Hezenci and Budak *Journal of Inequalities and Applications* (2024) 2024:45
https://doi.org/10.1186/s13660-024-03130-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Bullen-type inequalities for twice-differentiable functions by using conformable fractional integrals

Fatih Hezenci[1*] and Hüseyin Budak[1]

*Correspondence:
fatihezenci@gmail.com
[1]Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce 81620, Turkey

Abstract

In this paper, we prove an equality for twice-differentiable convex functions involving the conformable fractional integrals. Moreover, several Bullen-type inequalities are established for twice-differentiable functions. More precisely, conformable fractional integrals are used to derive such inequalities. Furthermore, sundry significant inequalities are obtained by taking advantage of the convexity, Hölder inequality, and power-mean inequality. Finally, we provide our results by using special cases of obtained theorems.

Mathematics Subject Classification: 26D07; 26D10; 26D15; 26A33

Keywords: Bullen-type inequalities; Convex functions; Fractional integrals; Riemann–Liouville fractional integrals; Conformable fractional integrals

1 Introduction

It is well known that theory of inequalities and fractional calculus, which has received much attention recently, has been the topic of many studies in the literature. Therefore, it has offered solutions to many problems in several disciplines. The most famous of the fractional approaches that are developing day by day are the Riemann–Liouville, conformable fractional approaches, Caputo, and many types of fractional integrals. In addition to this, convexity theory is an important subject that has been used in many fields of optimization theory, energy systems, engineering applications, and physics. Moreover, convexity theory is an available way to solve a large number of problems from different branches of mathematics. That is why convexity theory has an important place in these branches of mathematics, especially in inequalities such as Hermite–Hadamard, Simpson, Newton, and Bullen-type inequalities are the most well known of these inequalities.

Although classical derivative, classical integral, and differential concepts solve most of the problems that arise in many areas of technology, these concepts are insufficient. Hence, fractional calculus offers new solutions to such problems. Two fundamental approaches are used to do this fractional calculation. The first approach is called the Riemann–Liouville approach. In addition to repeating the integral operator n times, the authors made it possible to convert it to an integral with the Cauchy formula where $n!$ is changed to the Gamma function. Thus, the fractional integral operator of noninteger order is described. These operators were then used to find the Riemann–Liouville and Caputo fractional derivatives. The second approach is the Grünwald–Letnikov approach which is iterating the derivative n times and then fractionalizing including the Gamma function in the binomial coefficients. Taking advantage of the results obtained with these approaches, the calculations become complicated as the product and chain rules are lost from the properties of the derivative. That is why the conformable fractional approach was developed in [20], which depends on the basic definition of the derivative. The author of [1] established that the conformable approach in [20] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by several extensions of the conformable approach [15, 30]. With the help of these approaches, Jarad presented the definitions of conformable fractional integrals in [18]. Considering all these studies, fractional calculus attracts mathematicians day by day.

Bullen introduced Bullen-type inequalities in [3]. Dragomir and Wang [8] acquired a natural generalization of Bullen-type inequalities. Sarikaya et al. obtained generalized Bullen-type inequalities in [27]. Erden and Sarikaya established a few generalized inequalities of Bullen-type by using the local fractional integrals on fractal sets in [12]. Moreover, Du et al. presented the generalized fractional integrals to discover Bullen-type inequalities in [9]. Hwang et al. [14] investigated several new Hermite–Hadamard-, Bullen-, and Simpson-type inequalities with the help of the fractional integrals. In [16], İşcan et al. acquired several Hermite–Hadamard- and Bullen-type inequalities via functions whose derivatives in modulus at certain power are convex. Tseng et al. established several Hadamard- and Bullen-type inequalities via Lipschitz functions and presented some applications by using the special means in [28]. With the aid of the several Euler-type equalities, Matic et al. [22] proved a generalization of Bullen–Simpson's inequality based on $(2r)$-convex functions. In [4], Çakmak acquired several Bullen-type inequalities for differentiable functions by using the s-convexity and Riemann–Liouville fractional integral operators via Gauss hypergeometric function. What is more, several Bullen-type inequalities via differentiable mappings with the aid of the h-convex functions are given in [7]. Furthermore, Kara et al. [19] established the upper and lower bounds for parameterized inequalities with the help of the Riemann–Liouville fractional integral operators. By using the specific choices of the parameter, the authors presented several new Bullen-type inequalities. For further information and unexplained subjects about such inequalities involving different types of fractional integral operators, one can refer the reader to [5, 6, 10, 11, 23–25, 29] and the references cited therein.

