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RESEARCH

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Statistical convergence of integral form of modified Szász–Mirakyan operators: an algorithm and an approach for possible applications

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Abstract

In this study, we take into account the of modified Szász–Mirakyan–Kantorovich operators to obtain their rate of convergence using the modulus of continuity and for the functions in Lipschitz space. Then, we obtain the statistical convergence of this form. In addition, we determine the weighted statistical convergence and compare it with the statistical one for the same operator. Medical applications and traditional mathematics; one way to get a close approximation of the Riemann integrable functions is through the use of the Kantorovich modification of positive linear operators. The use of Kantorovich operators is tremendously helpful from a medical point of view. Their application is shown as an approximation of the rate of convergence in respect of modulus of continuity.

Mathematics Subject Classification: 26A15; 41A10; 41A25; 41A30; 41A63

Keywords: Korovkin theorem; Szász; Mirakyan operators; Weighted statistical convergence in Lipschitz space; Statistical convergence

1 Introduction

1.1 Problem context

First introduced by Fast in [1], statistical convergence can be applied to either real or complex number sequences. This is strongly connected to the idea of the natural/ asymptotic density of subsets of positive integers $\mathbb{N}$. In [2], Zygmund refers to this as "almost convergence" and identifies the association between statistical convergence and strong summability. This notation has been further investigated in the number of papers [3–7]. In [7], $D-\lim \varkappa_{\mathfrak{r}} = L$ represents the statistical convergence and is called D-convergence. Salat [5] showed that if $D-\lim \varkappa_{\mathfrak{r}} = L$ holds, then the number L is unique. Conversely, if $\lim_{\mathfrak{r}\to\infty} \varkappa_{\mathfrak{r}} = L$ holds, the $D-\lim \varkappa_{\mathfrak{r}} = L$ holds too, since the set $\{\mathfrak{r} \leq n : |\varkappa_{\mathfrak{r}} - L| \geq \epsilon\}$ is finite in this case for all $\epsilon > 0$. As stated in [8], this approach has significant applications in the theory of approximating polynomials, functional analysis, numerical solutions to differential equations, integral equations, and others. We used the work to study the following:

(1) apply statistical convergence on Kantorovich form of Szász–Mirakyan operators,
(2) statistical convergence for approximation theorem of the Korovkin type,
(3) the convergence of the same operators in weighted statistical sense.

1.2 Limitations of existing literature and algorithms

Statistical convergence was introduced as a solution to problems arising from series summation. This concept has been widely applied in diverse branches of mathematics, with a particular focus on estimating the characteristics of linear positive operators. The limitations of the existing literature and algorithms are the following: First, classical convergence and statistical convergence are not compared for the same operators in the existing literature. Second, the main drawback of statistical convergence is that it does not ensure convergence of the sequence, while the converse is true. In relation to issues of series summation, statistical convergence was introduced to the theory of approximation to be applied in several areas of pure and applied mathematics to estimate the properties of linear positive operators, and the present study refines this motive for better application in the different areas of mathematics where the rate of convergence needs to be significant. The replacement of uniform convergence with statistical convergence has the benefit of modeling and enhancing the signal approximation technique in different function spaces.

Using summability and sequence to function methods, in various ways, the Cauchy Convergence classical notion has been generalized. In 1932, the first generalization was initiated by Banach, which was later studied in detail by Lorentz (1948). Many authors have used statistical convergence for the operators, but the convergence for the same operators is missing. Unless the two convergences are compared, the purpose of using statistical convergence is of no use. Main limitation of statistical convergence is that any convergent sequence is statistically convergent, but the statistically convergent sequence is not always convergent. Nevertheless, neither the limits nor the statistical limits can be computed or quantified with exact precision in the general case. Many mathematical approaches have been developed in order to model this imprecision using mathematical structures and to account for its imprecision. As is well-known, real-world sequences are not convergent in a strict mathematical sense. The algorithm used here is to calculate the auxiliary results for the test functions using the Korovkin theorem, which proves it as a linear positive operator, then to find the rate of convergence in the classical, statistical sense and also in weighted space.

1.3 Motivation and objectives of this paper

Different approaches have been applied by various methods to reduce rate of convergence. Therefore, this study uses modulus of continuity to get an efficient rate of convergence and compares classical, statistical and weighted statistical convergence. Most of the problem arises when we do not have a proper algorithm / methodology to reach the required result. The present work provides this for statistical convergence, where the behavior of the elements become irrelevant.

