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A surface area formula for compact hypersurfaces in $\mathbb{R}^n$

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Abstract

The classical Cauchy surface area formula states that the surface area of the boundary $\partial K = \Sigma$ of any n-dimensional convex body in the n-dimensional Euclidean space $\mathbb{R}^n$ can be obtained by the average of the projected areas of Σ along all directions in $\mathbb{S}^{n-1}$. In this note, we generalize the formula to the boundary of arbitrary n-dimensional submanifold in $\mathbb{R}^n$ by introducing a natural notion of projected areas along any direction in $\mathbb{S}^{n-1}$. This surface area formula derived from the new notion coincides with not only the result of the Crofton formula but also with that of De Jong (Math. Semesterber. 60(1):81–83, 2013) by using a tubular neighborhood. We also define the projected r-volumes of Σ onto any r-dimensional subspaces and obtain a recursive formula for mean projected r-volumes of Σ.

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1 Introduction and main results

Finding the volume of k-dimensional embedded submanifolds K with boundary $\Sigma = \partial K$ in the Euclidean space $\mathbb{R}^n$ is an interesting research topic in differential geometry. There have been abundant results to measure the volumes and surface areas of K. For instance, the surface area of K can be measured by restricting the Euclidean metrics to Σ and integrating the $(k-1)$-volumes of the infinitesimal parallelotopes in the tangent space of K [1, Sect. 2.2]; it can also be obtained by limiting the ratio for the volume of an $(n-k)$-ball and the n-dimensional Lebesgue measure of the tubular neighborhood of Σ derived by De Jong [4].

In this note, we restrict our attention to the method developed in integral geometry [11] and convex geometry [12]. The efficiency of the method has been validated on the applications of geometric tomography and other scientific fields (for more detail, see [6]). The basic assumption in this approach is that K must be convex, since the original proof was established by a limiting process of convex polyhedrons inscribed in K. When $k = n$, we observe that the convexity assumption for K may be relaxed by introducing a natural notion of projected surface areas of Σ onto any r-dimensional subspaces, $r \leq n-1$ (see Definition 1). The new notion also gives an alternative proof for Crofton's formula, which states that the surface area of Σ can be obtained by "counting" the number of intersections of all lines with Σ (Lemma 2 and Theorem 2), and it results in the generalization of the Cauchy surface area formula for arbitrary K (Theorem 3). We also generalize the projected surface area to higher-codimensional subspaces, namely, the projected r-volume ($1 \le r \le n-1$) of Σ onto r-subspaces (Definition 2), and derive a recursive formula of mean projected area of Σ (Theorem 4).

It seems that there are only a few results in the literature concerning the relation between projected areas and surface areas for nonconvex boundaries. Two closer notions probably are the integral geometric measure defined by Favard [5] and more recently by Bouafia and Pauw [2]; both are based on the approach of geometric measure theory. The kinematic formula has similar results without introducing the notion of projections; see, for instance, [14, 15]. As mentioned before, De Jong [4] gave a geometric definition of the volume of an r-dimensional submanifold in $\mathbb{R}^n$ ($1 \le r \le n-1$) and derived an r-volume formula by considering the ratio of r-volume of the tubular neighborhood to the volume of the unit ball in the normal bundle. He claimed that the formula holds when the submanifold is of dimension n with C^1-boundary. However, we will take parallel transformations of Lie groups and the method of moving frames to construct the r-volume forms on the r-subspaces to represent the r-projected volumes of Σ (see the discussion in *Case 1* and *Case 2* of Sect. 2). In Proposition 1, we will also prove that the surface area we derived (see Theorem 3) coincides with that in his work [4].

The Cauchy surface area formula in $\mathbb{R}^n$ states that the surface area of a convex hypersurface Σ in $\mathbb{R}^n$ can be represented by the average of the projected areas of Σ along all normal directions of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$:

Theorem 1 ([9], Theorem 5.5.2) *Let $K \subset \mathbb{R}^n$ be an n-dimensional convex body* (*i.e., a convex set with nonempty interior*) *with rectifiable boundary $\partial K = \Sigma$. The surface area* (*or the $(n-1)$-dimensional volume*) *of K, denoted by $\mathcal{V}_{n-1}(\Sigma)$, is given by*

$$\mathcal{V}_{n-1}(\Sigma) = \frac{1}{\omega_{n-1}} \int_{\nu \in \mathbb{S}^{n-1}} \mathcal{V}_{n-1}\big(\Sigma|\nu^{\perp}\big)\, dS_{\nu}. \tag{1}$$

Here dS_ν is the surface area element at $\nu \in \mathbb{S}^{n-1}$, ω_{n-1} is the $(n-1)$-dimensional volume of the unit ball in $\mathbb{R}^{n-1}$, and $\mathcal{V}_{n-1}(\Sigma|\nu^{\perp})$ is the $(n-1)$-dimensional volume of the orthogonal projection of Σ onto the $(n-1)$-dimensional subspace $\nu^{\perp}$ perpendicular to the unit outward normal ν.

The orthogonal projected area $\mathcal{V}_{n-1}(\Sigma|\nu^{\perp})$ of Σ can be explicitly represented by the integral formula

$$\mathcal{V}_{n-1}\big(\Sigma|\nu^{\perp}\big) = \frac{1}{2} \int_{\tilde{\nu} \in \mathbb{S}^{n-1}} |\nu \cdot \tilde{\nu}|\, dS_{\tilde{\nu}}, \tag{2}$$

where $dS_{\tilde{\nu}}$ is the surface area element at $\tilde{\nu} \in \mathbb{S}^{n-1}$. Identities (1) and (2) both can be proved by inscribing a convex m-polyhedron in Σ and calculating the limit of the surface area of the polyhedron as m goes to infinity. The convexity of the polyhedron ensures that at any point in $\nu^{\perp}$, the counting multiplicity of the projected areas of Σ is almost everywhere two (we may assume that the polyhedron contains the origin and each point in $\nu^{\perp}$ is projected for twice: from the "front" and the "back" of $\nu^{\perp}$ respectively). See the proof of

Theorem 5.5.2 in [9] or [11, p. 217] for details. However, such a method does not work for nonconvex K since it may not necessarily have the inscribed convex polyhedron.

Although, in general, Theorem 1 does not work for arbitrary hypersurface Σ, formula (1) still holds in the particular nonconvex case where $\Sigma = \partial K$ is such that its complement $K^c = \mathbb{R}^n \setminus K$ is convex. Indeed, let K be a nonconvex set with convex complement K^c. By Lemma 3 (see below) we will show that the closure $\overline{K^c}$ of K^c is convex. Since Σ shares the same boundary with K and K^c, the surface area of $\Sigma = \partial K$ can be obtained by applying formula (1) to the convex part ∂K^c.

As a result, if the hypersurface Σ obtained from a nonconvex subset K such that the complement K^c is also nonconvex, then the argument in the previous paragraph fails. A simple example is that when Σ is the boundary of a star domain S, its complement $\mathbb{R}^n \setminus S$ is again nonconvex. The main aim of the present paper is to generalize the notion of projected areas for nonconvex boundary and derive the Cauchy surface area formula (1) for arbitrary boundary.

Let us make some remarks about Theorem 1. First, when $K = \mathbb{S}^{n-1}$, the following lemma shows that the surface area of the orthogonal projection of the unit sphere is independent of the choice of the projected direction.

