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Matrix representation of Toeplitz operators on Newton spaces

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Abstract

In this paper, we study several properties of an orthonormal basis $\{N_n(z)\}$ for the Newton space $N^2(\mathbb{P})$. In particular, we investigate the product of N_m and N_m and the orthogonal projection P of $\overline{N_n}N_m$ that maps from $L^2(\mathbb{P})$ onto $N^2(\mathbb{P})$. Moreover, we find the matrix representation of Toeplitz operators with respect to such an orthonormal basis on the Newton space $N^2(\mathbb{P})$.

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1 Introduction

For any $n \in \mathbb{N} \cup 0$, let $N_n(z)$ denote the nth Newton polynomial, which is determined by the coefficients in the expansion:

$$(1-w)^z = \sum_{n=0}^{\infty} N_n(z) w^n,$$

where $|w| < 1$ and z is any complex number in the complex plane $\mathbb{C}$. Using the notations in [1, 12–14], $N_n(z)$ has the following expression

$$N_n(z) := \frac{(-z)_n}{n!} = (-1)^n \binom{z}{n},$$

where $\binom{z}{n} = \frac{z(z-1)(z-2)\cdots(z-(n-1))}{n!}$ for $n \geq 1$ and $\binom{z}{0} = 1$.

Consider a probability measure μ defined on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |z|^n \, d\mu(z) < \infty, \quad (n \in \mathbb{N}).$$

Let $\gamma(x)$ denote the discrete measure on $\mathbb{R}$ with unit masses at $\{-\frac{1}{2} + \frac{n}{2} : n \in \mathbb{N}\}$ and $\mathbb{P} := \{z \in \mathbb{C} : Re(z) > -\frac{1}{2}\}$. Set

$$d\mu(x+iy) = \frac{1}{2\pi} \frac{|\Gamma(x+iy)|^2}{\Gamma(2x+2)} \, dy \, d\gamma(x),$$

where Γ denotes the usual gamma function.

Let $N^2(\mathbb{P})$ be a Newton space as the closure of the set of polynomials in $L^2(\mathbb{C},\mu)$ (see [3]). In [11], Markett, Rosenblum, and Rovnyak verified that $N^2(\mathbb{P})$ is a Hilbert space and the Newton polynomials $\{N_n(z)\}_{n=0}^{\infty}$ form an orthonormal basis for $N^2(\mathbb{P})$. Note that

$$N^2(\mathbb{P}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n N_n(z) : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

On the positive real line define a measure μ by $d\mu(t) = e^{-t}\,dt$. The measure μ is finite and has total mass $\Gamma(1) = 1$. Consider the weighted Lebesgue space denoted by $L^2(\mu)$, which comprises measurable complex-valued functions f defined with

$$\|f(t)\|^2 = \int_0^{\infty} |f(t)|^2 e^{-t}\,dt < \infty$$

and let $L^\infty(\mu)$ be the set of all essentially bounded measurable functions in $\mathbb{P}$. For $f \in L^2(\mu)$, the weighted Mellin transform F on $N^2(\mathbb{P})$ of f is defined by

$$F(z) = \frac{1}{\Gamma(z+1)} \int_0^{\infty} f(t) e^{-t} t^z\,dt.$$

For $f, g \in L^\infty(\mu)$, an inner product on $N^2(\mathbb{P})$ is defined by

$$\langle F(z), G(z) \rangle = \int_0^{\infty} f(t)\overline{g(t)} e^{-t}\,dt,$$

where F and G are weighted Mellin transforms of f and g, respectively, (see [11]).

Let P denote the orthogonal projection that maps $L^2(\mu)$ onto $N^2(\mathbb{P})$ defined by

$$Pf(z) = \int_{\mathbb{H}} K(z,w) f(w)\,dA(w),$$

where dA is an area measure on $\mathbb{P}$. The reproducing kernel of $N^2(\mathbb{P})$ has the following form:

$$K(\lambda, z) = \frac{\Gamma(z + \overline{\lambda} + 1)}{\Gamma(z+1)\Gamma(\overline{\lambda}+1)}, \quad z \in \mathbb{H}.$$

For $\varphi \in L^\infty(\mu)$, the *Toeplitz operator* T_φ on $N^2(\mathbb{P})$ is defined by

$$T_\varphi f := P(\varphi \cdot f) \quad \big(f \in N^2(\mathbb{P})\big).$$

From [9], it is known that the following properties of the Toeplitz operators T_φ on $N^2(\mathbb{P})$ hold:

(i) $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$ for $\varphi, \psi \in L^\infty(\mu)$.

(ii) $T_\varphi^* = T_{\overline{\varphi}}$ for $\varphi \in L^\infty(\mu)$.

(iii) $T_\varphi = 0$ if and only if $\varphi = 0$ a.e.

(iv) $T_\varphi T_\psi = T_{\varphi\psi}$ and $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$ for $\varphi \in L^\infty(\mu)$ and $\psi \in H^\infty(\mu)$.

In [6, 10], the authors studied the properties of composition operators on Newton space $N^2(\mathbb{P})$. Recently, Han [3] focused on the complex symmetric composition operators on

Newton space $N^2(\mathbb{P})$ with respect to the specific conjugation. Furthermore, in [7, 8], we considered the properties of Toeplitz operators on Newton space. From the above research point of view, we further investigate the properties of Newton space and Newton basis and study the matrix of Toeplitz operators on Newton space in this paper.

This paper is organized as follows. First, we study several properties of an orthonormal basis $\{N_n(z)\}$ for Newton space $N^2(\mathbb{P})$. In particular, we investigate the product of N_m and N_m and the orthogonal projection P of $\overline{N_n}N_m$ that maps from $L^2(\mathbb{P})$ onto $N^2(\mathbb{P})$. Next, we consider the matrix representation of Toeplitz operators with respect to such an orthonormal basis on Newton space $N^2(\mathbb{P})$.