1.4 Contribution of this study

The following are the primary contributions of this study:

(i) In this study, the rate of convergence of the operators is identified both in the classical sense and from a statistical perspective.

(ii) The idea that the majority, or something close to it, of a sequence's elements, will converge via statistical convergence; this means that the behavior of the sequence's other elements becomes less significant to the analysis. It was understood at the time that sequences originating from sources in the real world do not converge mathematically, and the work proposes approximation results of weighted statistical convergence.

(iii) We deployed different techniques to find the rate of convergence in classical, statistical and weighted statistical sense and hence thereby giving a comparative look of all three convergence.

(iv) Providing a proper methodology and algorithm gives a way out to reach the result using the modulus of continuity tool.

1.5 Paper organization

The remaining components of this investigation are laid out in the following manner: Sect. 2 examines the related articles that already exist in the literature relevant to our research. Section 3 focuses on preliminaries required for understanding the results obtained in the paper. Section 4 concentrates on determining the rate of convergence used in classical sense and for functions in Lipschitz class. Section 5 presents the proposed methodology to reach the required result and the specified smart algorithm used in the paper to reach at the desired result. Section 6 discusses some previous applications of Kantorovich form of modified Szász–Mirakyan operators to show the possible cases where our result may be applied. Section 6.1 explores application from previous publications in the area of convergence in sustainability, whereas Sect. 6.2 discusses applications in the area of medical diagnosis. Section 7 focuses the findings of this study and a comparative analysis with the existing work. Section 8 concludes the research with research scope.

2 Literature review

In the field of approximation theory, Szász–Mirakyan operators have been used as a fundamental tool for the approximation of functions. In particular, the integral form of modified Szász–Mirakyan operators has been analyzed in several studies.

Duman et al. [9] proposed a new approach to investigate the statistical convergence of these operators. The authors derived the conditions to prove the modified Szász–Mirakyan operators integral form convergence in a probabilistic sense. They also presented an algorithm for computing the convergence in statistical sense for these operators. Kizmaz and Karagoz [10] considered the statistical convergence of the integral form of modified Szász–Mirakyan operators for the continuous functions. The authors proved that these operators converge to the functions uniformly on compact subsets of the interval $[0,1]$ and established the order of the convergence.

Altin and Karacik [11] have introduced the statistical convergence of modified Szász–Mirakyan operators with respect to a new sequence of weights. They obtained some approximation results for the operators. They also proved that the sequence of modified Szász–Mirakyan operators is statistically convergent with respect to the new sequence of weights. Pehlivan and Duman [12] introduced a new type of Szász–Mirakyan operators, namely the exponential form of the same. They studied the convergence properties in statistical sense for these operators and proved that the sequence of exponential Szász–Mirakyan operators converges statistically to the function f for the functions continuous on $[0,1]$.

Several mathematicians extended Korovkin-type approximation theorems by incorporating Banach spaces, Banach algebras, function spaces and abstract Banach lattices, as well as utilising other test functions as in [13–17]. Statistical convergence plays a very important part in approximation theory out of all the approaches available to determine the rate of convergence of different linear positive operators. Statistical convergence is a conventional method for achieving sequence summability.

We recall the definition of statistical convergence: A sequence $\varkappa$ is statistically convergent to a real number L if for each $\epsilon > 0$,

$$\lim_{\mathfrak{i}\to\infty} \frac{1}{\mathfrak{i}} \left|\left\{\mathfrak{r} \leq \mathfrak{i} : |\varkappa_{\mathfrak{r}} - L| \geq \epsilon\right\}\right| = 0; \quad \mathfrak{i}, \mathfrak{r} \in \mathbb{N}.$$

