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Multiplicity of solutions for fractional $p(z)$-Kirchhoff-type equation

Tahar Bouali[1], Rafik Guefaifia[2] and Salah Boulaaras[2*]

*Correspondence:
s.boularas@qu.edu.sa
[2]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This work deals with the existence and multiplicity of solutions for a class of variable-exponent equations involving the Kirchhoff term in variable-exponent Sobolev spaces according to some conditions, where we used the sub-supersolutions method combined with the mountain pass theory.

Mathematics Subject Classification: 35J60; 35B30; 35B40

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1 Introduction and main results

The solutions to mathematical systems contribute to our ability to understand, predict, and manipulate the world around us. They are essential tools for addressing complex problems, making informed decisions, and driving progress in various fields of study and application [5, 6, 9, 11, 19, 20]. In this work, we will focus on the existence of solutions for the fractional Kirchhoff $p(z)$-Laplacian problem

$$\begin{cases} \mathrm{K}\big(\int_\Delta \frac{1}{p(z)} |{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu|^{p(z)}\, dz\big)\mathcal{R}_{p(z)}^{\alpha,\beta;\psi} \nu & \\ \quad = \gamma(z)\nu^{q(z)-1} + g(z,\nu), & \text{in } \Delta = [0,T]\times[0,T], \\ \nu \geq 0, & \text{in } \Delta = [0,T]\times[0,T], \\ \nu = 0, & \text{on } \partial\Delta, \end{cases} \tag{1.1}$$

where

$$\mathcal{R}_{p(z)}^{\alpha,\beta;\psi}\nu = {}^{\mathbb{H}}\mathbb{D}_{T}^{\alpha,\beta;\psi}\big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\big|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\big),$$

is the ψ-Hilfer fractional operator with variable exponent, ${}^{\mathbb{H}}\mathbb{D}_{T}^{\alpha,\beta;\psi}(\cdot)$ and ${}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}$ are ψ-Hilfer fractional partial derivatives of order $\frac{1}{p(z)} < \alpha < 1$ and type $0 \leq \beta \leq 1$, where $p \in C^1(\overline{\Delta})$, with $1 \leq p^- \leq p^+ < 2$, where $p^- := \operatorname{ess\,inf}_\Delta p$, $p^+ := \operatorname{ess\,sup}_\Delta p, q \in C(\overline{\Delta}, (1,+\infty))$, and $\gamma \in L^\infty(\Delta)$ in which $\gamma(z) > 0$ a.e., $z \in \Delta$. Define the function $p^\star(z) := \frac{2p(z)}{2-p(z)}$ if $p(z) < 2$ and $p^\star(z) := +\infty$ if $2 \geq p(z)$.

Equations characterized by variable exponent growth conditions have been the subject of extensive research in the past decade, with notable advances documented in recent works such as [8, 11, 13, 17]. The substantial volume of literature dedicated to problems involving variable exponent growth conditions is driven by the recognition that such equations can serve as effective models in various fields, including the theory of electrorheological fluids [1, 20], image processing [23], and the theory of elasticity [4]. Elliptic equations with variable exponent growth conditions typically rely on the utilization of the so-called $p(z)$-Laplace operator, i.e., $\sum_{i=1}^{N} \frac{\partial}{\partial z_i}(|\frac{\partial \nu}{\partial z_i}|^{p(z)-2}\frac{\partial \nu}{\partial z_i}) := \triangle_{p(z)}$, in which $p(z)$ is a function and for all z, $1 < p(z)$.

Problem (1.1) is $p(z)$-version associated with the following

$$\rho \frac{\partial^2 \nu}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial \nu}{\partial z}\right|^2 dz\right)\frac{\partial^2 \nu}{\partial z^2} = 0.$$

The above was introduced for the first time by Kirchhoff [14] and served as a generalization of the classical D'Alembert wave equation, accounting for alterations in the length of strings caused by transverse vibrations. Additionally, Woinowsky-Krieger [21] put forth the evolution equation of the Kirchhoff-type as given below

$$\nu_{tt} + \Delta^2 \nu - \mathrm{K}\big(\|\nabla \nu\|_2^2\big)\Delta \nu = f(z, \nu). \tag{1.2}$$

It serves as a model representing the deflection of an extensible beam; for a deeper understanding of the physics background and related models, refer to [2, 3]. From a mathematical standpoint, the existence and multiplicity of solutions for Kirchhoff-type problems involving the $p(z)$-Laplacian have undergone comprehensive investigation, as detailed in [13] and references therein. In [13], the authors successfully established the existence of solutions to a broad category of problems featuring variable exponents. Additional conditions were employed to derive the multiplicity of solutions. The paper also includes illustrative examples showcasing the applicability of the results. The methodology relies on the utilization of sub-supersolutions and suitable L^∞ estimates within the framework of variable spaces.

In [7], the authors studied existence and multiplicity problems for a new class of $\kappa(\xi)$-Kirchhoff-type equation of the form

$$\begin{cases} \Re(\int_\Lambda \frac{1}{\kappa(\xi)}|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\phi|^{\kappa(\xi)}\,d\xi)\boldsymbol{L}_{\kappa(\xi)}^{\mu,\upsilon,\psi}\phi = g(x,\xi), & \text{in } \Lambda = [0,T]\times[0,T], \\ \phi = 0, & \text{on } \partial\Lambda, \end{cases}$$

where

$$\boldsymbol{L}_{\kappa(\xi)}^{\mu,\upsilon,\psi}\phi := \mathcal{R}_{p(z)}^{\alpha,\beta;\psi}\nu,$$

where $\Re(t)$ is a continuous function, and $g(x,\xi) : \Lambda \times \mathbb{R} \to \mathbb{R}$ is the Caratheodory function that satisfies certain conditions. Using a variational approach, they investigated results from the theory of variable exponent Sobolev spaces and the theory of the ψ-fractional space $\mathcal{H}_{\kappa(\xi)}^{\mu,\upsilon,\psi}(\Lambda)$. In this sense, some new findings have been explored.

In this research, our primary objective is to investigate the existence and multiplicity of solutions to the problem (1.1). The outcomes of this study can be seen as an extension of previous findings [13] and [7], focusing on the $p(z)$-Laplacian problem with $\mathrm{K} \equiv 1$. This paper delves into the realm of fractional Kirchhoff-type problems with a variable exponent, specifically considering scenarios where K is not fixed. Leveraging the sub-supersolution method and a specialized weak fractional comparison principle, we establish the existence of a solution to the problem (1.1). Furthermore, employing the mountain pass theorem, we derive the multiplicity of solutions to the problem (1.1). Notably, these results represent novel contributions to fractional Kirchhoff-type variable exponent boundary value problems.

Here, we take the Kirchhoff function K and the nonlinearity g with some assumptions.