This article is organized as follows: In Sect. 2, we will recall the gamma, beta, and incomplete beta functions, which are well known in the literature. Moreover, the basic definitions of Riemann–Liouville integral operators and conformable integrals will be explained for building our main results. In Sect. 3, an equality will be proved for twice-differentiable convex functions involving the conformable fractional integral operators. By using this equality, we establish several Bullen-type inequalities via convex mappings with the help of conformable fractional integrals. To be more precise, Hölder and power-mean inequalities, which are well known in the literature, will be used in some of the proven inequalities. Furthermore, we also give some corollaries and remarks. Finally, in Sect. 4, summary and concluding remarks are noted.

2 Preliminaries

Let us put forth some preliminaries which will be utilized in the sequel. More precisely, definitions of Riemann–Liouville and conformable integrals, which are well known in the literature, are given. From the fractional calculus theory, mathematical preliminaries are given as follows:

Definition 1 The gamma, beta, and incomplete beta functions are defined by:

$$\Gamma(x) := \int_0^\infty \mu^{x-1} e^{-\mu}\, d\mu,$$

$$\mathcal{B}(x,y) := \int_0^1 \mu^{x-1}(1-\mu)^{y-1}\, d\mu,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r \mu^{x-1}(1-\mu)^{y-1}\, d\mu,$$

respectively, for $0 < x, y < \infty$ and $x, y \in \mathbb{R}$.

In [21], Kilbas et al. described fractional integrals, also called Riemann–Liouville integrals as follows:

Definition 2 ([21]) The Riemann–Liouville integrals $J_{\sigma+}^{\beta}\mathcal{F}(x)$ and $J_{\delta-}^{\beta}\mathcal{F}(x)$ of order $\beta > 0$ are given by

$$J_{\sigma+}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x (x-\mu)^{\beta-1}\mathcal{F}(\mu)\, d\mu, \quad x > \sigma \tag{1}$$

and

$$J_{\delta-}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta (\mu-x)^{\beta-1}\mathcal{F}(\mu)\, d\mu, \quad x < \delta, \tag{2}$$

respectively, for $\mathcal{F} \in L_1[\sigma,\delta]$. Here, Γ denotes the gamma function. For $\beta = 1$, the Riemann–Liouville integrals are equal to the classical integrals.

Jarad et al. [18] introduced the following fractional conformable integral operators. They also derived certain characteristics and relationships between these operators and some other fractional operators in the literature. The fractional conformable integral operators are described as follows:

Definition 3 ([18]) The fractional conformable integral operators ${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x)$ and ${}^{\beta}\mathcal{J}_{\delta-}^{\alpha} \times \mathcal{F}(x)$ of order $\beta \in \mathbb{R}^+$ and $\alpha \in (0,1]$ are presented by

$${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x \left(\frac{(x-\sigma)^\alpha - (\mu-\sigma)^\alpha}{\alpha}\right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}\, d\mu, \quad \mu > \sigma, \tag{3}$$

and

$${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta \left(\frac{(\delta-x)^\alpha - (\delta-\mu)^\alpha}{\alpha}\right)^{\beta-1} \frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}}\, d\mu, \quad \mu < \delta, \tag{4}$$

respectively, for $\mathcal{F} \in L_1[\sigma,\delta]$.