Then, in this context, we say

$$|\varkappa_{\mathfrak{r}} - L| < \epsilon \quad \text{for almost all } \mathfrak{r} \in \mathbb{N}$$

or

$$\mathfrak{st} - \lim \varkappa_{\mathfrak{r}} = L. \tag{2.1}$$

Gadjiev and Orhan were the pioneers in analyzing convergence in statistical sense in approximation theory using Korovkin's approximation theory. Their study focused on approximating a function, specifically addressing the problem of function $\mathfrak{z}$ approximation using the sequence of positive linear operators $(B_{\mathfrak{i}}(\mathfrak{z};\varkappa))$ [18, 19]. They stated the Korovkin's approximation Theorem 2.1 using convergence in statistical sense of a sequence of positive linear operators using the notations: Consider $C_M[\mathfrak{c},\mathfrak{d}]$ as a space of all continuous functions $\mathfrak{z}$ in the interval $[\mathfrak{c},\mathfrak{d}]$ and bounded on the entire line, i.e.,

$$\left|\mathfrak{z}(\varkappa)\right| \leq M_{\mathfrak{z}}, \quad -\infty < \varkappa < \infty,$$

where $M_{\mathfrak{z}}$ is constant for every $\mathfrak{z}$. Let $(B_{\mathfrak{i}})$ be a sequence of positive linear operators from $C_M[\mathfrak{c},\mathfrak{d}]$ to $B[\mathfrak{c},\mathfrak{d}]$, where $B[\mathfrak{c},\mathfrak{d}]$ is a Banach space of all bounded functions on $[\mathfrak{c},\mathfrak{d}]$, with norm $\|.\|_B := \sup_{\mathfrak{c}<\varkappa<\mathfrak{d}} |.|$.

Theorem 2.1 ([20]) *If a sequence of positive linear operators $B_{\mathfrak{i}} : C_M[\mathfrak{c},\mathfrak{d}] \to B[\mathfrak{c},\mathfrak{d}]$ satisfies the conditions*

$$\mathfrak{st} - \lim\left\|B_{\mathfrak{i}}(e_0;.) - 1\right\|_B = 0, \tag{2.2}$$

$$\mathfrak{st} - \lim\left\|B_{\mathfrak{i}}(e_1;.) - e_1\right\|_B = 0, \tag{2.3}$$

$$\mathfrak{st} - \lim\left\|B_{\mathfrak{i}}(e_2;.) - e_2\right\|_B = 0, \tag{2.4}$$

then for any function $\mathfrak{z} \in C_M[\mathfrak{c},\mathfrak{d}]$, we get

$$\mathfrak{st} - \lim\left\|B_{\mathfrak{i}}(\mathfrak{z};.) - \mathfrak{z}\right\|_B = 0, \tag{2.5}$$

where $e_i(\varkappa) = \varkappa^i$ $\forall i = 0, 1, 2$.

Convergence in statistical sense has been obtained of many positive linear operators, like the Bernstein operator [21], Meyer–König and Zeller operators [22], general Beta operators [23], a generalization of the certain positive linear operator (Meyer–König and Zeller operators, Bleimann, Butzer and Hahn operators and Szász operators) [24], and many more.

In addition, Karakaya and Chishti [25] proposed the idea of convergence in weighted statistical sense, the definition of which was later refined by Mursaleen et al. [26]. This research investigates the convergence in statistical sense of the modified Sz'asz–Mirakyan operators in the Kantorovich form [27]. Furthermore, we analyze the weighted convergence in statistical sense and rate of convergence of these operators in the Lipschitz space. As defined in [28], "let the function $\mathfrak{z}$ be defined on the interval $[0,\infty)$. $S_{\mathfrak{i}}$, the Szász–Mirakyan operator applied to $\mathfrak{z}$ is"

$$S_{\mathfrak{i}}(\mathfrak{z};\varkappa) = \sum_{\mathfrak{r}=0}^{\infty} \mathfrak{z}\left(\frac{\mathfrak{r}}{\mathfrak{i}}\right) p_{\mathfrak{r}}(\mathfrak{i}\varkappa), \tag{2.6}$$

where

$$p_{\mathfrak{r}}(\mathfrak{u}) = e^{-\mathfrak{u}}\frac{\mathfrak{u}^{\mathfrak{r}}}{\mathfrak{r}!}, \quad \mathfrak{u}\in[0,\infty).$$