(K_0) Let $\mathrm{K} : [0,+\infty) \to [k_0,+\infty)$ be a nondecreasing and continuous function for some positive constant k_0;

(K_1) One can find $\theta \in (0,1)$ in a way that

$$\widehat{\mathrm{K}}(t) := \int_0^t \mathrm{K}(r)\,dr \geq (1-\theta)\mathrm{K}(t)t \quad \text{for all } t \geq 0;$$

(g_1) $g \in C(\Delta \times [0,+\infty),\mathbb{R})$ and there exists $\eta > 0$ in a manner that

$$g(z,t) \geq \gamma(z)\bigl(1 - t^{q(z)-1}\bigr) \quad \text{for all } (z,t) \in \Delta \times [0,\eta];$$

(g_2) One can find $s \in C(\overline{\Delta},(1,+\infty))$ in a way that

$$\bigl|g(z,t)\bigr| \leq \gamma(z)\bigl(1 + t^{s(z)-1}\bigr) \quad \text{for all } (z,t) \in \Delta \times [0,+\infty);$$

(g_3) One can find $\mu > \frac{p^+}{1-\theta}$ in a manner that

$$0 < \mu G(z,t) := \mu \int_0^t g(z,r)\,dr \leq g(z,t)t \quad \text{a.e. } z \in \Delta \text{ and for all } 0 < T < t.$$

After that, we present the result in the following way:

Theorem 1.1 *Let us consider that (K_0) and (g_1)–(g_2) are satisfied. Then, one can find that $\sigma_\star > 0$ in a way that the problem (1.1) provided at least one solution with $\|\gamma\|_\infty < \sigma_\star$.*

Theorem 1.2 *Let us consider that (K_0)–(K_1) and (g_1)–(g_3) are satisfied. If $q^+, r^+ < (p^\star)^-$ and ($q^- > \frac{p^+}{1-\theta}$ or $q^+ < p^-$), then one can find that $\sigma^\star > 0$, in a way that problem (1.1) has two solutions with the condition $\|\gamma\|_\infty < \sigma^\star$.*

The paper has the following structure: Sect. 2 introduces results related to variable exponentiated distances, Sect. 3 establishes the auxiliary L^∞ estimate, and Sects. 4 and 5 are devoted to proving Theorems 1.1 and 1.2, respectively.

2 Fundamental theory

In this section, the basic concepts and idea of variable exponent Lebesgue spaces will be presented and will be used to prove the main results (see [12]). Let us indicate the set of all continuous functions by $C_+(\overline{\Delta})$ and $q : \overline{\Delta} \to (1,+\infty)$. For $q \in C_+(\overline{\Delta})$, we express

$$q^+ := \max_{\overline{\Delta}} q(x) \quad \text{and} \quad q^- := \min_{\overline{\Delta}} q(x).$$

Suppose $\Theta(\Delta)$ is the set of all measurable real functions defined on Δ.

In the next step, we define the variable exponent Lebesgue space as

$$L^{q(z)}(\Delta) = \left\{ \nu \in \Theta(\Delta) : \int_{\Delta} |\nu|^{q(z)}\, dz < \infty \right\},$$

with the norm

$$\|\nu\|_{q(z)} = \inf\left\{ \tau > 0 : \int_{\Delta} \left| \frac{\nu}{\tau} \right|^{q(z)} dz \leq 1 \right\}.$$

Assume that $L^{q'(z)}(\Delta)$ is the conjugate space of $L^{q(z)}(\Delta)$ with $\frac{1}{q(z)} + \frac{1}{q'(z)} = 1$. Then, the inequality below of the Holder type is satisfied.

Lemma 2.1 ([12]): *Let $\nu \in L^{q(z)}(\Delta)$ and $w \in L^{q'(z)}(\Delta)$. Then,*

$$\int_{\Delta} |\nu w|\, dz \leq \left(\frac{1}{q-} + \frac{1}{(q'^{-})} \right) \|\nu\|_{q(z)} \|w\|_{q'(z)}.$$

The modular function in the space $L^{q(x)}$ is considered as follows

$$\rho_{q(z)}(\nu) = \int_{\Delta} |\nu|^{q(z)}\, dz.$$

Lemma 2.2 ([12]): *For any $\nu \in L^{q(z)}(\Delta)$, we have*

$$\min\left(\|\nu\|_{q(z)}^{q^-}, \|\nu\|_{q(z)}^{q^+}\right) \leq \rho_{q(z)}(\nu) \leq \max\left(\|\nu\|_{q(z)}^{q^-}, \|\nu\|_{q(z)}^{q^+}\right).$$

Lemma 2.3 ([12]): *Let $\nu \in L^{q(z)}(\Delta)$ and $\{\nu_n\} \subset L^{q(z)}(\Delta)$. Then, the following properties are equivalent*:

(1) $\lim_{n\to+\infty} \|\nu_n - \nu\|_{q(z)} = 0$;

(2) $\lim_{n\to+\infty} \rho_{q(z)}(\nu_n - \nu) = 0$.

Next, we define the ψ-fractional space given by [7]

$$\mathcal{H}_{p(z)}^{\alpha,\beta;\psi}(\Delta) = \left\{ \nu \in L^{p(z)}(\Delta) : \left| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \right| \in L^{p(z)}(\Delta) \right\}$$

and equipped with a norm

$$\|\nu\|_{1,p(z)} = \|\nu\|_{\mathcal{H}_{p(z)}^{\alpha,\beta;\psi}(\Delta)} = \|\nu\|_{p(z)} + \left\| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \right\|_{p(z)}.$$

Then, $(\mathcal{H}_{p(z)}^{\alpha,\beta;\psi}(\Delta), \|\cdot\|_{1,p(x)})$ is a Banach reflexive space [7]. Let $X_0 := \mathcal{H}_{p(z),0}^{\alpha,\beta;\psi}(\Delta)$ be the closure of $C_0^{\infty}(\Delta)$ in $\mathcal{H}_{p(z)}^{\alpha,\beta;\psi}(\Delta)$. After all $p(z) < p^\star(z)$ for all $z \in \overline{\Delta}$,

$$\|\nu\|_{p(z)} \leq C \left\| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \right\|_{p(z)} \quad \text{for all } \nu \in X_0 \text{ (a Poincaré-type inequality),}$$

where C is a positive constant independent of u, and $\|\cdot\|_{X_0}$ is the norm of the space X_0 of the form

$$\|\nu\|_{X_0} := \left\| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \right\|_{p(z)} \quad \text{for all } \nu \in X_0.$$

Lemma 2.4 ([7]): *Assume that $q \in C_+(\overline{\Delta})$ in a manner that $p^\star(z) > q(z) \geq 1$ for all $z \in \overline{\Delta}$. Then, we have compact and continuous embedding $\mathcal{H}^{\alpha,\beta;\psi}_{p(z)}(\Delta) \hookrightarrow L^{q(x)}(\Delta)$.*

3 Fractional comparison principle

Here, we will provide an estimate for a weak fractional comparison principle for (1.1) and an L^∞, which will be used in generating the corresponding sub-supersolutions.