Note that the fractional integral in (3) coincides with the Riemann–Liouville fractional integral in (1) if $\sigma = 0$ and $\alpha = 1$. Furthermore, the fractional integral in (4) reduces to the Riemann–Liouville fractional integral in (2) if $\delta = 0$ and $\alpha = 1$. For several recent results connected with fractional integral inequalities, see [2, 13, 17] and the references therein.

3 Main results

In this section, we establish an identity for twice-differentiable convex functions involving the conformable fractional integrals. Moreover, sundry Bullen-type inequalities are proved for twice-differentiable functions. To be more precise, conformable fractional integrals are used to derive these inequalities. Moreover, several important inequalities are obtained by taking advantage of the convexity, Hölder inequality, and power-mean inequality.

Lemma 1 *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ denote a twice-differentiable function on (σ,δ) so that $\mathcal{F}''\in L_1[\sigma,\delta]$. Then, the following equality holds*:

$$\begin{aligned}
&\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right] \\
&\quad-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right] \\
&=\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left\{\int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right. \\
&\quad\left.+\int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right\}.
\end{aligned} \tag{5}$$

Proof With the help of integration by parts, we have

$$\begin{aligned}
I_1&=\int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu \\
&=\frac{2}{\delta-\sigma}\left(\int_\mu^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\Bigg|_0^1 \\
&\quad+\frac{2}{\delta-\sigma}\int_0^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\mathcal{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu \\
&=-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}'\left(\frac{\sigma+\delta}{2}\right) \\
&\quad+\frac{2}{\delta-\sigma}\left\{\frac{2}{\delta-\sigma}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\Bigg|_0^1\right. \\
&\quad\left.-\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right\}
\end{aligned}$$

If we use the change of variables $x=\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta$, then we obtain

$$I_1=-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right)\mathcal{F}'\left(\frac{\sigma+\delta}{2}\right) \tag{6}$$

$$
\begin{aligned}
&+\frac{2}{(\delta-\sigma)^2\alpha^\beta}\left[\mathcal{F}(\delta)+\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\\
&-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+2}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{(\frac{\delta-\sigma}{2})^\alpha-(\delta-x)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(\delta-x)^{1-\alpha}}\,dx\\
=&-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right)\\
&\times\mathcal{F}'\left(\frac{\sigma+\delta}{2}\right)+\frac{2}{(\delta-\sigma)^2\alpha^\beta}\left[\mathcal{F}(\delta)+\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\\
&-\left(\frac{2}{\delta-\sigma}\right)^{2+\alpha\beta}\Gamma(\beta+1)^\beta\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right).
\end{aligned}
$$

Similarly, we can easily get

$$
\begin{aligned}
I_2=&\int_0^1\left(\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right)\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu \qquad (7)\\
=&\frac{2}{\delta-\sigma}\left(\int_0^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right)\mathcal{F}'\left(\frac{\sigma+\delta}{2}\right)\\
&+\frac{2}{(\delta-\sigma)^2\alpha^\beta}\left[\mathcal{F}(\sigma)+\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\\
&-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+2}\Gamma(\beta+1)\,{}^\beta\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right).
\end{aligned}
$$

If (6) and (7) are added and then multiplied by $\frac{(\delta-\sigma)^2\alpha^\beta}{8}$ simultaneously, then the proof of Lemma 1 is finished. □

Theorem 1 *Consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a twice-differentiable function on (σ,δ) such that $\mathcal{F}''\in L_1[\sigma,\delta]$. Assume also that $|\mathcal{F}''|$ is convex on $[\sigma,\delta]$. Then, the following inequality holds*:

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right.\\
&\left.\qquad-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)^2\alpha^\beta}{8}\psi_1(\alpha,\beta)\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].
\end{aligned}
$$

Here,

$$
\begin{aligned}
\psi_1(\alpha,\beta)=&\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|d\mu \qquad (8)\\
=&\frac{1}{\alpha^\beta}\int_0^1\left|\frac{1}{\alpha}\left(\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-\mu)^\alpha\right)\right)-\frac{1-\mu}{2}\right|d\mu,
\end{aligned}
$$

where $\mathcal{B}$ and $\mathscr{B}$ denote the beta and incomplete beta functions, respectively.