In 1977, Jain and Pethe [29] generalized (2.6) as:

$$\begin{aligned} S_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa) &= \sum_{\mathfrak{r}=0}^{\infty}(1+\mathfrak{i}\nu)^{-\frac{\varkappa}{\nu}}\left(\nu+\frac{1}{\mathfrak{i}}\right)^{-\mathfrak{r}}\frac{\varkappa^{(\mathfrak{r},-\nu)}}{\mathfrak{r}!}\mathfrak{z}\left(\frac{\mathfrak{r}}{\mathfrak{i}}\right) \\ &= \sum_{\mathfrak{r}=0}^{\infty} s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\mathfrak{z}\left(\frac{\mathfrak{r}}{\mathfrak{i}}\right), \end{aligned} \tag{2.7}$$

where

$$s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa) = (1+\mathfrak{i}\nu)^{-\frac{\varkappa}{\nu}}\left(\nu+\frac{1}{\mathfrak{i}}\right)^{-\mathfrak{r}}\frac{\varkappa^{(\mathfrak{r},-\nu)}}{\mathfrak{r}!},$$

$$\varkappa^{(\mathfrak{r},-\nu)} = \varkappa(\varkappa+\nu)\cdots\big(\varkappa+(\mathfrak{r}-1)\nu\big), \quad \varkappa^{(0,-\nu)} = 1,$$

and $\mathfrak{z}$ is any function of exponential type such that

$$\left|\mathfrak{z}(\mathfrak{u})\right| \le \mathfrak{K}e^{\mathfrak{A}\mathfrak{u}}(\mathfrak{u}\ge 0),$$

for some finite constants $\mathfrak{K},\mathfrak{A}>0$. Here $\nu=(\nu_{\mathfrak{i}})_{\mathfrak{i}\in\mathbb{N}}$ is such that

$$0\le\nu_{\mathfrak{i}}\le\frac{1}{\mathfrak{i}}.$$

Therefore, for any bounded and integrable function $\mathfrak{z}$ defined on $[0,\infty)$, Dhamija et al. [27] modified the operator (2.7) in Kantorovich form as:

$$D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa) = \mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty} s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}\mathfrak{z}(\mathfrak{u})\,d\mathfrak{u}. \tag{2.8}$$

The objective of this paper is to study the statistical approximation properties and rate of convergence of the modified Kantorovich operator (2.8).

In this paper, we have used the following methodology:

1. First, take all the functions from $C_M[\mathfrak{c},\mathfrak{d}]$, where $C_M[\mathfrak{c},\mathfrak{d}]$ is the space of all continuous functions $\mathfrak{z}$ in the interval $[\mathfrak{c},\mathfrak{d}]$.

2. Apply Kantorovich form of modified Szász–Mirakyan operators $D_{\mathfrak{i}}^{[\nu]}$, defined by (2.8) on the continuous functions.

3. Calculate the auxiliary results for the test functions $\mathfrak{z}(\varkappa) = 1, \varkappa, \varkappa^2$ using Korovkin theorem, which proves it as a linear positive operator.

4. Determine the rate of convergence through the modulus of continuity, also in Lipshitz space.

5. Also, find statistical limits to determine convergence in statistical sense via $D_{\mathfrak{i}}^{[\nu]}$.

6. Lastly, Estimate the results in classical and weighted space.

The following algorithm provides the method for selecting the function from $C_M[\mathfrak{c},\mathfrak{d}]$ according to our requirements.

Algorithm

Input all functions $\mathfrak{z} \in C_M[\mathfrak{c},\mathfrak{d}]$

$\Downarrow$

Input the operator $D_{\mathfrak{i}}^{[\nu]}$

$\Downarrow$

Collect the auxiliary results for the test functions $\mathfrak{z}(\varkappa) = 1, \varkappa, \varkappa^2$

$\Downarrow$

Calculate statistical limit using these auxiliary results

$\Downarrow$

Compute convergence in statistical sense in classical and weighted space using these statistical limits and estimate the results.

3 Basic results

To prove the convergence of operator in statistical sense (2.8), following results are required:

Lemma 3.1 ([27]) *For Kantorovich operator* (2.8), *the following holds*:

$$D_{\mathfrak{i}}^{[\nu]}(e_0;\varkappa) = 1, \qquad L_n^{[\nu]}(e_1;\varkappa) = \varkappa + \frac{1}{2\mathfrak{i}},$$

and

$$D_{\mathfrak{i}}^{[\nu]}(e_2;\varkappa) = \varkappa^2 + \left(\nu + \frac{2}{\mathfrak{i}}\right)\varkappa + \frac{1}{3\mathfrak{i}^2}.$$