Definition 3.1 Take $\nu, w \in X_0$. We say that

$$\mathrm{K}\big(I^{\alpha,\beta}(\nu)\big)\mathcal{R}^{\alpha,\beta;\psi}_{p(z)}\nu \leq \mathrm{K}\big(I^{\alpha,\beta}(w)\big)\mathcal{R}^{\alpha,\beta;\psi}_{p(z)}w$$

if for all nonnegative functions $\varphi \in X_0$.

$$\begin{aligned}&\mathrm{K}\big(I^{\alpha,\beta}(\nu)\big)\int_\Delta \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big).{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\varphi\,dz\\ &\quad\leq \mathrm{K}\big(I^{\alpha,\beta}(w)\big)\int_\Delta \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w\big|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w\big).{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\varphi\,dz,\end{aligned}$$

where

$$I^{\alpha,\beta}(\nu) = \int_\Delta \frac{1}{p(z)}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)}\,dz.$$

Lemma 3.2 *Let (K_0) be satisfied. Then, $\phi: X_0 \to X_0^*$ of the form*

$$\langle\phi(\nu),\chi\rangle = \mathrm{K}(I^{\alpha,\beta}(\nu)\int_\Delta \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big).{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\chi\,dz,\quad \nu,\chi\in X_0, \tag{3.1}$$

is strictly monotone and continuous.

Proof It is clear that the operator ϕ is continuous. Assume that $\nu \neq w \in X_0$. Without loss of generality, one can suppose that $I^{\alpha,\beta}(\nu) \geq I^{\alpha,\beta}(w)$. Further, the nondecreasing property of K implies that

$$\mathrm{K}\big(I^{\alpha,\beta}(\nu)\big) \geq \mathrm{K}\big(I^{\alpha,\beta}(w)\big). \tag{3.2}$$

Further, we have

$${}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu.{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w \leq \frac{1}{2}\big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^2 + \big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w\big|^2\big). \tag{3.3}$$

Thus,

$$\int_\Delta \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)} - \big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu.{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w\big)\,dz \tag{3.4}$$

$$\geq \int_\Delta \frac{1}{2}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)-2}\big(\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^2 - \big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}w\big|^2\big)\,dz \tag{3.5}$$

and

$$\begin{aligned}
&\int_{\Delta}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right)dz \qquad (3.6)\\
&\quad\geq\int_{\Delta}\frac{1}{2}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}\right)dz, \qquad (3.7)
\end{aligned}$$

we put

$$\Delta_a=\left\{z\in\Delta:\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|\geq\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|\right\}$$

and

$$\Delta_b=\left\{z\in\Delta:\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|<\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|\right\}.$$

By (3.2), (3.4)–(3.6) and (K_0), we get

$$\begin{aligned}
A:={}&\mathrm{K}\left(I^{\alpha,\beta}(v)\right)\int_{\Delta_a}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz \qquad (3.8)\\
&+\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_a}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz\\
\geq{}&\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(v)\right)\int_{\Delta_a}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
&-\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_a}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
\geq{}&\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_a}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\\
&\times\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
\geq{}&\frac{k_0}{2}\int_{\Delta_a}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
\geq{}&0.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
B:={}&\mathrm{K}\left(I^{\alpha,\beta}(v)\right)\int_{\Delta_b}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz \qquad (3.9)\\
&+\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_b}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz\\
\geq{}&\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(v)\right)\int_{\Delta_b}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
&-\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_b}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz\\
\geq{}&\frac{1}{2}\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta_b}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\\
&\times\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz
\end{aligned}$$

$$\geq \frac{k_0}{2}\int_{\Delta_b}\left(|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu|^{p(z)-2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\left(|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu|^{2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz$$
$$\geq 0.$$

It follows that

$$\begin{aligned}
&\langle\phi(\nu)-\phi(w),\nu-w\rangle\\
&\quad=\langle\phi(\nu),\nu-w\rangle-\langle\phi(w),\nu-w\rangle\\
&\quad=\langle\phi(\nu),\nu\rangle-\langle\phi(\nu),w\rangle+\langle\phi(w),w\rangle-\langle\phi(w),\nu\rangle\\
&\quad=\mathrm{K}\left(I^{\alpha,\beta}(\nu)\right)\int_{\Delta}\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{p(z)}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz\\
&\qquad+\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\int_{\Delta}\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)dz\\
&\quad=(A+B)\geq 0.
\end{aligned}$$

This implies that $\langle\phi(\nu)-\phi(w),\nu-w\rangle>0$. On the other hand, using (3.8)–(3.9), we have

$$0=\langle\phi(\nu)-\phi(w),\nu-w\rangle=(A+B) \tag{3.10}$$
$$\geq\frac{k_0}{2}\int_{\Delta}\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{p(z)-2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz \tag{3.11}$$
$$\geq 0,$$

which gives the following

$$\int_{\Delta}\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{p(z)-2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(z)-2}\right)\left(\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{2}-\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{2}\right)dz,$$

hence $|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu|=|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w|$. After that, $\mathrm{K}(I^{\alpha,\beta}(\nu))=\mathrm{K}(I^{\alpha,\beta}(w))$ and from (3.11), the following is obtained

$$\begin{aligned}
0&=\langle\phi(\nu)-\phi(w),\nu-w\rangle\\
&=\mathrm{K}\left(I^{\alpha,\beta}(\nu)\right)\int_{\Delta}\left|^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu\right|^{p(z)-2}\left({}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu-{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w\right)^2dz,
\end{aligned}$$

so, ${}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\nu={}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}w$ a.e., in Δ, as a result $\nu=w$ in X_0. This leads to a contradiction, and thus, $\langle\phi(\nu)-\phi(w),\nu-w\rangle>0$. As a result, it can be affirmed that ϕ is strictly monotonic. □

Lemma 3.3 (*Fractional comparison principle*): *Let* (K_0) *be satisfied and assume that* $u,w\in X_0$ *verify*

$$\mathrm{K}\left(I^{\alpha,\beta}(\nu)\right)\mathcal{R}_{p(z)}^{\alpha,\beta;\psi}\nu\leq\mathrm{K}\left(I^{\alpha,\beta}(w)\right)\mathcal{R}_{p(z)}^{\alpha,\beta;\psi}w \tag{3.12}$$

and $\nu\leq w$ on $\partial\Delta$, i.e., $(\nu-w)^+\in X_0$. Then, $\nu\leq w$ a.e., in Δ.