Lemma 1 ([9], Lemma 5.5.1) *For any* $\nu \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$,

$$\mathcal{V}_{n-1}\left(\mathbb{S}^{n-1}\middle|\nu^{\perp}\right) = \frac{1}{2}\int_{\tilde{\nu}\in\mathbb{S}^{n-1}} |\nu \cdot \tilde{\nu}|\, dS_{\tilde{\nu}} = \omega_{n-1}. \tag{3}$$

This natural property for $\mathbb{S}^{n-1}$ plays the key role for the proof of Theorem 1 (see [9] or the proof of Theorem 3). We also notice that the converse of Lemma 1 is not true in general. There exist compact hypersurfaces in $\mathbb{R}^n$ that are not the standard spheres but have constant projected areas. For instance, Reuleaux triangles in $\mathbb{R}^2$ and bodies of constant width in $\mathbb{R}^n$ (see [3]) both are not round spheres but have constant projected areas.

Secondly, it is known that the Gauss map $\nu : \Sigma = \partial K \to \mathbb{S}^{n-1}$ is bijective if and only if K is convex. Moreover, if K is strictly convex, then the Gauss map is a diffeomorphism. Since the surfaces Σ considered in Theorem 1 and Lemma 1 are convex, the domains over which both integrations are performed are $\mathbb{S}^{n-1}$, which can be identified (via the diffeomorphism ν) with Σ. In consequence, a natural generalization of Theorem 1 to nonconvex hypersurfaces Σ can be considered as an integral over Σ itself.

We recall some fundamental background for the n-dimensional Euclidean space $\mathbb{R}^n$. The space $\mathbb{R}^n$ can be regarded as an n-dimensional Lie group with natural left translation L_q defined by $L_q p = q + p$ for all $q, p \in \mathbb{R}^n$. Its inverse translation and compositions are as be $(L_q)^{-1} r = L_{q^{-1}} r$ and $(L_q \circ L_{p^{-1}}) r = L_{qp^{-1}} r = q - p + r$, respectively, for all $r \in \mathbb{R}^n$. For each point $p \in \mathbb{R}^n$, we may identify the whole space $\mathbb{R}^n$ with the tangent space $T_p\mathbb{R}^n$. In this paper, we will abuse the notations for any point $q \in \mathbb{R}^n$ and the vector $\vec{Oq}$ starting from the origin O and ending at q if the content is clear. Thus, for any point $p \in \mathbb{R}^n$ and any tangent vector $q \in T_0\mathbb{R}^n$, the push-forward of the left translation $L_{p_*}q$ is the vector obtained by moving the vector $\vec{Oq}$ to the vector starting from p and parallel to $\vec{Oq}$. Similarly, $(L_p^*)(\omega(q))$ denotes the pull-back of the r-form ω at q. We insist using the Lie group notations in $\mathbb{R}^n$ (for instance, using $L_p q$ instead of $p + q$ for vector addition) because we believe that all contents in the paper can be applied to any smooth Lie group without change. In particular, recently, the author also proved the Cauchy surface area formula for domains in the Heisenberg groups, which is regarded as a strictly pseudoconvex CR manifold with Tanaka–Webster curvature vanished (see [7]). We also denote by $n(p) \in \mathbb{S}^{n-1}$ the outward unit normal at $p \in \mathbb{S}^{n-1}$ and by $n(p)^{\perp}$ the subspace perpendicular to $n(p)$.

Let K be an n-dimensional compact submanifold embedded in $\mathbb{R}^n$ ($n \geq 1$) with rectifiable boundary $\Sigma = \partial K$ in the usual topology of $\mathbb{R}^n$. Next, we give the definition of projected areas of compact hypersurfaces (not necessarily convex).

Definition 1 The (orthogonal) weighted projected area of a compact hypersurface Σ in $\mathbb{R}^n$ along the direction $n(p) \in \mathbb{S}^{n-1}$ onto the subspace $n(p)^{\perp}$ is given by

$$\mathcal{V}_{n-1}\left(\Sigma | n(p)^{\perp}\right) = \frac{1}{2} \int_{q \in \Sigma} \left| L_{qp^{-1}{}_*} n(p) \cdot \tilde{n}(q) \right| d\Sigma_q, \tag{4}$$

where $L_{qp^{-1}{}_*} n(p)$ is the push-forward of the outward unit normal $n(p)$ at $p \in \mathbb{S}^{n-1}$, $\tilde{n}(q)$ is the outward unit normal at $q \in \Sigma$, and $d\Sigma_q$ is the area element of Σ at $q \in \Sigma$.

Remark 1 We give a geometric interpretation of (4) as follows. At each point $q \in \Sigma$, the integrand with area element at q, $|L_{qp^{-1}{}_*} n(p) \cdot \tilde{n}(q)| d\Sigma_q$, is the projected infinitesimal area element of $d\Sigma_q$ onto the projected point of q on the plane $n(p)^{\perp}$, and the integral becomes the projected area of Σ onto $n(p)^{\perp}$ counted with multiplicity. In other words, suppose ℓ is a line parallel to $n(p)$ such that $\ell \cap \Sigma \neq \emptyset$ and $\ell \cap n(p)^{\perp} \neq \emptyset$. Then all points at $\ell \cap \Sigma$ are projected along ℓ onto one point at $\ell \cap n(p)^{\perp}$ with multiple times depending on the number of $\ell \cap \Sigma$. This is the reason we call the integral the weighted projected area. When Σ is convex, the number of the points on $\ell \cap \Sigma$ is almost everywhere two, but in general the number depends on the projected direction and Σ.

According to Remark 1, a geometric definition equivalent to Definition 1 is given in the following lemma, which states that the value $\mathcal{V}_{n-1}(\Sigma | n(p)^{\perp})$ is equal to the integral of the number of intersections over all lines parallel to n.

Lemma 2 *Let Σ be a compact hypersurface in $\mathbb{R}^n$, and let $n(p) \in \mathbb{S}^{n-1}$ ve a unit vector. Suppose ℓ is any line parallel to $n(p)$ and intersects with the orthogonal complement $n^{\perp}(p)$ at the point u. Then the projected area of Σ onto the subspace $n(p)^{\perp}$ in Definition 1 can be obtained by*

$$\mathcal{V}_{n-1}\left(\Sigma | n(p)^{\perp}\right) = \frac{1}{2} \int_{\ell \perp n(p)^{\perp}} \#(\ell \cap \Sigma)\, dn_u^{\perp}, \tag{5}$$

where $\#(\ell \cap \Sigma)$ is the number of intersections of ℓ and Σ, and $dn_u^{\perp}$ is the area element of $n(p)^{\perp}$ at $u \in n(p)^{\perp} \cap \ell$.

Proof For simplicity, we fix any point $p \in \mathbb{S}^{n-1}$ and write $n = n(p)$ and $n^{\perp} = n(p)^{\perp}$. Suppose the line ℓ parallel to n intersects Σ at the point q. Then ℓ must intersect the subspace $n^{\perp}$ at the unique point u. We may choose two orthonormal frames $\{q, e_i\}$ and $\{u, \bar{e}_j\}$, $1 \leq i, j \leq n$, at q and u, respectively, satisfying the conditions

$$\begin{cases} \text{at the point } q: & e_1 \in T_q\Sigma^{\perp}, e_2, e_\alpha \in T_q\Sigma, \\ \text{at the point } u: & \bar{e}_1 \in \ell, \bar{e}_2, \bar{e}_\alpha \in T_u n^{\perp}, \\ e_\alpha = \bar{e}_\alpha & \text{for } \alpha = 3, \dots, n. \end{cases} \tag{6}$$