2 Main results

In this section, we first study an orthonormal basis for the Newton space $N^2(\mathbb{P})$. We begin with the following lemma.

Lemma 2.1 ([10]) *Let the map* $\Delta : N^2(\mathbb{P}) \to N^2(\mathbb{P})$ *defined by*

$$\Delta F(z) := F(z) - F(z+1)$$

be the backwards unilateral shift on the orthonormal basis $\{N_n(z)\}_{n=0}^{\infty}$. *Then,* Δ^* *is the unilateral shift on the orthonormal basis* $\{N_n(z)\}_{n=0}^{\infty}$, *i.e.,* $\Delta^* N_n(z) = N_{n+1}(z)$ *holds.*

Lemma 2.2 *For any* $m, n \geq 0$, *the following equation holds*

$$N_1(z)N_n(z) = (n+1)N_{n+1}(z) - nN_n(z).$$

Proof Since $N_n(z) = (-1)^n \frac{z(z-1)\cdots(z-(n-1))}{n!}$ for $n \geq 1$, it follows that

$$\begin{aligned}
(n+1)N_{n+1}(z) - nN_n(z) &= (n+1)(-1)^{n+1}\frac{z(z-1)\cdots(z-n)}{(n+1)!} - nN_n(z) \\
&= (-1)^{n+1}\frac{z(z-1)\cdots(z-n)}{n!} - nN_n(z) \\
&= (-1)(z-n)N_n(z) - nN_n(z) \\
&= -zN_n(z) = N_1(z)N_n(z).
\end{aligned}$$

Hence, we complete the proof. □

Lemma 2.3 *Let* $\mathcal{N}$ *be an* $(n+1) \times (n+1)$ *matrix given by*

$$\mathcal{N} = \begin{pmatrix}
N_m(m) & 0 & 0 & 0 & \cdots & 0 \\
N_m(m+1) & N_{m+1}(m+1) & 0 & 0 & \cdots & 0 \\
N_m(m+2) & N_{m+1}(m+2) & N_{m+2}(m+2) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
N_m(m+n) & N_{m+1}(m+n) & N_{m+2}(m+n) & N_{m+3}(m+n) & \cdots & N_{m+n}(m+n)
\end{pmatrix}.$$

Then, $\mathcal{N}^{-1} = \mathcal{N}$.

Proof Put

$$x_j = \big(N_m(m+j), N_{m+1}(m+j), \ldots, N_{m+j}(m+j), 0, \ldots, 0\big)$$

and

$$y_k = \big(0, \ldots, 0, N_{m+k}(m+k), N_{m+k}(m+k+1), \ldots, N_{m+k}(m+n)\big)^T,$$

where A^T is the transpose of the matrix A. If $j < k$, then $x_j \cdot (y_k)^T = 0$. If $j = k$, then

$$x_j \cdot (y_j)^T = N_{m+j}(m+j)N_{m+j}(m+j) = (-1)^{m+j}(-1)^{m+j} = 1.$$

Let $j > k$. Note that $\sum_{i=0}^{j-k}(-1)^i\binom{j-k}{i} = 0$. Since

$$\begin{aligned} N_m(j) &= (-1)^m \frac{j(j-1)(j-2)\cdots(j-(m-1))}{m!} \\ &= (-1)^m \frac{j!}{m!(j-m)!} \\ &= (-1)^m \binom{j}{m}, \end{aligned}$$

we have

$$\begin{aligned} x_j \cdot (y_k)^T &= N_{m+k}(m+j)N_{m+k}(m+k) + N_{m+k+1}(m+j)N_{m+k}(m+k+1) + \cdots \\ &\quad + N_{m+j}(m+j)N_{m+k}(m+j) \\ &= \sum_{\ell=0}^{j-k} N_{m+k+\ell}(m+j)N_{m+k}(m+k+\ell) \\ &= (-1)^{2m+2k}\sum_{\ell=0}^{j-k}(-1)^\ell \binom{m+j}{m+k+\ell}\binom{m+k+\ell}{m+k} \\ &= \sum_{\ell=0}^{j-k}(-1)^\ell \frac{(m+j)!}{(j-k-\ell)!\ell!(m+k)!} \\ &= \frac{(m+j)!}{(j-k)!(m+k)!}\sum_{\ell=0}^{j-k}(-1)^\ell \frac{(j-k)!}{(j-k-\ell)!\ell!} \\ &= \frac{(m+j)!}{(j-k)!(m+k)!}\sum_{\ell=0}^{j-k}(-1)^\ell \binom{j-k}{\ell} = 0. \end{aligned}$$

Hence, it means that $\mathcal{N}^2 = I$ and so $\mathcal{N}^{-1} = \mathcal{N}$. □

Let $m, n \geq 0$ be nonnegative integers. Then, $z^m z^n = z^{m+n}$ holds on the Hardy space H^2. However, $N_m N_n$ is not N_{m+n} on Newton space $N^2(\mathbb{P})$, in general. We next show that the product of N_m and N_n is the linear combination of $\{N_j\}_{j=\max\{m,n\}}^{(m+n)}$.