Proof If we start by taking the absolute value of both sides of (5), then we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right. \\
&\qquad \left. -\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad \leq \frac{(\delta-\sigma)^{2}\alpha^{\beta}}{8}\left[\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\right. \\
&\qquad \left.+\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\right].
\end{aligned} \tag{9}$$

From the fact that $|\mathcal{F}''|$ is convex on $[\sigma,\delta]$, we obtain

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right. \\
&\qquad \left. -\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad \leq \frac{(\delta-\sigma)^{2}\alpha^{\beta}}{8}\left[\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|\right. \\
&\qquad \times\left(\frac{1-\mu}{2}\left|\mathcal{F}''(\sigma)\right|+\frac{1+\mu}{2}\left|\mathcal{F}''(\delta)\right|\right)d\mu \\
&\qquad \left.+\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|\left(\frac{1+\mu}{2}\left|\mathcal{F}''(\sigma)\right|+\frac{1-\mu}{2}\left|\mathcal{F}''(\delta)\right|\right)d\mu\right] \\
&\quad = \frac{(\delta-\sigma)^{2}\alpha^{\beta}}{8}\left(\int_{0}^{1}\left|\int_{\mu}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|d\mu\right)\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].
\end{aligned}$$

Therefore, the proof of Theorem 1 is completed. □

Corollary 1 *Let us consider $\alpha=1$ in Theorem 1. Then, we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J_{\delta-}^{\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^{\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad \leq \frac{(\delta-\sigma)^{2}}{8}\psi_{1}(1,\beta)\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].
\end{aligned}$$

Here,

$$\psi_{1}(1,\beta)=\int_{0}^{1}\left|\int_{\mu}^{1}\left[\mu^{\beta}-\frac{1}{2}\right]d\mu\right|d\mu=\int_{0}^{1}\left|\frac{\mu-1}{2}+\frac{1-\mu^{\beta+1}}{\beta+1}\right|d\mu. \tag{10}$$

Remark 1 Consider $\alpha=1$ and $\beta=1$ in Theorem 1. Then,

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx\right|\leq\frac{(\delta-\sigma)^{2}}{96}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right],$$

which is given in [26, Proposition 4].

Theorem 2 *Suppose that* $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ *is a twice-differentiable function on* (σ,δ) *such that* $\mathcal{F}''\in L_1[\sigma,\delta]$ *and* $|\mathcal{F}''|^q$ *is convex on* $[\sigma,\delta]$ *with* $q>1$. *Then, the following inequalities*:

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right.\\
&\quad\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\varphi^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}\right]\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(4\varphi^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right]
\end{aligned}$$

are valid. Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\varphi^{\beta}_{\alpha}(p)=\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|^p d\mu.$$

Proof If we use Hölder inequality in (9), then we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right.\\
&\quad\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left[\left(\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|^p d\mu\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|^p d\mu\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\int_0^1\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is known that $|\mathcal{F}''|^q$ is convex on $[\sigma,\delta]$. Then, we get

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right.\\
&\quad\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]d\mu\right|^p d\mu\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left(\frac{1-\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathcal{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left(\frac{1+\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathcal{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Consider $\eta_1 = |\mathcal{F}''(\sigma)|^q$, $\varrho_1 = 3|\mathcal{F}''(\delta)|^q$, $\eta_2 = 3|\mathcal{F}''(\sigma)|^q$, and $\varrho_2 = |\mathcal{F}''(\delta)|^q$. If we apply the inequality $\sum_{k=1}^n(\eta_k+\varrho_k)^\mu \le \sum_{k=1}^n \eta_k^\mu + \sum_{k=1}^n \varrho_k^\mu$, with $0 \le \mu < 1$, then the proof of Theorem 2 is completed. □