Proof Let us assume that a test function $\chi = (\nu - w)^+$ in (3.12), then, by (3.1), the following is obtained

$$\begin{aligned}
&\langle \phi(\nu) - \phi(w), (\nu - w)^+ \rangle \\
&= \mathrm{K}\big(I^{\alpha,\beta}(\nu)\big) \int_{\Delta \cap [\nu > w]} \big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \big|^{p(z)-2} \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} (\nu - w) \, dz \\
&\quad - \mathrm{K}\big(I^{\alpha,\beta}(w)\big) \int_{\Delta \cap [\nu > w]} \big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} w \big|^{p(z)-2} \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} w \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} (\nu - w) \, dz \\
&\le 0.
\end{aligned}$$

Otherwise, we have the following from (3.11):

$$\begin{aligned}
&\langle \phi(\nu) - \phi(w), (\nu - w)^+ \rangle \\
&\ge \frac{k_0}{2} \int_{\Delta \cap [\nu > w]} \Big(\big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \big|^{p(z)-2} - \big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} w \big|^{p(z)-2} \Big) \Big(\big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu \big|^{2} - \big| {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} w \big|^{2} \Big) \, dz \\
&\ge 0.
\end{aligned}$$

Hence, $\langle \phi(\nu) - \phi(w), (\nu - w)^+ \rangle = 0$. Applying Lemma 3.2, we deduce that $(\nu - w)^+ = 0$, and hence the proof is completed. □

Lemma 3.4 *Let us assume that* (K_0) *is satisfied and assume that* $\gamma \in L^\infty(\Delta)$. *Then, there exists a unique solution to*

$$\begin{cases} \mathrm{K}(I^{\alpha,\beta}(\nu)) \mathcal{R}_{p(z)}^{\alpha,\beta;\psi} \nu = \gamma(z) & \text{in } \Delta \\ \nu = 0 & \text{on } \partial\Delta \end{cases} \tag{3.13}$$

in the space $X_0 := \mathcal{H}_{p(z),0}^{\alpha,\beta;\psi}(\Delta)$.

Proof The operator defined in (3.1) has the following characteristics: see Proposition 2.7 in [7]

1. $\mathcal{L}^{\alpha,\beta}$: is a continuous, bounded and strictly monotone operator;
2. $\mathcal{L}^{\alpha,\beta}$: is a mapping of type (S^+);
3. $\mathcal{L}^{\alpha,\beta}$: is a homeomorphism.

Hence by (K_0) and Lemma 2.2, the following is obtained

$$\lim_{n \to +\infty} \frac{\langle \mathcal{L}^{\alpha,\beta}(\nu), \nu \rangle}{\|\nu\|_{X_0}} \ge \lim_{n \to +\infty} \frac{k_0 \int_\Delta |{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu|^{p(z)} \, dz}{\|\nu\|_{X_0}} = +\infty, \tag{3.14}$$

$\mathcal{L}^{\alpha,\beta}$ is coercive, thus $\mathcal{L}^{\alpha,\beta}$ is a surjection. Applying the Minty-Browder theorem [22], we conclude that the equation $\phi(\nu) = \gamma$ possesses a unique solution in X_0. □

Let us indicate the best constant of the continuous embedding $X_0(\Delta) \hookrightarrow L^2$ by C_0. Then,

$$\|\nu\|_{L^2(\Delta)} \le C_0 \|\nu\|_{X_0(\Delta)} \quad \text{for all } \nu \in X_0. \tag{3.15}$$

Lemma 3.5 *Let (K_0) be satisfied. Assume that $\varrho > 0$ and ν_ϱ is a unique solution to the following*

$$\begin{cases} -\mathrm{K}(I^{\alpha,\beta}(\nu))\mathcal{R}^{\alpha,\beta;\psi}_{p(z)}\nu = \varrho & \text{in } \Delta \\ \nu = 0 & \text{on } \partial\Delta. \end{cases} \tag{3.16}$$

Put $\delta = \frac{k_0 p^-}{2C_0|\Delta|^{\frac{1}{2}}}$. Then, when $\varrho \geq \delta$, $\nu_\varrho \in L^\infty(\Delta)$ with

$$\|\nu_\varrho\|_\infty \leq C_1^\star \mathrm{K}\big(C_2^\star \varrho^{(p^-)'}\big)\varrho^{\frac{1}{p^- -1}},$$

and when $\varrho < \delta$,

$$\|\nu_\varrho\|_\infty \leq C_\star \varrho^{\frac{1}{p^+ -1}},$$

where $C_1^\star$, $C_2^\star$, and $C_\star$ are positive constants depending only on Δ, k_0 and p.

Proof To prove the required result, let $\zeta \geq 0$ be fixed and put $\Delta_\zeta = \{z \in \Delta : \nu_\varrho(z) > \zeta\}$ and $\nu_\varrho \geq 0$ and by comparison principle. Testing equation (3.16) with $(\nu_\varrho - \zeta)^+$, it follows from (3.15) and the Young inequality that

$$\begin{aligned} \int_\Delta \big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu\big|^{p(z)}\,dz &= \frac{\varrho}{\mathrm{K}(I^{\alpha,\beta}(\nu_\varrho))}\int_{\Delta_\zeta}(\nu_\varrho - \zeta)\,dz \qquad (3.17) \\ &\leq \frac{\varrho|\Delta_\zeta|^{\frac{1}{2}}}{\mathrm{K}(I^{\alpha,\beta}(\nu_\lambda))}\big\|(\nu_\varrho - \zeta)^+\big\|_{L^2(\Delta)} \\ &\leq \frac{\varrho|\Delta_\zeta|^{\frac{1}{2}}C_0}{k_0}\int_{\Delta_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho\big|\,dz \\ &\leq \frac{\varrho|\Delta_\zeta|^{\frac{1}{2}}C_0}{k_0}\int_{\Delta_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho\big|^{p(z)}\,dz \\ &\leq \frac{\varrho|\Delta_\zeta|^{\frac{1}{2}}C_0}{k_0}\bigg(\int_{\Delta_\zeta}\frac{\varepsilon^{p(z)}|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho|^{p(z)}}{p(z)}\,dz + \int_{\Delta_\zeta}\frac{\varepsilon^{-p'(z)}}{p'(z)}\,dz\bigg). \qquad (3.18) \end{aligned}$$

When $\varrho \geq \delta$, taking

$$\epsilon = \left(\frac{k_0 p^-}{2\varrho|\Delta|^{\frac{1}{N}}C_0}\right)^{\frac{1}{p^-}} = \left(\frac{\delta}{\varrho}\right)^{\frac{1}{p^-}}, \tag{3.19}$$

we have $\epsilon \leq 1$, thus

$$\begin{aligned} \frac{\varrho|\Delta_\zeta|^{\frac{1}{2}}C_0}{k_0}\int_{\Delta_\zeta}\frac{\varepsilon^{p(z)}|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho|^{p(z)}}{p(z)}\,dz &\leq \frac{\varrho|\Delta|^{\frac{1}{2}}C_0\epsilon^{p^-}}{k_0 p^-}\int_{\Delta_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho\big|^{p(z)}\,dz, \qquad (3.20) \\ &= \frac{1}{2}\int_{\Delta_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\alpha,\beta;\psi}_{0^+}\nu_\varrho\big|^{p(z)}\,dz. \end{aligned}$$