According to the construction of the frames, the transition matrix $[a^i_j]$ is given by $\bar{e}_j = \Sigma^n_{i=1} e_i a^i_j$, where

$$\begin{cases} a^1_1 = a^2_2 = \cos\theta, \\ a^1_2 = -a^2_1 = -\sin\theta, \\ a^\alpha_\beta = \delta^\alpha_\beta, \quad \text{the Kronecker delta}, 3 \le \alpha, \beta \le n, \end{cases}$$

and θ is the angle between the planes $T_q\Sigma$ and $n^\perp$. Suppose $\{\omega^1, \omega^2, \omega^\alpha\}$ and $\{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^\alpha\}$ are the dual forms of $\{e_i\}$ and $\{\bar{e}_i\}$, respectively. It can be shown that $\bar{\omega}^i = \Sigma^n_{j=1} b^i_j \omega^j$, where the matrix $[b^j_i] = [a^j_i]^{-1}$. Therefore by identifying the point $q \in \Sigma$ and $u \in n^\perp$ the relation between the area elements $d\Sigma_q$ and $dn^\perp_u$ at q and u, respectively, can be obtained as follows:

$$\begin{aligned} dn^\perp_u &= \bar{\omega}^2 \wedge \bar{\omega}^3 \wedge \cdots \wedge \bar{\omega}^n \\ &= \left(\sin\theta\omega^1 + \cos\theta\omega^2\right) \wedge \omega^3 \wedge \cdots \wedge \omega^n \\ &= \sin\theta\omega^1 \wedge \omega^3 \wedge \cdots \wedge \omega^n + \cos\theta\omega^2 \wedge \omega^3 \wedge \cdots \wedge \omega^n \\ &= \sin\theta\omega^1 \wedge \omega^3 \wedge \cdots \wedge \omega^n + \cos\theta\, d\Sigma_q. \end{aligned} \tag{7}$$

When restricted on Σ, by (7) we have the projected formula

$$dn^\perp_u|_\Sigma = |\cos\theta|\, d\Sigma_q \tag{8}$$

(here we have put the absolute value on the cosine to get the positive surface area). By integrating over all lines ℓ parallel to n we have

$$\begin{aligned} \mathcal{V}_{n-1}\left(\Sigma|n^\perp\right) &= \frac{1}{2}\int_{q\in\Sigma} \left|L_{qp^{-1}}{}_* n(p) \cdot \tilde{n}(q)\right| d\Sigma_q \\ &= \frac{1}{2}\int_{q\in\Sigma} |\cos\theta|\, d\Sigma_q = \frac{1}{2}\int_{u\in\ell\cap n^\perp} \#(\ell\cap\Sigma)\, dn^\perp_u, \end{aligned}$$

which completes the proof. □

Recall that the Crofton formula in $\mathbb{R}^2$ states that the perimeter of a rectifiable plane curve γ is equal to the integral of the number $\#(\ell\cap\gamma)$ of intersections of γ and any line ℓ. It has been generalized to higher dimensions with a variety of versions (see [11, Chap. 14] or [13]). Notice that, in contrast to the Cauchy surface formula, the Crofton formula does not need the convexity assumption for curves and hypersurfaces, and hence $\mathcal{V}_{n-1}(\Sigma|n^\perp(p))$ in Lemma 2 seems a reasonable definition connecting both Cauchy's and Crofton's formulas.

The previous result, Lemma 2, will give a simpler proof for the Crofton formula in $\mathbb{R}^n$ for $n \ge 1$. Before that, let us introduce some basic settings. Let $\mathcal{L}$ be the set of oriented lines in $\mathbb{R}^n$. Suppose $\ell \in \mathcal{L}$ and $\ell^\perp$ is its orthogonal complement through the origin. The line ℓ can be uniquely determined by the following process: first, choose a line ℓ' parallel to ℓ through the origin and use the unit vector $n(p) \in \mathbb{S}^{n-1}$ to represent the direction of ℓ' for some $p \in \mathbb{S}^{n-1}$. Second, parallelly move ℓ' to ℓ at the point $u = \ell\cap\ell^\perp$. By identifying $\ell^\perp$ with

$\mathbb{R}^{n-1}$, there is a natural one-to-one correspondence between the set $\mathcal{L}$ and $\mathbb{S}^{n-1} \times \mathbb{R}^{n-1}$,

$$\mathcal{L} \longleftrightarrow \mathbb{S}^{n-1} \times \mathbb{R}^{n-1},$$
$$\ell \longleftrightarrow \big(n(p), u\big).$$

Thus we may take the $(2n-2)$-form $d\ell := dS_p \wedge d\ell_u^{\perp}$ as an invariant measure on $\mathcal{L}$, where dS_p is the area element at $p \in \mathbb{S}^{n-1}$, and $d\ell_u^{\perp}$ is the area element of $\ell^{\perp}$ at $u = \ell \cap \ell^{\perp}$. Notice that $dS_p \wedge d\ell_u^{\perp}$ is invariant under the rigid motions (rotations and translations) in $\mathbb{R}^n$.

Theorem 2 (The Crofton formula in $\mathbb{R}^n$) *Let K be an n-dimensional compact submanifold with rectifiable boundary $\partial K = \Sigma$. Let $\mathcal{L}$ be the set of all oriented lines in $\mathbb{R}^n$ and denote by $\#(\ell \cap \Sigma)$ the number of intersections of $\ell \in \mathcal{L}$ and Σ. Then the surface area $\mathcal{V}_{n-1}(\Sigma)$ of K is given by*

$$\mathcal{V}_{n-1}(\Sigma) = \frac{1}{2\omega_{n-1}} \int_{\ell \in \mathcal{L}} \#(\ell \cap \Sigma)\, d\ell, \tag{9}$$

where $d\ell = dS_p \wedge d\ell_u^{\perp}$ is the invariant measure in $\mathcal{L}$ consisting of the area elements dS_p at $p \in \mathbb{S}^{n-1}$ and $d\ell_u^{\perp}$ at $u = \ell \cap \ell^{\perp}$, with the orthogonal complement $\ell^{\perp}$ of ℓ through the origin.

Proof For any line $\ell = \ell_p \in \mathcal{L}$ with direction $n(p) \in \mathbb{S}^{n-1}$ and $\ell \cap \Sigma = q \neq \emptyset$, let $\phi_{p,q}$ be the angle between ℓ_p and the normal vector at $q \in \Sigma$. Notice that $d\ell_u^{\perp} = dn_u^{\perp}$ in Lemma 2. Using Lemma 1, Definition 1, and Lemma 2, a straight-forward derivation implies that

$$\begin{aligned}
2\omega_{n-1}\mathcal{V}_{n-1}(\Sigma) &= 2\int_{q \in \Sigma} \omega_{n-1}\, d\Sigma_q = \int_{q \in \Sigma} \int_{p \in \mathbb{S}^{n-1}} |\cos\phi_{p,q}|\, dS_p\, d\Sigma_q \\
&= \int_{p \in \mathbb{S}^{n-1}} \int_{q \in \Sigma} |\cos\phi_{p,q}|\, d\Sigma_q\, dS_p \\
&= \int_{p \in \mathbb{S}^{n-1}} 2\mathcal{V}_{n-1}\big(\Sigma | n(p)^{\perp}\big)\, dS_p \\
&= \int_{p \in \mathbb{S}^{n-1}} \int_{\ell \perp \ell_p^{\perp}} \#(\ell \cap \Sigma)\, d\ell_u^{\perp}\, dS_q \\
&= \int_{\ell \in \mathcal{L}} \#(\ell \cap \Sigma)\, d\ell.
\end{aligned}$$

□

The following lemma is one of the motivations for the paper.