Corollary 2 *If we choose* $\alpha = 1$ *in Theorem* 2, *then*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^\beta}\left[J_{\delta-}^\beta\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^\beta\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$\le \frac{(\delta-\sigma)^2}{8}\left(\varphi_1^\beta(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}\right]$$
$$\le \frac{(\delta-\sigma)^2}{8}\left(4\varphi_1^\beta(p)\right)^{\frac{1}{p}}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right],$$

where

$$\varphi_1^\beta(p) = \int_0^1\left|\frac{\mu-1}{2}+\frac{1-\mu^{\beta+1}}{\beta+1}\right|^p d\mu.$$

Corollary 3 *If we assign* $\alpha = 1$ *and* $\beta = 1$ *in Theorem* 2, *then the following double inequality holds*:

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{1}{\delta-\sigma}\int_\sigma^\delta\mathcal{F}(x)\,dx\right|$$
$$\le \frac{(\delta-\sigma)^2}{16}\left(\mathcal{B}(p+1,p+1)\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|\mathcal{F}''(\sigma)|^q+3|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}+\left(\frac{3|\mathcal{F}''(\sigma)|^q+|\mathcal{F}''(\delta)|^q}{4}\right)^{1/q}\right]$$
$$\le \frac{(\delta-\sigma)^2}{16}\left(4\mathcal{B}(p+1,p+1)\right)^{\frac{1}{p}}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].$$

Theorem 3 *If* $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ *is a twice-differentiable function on* (σ,δ) *such that* $\mathcal{F}'' \in L_1([\sigma,\delta])$ *and* $|\mathcal{F}''|^q$ *is convex on* $[\sigma,\delta]$ *with* $q \ge 1$, *then the following inequality holds*:

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\right.$$
$$\left.-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{\delta-}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}_{\sigma+}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$\le \frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\psi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{\psi_1(\alpha,\beta)-\psi_2(\alpha,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{\psi_1(\alpha,\beta)+\psi_2(\alpha,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\psi_1(\alpha,\beta)+\psi_2(\alpha,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{\psi_1(\alpha,\beta)-\psi_2(\alpha,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right].$$

Here, $\psi_1(\alpha,\beta)$ *is defined as in* (8) *and*

$$\psi_2(\alpha,\beta)=\int_0^1 \mu\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|d\mu$$

$$=\frac{1}{\alpha^\beta}\int_0^1 \mu\left|\frac{1}{\alpha}\left(\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-\mu)^\alpha\right)\right)-\frac{1-\mu}{2}\right|d\mu,$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta and incomplete beta functions, respectively.*

Proof If we apply the power-mean inequality in (9), then we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right] \\
&\quad-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{\delta-}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}_{\sigma+}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\Bigg| \\
&\quad\le\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|d\mu\right)^{1-\frac{1}{q}}\right. \\
&\quad\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|\left|\mathcal{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\left.\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|\left|\mathcal{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|\mathcal{F}''|^q$ is convex on $[\sigma,\delta]$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right] \\
&\quad-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{\delta-}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}_{\sigma+}^\alpha\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\Bigg| \\
&\quad\le\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|\right.\right. \\
&\quad\left.\times\left[\frac{1-\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathcal{F}''(\delta)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right]d\mu\right|\right. \\
&\quad\left.\left.\times\left[\frac{1+\mu}{2}\left|\mathcal{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathcal{F}''(\delta)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is clearly seen that