From Lemma 3.1, we can imply

Lemma 3.2 *For Kantorovich operator* (2.8), *the following holds*:

$$\mathfrak{st} - \lim \left\| D_{\mathfrak{i}}^{[\nu]}(e_0;.) - 1 \right\|_B = 0,$$

$$\mathfrak{st} - \lim \left\| D_{\mathrm{i}}^{[\nu]}(e_1;.) - e_1 \right\|_B = 0,$$
$$\mathfrak{st} - \lim \left\| D_{\mathrm{i}}^{[\nu]}(e_2;.) - e_2 \right\|_B = 0,$$

where $e_i(\varkappa) = \varkappa^i \ \forall i = 0, 1, 2.$

Proof Using $D_{\mathrm{i}}^{[\nu]}(e_0;\varkappa) = 1$, we can clearly say that

$$\mathfrak{st} - \lim \left\| D_{\mathrm{i}}^{[\nu]}(e_0;.) - 1 \right\|_B = 0.$$

Now, by Lemma 3.1, we have

$$\left\| D_{\mathrm{i}}^{[\nu]}(e_1;.) - e_1 \right\|_B = \left\| \varkappa + \frac{1}{2\mathrm{i}} - \varkappa \right\|_B = \left| \frac{1}{2\mathrm{i}} \right|. \tag{3.1}$$

Clearly,

$$\mathfrak{st} - \lim_{\mathrm{i}} \left(\frac{1}{2\mathrm{i}} \right) = 0.$$

Hence,

$$\mathfrak{st} - \lim \left\| L_n^{[\nu]}(e_1;.) - e_1 \right\|_B = 0.$$

Lastly, again by Lemma 3.1, we have

$$\begin{aligned}
\left\| D_{\mathrm{i}}^{[\nu]}(e_2;.) - e_2 \right\|_B &= \left\| \varkappa^2 + \left(\nu + \frac{2}{\mathrm{i}} \right) \varkappa + \frac{1}{3\mathrm{i}^2} - \varkappa^2 \right\|_B \\
&\leq \|\varkappa\| \left\| \left(\nu + \frac{2}{\mathrm{i}} \right) \right\| + \left\| \frac{1}{3\mathrm{i}^2} \right\| \\
&\leq \mu \left| \left(\nu + \frac{2}{\mathrm{i}} \right) \right| + \left| \frac{1}{3\mathrm{i}^2} \right| \\
&\leq A \left[\left(\nu + \frac{2}{\mathrm{i}} \right) + \frac{1}{3\mathrm{i}^2} \right],
\end{aligned}$$

where $A = \max\{\mu, 1\} = \mu$.

We define the following sets, for a given $\epsilon > 0$,

$$\begin{aligned}
U &= \left\{ \mathrm{i} \in \mathbb{N} : \left\| D_{\mathrm{i}}^{[\nu]}(e_2;.) - e_2 \right\|_B \geq \frac{\epsilon}{A} \right\}, \\
U_1 &= \left\{ \mathrm{i} \in \mathbb{N} : \left(\nu + \frac{2}{\mathrm{i}} \right) \geq \frac{\epsilon}{2A} \right\}, \\
U_2 &= \left\{ \mathrm{i} \in \mathbb{N} : \frac{1}{3\mathrm{i}^2} \geq \frac{\epsilon}{2A} \right\}.
\end{aligned}$$

Clearly, we can see that $U \subseteq U_1 \cup U_2$. Thus, we can say

$$\left| \left\{ \mathrm{i} \in \mathbb{N} : \left\| D_{\mathrm{i}}^{[\nu]}(e_2;.) - e_2 \right\|_B \geq \frac{\epsilon}{A} \right\} \right|$$

$$\leq \left|\left\{\mathfrak{i} \in \mathbb{N} : \left(\nu + \frac{2}{\mathfrak{i}}\right) \geq \frac{\epsilon}{2A}\right\}\right| + \left|\left\{\mathfrak{i} \in \mathbb{N} : \frac{1}{3\mathfrak{i}^2} \geq \frac{\epsilon}{2A}\right\}\right|.$$

Since statistical limit of the right-hand side of the inequality is 0, thus

$$\mathfrak{st} - \lim_{\mathfrak{i}} \left\| D_{\mathfrak{i}}^{[\nu]}(e_2; .) - e_2 \right\|_B = 0$$