Lemma 3 *If a compact subset K and its complement K^c in $\mathbb{R}^n$ both are convex, then the boundary $\Sigma = \partial K$ is a hyperplane.*

Proof Since K and the closure $\overline{K^c}$ of its complement K^c both are closed and convex, they can be written as the intersections of families $\mathcal{F}$ and $\mathcal{G}$ of closed halfspaces, namely, $K = \bigcap_{f \in \mathcal{F}} f$ and $\overline{K^c} = \bigcap_{g \in \mathcal{G}} g$. We claim that all elements $f \in \mathcal{F}$ and $g \in \mathcal{G}$ are parallel. By parallel we mean that the hyperplanes of two halfplanes are parallel. Indeed, let us fix an element $f \in \mathcal{F}$ and assume that there exists $g \in \mathcal{G}$ such that g is not parallel to f. On the one hand, we have $f^c \cap g^c = (f \cup g)^c \neq \emptyset$. On the other, $\mathbb{R}^n = K \cup \overline{K^c} \subset f \cup g$ implies that $(f \cup g)^c = \emptyset$, and we get a contradiction. Thus, for fixed f, all elements in $\mathcal{G}$ are parallel to f.

Since f can be arbitrarily chosen, we conclude that all elements in $\mathcal{F}$ are parallel to those of $\mathcal{G}$, and Σ is a hyperplane. □

Next, we prove the main theorem, which states that the surface area of arbitrary compact hypersurface in $\mathbb{R}^n$ is the average of the integrals of weighted projected areas over the unit sphere.

Theorem 3 *Let K be a compact n-dimensional subset in $\mathbb{R}^n$ with boundary $\Sigma = \partial K$. Then its surface area is given by*

$$\mathcal{V}_{n-1}(\Sigma) = \frac{1}{\omega_{n-1}} \int_{q\in\mathbb{S}^{n-1}} \mathcal{V}_{n-1}\big(\Sigma|n(q)^{\perp}\big)\, dS_q.$$

Proof By using (2) and Definition 1 we immediately have

$$\begin{aligned}
\int_{p\in\mathbb{S}^{n-1}} \mathcal{V}_{n-1}\big(\Sigma|n(p)^{\perp}\big)\, dS_p &= \frac{1}{2}\int_{p\in\mathbb{S}^{n-1}}\int_{q\in\Sigma} \big|\tilde{n}(q)\cdot n(p)\big|\, d\Sigma_q\, dS_p \\
&= \frac{1}{2}\int_{q\in\Sigma}\int_{p\in\mathbb{S}^{n-1}} \big|\tilde{n}(q)\cdot n(p)\big|\, dS_p\, d\Sigma_q \\
&= \int_{q\in\Sigma} \omega_{n-1}\, d\Sigma_q \\
&= \omega_{n-1}\mathcal{V}_{n-1}(\Sigma),
\end{aligned}$$

and the result follows. □

An immediate application of Theorem 3 is that a hypersurface with the smaller projected areas onto all hyperplanes has the smaller surface area; particularly, we obtain a comparison theorem of projected surface areas between two compact hypersurfaces. Notice that by the Alexandrov projection theorem [6, p. 115, Theorem 3.3.6], even if two convex bodies in $\mathbb{R}^n$ have the same projected areas in all directions, they may be completely different. In fact, there exist noncongruent convex polytopes Σ_i, $i = 1,2$, with $\mathcal{V}_{n-1}(\Sigma_1|n(p)^{\perp}) = \mathcal{V}_{n-1}(\Sigma_2|n(p)^{\perp})$ for all $p \in \mathbb{S}^{n-1}$ (see [6, p. 121, Theorem 3.3.17]). Thus the following corollary gives a necessary condition to determine the consistency of surface areas for two hypersurfaces, but not their congruence.

Corollary 1 *Given two compact hypersurfaces Σ_i, $i = 1,2$, in $\mathbb{R}^n$, if $\mathcal{V}_{n-1}(\Sigma_1|n(p)^{\perp}) \le \mathcal{V}_{n-1}(\Sigma_2|n(p)^{\perp})$ for all $p \in \mathbb{S}^{n-1}$, then $\mathcal{V}_{n-1}(\Sigma_1) \le \mathcal{V}_{n-1}(\Sigma_2)$. In particular, if $\mathcal{V}_{n-1}(\Sigma_1|n(p)^{\perp}) = \mathcal{V}_{n-1}(\Sigma_2|n(p)^{\perp})$ for all $p \in \mathbb{S}^{n-1}$, then $\mathcal{V}_{n-1}(\Sigma_1) = \mathcal{V}_{n-1}(\Sigma_2)$.*

Finally, we will prove that the surface area formula in Theorem 3 is equivalent to that derived by De Jong [4]. More precisely, given a k-dimensional submanifold $M \subset \mathbb{R}^n$ and a compact subset $A \subset M$, De Jong gave a simpler method to prove that the limit

$$\lim_{\epsilon\downarrow 0} \frac{\mu_n(Tub_\epsilon A)}{\beta_{n-k}\epsilon^{n-k}} \tag{10}$$

exits and can be used to define the k-dimensional volume of A. Here $Tub_\epsilon A = \{p + a; p \in A, a \in N_pM, |a| < \epsilon\}$, N_pM denotes the orthogonal complement of the tangent space T_pM

at p, μ_n is the n-dimensional Lebesgue measure, and β_{n-k} is the $(n-k)$-dimensional volume of the unit ball in $\mathbb{R}^{n-k}$ (for instance, $\beta_1 = 2$, $\beta_2 = \pi$, $\beta_3 = \frac{4\pi}{3}$). In particular, when $A = M := \Sigma = \partial K$ for some smooth n-dimensional compact submanifold K in $\mathbb{R}^n$ (namely, $k = n-1$ in (10)), the surface area of K can be obtained by

$$\mathcal{V}_{n-1}(\Sigma) = \lim_{\epsilon\downarrow 0} \frac{\mu_n(Tub_\epsilon \Sigma)}{2\epsilon}. \tag{11}$$

Proposition 1 *The surface area obtained in* (11) *for an n-dimensional submanifold K with smooth boundary* $\partial K = \Sigma$ *is equal to that obtained in Theorem* 3.

Proof Since locally Σ can be represented by a smooth defining function, we may assume that for any open subset $W \subset \Sigma$, there exists a smooth function $\phi = (\phi_1, \dots, \phi_n) : U \subset \mathbb{R}^{n-1} \to W \subset \mathbb{R}^n$ such that $W = \{\phi_n(x_1, \dots, x_{n-1}) = 0;\ \text{any point } (x_1, \dots, x_{n-1}) \in U\}$. Let $B = (\frac{\partial \phi_i}{\partial x_j})$ for $1 \le i \le n$ and $1 \le j \le n-1$ be the $n \times (n-1)$-matrix. Then the surface area element $d\Sigma_q$ at $q \in W$ satisfies

$$d\Sigma_q = \sqrt{\det\bigl(B^T \cdot B\bigr)}\, dx_1 \dots dx_{n-1},$$

where B^T is the matrix transpose. Thus by Definition 1

$$\begin{aligned} \mathcal{V}_{n-1}\bigl(W|n(p)^\perp\bigr) &= \frac{1}{2}\int_{q\in W} \bigl|L_{qp^{-1}}{}_* n(p) \cdot \tilde{n}(q)\bigr|\, d\Sigma_q \\ &= \frac{1}{2}\int_U |\cos\theta| \sqrt{\det\bigl(B^T \cdot B\bigr)}\, dx_1 \dots dx_{n-1}, \end{aligned} \tag{12}$$

where θ is the angle between the unit normal vector $\tilde{n}(q)$ of W and $n(p)$. Therefore by (12)

$$\begin{aligned} \int_{p\in\mathbb{S}^{n-1}} \mathcal{V}_{n-1}\bigl(W|n(p)^\perp\bigr)\, dS_p &= \frac{1}{2}\int_{p\in\mathbb{S}^{n-1}}\int_U |\cos\theta| \sqrt{\det\bigl(B^T \cdot B\bigr)}\, dx_1 \dots dx_{n-1}\, dS_p \\ &= \frac{1}{2}\int_U 2\omega_{n-1} \sqrt{\det\bigl(B^T \cdot B\bigr)}\, dx_1 \dots dx_{n-1} \\ &= \omega_{n-1}\int_U \sqrt{\det\bigl(B^T \cdot B\bigr)}\, dx_1 \dots dx_{n-1} \\ &= \omega_{n-1} \lim_{\epsilon\downarrow 0} \frac{\mu_n(Tub_\epsilon W)}{2\epsilon}. \end{aligned} \tag{13}$$