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\leq\frac{(\delta-\sigma)^{2}\alpha^{\beta}}{8}\big(\psi_{1}(\alpha,\beta)\big)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\frac{\psi_{1}(\alpha,\beta)-\psi_{2}(\alpha,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^{q}+\frac{\psi_{1}(\alpha,\beta)+\psi_{2}(\alpha,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\frac{\psi_{1}(\alpha,\beta)+\psi_{2}(\alpha,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^{q}+\frac{\psi_{1}(\alpha,\beta)-\psi_{2}(\alpha,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]. \qquad \square
\end{aligned}$$

Corollary 4 *Let us consider $\alpha = 1$ in Theorem 3. Then, the following inequality holds*:

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J_{\delta-}^{\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^{\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad\leq\frac{(\delta-\sigma)^{2}}{8}\big(\psi_{1}(1,\beta)\big)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\frac{\psi_{1}(1,\beta)-\psi_{2}(1,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^{q}+\frac{\psi_{1}(1,\beta)+\psi_{2}(1,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\frac{\psi_{1}(1,\beta)+\psi_{2}(1,\beta)}{2}\left|\mathcal{F}''(\sigma)\right|^{q}+\frac{\psi_{1}(1,\beta)-\psi_{2}(1,\beta)}{2}\left|\mathcal{F}''(\delta)\right|^{q}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\psi_1(1,\beta)$ is defined in (10) and

$$\psi_{2}(1,\beta)=\int_{0}^{1}\mu\left|\int_{\mu}^{1}\left[\mu^{\beta}-\frac{1}{2}\right]d\mu\right|d\mu=\int_{0}^{1}\mu\left|\frac{\mu-1}{2}+\frac{1-\mu^{\beta+1}}{\beta+1}\right|d\mu.$$

Remark 2 If we take $\alpha = 1$ and $\beta = 1$ in Theorem 3, then

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx\right| \\
&\quad\leq\frac{(\delta-\sigma)^{2}}{96}\left[\left(\frac{|\mathcal{F}''(\sigma)|^{q}+3|\mathcal{F}''(\delta)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}''(\sigma)|^{q}+|\mathcal{F}''(\delta)|^{q}}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is given in [26, Theorem 5 (for $\lambda = 1/2$)].

4 Summary and concluding remarks

In the present paper, an equality for twice-differentiable convex functions involving the conformable fractional integrals is established. Moreover, sundry Bullen-type inequalities are proved for twice-differentiable functions. In addition, several important inequalities are obtained by taking advantage of the convexity, Hölder inequality, and power-mean inequality. Furthermore, we provide our results by using special cases of obtained theorems.

We expect that the ideas and techniques of this paper will inspire interested readers working in this field. With the techniques used in obtaining our inequalities, different fractional integrals can be used to obtain new inequalities in the future. In addition, new inequalities can be acquired by considering different order derivatives of the functions.

Furthermore, one can obtain sundry Bullen-type inequalities for convex functions by using quantum calculus.

Author contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding Info
There was no funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Received: 24 January 2024 Accepted: 26 March 2024 Published online: 02 April 2024

