigl|x(y,1,r,t)\bigr|^{m(y)-2}\,dr \biggr\}\,dy\,dt = 0. \end{aligned} \tag{5.11}$$

By (5.2) and the relation

$$\begin{aligned} &\frac{d}{dt}\biggl(\mathcal{E}^p(t) \int_\Omega \bigl(v|v_t|^\eta v_t + \nabla v \nabla v_t\bigr)\,dy\biggr) \\ &= p\mathcal{E}^{p-1}(t)\mathcal{E}'(t)\biggl(\int_\Omega v|v_t|^\eta v_t\,dy + \int_\Omega \nabla v \nabla v_t\,dy\biggr) \\ &\quad + \mathcal{E}^p(t)\frac{d}{dt}\biggl(\int_\Omega v|v_t|^\eta v_t\,dy + \int_\Omega \nabla v \nabla v_t\,dy\biggr), \end{aligned}$$

this implies

$$\begin{aligned} &(\eta+2)\int_r^T \mathcal{E}^{p+1}(t)\,dt \\ &= \underbrace{\int_r^T \frac{d}{dt}\biggl(\mathcal{E}^p(t)\int_\Omega v|v_t|^\eta v_t\,dy\biggr)\,dt}_{I_1} - \underbrace{p\int_r^T \biggl(\mathcal{E}^{p-1}(t)\mathcal{E}'(t)\int_\Omega v|v_t|^\eta v_t\,dy\biggr)\,dt}_{I_2} \end{aligned}$$

$$
\begin{aligned}
&+ \underbrace{(\eta+1)\int_r^T \frac{d}{dt}\left(\mathcal{E}^p(t)\int_\Omega \nabla v\nabla v_t\,dy\right)dt}_{I_3} \\
&- \underbrace{(\eta+1)p\int_r^T \left(\mathcal{E}^{p-1}(t)\mathcal{E}'(t)\int_\Omega \nabla v\nabla v_t\,dy\right)dt}_{I_4} \\
&- \underbrace{\frac{\eta}{2}\int_r^T \left(\mathcal{E}^p(t)\int_\Omega |\nabla v_t|^2\,dx\right)dt}_{I_5} + \underbrace{(\eta+2)\int_r^T \mathcal{E}^p(t)\mathcal{F}(x)\,dt}_{I_6} \\
&+ \underbrace{\frac{\eta+2}{2}\int_r^T \left(\mathcal{E}^p(t)\left(1-\int_0^t h_1(r)\,dr\right)\int_\Omega |\nabla v|^2\,dy\right)dt}_{I_7} \\
&+ \underbrace{\left((\eta+1)+\frac{\eta+2}{2(\gamma+1)}\right)\int_r^T \left(\mathcal{E}^p(t)\int_\Omega \|\nabla v\|_2^{2\gamma}|\nabla v|^2\,dy\right)dt}_{I_8} \\
&+ \underbrace{(\eta+1)\beta_1\int_r^T \left(\mathcal{E}^p(t)\int_\Omega v v_t\left|v_t(t)\right|^{m(y)-2}\,dy\right)dt}_{I_9} \\
&+ \underbrace{(\eta+1)\int_r^T \left(\mathcal{E}^p(t)\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(r)\right|vx(y,1,r,t)\left|x(y,1,r,t)\right|^{m(y)-2}\,dr\,dy\right)dt}_{I_{10}} \\
&+ \underbrace{\frac{\eta+2}{2}\int_r^T \left(\mathcal{E}^p(t)(h_1\circ\nabla v)(t)\right)dt}_{I_{11}} \\
&- \underbrace{(\eta+1)\int_r^T \left(\mathcal{E}^p(t)\int_0^t h_1(t-r)\int_\Omega \nabla v\nabla v(r)\,dy\,dr\right)dt}_{I_{12}}.
\end{aligned}
\tag{5.12}
$$

At this point, we estimate I_i, $i = 1,\dots,12$, of the RHS in (5.12), we have

$$
\begin{aligned}
I_1 &= \mathcal{E}^p(T)\int_\Omega v|v_t|^\eta v_t(y,T)\,dy - \mathcal{E}^p(r)\int_\Omega v|v_t|^\eta v_t(y,r)\,dy \\
&\le c\mathcal{E}^p(T)\left\{\left\|v(y,T)\right\|_2^2 + \left\|v_t(y,T)\right\|_{\eta+2}^{\eta+2}\right\} \\
&\quad + c\mathcal{E}^p(r)\left\{\left\|v(y,r)\right\|_2^2 + \left\|v_t(y,r)\right\|_{\eta+2}^{\eta+2}\right\} \\
&\le c\mathcal{E}^p(T)\left\{c_*\left\|\nabla v(T)\right\|_2^2 + \mathcal{E}(T)\right\} \\
&\quad + c\mathcal{E}^p(r)\left\{c_*\left\|\nabla v(r)\right\|_2^2 + \mathcal{E}(r)\right\} \\
&\le c_1\left(\mathcal{E}^{p+1}(T) + \mathcal{E}^{p+1}(r)\right).
\end{aligned}
\tag{5.13}
$$