Theorem 2.4 *For any $m \geq n \geq 0$, it holds that*

$$N_m(z)N_n(z) = \sum_{j=\max\{m,n\}}^{(m+n)} b_j(m,n)N_j(z), \tag{2.1}$$

where

$$\begin{pmatrix} b_m(m,n) \\ b_{m+1}(m,n) \\ b_{m+2}(m,n) \\ \vdots \\ b_{m+n}(m,n) \end{pmatrix} = \mathcal{N} \begin{pmatrix} N_m(m)N_n(m) \\ N_m(m+1)N_n(m+1) \\ N_m(m+2)N_n(m+2) \\ \vdots \\ N_m(m+n)N_n(m+n) \end{pmatrix} \tag{2.2}$$

for $b_j(m,n) \in \mathbb{R}$ and $\mathcal{N}$ is denoted as in Lemma 2.3.

Proof By the definition of $N_n(z)$, we have

$$\begin{aligned} &N_m(z)N_n(z) \\ &\quad = (-1)^{n+m}\left(\frac{z(z-1)\cdots(z-(m-1))}{m!}\right)\left(\frac{z(z-1)\cdots(z-(n-1))}{n!}\right). \end{aligned}$$

Thus, we can write $N_m(z)N_n(z)$ as follows:

$$N_m(z)N_n(z) = \sum_{j=0}^{m+n} b_j(m,n)N_j(z) \tag{2.3}$$

for some $b_j(m,n)$. Substituting 0 to $m-1$ into (2.3), we obtain that $b_0(m,n) = b_1(m,n) = \cdots = b_{m-1}(m,n) = 0$. If we put m into (2.3), then we have

$$N_m(m)N_n(m) = \sum_{j=m}^{m+n} b_j(m,n)N_j(m) = b_m(m,n)N_m(m)$$

and so $N_n(m) = b_m(m,n)$. Next, we put $m+1$ into (2.3) and we have

$$N_n(m+1)N_n(m+1) = b_m(m,n)N_m(m+1) + b_{m+1}(m,n)N_{m+1}(m+1).$$

Set

$$\mathcal{N} = \begin{pmatrix} N_m(m) & 0 & 0 & 0 & \cdots & 0 \\ N_m(m+1) & N_{m+1}(m+1) & 0 & 0 & \cdots & 0 \\ N_m(m+2) & N_{m+1}(m+2) & N_{m+2}(m+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ N_m(m+n) & N_{m+1}(m+n) & N_{m+2}(m+n) & N_{m+3}(m+n) & \cdots & N_{m+n}(m+n) \end{pmatrix}$$

as in Lemma 2.3. By repeating this method, we deduce that

$$\mathcal{N}\begin{pmatrix} b_m(m,n) \\ b_{m+1}(m,n) \\ b_{m+2}(m,n) \\ \vdots \\ b_{m+n}(m,n) \end{pmatrix} = \begin{pmatrix} N_m(m)N_n(m) \\ N_m(m+1)N_n(m+1) \\ N_m(m+2)N_n(m+2) \\ \vdots \\ N_m(m+n)N_n(m+n) \end{pmatrix}.$$

Therefore, we conclude that $N_m(z)N_n(z) = \sum_{j=m}^{(m+n)} b_j(m,n)N_j(z)$, where

$$\begin{pmatrix} b_m(m,n) \\ b_{m+1}(m,n) \\ b_{m+2}(m,n) \\ \vdots \\ b_{m+n}(m,n) \end{pmatrix} = \mathcal{N}^{-1}\begin{pmatrix} N_m(m)N_n(m) \\ N_m(m+1)N_n(m+1) \\ N_m(m+2)N_n(m+2) \\ \vdots \\ N_m(m+n)N_n(m+n) \end{pmatrix}.$$

By Lemma 2.3, we have the results. □

From (2.2), we obtain the exact value of $b_j(m,n)$ as follows for the given m, n. We obtain the specific value of $b(m,n)$ through a simple calculation.

Remark 2.5 (a) If $m = n = 2$, then we have $N_2(z)N_2(z) = \sum_{j=1}^{4} b_j(2,2)N_j(z)$ gives that

$$\frac{N_2(z)N_2(z)}{N_1(z)} = b_1(2,2) + \sum_{j=2}^{4} b_j(2,2)\frac{N_j(z)}{N_1(z)}.$$

Take $z = 1$, then $b_1(2,2) = 0$ and so $N_2(z)N_2(z) = \sum_{j=2}^{4} b_j(2,2)N_j(z)$. Thus,

$$\frac{N_2(z)N_2(z)}{N_2(z)} = b_2(2,2) + \sum_{j=3}^{4} b_j(2,2)\frac{N_j(z)}{N_2(z)}.$$

Take $z = 2$, then $b_2(2,2) = N_2(2) = 1$ and hence

$$N_2(z)N_2(z) = N_2(z) + b_3(2,2)N_3(z) + b_4(2,2)N_4(z).$$

Therefore,

$$N_2(z) = 1 + b_3(2,2)\frac{N_3(z)}{N_2(z)} + b_4(2,2)\frac{N_4(z)}{N_2(z)}. \tag{2.4}$$

If $z = 3$ in (2.4), then $b_3(2,2) = -6$. If $z = 4$ in (2.4), then $6 = 3 + \frac{b_4(2,2)}{2}$ and so $b_4(2,2) = 6$. Hence, $b_1(2,2) = 0$, $b_2(2,2) = 1$, $b_4(2,2) = -b_3(2,2) = 6$. Hence,

$$N_2(z)N_2(z) = N_2(z) - 6N_3(z) + 6N_4(z).$$

From (2.2), for $m = 2$ and $n = 2$, we have

$$\begin{pmatrix} b_2(2,2) \\ b_3(2,2) \\ b_4(2,2) \end{pmatrix} = \begin{pmatrix} N_2(2) & 0 & 0 \\ N_2(3) & N_3(3) & 0 \\ N_2(4) & N_3(4) & N_4(4) \end{pmatrix}^{-1} \begin{pmatrix} N_2(2)N_2(2) \\ N_2(3)N_2(3) \\ N_2(4)N_2(4) \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 6 \end{pmatrix}.$$