Here we have used the fact in the last identity derived in [4, p. 83]. Finally, the smoothness of Σ implies that Σ can be covered by such summable open subsets W. By the standard argument of partitions of unity we conclude that

$$\mathcal{V}_{n-1}(\Sigma) = \frac{1}{\omega_{n-1}} \int_{p\in\mathbb{S}^{n-1}} \mathcal{V}_{n-1}\bigl(\Sigma|n^\perp(p)\bigr)\, dS_p = \lim_{\epsilon\downarrow 0} \frac{\mu_n(Tub_\epsilon \Sigma)}{2\epsilon},$$

which completes the proof. □

2 Generalization to higher codimensions

In this section, we will project the compact hypersurface Σ in $\mathbb{R}^n$ to the lower-dimensional subspaces and consider their projected volumes. To be precise, we plan to define the projected r-dimensional volume (projected r-volume, in short) of Σ, $1 \leq r \leq n-1$, onto any r-dimensional subspace in $\mathbb{R}^n$ and find the recursive formula (see (24)) for the average projected r-volume. Using the same notation as Santaló [11], for any point $q \in \mathbb{R}^n$, we denote by $L_{r[q]}$ the r-dimensional plane (shortly, r-plane) in $\mathbb{R}^n$ through q and by $L^{\perp}_{r[q]}$ its orthogonal complement. Notice that for any point $q \in L_{r[0]}$, there exists a unique affine $(n-r)$-plane $L^{\perp}_{n-r[q]}$ through q and perpendicular to $L_{r[0]}$. Indeed, the uniqueness and existence of the affine orthogonal complement can be obtained by parallel shifting $L^{\perp}_{n-r[0]}$ to q, namely, $L^{\perp}_{n-r[q]} = \{q + v \text{ for any } v \in L^{\perp}_{n-r[0]}\}$.

Let $L_{r[0]}$ be an r-plane, and let $q \in \Sigma$ be a fixed point. Denote u by the orthogonal projection of q onto $L_{r[0]}$. Then there exists the unique $(n-r)$-plane containing p and q and perpendicular to $L_{r[0]}$. Let us denote the $(n-r)$-plane by $L^{\perp}_{n-r[q]}$. Clearly, $q \in L_{r[0]} \cap L^{\perp}_{n-r[q]}$. Now we discuss the dimension of the intersection $T_p\Sigma \cap L^{\perp}_{n-r[q]}$. Since the following discussion also holds for any vector subspaces, we may simplify the notations by setting

$$M = T_p\Sigma, \qquad V = L_{r[0]}, \qquad V^{\perp} = L^{\perp}_{n-r[q]}.$$

It is clear that $\dim(M \cap V^{\perp}) \leq \min\{\dim(M), \dim(V^{\perp})\} = \dim(V^{\perp}) = n-r$. Moreover, since the dimension of the sum of M and $V^{\perp}$, $M + V^{\perp} = \{m + v, m \in M, v \in V^{\perp}\}$, is at most n, we have

$$n - r - 1 \leq \dim(M) + \dim\left(V^{\perp}\right) - \dim\left(M + V^{\perp}\right) = \dim\left(M \cap V^{\perp}\right) \leq n - r.$$

Thus there are two cases concerning the dimension $\dim(M \cap V^{\perp})$, $n-r-1$ and $n-r$. We will construct the projected r-volume forms of M onto V for the first case and show that the all r-forms vanish in the second case, that is, they are of measure zero when considering the integral over such points.

Case 1. When $\dim(M \cap V^{\perp}) = n-r-1$, let ν be the unit normal to M. We may choose the orthonormal basis $\{e_\alpha, e, e_\beta\}$ in the space $M \oplus span\{\nu\} = \mathbb{R}^n$ and orthonormal basis $\{e_\alpha, u_\delta\}$ in M satisfying the following conditions: ($1 \leq \alpha \leq n-r-1$, $n-r+1 \leq \beta \leq n$, and $n-r \leq \delta \leq n-1$):

(1) $span\{e_\alpha\} = M \cap V^{\perp}$,
(2) $span\{e_\alpha, e\} = V^{\perp}$,
(3) $span\{e_\beta\} = V$,
(4) $span\{e_\alpha, u_\delta\} = M$.

Notice that since $\dim(V^{\perp} \setminus (M \cap V^{\perp})) = 1$, the unit vector e is uniquely determined (up to a sign), independent of the choice of the vectors e_α, e_β, and u_δ. Since for the point $p \in M$, the infinitesimal change dp is still contained in M, so we have the vector-valued one-form

$$dp = \sum_{\alpha=1}^{n-r-1} A^\alpha e_\alpha + \sum_{\delta=n-r}^{n-1} B^\delta u_\delta$$

for some connection 1-forms A^α and B^δ. In addition, dp is a vector in $\mathbb{R}^n$, so it can be written in terms of the linear combination of the basis $\{e_\alpha, e, e_\beta\}$, namely,

$$dp = \left(\sum_{\alpha=1}^{n-r-1} \omega^\alpha e_\alpha\right) \wedge \omega e \wedge \left(\sum_{\beta=n-r+1}^{n} \omega^\beta e_\beta\right)$$

for some 1-forms ω^α, ω, and ω^β. Thus, for any $\beta = n-r+1,\dots,n$, we have

$$\omega^\beta = dp \cdot e_\beta = \sum_{\delta=n-r}^{n-1} B^\delta (u_\delta \cdot e_\beta),$$

and so

$$\bigwedge_{\beta=n-r+1}^{n} \omega^\beta = \Delta \cdot \bigwedge_{\delta=n-r}^{n-1} B^\delta, \tag{14}$$

where $\Delta = \det(u_\delta \cdot e_\beta)$ is the determinant of the $(r \times r)$-matrix with entries $u_\delta \cdot e_\beta$. Notice that the value Δ satisfies $-1 \le \Delta \le 1$, and it measures the cosine of the angle between $M \setminus V^\perp$ and V, equivalently, the angle between ν and e. Indeed, recall that the Hodge star operator $*$ satisfies the property

$$*\big(\eta \wedge (*\zeta)\big) = \langle \eta, \zeta \rangle$$

for any exterior r-forms η, ζ, where $\langle \cdot,\cdot \rangle$ is the inner product for r-forms. Substituting $\eta = \bigwedge_\delta u_\delta$ and $\zeta = \bigwedge_\beta e_\beta$ into the identity and using the orthogonal decomposition

$$\mathbb{R}^n = M \oplus span\{\nu\} = span\{e_\alpha\} \oplus span\{u_\delta\} \oplus span\{\nu\} \ni e,$$

a straightforward computation shows that

$$\begin{aligned} \Delta &= \det(u_\delta \cdot e_\beta) \\ &= \left\langle \bigwedge_\delta u_\delta, \bigwedge_\beta e_\beta \right\rangle \\ &= *\left(\bigwedge_\delta u_\delta \wedge \left(* \bigwedge_\beta e_\beta \right) \right) \\ &= *\left(\bigwedge_\delta u_\delta \wedge e \bigwedge_\alpha e_\alpha \right) \\ &= (e \cdot \nu) * \left(\bigwedge_\delta u_\delta \wedge \nu \bigwedge_\alpha e_\alpha \right) \\ &= e \cdot \nu, \end{aligned} \tag{15}$$

as desired. For our purpose (see (18)), we will take the absolute value $|\Delta|$ of Δ in (14) such that the r-volume form $\bigwedge_\beta \omega^\beta$ is a positive measure. For more detail about the angles between two subspaces of arbitrary dimensions, we refer the reader to [8, Theorem 1] and [10].