Since $\mathcal{E}$ is decreasing, this implies

$$
I_1 \le c\mathcal{E}^{p+1}(r) \le \mathcal{E}^p(0)\mathcal{E}(r) \le c\mathcal{E}(r).
\tag{5.14}
$$

Similarly, we find

$$
\begin{aligned}
I_2 &\le -p\int_r^T \mathcal{E}^{p-1}(t)\mathcal{E}'(t)\big(c_*\mathcal{E}(t)+\mathcal{E}(t)\big)\,dt \\
&\le -c\int_r^T \mathcal{E}^{p}(t)\mathcal{E}'(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r), \qquad (5.15)\\
I_3 &\le c\int_r^T \mathcal{E}^{p}(t)\big(\|\nabla v\|_2^2+\|\nabla v_t\|_2^2\big)\,dt \\
&\le c\mathcal{E}^{p+1}(r) \le \mathcal{E}^{p}(0)\mathcal{E}(r) \le c\mathcal{E}(r), \qquad (5.16)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\le -(\eta+1)p\int_r^T \mathcal{E}^{p-1}(t)\mathcal{E}'(t)\big(c\mathcal{E}(t)\big)\,dt \\
&\le -c\int_r^T \mathcal{E}^{p}(t)\mathcal{E}'(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r). \qquad (5.17)
\end{aligned}
$$

Next, we get

$$
\begin{aligned}
I_5 &= -\frac{\eta}{2}c\int_r^T \big(\mathcal{E}^{p}(t)\|\nabla v_t\|_2^2\big)\,dt \\
&\le c\int_r^T \mathcal{E}^{p}(t)\mathcal{E}(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r). \qquad (5.18)
\end{aligned}
$$

The other terms are estimated as follows:

$$
\begin{aligned}
I_6 &= (\eta+2)\int_r^T \mathcal{E}^{p}(t)\mathcal{F}(x)\,dt \\
&\le (\eta+2)\int_r^T \mathcal{E}^{p}(t)\mathcal{E}(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r), \qquad (5.19)\\
I_7 &\le (\eta+2)\int_r^T \mathcal{E}^{p}(t)\mathcal{E}(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r). \qquad (5.20)
\end{aligned}
$$

For the next term, we have

$$
\begin{aligned}
I_8 &= \big(2(\gamma+1)(\eta+1)+(\eta+2)\big)\int_r^T \left(\mathcal{E}^{p}(t)\frac{\|\nabla v\|_2^{2(\gamma+1)}}{2(\gamma+1)}\right)dt \\
&\le c\int_r^T \mathcal{E}^{p}(t)\mathcal{E}(t)\,dt \le c\mathcal{E}^{p+1}(r) \le c\mathcal{E}(r), \qquad (5.21)
\end{aligned}
$$

and applying the inequality of Young, the following is obtained:

$$
\begin{aligned}
I_9 &= (\eta+1)\beta_1\int_r^T \left(\mathcal{E}^{p}(t)\int_\Omega v v_t\big|v_t(t)\big|^{m(y)-2}\,dy\right)dt \\
&\le \varepsilon\int_r^T \left(\mathcal{E}^{p}(t)\int_\Omega \big|v(t)\big|^{m(y)}\,dy\right)dt \\
&\quad + c\int_r^T \left(\mathcal{E}^{p}(t)\int_\Omega c_\varepsilon(y)\big|v_t(t)\big|^{m(y)}\,dy\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varepsilon \int_r^T \mathcal{E}^p(t)\Big[\int_{\Omega_+} \big|v(t)\big|^{m^+}\,dy + \int_{\Omega_-} \big|v(t)\big|^{m^-}\,dy\Big]\,dt \\
&\quad + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|v_t(t)\big|^{m(y)}\,dy\Big)\,dt.
\end{aligned}
$$

Here, utilizing $H_0^1(\Omega) \hookrightarrow L^{m^-}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we get

$$
\begin{aligned}
I_9 &\leq \varepsilon \int_r^T \mathcal{E}^p(t)\big[c\big\|\nabla v(t)\big\|_2^{m^+} + c\big\|\nabla v(t)\big\|_2^{m^-}\big]\,dt \\
&\quad + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|v_t(t)\big|^{m(y)}\,dy\Big)\,dt \\
&\leq \varepsilon \int_r^T \mathcal{E}^p(t)\big[c\mathcal{E}^{\frac{m^+-2}{2}}(0)\mathcal{E}(t) + c\mathcal{E}^{\frac{m^--2}{2}}(0)\mathcal{E}(t)\big]\,dt \\
&\quad + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|v_t(t)\big|^{m(y)}\,dy\Big)\,dt \\
&\leq c\varepsilon \int_r^T \mathcal{E}^{p+1}(t)\,dt + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|v_t(t)\big|^{m(y)}\,dy\Big)\,dt.
\end{aligned} \tag{5.22}
$$