(b) From (2.2), for $m = 3$ and $n = 2$, we have

$$\begin{pmatrix} b_3(3,2) \\ b_4(3,2) \\ b_5(3,2) \end{pmatrix} = \begin{pmatrix} N_3(3) & 0 & 0 \\ N_3(4) & N_4(4) & 0 \\ N_3(5) & N_4(5) & N_5(5) \end{pmatrix}^{-1} \begin{pmatrix} N_3(3)N_2(3) \\ N_3(4)N_2(4) \\ N_3(5)N_2(5) \end{pmatrix} = \begin{pmatrix} 3 \\ -12 \\ 10 \end{pmatrix}.$$

(c) From (2.2), for $m = 4$ and $n = 2$, we have

$$\begin{pmatrix} b_4(4,2) \\ b_5(4,2) \\ b_6(4,2) \end{pmatrix} = \begin{pmatrix} N_4(4) & 0 & 0 \\ N_4(5) & N_5(5) & 0 \\ N_4(6) & N_5(6) & N_6(6) \end{pmatrix}^{-1} \begin{pmatrix} N_4(4)N_2(4) \\ N_4(5)N_2(5) \\ N_4(6)N_2(6) \end{pmatrix} = \begin{pmatrix} 3 \\ -35 \\ -105 \end{pmatrix}.$$

In the Hardy space $H^2(\mathbb{T})$, $\bar{z}^n z^m$ is equal to z^{m-n}, but in the weighted Bergmann space $A^2_\alpha(\mathbb{D})$, $\bar{z}^n z^m \neq z^{m-n}$ since $z \in \mathbb{D}$. In addition, $\overline{N_n(z)}N_m(z)$ is not $N_{m-n}(z)$ in the Newton space $N^2(\mathbb{P})$.

Lemma 2.6 [10, *Theorem* 1.2] *The* $N^2(\mathbb{P})$ *is a Hilbert space that includes the Newton polynomials as a complete orthogonal set.*

Lemma 2.7 *The set of functions* $\{N_n(z)\} \cup \{\overline{N_m(z)}\}$ *forms an orthonormal basis for* $L^2(\mathbb{P})$ *for* $n \in \mathbb{N} \cup \{0\}$ *and* $m \in \mathbb{N}$.

Proof For positive integers m, n with $m \neq n$,

$$\langle \overline{N_n}, \overline{N_m} \rangle = \langle N_m, N_n \rangle = 0$$

and $\langle \overline{N_m}, \overline{N_m} \rangle = \langle N_m, N_m \rangle = 1$. Now, we want to show that Parseval's identity

$$\sum_{n=0}^{\infty} \left| \langle f, N_n \rangle \right|^2 + \sum_{n=1}^{\infty} \left| \langle f, \overline{N}_n \rangle \right|^2 = \|f\|_2^2$$

holds for every $f \in L^2(\mathbb{P})$. Let $f(z) = \sum_{k=0}^{\infty} a_k N_k(z) + \sum_{k=1}^{\infty} a_{-k} \overline{N_k(z)}$. Then, $\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |a_k|^2$ and for any $n \geq 0$,

$$\langle f(z), N_n(z) \rangle = \left\langle \sum_{k=0}^{\infty} a_k N_k(z) + \sum_{k=1}^{\infty} a_{-k} \overline{N_k(z)}, N_n(z) \right\rangle = a_n$$

and for any $n > 0$,

$$\langle f(z), \overline{N}_n(z) \rangle = \left\langle \sum_{k=0}^{\infty} a_k N_k(z) + \sum_{k=1}^{\infty} a_{-k} \overline{N_k(z)}, \overline{N_n(z)} \right\rangle = a_{-n}.$$

Therefore,

$$\sum_{n=0}^{\infty}\left|\langle f,N_n\rangle\right|^2+\sum_{n=1}^{\infty}\left|\langle f,\overline{N_n}\rangle\right|^2=\sum_{n=-\infty}^{\infty}|a_n|^2=\|f\|_2^2.$$

Since Parseval's identity holds, we obtain that $\{N_n(z)\}\cup\{\overline{N_m(z)}\}$ is complete (see [2]). Hence, for $n\in\mathbb{N}\cup\{0\}$ and $m\in\mathbb{N}$, $\{N_n(z)\}\cup\{\overline{N_m(z)}\}$ forms an orthonormal basis for $L^2(\mathbb{P})$. □

In [4], let P be an orthogonal projection of $L^2(\mathbb{D})$ onto the Bergamm space $A^2(\mathbb{D})$. Then, for nonnegative integers n, m,

$$P\left(z^n\overline{z}^m\right)=\begin{cases}\frac{\Gamma(n+1)\Gamma(n-m+\alpha+2)}{\Gamma(n+\alpha+2)\Gamma(n-m+1)}z^{n-m} & \text{if } n\geq m;\\ 0 & \text{if } n<m.\end{cases}$$

Next, we investigate the orthogonal projection P of $\overline{N_n}N_m$.

Theorem 2.8 *For any nonnegative integers m, n,*

$$P(\overline{N_n}N_m)=\begin{cases}\sum_{j=0}^{n}b_m(n,m-n+j)N_{m-n+j}(z) & \textit{if } m\geq n\\ 0 & \textit{if } m<n,\end{cases}\tag{2.5}$$

where $b_m(n,m-n+j)$ is the solution of the matrix equation as in Theorem 2.4 for $0\leq j\leq n$.