The geometric meaning of (14) can be interpreted as follows: by identifying the origin $O \in V$ (and so q) and p via the parallel transport in $\mathbb{R}^n$, $\bigwedge_\beta \omega^\beta$ (resp., $\bigwedge_\delta B^\delta$) is the r-dimensional volume element in V at q (resp., in $M \setminus V^\perp$ at p, the subspace in M that is *not* perpendicular to V). Formula (14) describes that the projected r-form $\bigwedge_\beta \omega^\beta$ is the orthogonal projection of $\bigwedge_\delta B^\delta$ onto the plane V, and the projection is independent of the choice of the vectors e_α, e, e_β, and u_δ. As a consequence, we have constructed a natural projected r-volume element of M onto V and finish the discussion for the first case.

Before advancing to the second case, let us implement the previous construction to the compact hypersurface in $\mathbb{R}^n$.

Proposition 2 *For any r-plane $L_{r[0]}$ through the origin and any point $q \in L_{r[0]}$, there exists a unique $(n-r)$-plane $L^\perp_{n-r[q]}$ through q satisfying*

(1) *the orthogonal decomposition $L_{r[0]} \oplus L^\perp_{n-r[q]} = \mathbb{R}^n$, and*
(2) *if $L^\perp_{n-r[q]} \cap \Sigma \neq \emptyset$, then for any point $p \in L^\perp_{n-r[q]} \cap \Sigma$ with $L^\perp_{n-r[q]} \nsubseteq T_p\Sigma$, there exists the unique (up to a sign) unit vector $e_{p,q}$ in $L^\perp_{n-r[q]} \setminus T_p\Sigma$.*

Proof

(1) It is clear by the assumption.
(2) Suppose $p \in L^\perp_{n-r[q]} \cap \Sigma$. Since $L^\perp_{n-r[q]} \nsubseteq T_p\Sigma$, $\dim(L^\perp_{n-r[q]} \cap T_p\Sigma) = n-r-1$. Then at p, we have the orthogonal decomposition $L^\perp_{n-r[q]} = (L^\perp_{n-r[q]} \cap T_p\Sigma) \oplus U$ for some orthogonal complement U of $(L^\perp_{n-r[q]} \cap T_p\Sigma)$ in $L^\perp_{n-r[q]}$. Besides, since $\dim(U) = \dim(L^\perp_{n-r[q]}) - \dim(L^\perp_{n-r[q]} \cap T_p\Sigma) = 1$, we may choose the unique vector starting from the point p with length one (up to a sign) in the one-dimensional affine subspace U, and the result follows. □

We point out that in (2) of Proposition 2, in general, the cross-section $L^\perp_{n-r[q]} \cap \Sigma$ may be comprised of infinitely or finitely many connected components. For our purpose, we only consider the hypersurface Σ with finitely many cross-sections for all $(n-r)$-planes through any point on Σ.

Proposition 3 *For any $p \in \Sigma$ and any $(n-r)$-plane $U \nsubseteq T_p\Sigma$ through p, there exist a unique r-plane $L_{r[0]}$ through the origin and a unique point $q \in L_{r[0]}$ such that*

(1) *$q \in U \cap L_{r[0]}$ and $U \oplus L_{r[0]} = \mathbb{R}^n$ (thus we may denote $U = L^\perp_{n-r[q]}$ as shown in Proposition 2), and*
(2) $\dim(U \setminus T_p\Sigma) = 1$.

Moreover, by (2) there exists the unique unit vector $e \in U \setminus T_p\Sigma$ coinciding with the vector $e_{p,q}$ constructed in (2) of Proposition 2.

Proof

(1) The r-plane $L_{r[0]}$ can be uniquely obtained by the orthogonal affine subspace to U, namely, $L_{r[0]} = \{x \in \mathbb{R}^n, x \cdot y = 0 \text{ for all } y \in U\}$, and so the point q is given by the unique point at $L_{r[0]} \cap U$.
(2) The transversal assumption $U \nsubseteq T_p\Sigma$ immediately implies the result.

Finally, by (2) we may have a unique (up to a sign) unit vector $e \in U \setminus T_p\Sigma$. Since the vector e is uniquely determined by U, $L_{r[0]}$, p, and q, by setting $U = L^\perp_{n-r[q]}$ in Proposition 2 the unique vector $e_{p,q}$ coincides with the vector e. □

Remark 2 For any fixed point $p \in \mathbb{R}^n$, the natural orthogonal decomposition for $\mathbb{R}^n$ implies that there exists a one-to-one correspondence that assigns the r-plane $L_{r[0]}$ an $(n-r)$-plane $L^\perp_{n-r[q]}$ through p, where q is the orthogonal projection of p onto $L_{r[0]}$. Let $G_{n,r}$ be the Grassmannian, the set of all r-subspaces in $\mathbb{R}^n$. According to Proposition 3, if $p \in \Sigma$, then the map

$$\begin{aligned} \phi_p : G_{n,r} &\to G_{n-r,1} \\ L_{r[0]} &\mapsto \phi(L_{r[0]}) = e \in L^{\perp}_{n-r[q]} \setminus T_p\Sigma \end{aligned} \tag{16}$$

is a bijection except for the r-planes $L_{r[0]}$ perpendicular to $T_p\Sigma$ (namely, $L^{\perp}_{n-r[q]} \subseteq T_p\Sigma$ for some point $q \in L_{r[0]}$ in Proposition 2 (2)).

Next, let us continue to discuss the second case of $\dim(M \cap V^{\perp})$.

Case 2. When $\dim(M \cap V^{\perp}) = n - r$, $V^{\perp}$ is contained in M. We claim that, by a similar construction in *Case 1*, the r-form $\bigwedge_{\beta} \omega^{\beta}$ vanishes. We may choose the orthonormal bases $\{e_{\alpha}, e_{\beta}\}$ in $M \oplus span\{\nu\} = \mathbb{R}^n$ and $\{e_{\alpha}, u_{\delta}\}$ in M satisfying ($1 \le \alpha \le n - r$, $n - r + 1 \le \beta \le n$, $n - r + 1 \le \delta \le n - 1$)

(1) $span\{e_{\alpha}\} = M \cap V^{\perp} = V^{\perp}$,

(2) $span\{e_{\beta}\} = V$,

(3) $span\{e_{\alpha}, u_{\delta}\} = M$.

Similarly to *Case 1*, on one hand, since $p = M \cap V^{\perp}$, $dp = \sum_{i=1}^{n} \omega^i e_i$ for some connection 1-forms ω^i. On the other hand, by writing

$$dp = \sum_{\alpha=1}^{n-r} A^{\alpha} e_{\alpha} + \sum_{\delta=n-r+1}^{n-1} B^{\delta} u_{\delta}$$

and taking the inner product with e_{β}, $n - r + 1 \le \beta \le n$, we have

$$\omega^{\beta} = dp \cdot e_{\beta} = \sum_{\delta=n-r+1}^{n-1} B^{\delta}(u_{\delta} \cdot e_{\beta}).$$

We deduce

$$\bigwedge_{\beta=n-r+1}^{n} \omega^{\beta} = \bigwedge_{\beta=n-r+1}^{n} \left(\sum_{\delta=n-r+1}^{n-1} B^{\delta}(u_{\delta} \cdot e_{\beta}) \right) = 0. \tag{17}$$

The last equality holds since the wedge product makes an r-form from $(r - 1)$ one-forms B^{δ}, and there must be some B^{δ}s repeated. This finishes the discussion for *Case 2*.