References

1. Abdelhakim, A.A.: The flaw in the conformable calculus: it is conformable because it is not fractional. Fract. Calc. Appl. Anal. **22**, 242–254 (2019)
2. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
3. Bullen, P.S.: Error estimates for some elementary quadrature rules. Publ. Elektroteh. Fak. Univ. Beogr., Ser. Mat. Fiz. **602**(633), 97–103 (1978)
4. Çakmak, M.: Refinements of Bullen-type inequalities for μ-convex functions via Riemann–Liouville fractional integrals involving Gauss hypergeometric function. J. Interdiscip. Math. **22**(6), 975–989 (2019)
5. Çakmak, M.: On some Bullen-type inequalities via conformable fractional integrals. J. Sci. Perspect. **3**(4), 285–298 (2019)
6. Çakmak, M.: Some Bullen-type inequalities for conformable fractional integrals. Gen. Math. **28**(2), 3–17 (2020)
7. Çakmak, M.: The differentiable *h*-convex functions involving the Bullen inequality. Acta Univ. Apulensis, Mat.-Inform. **65**, 29 (2021)
8. Dragomir, S.S., Wang, S.O.N.G.: A generalization of Bullen's inequality for convex mappings and its applications. Soochow J. Math. **24**(2), 97–103 (1998)
9. Du, T.S., Luo, C., Cao, Z.: On the Bullen-type inequalities via generalized fractional integrals and their applications. Fractals **29**(07) (2021)
10. Du, T.S., Peng, Y.: Hermite–Hadamard type inequalities for multiplicative Riemann–Liouville fractional integrals. J. Comput. Appl. Math. **440**, 115582 (2024)
11. Du, T.S., Zhou, T.C.: On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings. Chaos Solitons Fractals **156**, 111846 (2022)
12. Erden, S., Sarikaya, M.Z.: Generalized Bullen-type inequalities for local fractional integrals and its applications. Palest. J. Math. **9**(2), 945 (2020)
13. Hezenci, F., Budak, H., Kara, H.: New version of fractional Simpson type inequalities for twice differentiable functions. Adv. Differ. Equ. **2021**, 460 (2021)
14. Hwang, H.R., Tseng, K.L., Hsu, K.C.: New inequalities for fractional integrals and their applications. Turk. J. Math. **40**(3), 471–486 (2016)
15. Hyder, A., Soliman, A.H.: A new generalized θ-conformable calculus and its applications in mathematical physics. Phys. Scr. **96**, 015208 (2020)
16. İşcan, I., Toplu, T., Yetgin, F.: Some new inequalities on generalization of Hermite–Hadamard and Bullen type inequalities, applications to trapezoidal and midpoint formula. Kragujev. J. Math. **45**(4), 647–657 (2021)
17. Jarad, F., Abdeljawad, T., Baleanu, D.: On the generalized fractional derivatives and their Caputo modification. J. Nonlinear Sci. Appl. **10**(5), 2607–2619 (2017)
18. Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, 247 (2017)
19. Kara, H., Budak, H., Hezenci, F.: New extensions of the parameterized inequalities based on Riemann–Liouville fractional integrals. Mathematics **10**(18), 3374 (2022)
20. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
21. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Sci. B.V., Amsterdam (2006)
22. Matić, M., Pečarić, J., Vukelić, A.: On generalization of Bullen–Simpson's inequality. Rocky Mt. J. Math. **35**, 1727–1754 (2005)
23. Qi, Y., Li, G.: Fractional Ostrowski type inequalities for (s,m)-convex function with applications. Fractals **31**(9), 2350128 (2023)
24. Qi, Y., Li, G., Wang, S., Wen, Q.Z.: Hermite–Hadamard–Fejer type inequalities via Katugampola fractional integrals for *s*-convex functions in the second sense. Fractals **30**(7), 2250131 (2022)
25. Qi, Y., Wen, Q., Li, G., Xiao, K., Wang, S.: Discrete Hermite–Hadamard type inequalities for (s,m)-convex function. Fractals **30**(7), 2250160 (2022)

26. Sarikaya, M.Z., Aktan, N.: On the generalization of some integral inequalities and their applications. Math. Comput. Model. **54**(9–10), 2175–2182 (2011)
27. Sarikaya, M.Z., Budak, H.: Some integral inequalities for local fractional integrals. Int. J. Anal. Appl. **14**(1), 9–19 (2017)
28. Tseng, K.L., Hwang, S.R., Hsu, K.C.: Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications. Comput. Math. Appl. **64**(4), 651–660 (2012)
29. Zhang, L.L., Peng, Y., Du, T.S.: On multiplicative Hermite–Hadamard- and Newton-type inequalities for multiplicatively (P,m)-convex functions. J. Math. Anal. Appl. **534**(2), 128117 (2024)
30. Zhao, D., Luo, M.: General conformable fractional derivative and its physical interpretation. Calcolo **54**, 903–917 (2017)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.