Proof For any nonnegative integer k, we obtain from Theorem 2.4 that

$$\begin{aligned}\left\langle P(\overline{N_n}N_m),N_k\right\rangle&=\langle\overline{N_n}N_m,N_k\rangle=\langle N_m,N_nN_k\rangle\\&=\left\langle N_m,\sum_{j=\max\{n,k\}}^{n+k}b_j(n,k)N_j(z)\right\rangle.\end{aligned}\tag{2.6}$$

(a) If $m<n$, then (2.6) implies

$$\left\langle P(\overline{N_n}N_m),N_k\right\rangle=\left\langle N_m,\sum_{j=n}^{n+k}b_j(n,k)N_j(z)\right\rangle=0$$

for $n\geq k$ and

$$\left\langle P(\overline{N_n}N_m),N_k\right\rangle=\left\langle N_m,\sum_{j=n}^{n+k}b_j(n,k)N_j(z)\right\rangle=0$$

for $n<k$ since the Newton polynomials $\{N_n(z)\}_{n=0}^{\infty}$ are orthonormal.

(b) If $m\geq n$, then

$$\left\langle N_m,\sum_{j=\max\{n,k\}}^{n+k}b_j(n,k)N_j(z)\right\rangle=\begin{cases}b_m(n,k) & \text{if } m-n\leq k\leq m,\\ 0 & \text{if } 0\leq k<m-n \text{ or } k>m.\end{cases}$$

Hence, if $m \geq n$, then (2.6) becomes

$$P(\overline{N_n}N_m) = b_m(n, m-n)N_{m-n} + b_m(n, m-n+1)N_{m-n+1} + \cdots + b_m(n, m)N_m$$
$$= \sum_{j=0}^{n} b_m(n, m-n+j)N_{m-n+j}(z)$$

and if $m < n$, then $P(\overline{N_n}N_m) = 0$. □

Remark 2.9 Note that $b_m(m, n) = b_m(n, m)$ and $b_j(m, 0) = 1$ for any nonnegative integer m, n, j.

(i) Since

$$N_1(z)N_1(z) = \sum_{j=1}^{2} b_j(1, 1)N_j(z) = b_1(1, 1)N_1(z) + b_2(1, 1)N_2(z)$$

and

$$N_2(z)N_1(z) = \sum_{j=2}^{3} b_j(2, 1)N_j(z) = b_2(2, 1)N_2(z) + b_3(2, 1)N_3(z)$$

by Theorem 2.4, we have $b_1(1, 1) = -1$, $b_2(1, 1) = 2$, $b_2(2, 1) = -2$, and $b_3(2, 1) = 3$. Then,

$$P(\overline{N_1}N_2) = \sum_{j=0}^{1} b_2(1, 1+j)N_{1+j}(z)$$
$$= b_2(1, 1)N_1 + b_2(1, 2)N_2$$
$$= 2(N_1 - N_2)$$

by Theorem 2.8.

(ii) By Theorem 2.8 and Remark 2.4, we also obtain that

$$P(\overline{N_2}N_3) = \sum_{j=0}^{2} b_3(2, 1+j)N_{1+j}(z)$$
$$= b_3(2, 1)N_1 + b_3(2, 2)N_2 + b_3(2, 3)N_3$$
$$= 3(N_1 - 2N_2 + N_3)$$

and

$$P(\overline{N_1}N_3) = \sum_{j=0}^{1} b_3(1, 2+j)N_{2+j}(z)$$
$$= b_3(1, 2)N_2 + b_3(1, 3)N_3$$
$$= 3(N_2 - N_3).$$

As an application of Theorem 2.8, we obtain the following corollary.

Corollary 2.10 *For any nonnegative integers m, k with $m \geq k$,*

$$P(\overline{N_k}N_m) = m\sum_{i=0}^{k}(-1)^i\binom{k}{i}N_{m-k+i}(z)$$

holds, where P denotes an orthogonal projection of $L^2(\mathbb{P})$ onto $N^2(\mathbb{P})$.

Proof By Theorem 2.8, we obtain that

$$\begin{aligned}P(\overline{N_k}N_m) &= b_m(n, m-k)N_{m-k} + b_m(k, m-k+1)N_{m-k+1} + \cdots + b_m(k, m)N_m \\ &= \sum_{j=0}^{k} b_m(k, m-k+j)N_{m-k+j}(z) \\ &= m\sum_{i=0}^{k}(-1)^i\binom{k}{i}N_{m-k+i}(z),\end{aligned}$$

where $b_m(n, m-k+j)$ denotes the solutions of the matrix equation as in Theorem 2.4 for $0 \leq j \leq n$. □

We finally find the matrices of Toeplitz operators T_φ with harmonic symbols φ on the Newton spaces by using Theorems 2.4 and 2.8. In Theorem 2.11, we explain the characteristics of the entries of the Toeplitz matrix in Newton space. Applying this, by specifically using the coefficient of $b_j(m, n)$ in Corollary 2.14, it was found that the entries of the Toeplitz matrix in Newton space are expressed as a linear combination of the binomial coefficients of the given entries.