Combining (3.18) and (3.20), we arrive at

$$\int_{\Delta_\zeta} \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} v_\varrho\right|^{p(z)} dz \le \frac{2\varrho|\Delta_\zeta|^{\frac{1}{2}} C_0}{k_0(p^+)'} \int_{\Delta_\zeta} \epsilon^{-(p^-)'} dz = \frac{4\varrho C_0 \epsilon^{-(p^-)'}}{k_0(p^+)'} |\Delta_\zeta|^{1+\frac{1}{2}}. \tag{3.21}$$

Similarly, from the test function in (3.16) with v_ϱ, the following is obtained

$$\int_{\Delta} \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} v_\varrho\right|^{p(z)} dz \le \frac{4\varrho C_0 \epsilon^{-(p^-)'}}{k_0(p^+)'} |\Delta|^{1+\frac{1}{2}}.$$

From (3.17), (3.20) and the monotonicity of K, the below is obtained

$$\begin{aligned}\int_{\Delta_\zeta} (v_\varrho - \xi)\, dz &= \frac{\mathrm{K}(I^{\alpha,\beta}(v))}{\varrho} \int_{\Delta_\zeta} \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} v_\varrho\right|^{p(z)} dz \\ &\le \mathrm{K}\left(\frac{4\varrho C_0 \epsilon^{-(p^-)'}}{k_0 p^-(p^+)'} |\Delta_\zeta|^{1+\frac{1}{2}}\right) \frac{4 C_0 \epsilon^{-(p^-)'}}{k_0(p^+)} |\Delta_\zeta|^{1+\frac{1}{2}}.\end{aligned}$$

Through Lemma 5.1 in [15], the below is achieved

$$\|v_\varrho\|_\infty \le \mathrm{K}\left(\frac{4\varrho C_0 \epsilon^{-(p^-)'}}{k_0 p^-(p^+)'} |\Delta_\zeta|^{1+\frac{1}{2}}\right) \frac{12 C_0 \epsilon^{-(p^-)'}}{k_0(p^+)'} |\Delta_\zeta|^{\frac{1}{2}}. \tag{3.22}$$

It follows from (3.19) and (3.22) that

$$\|v_\varrho\|_\infty \le C_1^\star \mathrm{K}\left(C_2^\star \varrho^{(p^-)'}\right) \varrho^{\frac{1}{p^- -1}},$$

where

$$C_1^\star := \frac{6(2C_0)^{(p^-)'}}{(p^+) k_0^{(p^-)'} (p^-)^{\frac{1}{p^- -1}}} |\Delta|^{\frac{(p^-)'}{2}},$$

and

$$C_2^\star := \frac{2(2C_0)^{(p^-)'}}{(p^+)' k_0^{(p^-)'} (p^-)^{p^-}} |\Delta|^{1+\frac{(p^-)'}{2}}.$$

When $\varrho < \delta$, taking

$$\varepsilon = \left(\frac{k_0 p^-}{2\varrho|\Delta|^{\frac{1}{2}} C_0}\right)^{\frac{1}{p^+}} = \left(\frac{\delta}{\varrho}\right)^{\frac{1}{p^+}},$$

we have $\epsilon < 1$. The following can be easily proved through the same argument:

$$\|v_\varrho\|_\infty \le C_\star \varrho^{\frac{1}{p^+ -1}},$$

where

$$C_* = \frac{6(2C_0)^{(p^+)'}}{(p^+)' k_0^{(p^+)'} (p^-)^{\frac{1}{p^+ -1}}} |\Delta|^{\frac{(p^+)'}{2}} \mathrm{K}\left(\frac{2(2\delta C_0)^{(p^+)'}}{(p^+)' k_0^{(p^+)'} (p^-)^{(p^+)'}} |\Delta|^{1+\frac{(p^+)'}{2}}\right).$$

□

4 Proof of Theorem 1.1

Taking the pair of sub-supersolutions $(\underline{\nu},\overline{\nu})$ to the problem (1.1), if $\underline{\nu},\overline{\nu} \in L^{\infty}(\Delta)$, $\underline{\nu} \leq \overline{\nu}$ a.e., in Δ and for all arbitrary nonnegative functions $\chi \in X_0$, the following is satisfied

$$\begin{cases} \mathrm{K}(I^{\alpha,\beta}(\underline{\nu}))\int_{\Delta}(|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{\nu}|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{\nu}).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\chi\,dz \\ \quad \leq \int_{\Delta}\gamma(z)\underline{\nu}^{q(z)-1}\chi\,dz + \int_{\Delta} g(z,\underline{\nu})\chi\,dz, \\ \mathrm{K}(I^{\alpha,\beta}(\overline{\nu}))\int_{\Delta}(|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\overline{\nu}|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\overline{\nu}).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\chi\,dz \\ \quad \geq \int_{\Delta}\gamma(z)\overline{\nu}^{q(z)-1}\chi\,dz + \int_{\Delta} g(z,\overline{\nu})\chi\,dz. \end{cases} \tag{4.1}$$

Lemma 4.1 *Assume that (K_0) and $(g_1)-(g_2)$ are satisfied. Then, there is $\sigma_\star > 0$ in a manner that (1.1) has a pair of sub-supersolutions $(\underline{\nu},\overline{\nu}) \in (X_0 \cap L^{\infty}(\Delta) \times (X_0 \cap L^{\infty}(\Delta))$ with $\|\nu\|_\infty \leq \eta$, provided that $\|\gamma\|_\infty < \sigma_\star$, where η is defined in (g_1).*

Proof Using Lemmas 3.2, 3.3, and 3.4, one can find that $\underline{\nu},\overline{\nu} \in X_0 \cap L^{\infty}(\Delta)$ is a unique nonnegative solution to the following

$$\begin{cases} -\mathrm{K}(I^{\alpha,\beta}(\underline{\nu})\mathcal{R}_{p(z)}^{\alpha,\beta;\psi}\underline{\nu} = \gamma(z) & \text{in } \Delta, \\ \underline{\nu} = 0 & \text{on } \partial\Delta \end{cases}$$

and

$$\begin{cases} -\mathrm{K}(I(\overline{\nu})\mathcal{R}_{p(z)}^{\alpha,\beta;\psi}\overline{\nu} = \gamma(z) + 1 & \text{in } \Delta, \\ \overline{\nu} = 0 & \text{on } \partial\Delta \end{cases} \tag{4.2}$$

such that

$$\|\underline{\nu}\|_\infty \leq \max\bigl(C_1^\star \mathrm{K}\bigl(C_2^\star\|\gamma\|_\infty^{(p^-)'}\bigr)\|\gamma\|^{\frac{1}{p^--1}}, C_\star\|\gamma\|^{\frac{1}{p^+-1}}\bigr),$$

where $C_1^\star$, $C_2^\star$, and $C_\star$ were mentioned in Lemma 3.4. Next, considering that K is nondecreasing, there exits $\sigma > 0$ relying only on $C_1^\star$, $C_2^\star$, and $C_\star$ such that $\|\underline{\nu}\|_\infty \leq \eta$, provided that $\|\gamma\|_\infty < \sigma$. Moreover, $\underline{\nu} \leq \overline{\nu}$ by Lemma 3.2.