In contrast to *Case 1*, *Case 2* shows that if $V^{\perp} \subset M$ (equivalently, V is perpendicular to M), then the orthogonal contribution of any r-volume element in M onto V is zero. As a consequence of both cases, when considering the integral over all projected r-volumes $\bigwedge_{\beta=n-r+1}^{n} \omega^{\beta}$, (14) and (17) suggest that we may ignore *Case 2* and only consider *Case 1*, $\dim(M \cap V^{\perp}) = n - r - 1$.

According to the previous discussion, we give a definition of the weighted projected area of any compact hypersurface Σ onto any subspace $L_{r[0]}$ of lower dimension.

Definition 2 Let Σ be a compact hypersurfacein $\mathbb{R}^n$, and let $L_{r[0]}$ be an r-plane through the origin, $1 \le r \le n - 1$. The (orthogonal) weighted projected r-volume $\mathcal{V}_r(\Sigma | L_{r[0]})$ of Σ onto $L_{r[0]}$ is defined by

$$\mathcal{V}_r(\Sigma | L_{r[0]}) = \frac{1}{2} \int_{p \in \Sigma} L^{*}_{pq^{-1}} \left(|\Delta(p)| \bigwedge_{\delta=1}^{r} B^{\delta}(p) \right), \tag{18}$$

where $L^*_{pq^{-1}}$ is the pullback of the left translation $L_{pq^{-1}}$, q is the orthogonal projection of $p \in \Sigma$ onto $L_{r[0]}$, $\Delta(p)$ is the angle between the unit normal ν to $T_p\Sigma$ and the unique vector e in $L^{\perp}_{n-r[q]}$ defined in (16), and $\bigwedge_{\delta=1}^{r} B^{\delta}(p)$ is the r-volume in $T_p\Sigma \setminus L^{\perp}_{n-r[q]}$. Also, the mean value of the projected r-volumes $\mathcal{V}_r(\Sigma|L_{r[0]})$ is defined by

$$E\big(\mathcal{V}_r(\Sigma|L_{r[0]})\big) = \frac{\int_{L_{r[0]}\in G_{n,r}} \mathcal{V}_r(\Sigma|L_{r[0]})\, dL_{r[0]}}{m(G_{n,r})}, \tag{19}$$

where $G_{n,r}$ is the Grassmannian consisting of all r-dimensional subspaces in $\mathbb{R}^n$, $dL_{r[0]}$ is the invariant density defined below in (20), and $m(G_{n,r})$ is the volume of the Grassmannian $G_{n,r}$.

We point out that when $r = n-1$, $T_pM \cap L^{\perp}_{1[q]} = \{p\}$, and so $\bigwedge_{\delta=1}^{n-1} B^{\delta}$ is the surface area element of Σ at p. Thus (18) coincides with (4). When $r = 1$, we have the mean width (19) for arbitrary Σ.

Recall [11, p. 202] that the invariant density of $L_{r[0]}$ is given by

$$dL_{r[0]} = \bigwedge_{\substack{1\le h\le n-r \\ n-r+1\le \beta\le n}} \omega^h_{\beta}, \tag{20}$$

where

$$\omega^h_{\beta} = \begin{cases} de_{\beta}\cdot e_{\alpha} & \text{if } 1 \le h \le n-r-1, \\ de_{\beta}\cdot e & \text{if } h = n-r \end{cases}$$

(use the same indices α, β as in *Case 1*), and identity (12.36) in [11, p. 203] gives

$$\int_{G_{r,q}} dL_{r[q]} = \frac{O_{n-q-1}O_{n-q-2}\cdots O_{n-r}}{O_{r-q-1}O_{r-q-r}\cdots O_1 O_0} \tag{21}$$

for $0 \le q < r \le n-1$. We also point out that the volume $m(G_{n,r}) = \frac{O_{n-1}\cdots O_{n-r}}{O_{r-1}\cdots O_1 O_0}$, where $O_r = \frac{2\pi^{(r+1)/2}}{\Gamma((r+1)/2)}$ is the surface area of the unit ball in $\mathbb{R}^{r+1}$, and Γ denotes the gamma function. For instance, $O_0 = 2$ (by convention), $O_1 = 2\pi$, and $O_2 = 4\pi$. Notice that the r-form $\bigwedge_{\beta=n-r+1}^{n} \omega^{\beta}$ (resp., $\bigwedge_{\delta=1}^{r} B^{\delta}$) is the r-dimensional volume of the infinitesimal parallelotope in $L_{r[q]}$ (resp., in $T_p\Sigma \setminus L^{\perp}_{n-r[q]}$). Moreover, if Σ is convex, then definitions (18) and (19) coincide with (13.1) and (13.2), respectively, in [11], and so we have had the generalized projected r-volumes for arbitrary hypersurface Σ.

The rest of this paper is devoted to the derivation of a recursive formula for the integral of projected r-volumes. Recall that [11, p, 216, (13.2)] the integral of the projected $(n-r)$-volume of a convex body K (shortly, the mean $(n-r)$-volume) is defined by

$$I_r(K) = \int_{G_{n,r}} V\big(K'_{n-r}\big)\, dL_{r[0]} = \int_{G_{n,n-r}} V\big(K'_{n-r}\big)\, dL_{n-r[0]}, \tag{22}$$

where K'_{n-r} is the convex set of all intersection points of $L_{n-r[0]}$ with the r-planes perpendicular to $L_{n-r[0]}$ through each point of K, and $V(K'_{n-r})$ is the $(n-r)$-volume of K'_{n-r}. Using our notation, this means that $V(K'_{n-r}) = \mathcal{V}_{n-r}(K|L_{n-r[0]})$. Definition (22) of $I_r(K)$ can be generalized to arbitrary compact submanifold K with smooth boundary $\Sigma = \partial K$ if the $(n-r)$-volume $V(K'_{n-r})$ is replaced by that of (18), namely, the mean $(n-r)$-volume of Σ is defined by

$$I_r(\Sigma) = \int_{G_{n,n-r}} \mathcal{V}_{n-r}(\Sigma | L_{n-r[0]})\, dL_{n-r[0]}, \tag{23}$$

which is exactly same as shown in (19) (up to a constant $m(G_{n,r})$.

Theorem 4 *Let K be an n-dimensional compact submanifold in $\mathbb{R}^n$ with boundary $\Sigma = \partial K$. Denote by $I_r(K)$ the mean $(n-r)$-volume as defined in (23). Then we have the recursive formula*

$$I_r(K) = \frac{2}{O_{r-1}} \int_{G_{n,n-1}} I_{r-1}^{(n-1)}\left(K'_{n-1}\right) dL_{n-1[0]}, \tag{24}$$

where O_{r-1} is the surface area of the unit ball in $\mathbb{R}^r$, and $I_{r-1}^{(n-1)}(K'_{n-1})$ is the mean $(r-1)$-volume of the projection K'_{n-1} of K onto $L_{n-1[0]}$.

Remark 3 In [11, p. 217] the author derived the same recursive formula (see identity (13.7) there) under the assumption that K is a convex body in $\mathbb{R}^n$. We observe that a similar argument can be applied even for nonconvex domains K when the new notion for projected r-volumes (i.e., (4) and (18)) is introduced. The main idea of the proof of Theorem 4 is based on identities (25) and (26), which are irrelevant to the convexity for K.