Theorem 2.11 *For the harmonic symbol $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} a_{-i}\overline{N}_i$, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_n\}_{n\geq 0}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0b_0(0,0) & a_{-1}b_1(1,0) & a_{-2}b_2(2,0) & a_{-3}b_3(3,0) & \cdots \\ a_1b_1(1,0) & \sum_{i=0}^{1} a_ib_1(i,1) & \sum_{i=1}^{2} a_{-i}b_2(i,1) & \sum_{i=2}^{3} a_{-i}b_3(i,1) & \cdots \\ a_2b_2(2,0) & \sum_{i=1}^{2} a_ib_2(i,1) & \sum_{i=0}^{2} a_ib_2(i,2) & \sum_{i=1}^{3} a_{-i}b_3(i,2) & \cdots \\ a_3b_3(3,0) & \sum_{i=2}^{3} a_ib_3(i,1) & \sum_{i=1}^{3} a_ib_3(i,2) & \sum_{i=0}^{3} a_ib_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the adjoint of the matrix of T_φ is given by

$$[T_\varphi]^*_{\mathcal{B}} = \begin{pmatrix} \overline{a}_0b_0(0,0) & \overline{a}_1b_1(1,0) & \overline{a}_2b_2(2,0) & \overline{a}_3b_3(3,0) & \cdots \\ \overline{a}_{-1}b_1(1,0) & \sum_{i=0}^{1} \overline{a}_{-i}b_1(i,1) & \sum_{i=1}^{2} \overline{a}_ib_2(i,1) & \sum_{i=2}^{3} \overline{a}_ib_3(i,1) & \cdots \\ \overline{a}_{-2}b_2(2,0) & \sum_{i=1}^{2} \overline{a}_{-i}b_2(i,1) & \sum_{i=0}^{2} \overline{a}_{-i}b_2(i,2) & \sum_{i=1}^{3} \overline{a}_ib_3(i,2) & \cdots \\ \overline{a}_{-3}b_3(3,0) & \sum_{i=2}^{3} \overline{a}_{-i}b_3(i,1) & \sum_{i=1}^{3} \overline{a}_{-i}b_3(i,2) & \sum_{i=0}^{3} \overline{a}_{-i}b_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_m(m, n) \in \mathbb{R}$ is denoted as in Theorem 2.4.

Proof For the harmonic symbol $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} a_{-i}\overline{N}_i$, the (m,n)th entry of the matrix of T_φ with respect to orthonormal basis $\{N_n\}_{n\geq 0}$ of $N^2(\mathbb{P})$ is given by

$$\begin{aligned}\langle T_\varphi N_n, N_m\rangle &= \langle P(\varphi N_n), N_m\rangle \\ &= \left\langle P\left(\sum_{i=0}^{\infty} a_i N_i N_n + \sum_{i=1}^{\infty} a_{-i}\overline{N}_i N_n\right), N_m\right\rangle \\ &= \left\langle \sum_{i=0}^{\infty} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j + \sum_{i=1}^{n} a_{-i} \sum_{j=0}^{i} b_n(i, n-i+j) N_{n-i+j}, N_m\right\rangle.\end{aligned} \tag{2.7}$$

Then, there are two cases to consider. If $m \geq n$, then

$$\begin{aligned}\sum_{i=0}^{\infty} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j &= \sum_{i=0}^{m-n-1} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j + \sum_{i=m-n}^{m} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j \\ &\quad + \sum_{i=m+1}^{\infty} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j.\end{aligned} \tag{2.8}$$

Thus, the first and third term of the right equation in (2.8) have no term of the form $a_i b_m(i,n) N_m$. Hence, (2.7) becomes

$$\begin{aligned}\langle T_\varphi N_n, N_m\rangle &= \left\langle \sum_{i=0}^{\infty} a_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j, N_m\right\rangle \\ &= \left\langle \sum_{i=m-n}^{m} a_i b_m(i,n) N_m, N_m\right\rangle \\ &= \sum_{i=m-n}^{m} a_i b_m(i,n).\end{aligned}$$

If $m < n$, then by a similar method, (2.7) gives

$$\langle T_\varphi N_n, N_m\rangle = \left\langle \sum_{i=n-m}^{n} a_{-i} b_n(i,m) N_m, N_m\right\rangle = \sum_{i=n-m}^{n} a_{-i} b_n(i,m).$$

Thus, we have

$$\langle T_\varphi N_n, N_m\rangle = \begin{cases} \sum_{i=m-n}^{m} a_i b_m(i,n) & \text{for } m \geq n \\ \sum_{i=n-m}^{n} a_{-i} b_n(i,m) & \text{for } m < n, \end{cases}$$

where m and n are nonnegative integers. Hence, the matrix of T_φ with respect to $\mathcal{B} = \{N_n\}_{n\geq 0}$ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 b_0(0,0) & a_{-1} b_1(1,0) & a_{-2} b_2(2,0) & a_{-3} b_3(3,0) & \cdots \\ a_1 b_1(1,0) & \sum_{i=0}^{1} a_i b_1(i,1) & \sum_{i=1}^{2} a_{-i} b_2(i,1) & \sum_{i=2}^{3} a_{-i} b_3(i,1) & \cdots \\ a_2 b_2(2,0) & \sum_{i=1}^{2} a_i b_2(i,1) & \sum_{i=0}^{2} a_i b_2(i,2) & \sum_{i=1}^{3} a_{-i} b_3(i,2) & \cdots \\ a_3 b_3(3,0) & \sum_{i=2}^{3} a_i b_3(i,1) & \sum_{i=1}^{3} a_i b_3(i,2) & \sum_{i=0}^{3} a_i b_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the adjoint of the matrix of T_φ is given by

$$[T_\varphi]^*_{\mathcal{B}} = \begin{pmatrix} \overline{a}_0 b_0(0,0) & \overline{a}_1 b_1(1,0) & \overline{a}_2 b_2(2,0) & \overline{a}_3 b_3(3,0) & \cdots \\ \overline{a}_{-1} b_1(1,0) & \sum_{i=0}^{1} \overline{a}_{-i} b_1(i,1) & \sum_{i=1}^{2} \overline{a}_i b_2(i,1) & \sum_{i=2}^{3} \overline{a}_i b_3(i,1) & \cdots \\ \overline{a}_{-2} b_2(2,0) & \sum_{i=1}^{2} \overline{a}_{-i} b_2(i,1) & \sum_{i=0}^{2} \overline{a}_{-i} b_2(i,2) & \sum_{i=1}^{3} \overline{a}_i b_3(i,2) & \cdots \\ \overline{a}_{-3} b_3(3,0) & \sum_{i=2}^{3} \overline{a}_{-i} b_3(i,1) & \sum_{i=1}^{3} \overline{a}_{-i} b_3(i,2) & \sum_{i=0}^{3} \overline{a}_{-i} b_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and, hence, we know that $[T_\varphi]^*_{\mathcal{B}} = [T_{\overline{\varphi}}]_{\mathcal{B}}$. □