For an arbitrary nonnegative function ψ in X_0. The above (4.2) and (g_1) imply that

$$\begin{aligned} &\mathrm{K}\bigl(I^{\alpha,\beta}(\underline{\nu})\bigr)\int_{\Delta}\bigl(\bigl|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{\nu}\bigr|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{\nu}\bigr).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\chi\,dz \\ &\qquad - \int_{\Delta}\gamma(z)\underline{\nu}^{q(z)-1}\chi\,dz - \int_{\Delta} g(z,\underline{\nu})\chi\,dz \\ &\quad \leq \int_{\Delta}\gamma(z)\chi\,dz - \int_{\Delta}\gamma(z)\underline{\nu}^{q(z)-1}\chi\,dz - \int_{\Delta}\gamma(z)\bigl(1-\underline{\nu}^{q(z)-1}\bigr)\chi\,dz \\ &\quad = 0. \end{aligned}$$

By (4.2) and (g_2), we have

$$\begin{aligned} &\mathrm{K}\bigl(I^{\alpha,\beta}(\overline{\nu})\bigr)\int_{\Delta}\bigl(\bigl|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\overline{\nu}\bigr|^{p(z)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\overline{\nu}\bigr).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\chi\,dz \\ &\qquad - \int_{\Delta}\gamma(z)\overline{\nu}^{q(z)-1}\chi\,dz - \int_{\Delta} g(z,\overline{\nu})\chi\,dz \end{aligned}$$

$$\geq \int_{\Delta} \left(1 - B_\infty \|\gamma\|_\infty\right) \chi \, dz,$$

where

$$B_\infty := \max\left(\|\nu\|_\infty^{q^+-1}, \|\nu\|_\infty^{q^--1}\right) + \max\left(\|\nu\|_\infty^{s^+-1}, \|\nu\|_\infty^{s^--1}\right).$$

Thus, choosing $\sigma_\star = \min(\sigma, \frac{1}{B_\infty})$, it yields

$$\int_{\Delta} \left(1 - B_\infty \|\gamma\|_\infty\right) \chi \, dx \geq 0 \quad \text{for } \|\gamma\|_\infty < \sigma_\star.$$

This completes the proof.

We give now the proof of Theorem 1.1:

Assume that $\underline{\nu}, \overline{\nu} \in X_0 \cap L^\infty(\Delta)$ as in the above lemma and introduce

$$h(z,t) = \begin{cases} \gamma(z)\overline{\nu}(z)^{q(z)-1} + g(z, \overline{\nu}(z)) & \text{if } t > \overline{\nu}(z), \\ \gamma(z) t^{q(z)-1} + g(z,t) & \text{if } \underline{\nu}(z) \leq t \leq \overline{\nu}(z), \\ \gamma(z)\underline{\nu}(z)^{q(z)-1} + g(z, \underline{\nu}(z)) & \text{if } t < \underline{\nu}(z). \end{cases}$$

Consider the problem

$$\begin{cases} -\mathrm{K}(I(\nu))\mathcal{R}_{p(z)}^{\alpha,\beta;\psi} \nu = h(z,\nu) & \text{in } \Delta, \\ \nu = 0 & \text{on } \partial\Delta \end{cases} \tag{4.3}$$

and the functional $\mathcal{I}^{\alpha,\beta} : X_0 \to \mathbb{R}$ defined by

$$\mathcal{I}^{\alpha,\beta}(\nu) = \widehat{\mathrm{K}}\left(I^{\alpha,\beta}(\nu)\right) - \int_{\Delta} H(z,\nu) \, dz,$$

where $G(z,t) = \int_0^t h(z,r) \, dr$. Then, $\mathcal{I}^{\alpha,\beta}$ belongs to the class C^1, and its critical points precisely correspond to the solutions to problem (4.3). Based on (K_0), it is evident that $\mathcal{I}^{\alpha,\beta}$ is both coercive and sequentially weakly lower semicontinuous. Consequently, $\mathcal{I}^{\alpha,\beta}$ achieves its minimum within the weakly closed subset $[\underline{\nu}, \overline{\nu}] \cap X_0$ at some ν_0, establishing it as a critical point of $\mathcal{I}^{\alpha,\beta}$. This concludes the proof of Theorem 1.1. □

5 Proof of Theorem 1.2

To prove the theorem, we introduce the following

$$f(z,t) = \begin{cases} \gamma(z) t^{q(z)-1} + g(z,t) & \text{if } \underline{\nu}(z) \leq t, \\ \gamma(z)\underline{\nu}(z)^{q(z)-1} + g(z, \underline{\nu}(z)) & \text{if } \underline{\nu}(z) \geq t, \end{cases}$$

and also take the following

$$\begin{cases} -\mathrm{K}(I^{\alpha,\beta}(\nu))\mathcal{R}_{p(z)}^{\alpha,\beta;\psi} \nu = f(z,\nu) & \text{in } \Delta, \\ \nu = 0 & \text{on } \partial\Delta. \end{cases} \tag{5.1}$$

Our approach to finding solutions to (5.1) involves identifying critical points of the functional $\mathcal{J}^{\alpha,\beta}: X_0 \to \mathbb{R}$, defined as:

$$\mathcal{J}^{\alpha,\beta}(v) = \widehat{\mathrm{K}}\big(I^{\alpha,\beta}(v)\big) - \int_{\Delta} F(z,v)\,dz,$$

where $F(z,t) = \int_0^t f(z,r)\,dr$. Obviously, $\mathcal{J}^{\alpha,\beta}$ is of class C^1.

Lemma 5.1 *The functional $\mathcal{J}^{\alpha,\beta}$ satisfies the Palais-Smale condition for the given assumptions of Theorem 1.2.*

Proof Assume that $\{v_n\} \subset X_0$ is a sequence in a way that

$$\mathcal{J}^{\alpha,\beta}(v_n) \to c \in \mathbb{R} \quad \text{and} \quad \mathcal{J}'^{\alpha,\beta}(v_n) \to 0 \quad \text{in } X_0^*.$$

Here, the boundedness of $\{v_n\}$ in X_0 is claimed.