Proof Let L_r, $1 \le r \le n-1$, be an r-plane in $\mathbb{R}^n$. Denote by $L_{i+1}^{(r)}$ the $(i+1)$-plane contained in L_r for $i+1 \le r \le n-1$. In [11, p. 207, (12.53)] the author considered the density for the sets of pairs of linear subspaces $(L_r, L_{i+1}^{(r)})$ and the identity

$$dL_{i+1}^{(r)} \wedge dL_r^* = dL_{r[i+1]} \wedge dL_{i+1}, \tag{25}$$

where dL_r^* is the density of the oriented r-plane L_r, and $dL_{r[i+1]}$ is the density for r-planes about a fixed $(i+1)$-plane. If we consider the linear spaces through the fixed origin O in $\mathbb{R}^n$, (25) still holds and may be written as

$$dL_{i+1[0]}^{(r)} \wedge dL_{r[0]}^* = dL_{r[i+1]} \wedge dL_{i+1[0]}. \tag{26}$$

In particular, when L_r is a hyperplane and $L_{i+1[0]}^{(r)}$ is of maximal dimension in L_r, namely, $r = n-1$, $i+1 = r$, (26) becomes

$$dL_{r[0]}^{(n-1)} \wedge dL_{n-1[0]}^* = dL_{n-1[r]} \wedge dL_{r[0]}. \tag{27}$$

Similarly, when restricting to the hyperplane $L_{n-1[0]}$ (namely, substitute n by $n-1$, and $i+1$ by $r-1$) in (26), we have

$$dL_{r-1[0]}^{(r)} \wedge dL_{r[0]}^{*(n-1)} = dL_{r[r-1]}^{(n-1)} \wedge dL_{r-1[0]}^{(n-1)}, \tag{28}$$

where the superscripts $(n-1)$ emphasize that the sub-lanes considered here are contained in the plane $L_{n-1[0]}$. For instance, $dL^{*(n-1)}_{r[0]}$ is the density of the oriented r-plane through the origin O contained in $L_{n-1[0]}$. Multiplying (27) by $dL^{r}_{r-1[0]}$ and (28) by $dL_{n-1[0]}$ and using the fact that an oriented plane is equivalent to two unoriented planes such that $dL^{*}_{n-1[0]} = 2dL_{n-1[0]}$ and $dL^{*(n-1)}_{r[0]} = 2dL^{(n-1)}_{r[0]}$, we arrive at

$$dL^{(n-1)}_{r[r-1]} \wedge dL^{(n-1)}_{r-1[0]} \wedge dL_{n-1[0]} = dL^{(r)}_{r-1[0]} \wedge dL_{n-1[r]} \wedge dL_{r[0]}. \tag{29}$$

Now let us integrate over all the pairs of $L_{r-1[0]}$ and $L^{(n-1)}_{r[0]}$. Let Σ'_{n-1} be the projected $(n-1)$-volume of Σ onto $L_{n-1[0]}$. Notice that the projected $(n-r)$-volume of Σ onto $L_{n-r[0]}$ is equal to the projected $(n-r)$-volume of Σ'_{n-1} onto $L_{n-r[0]}$ (counted for multiplicities). On the one hand, the integral of the left-hand side of (29) becomes

$$\begin{aligned}
&\int_{G_{n,r}} \int_{G_{r,r-1}} \int_{G_{n-1,r}} V\big(K'_{n-r}\big)\, dL^{(n-1)}_{r[r-1]} \wedge dL^{(n-1)}_{r-1[0]} \wedge dL_{n-1[0]} \\
&\quad= \int_{G_{n,r}} I^{(n-1)}_{r-1}\big(K'_{n-1}\big)\, dL_{n-1[0]} \wedge dL^{(n-1)}_{r[r-1]} \\
&\quad= \int_{G^{(n-1)}_{r,r-1}} \int_{G_{n,n-1}} I^{(n-1)}_{r-1}\big(K'_{n-1}\big)\, dL_{n-1[0]} \wedge dL^{(n-1)}_{r[r-1]} \\
&\quad= \int_{G_{n,n-1}} I^{(n-1)}_{r-1}\big(K'_{n-1}\big)\, dL_{n-1[0]} \cdot \int_{G^{(n-1)}_{r,r-1}} dL^{(n-1)}_{r[r-1]} \\
&\quad= \frac{O_{n-r-1}}{2} \int_{G_{n,n-1}} I^{(n-1)}_{r-1}\big(K'_{n-1}\big)\, dL_{n-1[0]},
\end{aligned} \tag{30}$$

where we have used that $\int_{G_{r,r-1}} \int_{G_{n-1,r}} = \int_{G_{n-1,r-1}}$, $\int_{G_{n,r}} = \int_{G_{n,n-1}} \int_{G^{(n-1)}_{r,r-1}}$ in the first two identities and (21) in the last identity $\frac{O_{n-r-1}}{2} = \int_{G^{(n-1)}_{r,r-1}} dL^{(n-1)}_{r[r-1]}$. On the other hand, we deduce the integral of the right-hand side of (29)

$$\begin{aligned}
&\int_{G_{n,r}} \int_{G_{r,r-1}} \int_{G_{n-1,r}} V\big(K'_{n-r}\big)\, dL^{(r)}_{r-1[0]} \wedge dL_{n-1[r]} \wedge dL_{r[0]} \\
&\quad= \int_{G_{r,r-1}} \int_{G_{n-1,r}} I_r(K)\, dL^{(r)}_{r-1[0]} \wedge dL_{n-1[r]} \\
&\quad= \frac{O_{r-1}}{2} \frac{O_{n-r-1}}{2} I_r(K).
\end{aligned} \tag{31}$$

Again, we have used (21) to get $\int_{G_{r,r-1}} dL^{(r)}_{r-1[0]} = \frac{O_{r-1}}{O_0}$ and $\int_{G_{n-1,r}} dL_{n-1[r]} = \frac{O_{n-r-1}}{O_0}$ in the last equality. Combining (30) and (31), we get the recursive formula (24). □

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References

1. Aminov, Y.: The Geometry of Submanifolds. CRC Press, Boca Raton (2001)
2. Bouafia, P., De Pauw, T.: Integral geometric measure in separable Banach space. Math. Ann. **363**(1), 269–304 (2015)
3. Chakerian, G.: Sets of constant width. Pac. J. Math. **19**(1), 13–21 (1966)
4. de Jong, T.: Volume of submanifolds. Math. Semesterber. **60**(1), 81–83 (2013)
5. Favard, J.: Une définition de la longueur et de l'aire. C. R. Math. Acad. Sci. Paris **194**, 344–346 (1932)
6. Gardner, R.J.: Geometric Tomography, vol. 6, 2nd edn. Cambridge University Press, Cambridge (2006)
7. Huang, Y.-C.: Cauchy's surface area formula in the Heisenberg groups. Rev. Mat. Iberoam. (2021)
8. Jiang, S.: Angles between Euclidean subspaces. Geom. Dedic. **63**(2), 113–121 (1996)
9. Klain, D.A., Rota, G.-C., et al.: Introduction to Geometric Probability. Cambridge University Press, Cambridge (1997)
10. Miao, J., Ben-Israel, A.: On principal angles between subspaces in R^n. Linear Algebra Appl. **171**, 81–98 (1992)
11. Santaló, L.A.: Integral Geometry and Geometric Probability. Cambridge university press, Cambridge (2004)
12. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory. Cambridge university press, Cambridge (2014)
13. Tsukerman, E., Veomett, E.: Brunn–Minkowski theory and Cauchy's surface area formula. Am. Math. Mon. **124**(10), 922–929 (2017)
14. Zeng, C., Bai, S., Tong, Y.: A new integral formula for the angle between intersected submanifolds. J. Inequal. Appl. **2016**(1), 83 (2016)
15. Zhang, G.: Dual kinematic formulas. Trans. Am. Math. Soc. **351**(3), 985–995 (1999)

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