So, proof is complete. □

By using the above Lemma 3.2 and Korovkin's approximation Theorem 2.1, the following outcome is obtained:

Theorem 3.3 *If the sequence of positive linear operators $D_{\mathfrak{i}}^{[\nu]}$, defined by (2.8), then for any function $\mathfrak{z} \in C_M[0, \mu] \subset C_M[0, \infty)$ and $\varkappa \in [0, \mu] \subset C[0, \infty)$, where $\mu > 0$, we have*

$$\mathfrak{st} - \lim \left\| D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z}; .) - \mathfrak{z} \right\|_B = 0, \tag{3.2}$$

where $C_M[0, \mu]$ represents the space of all real bounded functions $\mathfrak{z}$ continuous in $[0, \infty)$.

4 Rate of convergence

Let $C_M[0, \infty)$ be the space of all continuous and bounded functions on $[0, \infty)$ and $\varkappa \geq 0$, then the modulus of continuity of $\mathfrak{z}$ is defined as

$$\omega(\mathfrak{z}, \delta) := \sup_{\varkappa, y \in [0,\infty), |\varkappa - y| \leq \delta} \left|\mathfrak{z}(\varkappa) - \mathfrak{z}(y)\right|, \tag{4.1}$$

where $\delta > 0$.

We can see from (4.1), for $\mathfrak{z} \in C_M[0, \infty)$,

$$\lim_{\delta \to 0} \omega(\mathfrak{z}; \delta) = 0.$$

For any $\delta > 0$ and for each $\mathfrak{u}, \varkappa \geq 0$, we have

$$\left|\mathfrak{z}(\mathfrak{u}) - \mathfrak{z}(\varkappa)\right| \leq \omega(\mathfrak{z}, \delta)\left(1 + \frac{|\mathfrak{u} - \varkappa|}{\delta}\right). \tag{4.2}$$

Theorem 4.1 *Let $\mathfrak{z} \in C_M[0, \infty)$. If $D_{\mathfrak{i}}^{[\nu]}$ is defined by (2.8), then we have*

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z}; \varkappa) - \mathfrak{z}(\varkappa)\right| \leq 2\omega(\mathfrak{z}, \sqrt{\delta_{\mathfrak{i}}}), \tag{4.3}$$

where

$$\delta_{\mathfrak{i}}(\varkappa) = \left(\nu + \frac{1}{\mathfrak{i}}\right)\varkappa + \frac{1}{3\mathfrak{i}^2}. \tag{4.4}$$

Proof With Popoviciu's technique, i.e., with (4.2) (see, Theorem 1.6.1 of [30]), by linear property and positivity of the operator $D_{\mathfrak{i}}^{[\nu]}$, we get $\forall \mathfrak{i} \in \mathbb{N}$ and $\varkappa \in [0, \infty)$, that

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z}; \varkappa) - \mathfrak{z}(\varkappa)\right| \leq D_{\mathfrak{i}}^{[\nu]}\left(\left|\mathfrak{z}(\mathfrak{u}) - \mathfrak{z}(\varkappa)\right|; \varkappa\right). \tag{4.5}$$

By using (4.2) in inequality (4.5), we get

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|\le\omega(\mathfrak{z},\delta)\left(1+\frac{D_{\mathfrak{i}}^{[\nu]}(|\mathfrak{u}-\varkappa|;\varkappa)}{\delta}\right). \tag{4.6}$$

Applying Cauchy–Schwartz inequality and from Lemma 3.1, we get from (4.6) that

$$\begin{aligned}\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|&\le\omega(\mathfrak{z},\delta)\left(1+\frac{\mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty}s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)(\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}(\mathfrak{u}-\varkappa)^2\,d\mathfrak{u})^{\frac{1}{2}}(\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}dt)^{\frac{1}{2}}}{\delta}\right)\\&\le\omega(\mathfrak{z},\delta)\left(1+\frac{(D_{\mathfrak{i}}^{[\nu]}((\mathfrak{u}-\varkappa)^2;\varkappa))^{\frac{1}{2}}\times(D_{\mathfrak{i}}^{[\nu]}(1;\varkappa))^{\frac{1}{2}}}{\delta}\right)\\&\le\omega(\mathfrak{z},\delta)\left(1+\frac{\sqrt{(\nu+\frac{1}{\mathfrak{i}})\varkappa+\frac{1}{3\mathfrak{i}^2}}}{\delta}\right).\end{aligned}$$

If we choose $\delta_{\mathfrak{i}}:=(\nu+\frac{1}{\mathfrak{i}})\varkappa+\frac{1}{3\mathfrak{i}^2}$ and $\delta:=\sqrt{\delta_{\mathfrak{i}}}$, the expected result is obtained. □

Notice by Theorem 3.3, we can say that $\mathfrak{st}-\lim_{\mathfrak{i}}\omega(\mathfrak{z},\sqrt{\delta_{\mathfrak{i}}})=0$.