Case 1: $q^- > \frac{p^+}{1-\theta}$. Let $\mu_0 \in (\frac{p^+}{1-\theta}, \min(\mu, q))$. By $(K_0) - (K_1), (g_3)$, (3.14) and embedding theorem, for n large enough, the following is obtained

$$\begin{aligned}
1 + c + \|v_n\|_{X_0} &\geq \mathcal{J}^{\alpha,\beta}(v_n) - \frac{1}{\mu_0}\big\langle \mathcal{J}'^{\alpha,\beta}(v_n), v_n\big\rangle \\
&\geq (1-\theta)\mathrm{K}\big(I^{\alpha,\beta}(v_n)\big)I^{\alpha,\beta}(v_n) - \frac{1}{\mu_0}\mathrm{K}\big(I^{\alpha,\beta}(v_n)\big)\int_{\Delta}\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} v\big|^{p(z)}\,dz \\
&\quad + \int_{\Delta}\left(\frac{1}{\mu_0}f(z,v_n)v_n - F(z,v_n)\right)dz \\
&\geq k_0\left(\frac{1-\theta}{p^+} - \frac{1}{\mu_0}\right)\big(C_{p^-}\|v_n\|_{X_0}^{p^-} - 2\big) \\
&\quad + \int_{[v_n > \underline{v}]}\left(\frac{1}{\mu_0}g(z,z_n)z_n - H(z,v_n)\right)dz \\
&\quad + \int_{[v_n > \underline{v}]}\left(\frac{1}{\mu_0} - \frac{1}{q(z)}\right)\gamma(z)v_n^{q(z)}\,dz - C_3\|v_n\|_{X_0} - C_2 \\
&\geq k_0\left(\frac{1-\theta}{p_M^+} - \frac{1}{\mu_0}\right)\big(C_{p^-}\|v_n\|_{X_0}^{p^-} - 2\big) - C_4\big(\|v_n\|_{X_0}^{q^+} + \|v_n\|_{X_0}^{q^-}\big) \\
&\quad - C_3\|v_n\|_{X_0} - C_2,
\end{aligned}$$

where C_1 and C_2 are independent of n positive constants. Thus, the sequence $\{v_n\}$ is bounded in X_0 as $p^- > 1$.

Case 2: $q^+ < p^-$. Using $(K_0) - (K_1), (g_3)$, (3.14) and embedding theorem, the following is obtained

$$\begin{aligned}
1 + c + \|v_n\|_{X_0} &\geq \mathcal{J}^{\alpha,\beta}(v_n) - \frac{1}{\mu}\big\langle \mathcal{J}'^{\alpha,\beta}(v_n), v_n\big\rangle \\
&\geq k_0\left(\frac{1-\theta}{p^+} - \frac{1}{\mu}\right)\big(C_{p^-}\|v_n\|_{X_0}^{p^-} - 2\big) + \int_{[v_n > \underline{v}]}\left(\frac{1}{\mu}g(z,v_n)v_n - H(z,v_n)\right)dz \\
&\quad + \int_{[v_n > \underline{v}]}\left(\frac{1}{\mu} - \frac{1}{q(z)}\right)\gamma(z)v_n^{q(z)}\,dz - C_3\|v_n\|_{X_0} - C_2
\end{aligned}$$

$$\geq k_0\left(\frac{1-\theta}{p^+}-\frac{1}{\mu}\right)\left(C_{p^-}\|v_n\|_{X_0}^{p^-}-2\right)-C_4\left(\|v_n\|_{X_0}^{q^+}+\|v_n\|_{X_0}^{q^-}\right)$$
$$-C_3\|v_n\|_{X_0}-C_2,$$

where C_3 and C_4 are independent of n positive constants. Thus, the boundedness of $\{v_n\}$ in X_0 is proved. Further, we have

$$\begin{cases} v_n \rightharpoonup v & \text{in } X_0, \\ v_n \to v & \text{a.e. in } \Delta, \\ v_n \to v & \text{in } L^{w(z)}(\Delta) \text{ with } 1 < w^- \leq w^+ < (p^\star)^-. \end{cases} \tag{5.2}$$

Therefore,

$$\begin{aligned} o_n(1) &= \left\langle \mathcal{J}'^{\alpha,\beta}(v_n), v_n - v\right\rangle \\ &= \mathrm{K}\left(I^{\alpha,\beta}(v_n)\right)\int_\Delta \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(v_n-v)\right)dz \\ &\quad - \int_\Delta f(z,v_n)(v_n-v)\,dz. \end{aligned}$$

Now, from (g_2), (5.2), Lemmas 2.1 and 2.4, the following can be easily proved

$$\int_\Delta f(z,v_n)(v_n-v)\to 0$$

so that

$$\mathrm{K}\left(I^{\alpha,\beta}(v_n)\right)\int_\Delta \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n\right|^{p(z)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(v_n-v)\right)dz\to 0.$$

Through the assumption of (K_0), we have

$$\int_\Delta \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n\right|^{p(z)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(v_n-v)\right)dz\to 0.$$

Similarly, the following is obtained

$$\int_\Delta \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(v_n-v)\right)dz\to 0.$$

It holds that

$$\begin{aligned} &\frac{1}{2^{p^+-2}}\int_\Delta \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(v_n-v)\right|^{p(z)}dz \\ &\quad\leq \int_\Delta \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n\right|^{p(z)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right|^{p(z)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right) \\ &\quad\quad\times\left({}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v_n-{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}v\right)dz\to 0, \end{aligned}$$

combining with Lemma 2.3, we have $v_n\to v$ in X_0. □

Lemma 5.2 *For $\|\gamma\|_\infty$ sufficiently small with the assumptions of Theorem 1.2, the below is satisfied*

(i) *One can find $\kappa > 0$ and $\rho > \|\underline{v}\|_{X_0}$ in a way that*

$$\mathcal{J}^{\alpha,\beta}(\underline{v}) < 0 < \kappa \leq \inf_{v \in \partial B_\rho(0)} \mathcal{J}^{\alpha,\beta}(v);$$

(ii) *One can find $e \in X_0$ in a way that $\|e\|_{X_0} > 2\rho$ and $\mathcal{J}^{\alpha,\beta}(e) < \kappa$.*