Corollary 2.12 *If $[T_\varphi]_{\mathcal{B}}$ is self-adjoint, then $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} \overline{a}_i \overline{N}_i$.*

Proof The proof follows from Theorem 2.11. □

Corollary 2.13 (i) *For the harmonic symbol $\varphi(z) = a_1 N_1 + a_0 + a_{-1}\overline{N_1}$, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1\}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} \\ a_1 & a_0 - a_1 \end{pmatrix}.$$

(ii) *For the harmonic symbol $\varphi(z) = a_2 N_2 + a_1 N_1 + a_0 + a_{-1}\overline{N_1} + a_{-2}\overline{N_2}$, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1, N_2\}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 - a_1 & -2(a_{-1} + a_{-2}) \\ a_2 & -2(a_1 + a_2) & a_0 - 2a_1 + a_2 \end{pmatrix}.$$

(iii) *Let $\varphi(z) = a_3 N_3 + a_2 N_2 + a_1 N_1 + a_0 + a_{-1}\overline{N_1} + a_{-2}\overline{N_2} + a_{-3}\overline{N_3}$ be the harmonic symbol. Then, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1, N_2, N_3\}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 - a_1 & -2(a_{-1} + a_{-2}) & 3(a_{-2} - a_{-3}) \\ a_2 & -2(a_1 + a_2) & (a_0 - 2a_1 + a_2) & 3(a_{-1} - 2a_{-2} + a_{-3}) \\ a_3 & 3(a_2 - a_3) & 3(a_1 - 2a_2 + a_3) & a_0 - 3a_1 + 3a_2 - a_3 \end{pmatrix}.$$

Proof Since $N_0(z)N_0(z) = b_0(0,0)N_0(z)$ and $N_1(z)N_0(z) = b_1(1,0)N_1(z)$ by Theorem 2.4, we have $b_0(0,0) = 1$ and $b_1(1,0) = 1$. Moreover, since

$$N_1(z)N_1(z) = \sum_{j=1}^{2} b_j(1,1)N_j(z) = b_1(1,1)N_1(z) + b_2(1,1)N_2(z)$$

by Theorem 2.4, it follows that $b_1(1,1) = -1$. Since $N_2(z)N_0(z) = b_2(2,0)N_2(z)$, we have $b_2(2,0) = 1$ and $b_2(2,2) = 1$ by Remark 2.5. Since $b_m(m,n) = N_n(m)$, we obtain $b_2(2,1) = N_1(2) = -2$ and $b_2(1,1) = 2$ by Remark 2.9. Hence, the proof follows from Theorem 2.11. □

Corollary 2.14 *Let $\varphi(z) = \sum_{i=0}^{n} a_i N_i + \sum_{i=1}^{n} a_{-i}\overline{N}_i$ be the harmonic symbol for even n. Then, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_k\}_{k=0,1,2,\ldots,n}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & a_{-n} \\ a_1 & \sum_{i=0}^{1}(-1)^i a_i & -2\sum_{i=1}^{2} a_{-i} & 3\sum_{i=2}^{3}(-1)^i a_{-i} & \cdots & -n\sum_{i=n-1}^{n} a_{-i} \\ a_2 & -2\sum_{i=1}^{2} a_i & \sum_{i=0}^{2}(-1)^i\binom{2}{i}a_i & 3\sum_{i=1}^{3}(-1)^i\binom{2}{i}a_{-i} & \cdots & \vdots \\ a_3 & 3\sum_{i=2}^{3}(-1)^i a_i & 3\sum_{i=1}^{3}(-1)^i\binom{2}{i}a_i & \sum_{i=0}^{3}(-1)^i\binom{3}{i}a_i & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & -n\sum_{i=n-1}^{n} a_i & \cdots & \cdots & \cdots & \sum_{i=0}^{n}(-1)^i\binom{n}{i}a_i \end{pmatrix}.$$

Proof The proof follows from Theorem 2.11 and Corollary 2.13. □

Remark 2.15 Set

$$c_{m,n} = \langle T_\varphi N_n, N_m\rangle = \begin{cases} \sum_{i=m-n}^{m} a_i b_m(i,n) & \text{for } m \geq n \\ \sum_{i=n-m}^{n} a_{-i} b_n(i,m) & \text{for } m < n. \end{cases}$$

Then, for $m \geq n$, $c_{m,n} = \sum_{i=m-n}^{m} a_i b_m(i,n)$ and $c_{n,m} = \sum_{i=m-n}^{m} a_{-i} b_m(i,n)$. Hence, $[T_\varphi]_{\mathcal{B}}$ is self-adjoint if and only if $c_{m,n} = \overline{c_{n,m}}$ if and only if $a_{-i} = \overline{a_i}$.

Example 2.16 *(i)* Let $\varphi(z) = N_1 + 2 + i\overline{N_1}$ be the harmonic symbol. Then, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1\}$ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} \\ a_1 & a_0 - a_1 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 1 & 1 \end{pmatrix}.$$

(ii) Let $\varphi(z) = iN_2 - N_1 + 2 + i\overline{N_1} + 2\overline{N_2}$ be the harmonic symbol. Then, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1, N_2\}$ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 - a_1 & -2(a_{-1} + a_{-2}) \\ a_2 & -2(a_1 + a_2) & a_0 - 2a_1 + a_2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & i & 2 \\ -1 & 3 & -2i - 4 \\ i & 2 - 2i & 4 + i \end{pmatrix}.$$

A conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ that satisfies $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$.