Similarly, we find

$$
\begin{aligned}
I_{10} &= (\eta+1)\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega \int_{\tau_1}^{\tau_2} \big|\beta_2(r)\big|vx(y,1,r,t)\big|x(y,1,r,t)\big|^{m(y)-2}\,dr\}\,dy\Big)\,dt \\
&\leq \varepsilon \int_r^T \mathcal{E}^p(t)\big[c\big\|\nabla v(t)\big\|_2^{m^+} + c\big\|\nabla v(t)\big\|_2^{m^-}\big]\,dt \\
&\quad + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|x(y,1,r,t)\big|^{m(y)}\,dy\Big)\,dt \\
&\leq \varepsilon \int_r^T \mathcal{E}^p(t)\big[c\mathcal{E}^{\frac{m^+-2}{2}}(0)\mathcal{E}(t) + c\mathcal{E}^{\frac{m^--2}{2}}(0)\mathcal{E}(t)\big]\,dt \\
&\quad + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|x(y,1,r,t)\big|^{m(y)}\,dy\Big)\,dt \\
&\leq c\varepsilon \int_r^T \mathcal{E}^{p+1}(t)\,dt + c\int_r^T \Big(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\big|x(y,1,r,t)\big|^{m(y)}\,dy\Big)\,dt
\end{aligned} \tag{5.23}
$$

and

$$
I_{11} \leq (\eta+2)\int_r^T \mathcal{E}^p(t)\mathcal{E}(t)\,dt \leq c\mathcal{E}^{p+1}(r) \leq c\mathcal{E}(r). \tag{5.24}
$$

Now, by the inequality of Young from the last term, the following is obtained:

$$
\begin{aligned}
I_{12} &\leq (\eta+1)\int_r^T (\mathcal{E}^p(t)\big(c\|\nabla v\|_2^2 + c(h_1 \circ \nabla v)(t)\big)\,dt \\
&\leq c\int_r^T \mathcal{E}^p(t)\mathcal{E}(t)\,dt \leq c\mathcal{E}^{p+1}(r) \leq c\mathcal{E}(r).
\end{aligned} \tag{5.25}
$$

By substituting (5.14)–(5.25) into (5.12), we find

$$\int_r^T \mathcal{E}^{p+1}(t)\,dt \le c\varepsilon \int_r^T \mathcal{E}^{p+1}(t)\,dt + c\int_r^T \left(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\left|v_t(t)\right|^{m(y)}dy\right)dt + c\mathcal{E}(r) + c\int_r^T \left(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\left|x(y,1,r,t)\right|^{m(y)}dy\right)dt. \tag{5.26}$$

Now, choose ε so small that

$$\int_r^T \mathcal{E}^{p+1}(t)\,dt \le c\mathcal{E}(r) + c\int_r^T \left(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\left|v_t(t)\right|^{m(y)}dy\right)dt + c\int_r^T \left(\mathcal{E}^p(t)\int_\Omega c_\varepsilon(y)\left|x(y,1,r,t)\right|^{m(y)}dy\right)dt. \tag{5.27}$$

After, fix ε, $c_\varepsilon(y) \le M$ because $m(y)$ is bounded.

Hence, by (5.3),

$$\begin{aligned}\int_r^T \mathcal{E}^{p+1}(t)\,dt &\le c\mathcal{E}(r) + cM\int_r^T \left(\mathcal{E}^p(t)\int_\Omega \left|v_t(t)\right|^{m(y)}dy\right)dt \\ &\quad + cM\int_r^T \left(\mathcal{E}^p(t)\int_\Omega \left|x(y,1,r,t)\right|^{m(y)}dy\right)dt \\ &\le c\mathcal{E}(r) - \frac{cM}{C_0}\int_r^T \mathcal{E}^p(t)\mathcal{E}'(t)\,dt \\ &\le c\mathcal{E}(r) + \frac{cM}{C_0(p+1)}\left[\mathcal{E}^{p+1}(r) - \mathcal{E}^{p+1}(T)\right] \le c\mathcal{E}(r).\end{aligned} \tag{5.28}$$

Taking $T \to \infty$, we get

$$\int_r^\infty \mathcal{E}^{p+1}(t)\,dt \le c\mathcal{E}(r). \tag{5.29}$$

Finally, Komornik's Lemma 5.2 (with $\sigma = p = \frac{m^+-2}{2}$) implies our result. This completes the proof. □

6 Conclusion

In this research, we investigated the blow-up and growth of solutions in a coupled nonlinear viscoelastic Kirchhoff-type system with sources, distributed delay, and variable exponents. Additionally, we obtained a general decay result when $f_1 = f_2 = 0$ by leveraging an integral inequality introduced by Komornik [20]. Such problems are commonly encountered in various mathematical models of real-world problems. In future research, we plan to apply this approach to address similar problems, incorporating additional damping effects such as Balakrishnan–Taylor damping and logarithmic terms. We will also try to prove the general decay result in the case $(f_1, f_2 \neq 0)$.

Author contributions

All the authors contributed to the study. All authors read and approve the final manuscript, "S.B. and A.C. wrote the main manuscript text and DO and RJ. review and check. All authors reviewed the manuscript."

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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