Proof (i) For the proof, taking $\psi = \underline{v}$ in the first inequality of (4.1) and applying the fact that K is nondecreasing, we have

$$\begin{aligned}
\mathcal{J}^{\alpha,\beta}(\underline{v}) &= \mathrm{K}\big(I^{\alpha,\beta}(\underline{v})\big) - \int_\Delta F(z,v)\,dz \\
&\leq \mathrm{K}\big(I^{\alpha,\beta}(\underline{v})\big)I(\underline{v}) - \int_\Delta \gamma(z)\underline{v}^{q(z)}\,dz - \int_\Delta g(z,\underline{v})\underline{v}\,dz \\
&< \mathrm{K}\big(I^{\alpha,\beta}(\underline{v})\big)\int_\Delta \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{v}\big|^{p(z)}\,dz - \int_\Delta \alpha(z)\underline{v}^{q(z)}\,dz - \int_\Delta g(z,\underline{v})\underline{v}\,dz \\
&\leq 0.
\end{aligned}$$

Therefore, $\mathcal{J}^{\alpha,\beta}(\underline{v}) < 0$. Further, assume that $v \in X_0$ with $\|v\|_{X_0} \geq 1$. By (K_0), (g_2), (3.14) and embedding theorem, one has

$$\mathcal{J}^{\alpha,\beta}(v) \geq \frac{k_0}{p^+}\big(C_{p^-}\|v\|_{X_0}^{p^-} - 2\big) - C_5\|\gamma\|_\infty\big(\|v\|_{X_0} + \|v\|_{X_0}^{q^+} + \|v\|_{X_0}^{s^+}\big) - C_6,$$

where $C_5, C_6 > 0$. Observe that one can choose $\kappa > 0$ and $\rho > \|\underline{v}\|_{X_0}$ such that

$$\frac{k_0}{p^+}\big(C_{p^-}\|v\|_{X_0}^{p^-} - 2\big) - C_6 \geq 2\kappa.$$

Then, letting $\|\gamma\|_\infty \leq \frac{\kappa}{C_5(\rho + \rho^{q^+} + \rho^{s^+})}$ implies that $\mathcal{J}^{\alpha,\beta}(v) \geq \kappa$ for $\|u\|_{X_0} = \rho$.

(ii) By (K_1), there is $C_7 > 0$ such that

$$\mathrm{K}(t) \leq C_7 t^{\frac{1}{1-\theta}} \quad \text{for all } t > 1. \tag{5.3}$$

From (5.3) and (g_3), for all $t > 1$, the following is achieved

$$\begin{aligned}
\mathcal{J}^{\alpha,\beta}(t\underline{v}) &= \mathrm{K}\big(I(t\underline{v})\big) - \int_\Delta F(z,t\underline{v})\,dz \\
&\leq C_7 t^{\frac{p^+}{1-\theta}}\big(I(\underline{v})\big)^{\frac{1}{1-\theta}} - t^{q^-}\int_\Delta \gamma(z)\underline{v}^{q(z)}\,dz - C_8 t^\mu \int_\Delta \underline{v}^\mu\,dz + C_9.
\end{aligned}$$

Then, for some $t_0 > 1$ large enough, $\mathcal{J}^{\alpha,\beta}(t_0\underline{v}) < 0$ and $\|t_0\underline{v}\|_{X_0} > 2\rho$, due to $\frac{p^+}{1-\theta} < \mu$. Thus, we take $e = t_0\underline{v}$, which completes the proof.

Now, we give the proof of Theorem 1.2.

Let v_0 be the solution to problem (1.1) given in Theorem 1.1, which satisfies

$$\mathcal{I}^{\alpha,\beta}(v_0) = \inf_{v \in \Lambda} \mathcal{I}^{\alpha,\beta}(v),$$

with $\nu_0 \in \Lambda := [\underline{\nu}, \overline{\nu}] \cap X_0$. In a standard way, by Lemmas 5.1, 5.2 and the mountain pass theorem [16], the mountain pass level is defined by

$$c^* := \inf_{\lambda \in \Gamma} \max_{t \in [0,1]} \mathcal{J}^{\alpha,\beta}\big(\lambda(t)\big),$$

with

$$\Gamma := \big\{\lambda \in C\big([0,1], X_0\big); \lambda(0) = \underline{\nu}, \lambda(1) = e\big\}$$

is a critical value of $\mathcal{J}^{\alpha,\beta}$. This implies the existence of $\nu_1 \in X_0$ such that $\mathcal{J}'^{\alpha,\beta}(\nu_1) = 0$ and $\mathcal{J}'^{\alpha,\beta}(\nu_1) = c^*$. Considering that $\mathcal{I}^{\alpha,\beta}(\nu) = \mathcal{J}^{\alpha,\beta}(\nu)$ for all $\nu \in [0, \overline{\nu}] \cap X_0$, it follows that $\mathcal{J}^{\alpha,\beta}(\nu_0) \leq \mathcal{J}^{\alpha,\beta}(\underline{\nu})$. Now, we assert that $\nu_1 \geq \underline{\nu}$ a.e. in Δ. Utilizing $(\underline{\nu} - \nu_1)^+$ as a test function in $\mathcal{J}'^{\alpha,\beta}(\nu_1) = 0$ and in the first inequality of (4.1), we obtain:

$$\begin{aligned}
&\mathrm{K}\big(I^{\alpha,\beta}(\nu_1)\big) \int_\Delta \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu_1\big|^{p(z)-2} \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \nu_1 . {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} (\underline{\nu} - \nu_1)^+ \, dz \\
&\quad = \int_\Delta f(z, \nu_1)(\underline{\nu} - \nu_1)^+ \, dz \\
&\quad = \int_\Delta \gamma(z)\underline{\nu}(z)^{q(z)-1} + g(z, \underline{\nu}))(\underline{\nu} - \nu_1)^+ \, dz \\
&\quad \geq \mathrm{K}\big(I^{\alpha,\beta}(\underline{\nu})\big) \times \int_\Delta \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \underline{\nu}\big|^{p(z)-2} \, {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} \underline{\nu} . {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} (\underline{\nu} - \nu_1)^+ \, dz
\end{aligned}$$

that is

$$\big\langle \phi(\underline{\nu}) - \phi(\nu_1), (\underline{\nu} - \nu_1)^+ \big\rangle \leq 0.$$

So, basically, if ϕ is strictly monotone (check out Lemma 3.2), then $(\underline{\nu} - \nu_1)^+$ equals zero almost everywhere in Δ. This leads to ν_1 being greater than or equal to $\underline{\nu}$ almost everywhere in Δ. As a result, ν_0 and ν_1 emerge as two nonnegative solutions to problem (1.1) with

$$\mathcal{J}^{\alpha,\beta}(\nu_0) \leq \mathcal{J}^{\alpha,\beta}(\underline{\nu}) < 0 < \kappa \leq c^* = \mathcal{J}^{\alpha,\beta}(\nu_1).$$

Thus, the proof of Theorem 1.2 is completed. □

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Author details
[1]Laboratory of Mathematics, Informatics and Systems (LAMIS), Department of Mathematics and Computer Science, Echahid Cheikh Larbi Tebessi University, Tebessa, 12002, Algeria. [2]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia.

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