Corollary 2.17 *Assume that C and $C_{\mu,\lambda}$ are conjugations on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ and $C_{\mu,\lambda}f(z) = \mu\overline{f(\lambda\overline{z})}$ for $f \in N^2(\mathbb{P})$ with $|\lambda| = |\mu| = 1$, respectively. If for the harmonic symbol $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} a_{-i}\overline{N}_i$ and the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_n\}_{n\geq 0}$, then the following statements are equivalent*:

(i) $[T_\varphi]_{\mathcal{B}}$ *is complex symmetric with the conjugation C*;

(ii) $[T_\varphi]_{\mathcal{B}}$ *is complex symmetric with the conjugation $C_{\mu,\lambda}$*;

(iii) $a_i = a_{-i}$ *for $i = 0, 1, 2, \ldots$.*

Proof (i) ⇔ (iii) Let $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} a_{-i}\overline{N_i}$ be with respect to the basis $\mathcal{B} = \{N_n\}_{n=0}^{\infty}$. Since the matrix of $[T_\varphi]_{\mathcal{B}}$ is of the form as in Theorem 2.11, it follows that the matrix of $C[T_\varphi]_{\mathcal{B}}C$ is the following:

$$C[T_\varphi]_{\mathcal{B}}C = \begin{pmatrix} \overline{a_0}b_0(0,0) & \overline{a_{-1}}b_1(1,0) & \overline{a_{-2}}b_2(2,0) & \overline{a_{-3}}b_3(3,0) & \cdots \\ \overline{a_1}b_1(1,0) & \sum_{i=0}^{1}\overline{a_i}b_1(i,1) & \sum_{i=1}^{2}\overline{a_{-i}}b_2(i,1) & \sum_{i=2}^{3}\overline{a_{-i}}b_3(i,1) & \cdots \\ \overline{a_2}b_2(2,0) & \sum_{i=1}^{2}\overline{a_i}b_2(i,1) & \sum_{i=0}^{2}\overline{a_i}b_2(i,2) & \sum_{i=1}^{3}\overline{a_{-i}}b_3(i,2) & \cdots \\ \overline{a_3}b_3(3,0) & \sum_{i=2}^{3}\overline{a_i}b_3(i,1) & \sum_{i=1}^{3}\overline{a_i}b_3(i,2) & \sum_{i=0}^{3}\overline{a_i}b_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, $[T_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation C if and only if $a_i = a_{-i}$ for $i = 0, 1, 2, \ldots$.

(ii) ⇔ (iii) Let $\varphi(z) = \sum_{i=0}^{\infty} a_i N_i + \sum_{i=1}^{\infty} a_{-i}\overline{N_i}$ be with respect to the basis $\mathcal{B} = \{N_n\}_{n=0}^{\infty}$. It is known from [5] that $C_{\mu,\lambda}$ is unitarily equivalent to $C_{1,\lambda}$. Since the matrix of T_φ is of the form as in Theorem 2.11, it follows that the matrix of $C_{1,\lambda}T_\varphi C_{1,\lambda}$ is the following:

$$[C_{1,\lambda}T_\varphi C_{1,\lambda}]_{\mathcal{B}} = \lambda[CT_\varphi C]_{\mathcal{B}}.$$

Then, $[T_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation $C_{1,\lambda}$ if and only if $a_i = a_{-i}$ for $i = 0, 1, 2, \ldots$. □

Corollary 2.18 *Let C be a conjugation on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ for $f \in N^2(\mathbb{P})$. If for the harmonic symbol $\varphi(z) = \sum_{i=0}^{3} a_i(N_i + \overline{N_i})$, the matrix of T_φ with respect to orthonormal basis $\mathcal{B} = \{N_0, N_1, N_2, N_3\}$ is given by*

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 - a_1 & -2(a_1 + a_2) & 3(a_2 - a_3) \\ a_2 & -2(a_1 + a_2) & (a_0 - 2a_1 + a_2) & 3(a_1 - 2a_2 + a_3) \\ a_3 & 3(a_2 - a_3) & 3(a_1 - 2a_2 + a_3) & a_0 - 3a_1 + 3a_2 - a_3 \end{pmatrix},$$

then $[T_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation C.

Example 2.19 Let C be a conjugation on L^2 given by $Cf(z) = \overline{f(\overline{z})}$ for $f \in N^2(\mathbb{P})$ and let $\mathcal{B} = \{N_0, N_1, N_2, N_3\}$.

(i) Let

$$\varphi(z) = 2iN_3 + 2iN_2 + 3N_1 - 7i + 3\overline{N_1} + 2i\overline{N_2} + 2i\overline{N_3}$$

be the harmonic symbol. If the matrix of T_φ with respect to orthonormal basis $\mathcal{B}$ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} -7i & 3 & 2i & 2i \\ 3 & -7i - 3 & -2(2i + 3) & 0 \\ 2i & -2(2i + 3) & -5i - 6 & 3(3 - 4i) \\ 2i & 0 & 3(3 - 4i) & -3i - 9 \end{pmatrix},$$

then $[T_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation C.

(ii) If for the harmonic symbol $\varphi(z) = \sum_{i=0}^{3}(N_i + \overline{N_i})$, the matrix of T_φ with respect to orthonormal basis $\mathcal{B}$ is given by

$$[T_\varphi]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then $[T_\varphi]_{\mathcal{B}}$ is complex symmetric with the conjugation C.

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The authors declare no competing interests.

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