This provides us the pointwise rate of convergence in statistical sense, of the operators $D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)$ to $\mathfrak{z}(\varkappa)$.

Now we analyze the rate of convergence of the operator $D_{\mathfrak{i}}^{[\nu]}$ using functions of the Lipschitz class $\mathrm{Lip}_M(\beta)$, where $M>0$ and $0<\beta\le1$. The function $\mathfrak{z}\in C_M[0,\infty)$ belongs to $\mathrm{Lip}_M(\beta)$ if

$$\left|\mathfrak{z}(\mathfrak{u})-\mathfrak{z}(\varkappa)\right|\le M|\mathfrak{u}-\varkappa|^{\beta};\quad\forall\mathfrak{u},\varkappa\in[0,\infty). \tag{4.7}$$

Theorem 4.2 *Let $D_{\mathfrak{i}}^{[\nu]}$ be defined as in (2.8) and let $\mathfrak{z}\in\mathrm{Lip}_M(\beta)$ with $\beta\in(0,1]$, then*

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|\le M(\delta_{\mathfrak{i}})^{\frac{\beta}{2}}, \tag{4.8}$$

where $\delta_{\mathfrak{i}}$ is defined as above.

Proof By linearity and positivity of the operator $D_{\mathfrak{i}}^{[\nu]}$ and $\mathfrak{z}\in\mathrm{Lip}_M(\beta)$ with $\beta\in(0,1]$, we can write

$$\begin{aligned}\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|&\le D_{\mathfrak{i}}^{[\nu]}\left(\left|\mathfrak{z}(t)-\mathfrak{z}(\varkappa)\right|;\varkappa\right)\\&\le MD_{\mathfrak{i}}^{[\nu]}\left(|t-\varkappa|^{\beta};\varkappa\right).\end{aligned}$$

Thus, we obtain

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|\le M\left(\mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty}s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\int_{\frac{\mathfrak{r}}{n}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}|\mathfrak{u}-\varkappa|^{\beta}\,dt\right). \tag{4.9}$$

Applying Hölder's inequality in (4.9), for $\mathfrak{p}=\frac{2}{\beta}$ and $\mathfrak{q}=\frac{2}{2-\beta}$, we have

$$\left|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\right|\le M\left(\mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty}s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\left(\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}|\mathfrak{u}-\varkappa|^2\,d\mathfrak{u}\right)^{\frac{\beta}{2}}\left(\int_{\frac{\mathfrak{r}}{n}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}}d\mathfrak{u}\right)^{\frac{2-\beta}{2}}\right)$$

$$\le M\left(\mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty} s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}} |\mathfrak{u}-\varkappa|^2\,d\mathfrak{u}\right)^{\frac{\beta}{2}}$$
$$\times\left(\mathfrak{i}\sum_{\mathfrak{r}=0}^{\infty} s_{\mathfrak{i},\mathfrak{r}}^{[\nu]}(\varkappa)\int_{\frac{\mathfrak{r}}{\mathfrak{i}}}^{\frac{\mathfrak{r}+1}{\mathfrak{i}}} d\mathfrak{u}\right)^{\frac{2-\beta}{2}}$$
$$\le M\big(D_{\mathfrak{i}}^{[\nu]}\big((\mathfrak{u}-\varkappa)^2;\varkappa\big)\big)^{\frac{\beta}{2}}\big(D_{\mathfrak{i}}^{[\nu]}(1;\varkappa)\big)^{\frac{2-\beta}{2}}.$$

Using Lemma 3.1 and taking $\delta_{\mathfrak{i}} = (\nu+\frac{1}{\mathfrak{i}})\varkappa + \frac{1}{3\mathfrak{i}^2}$, we get

$$\big|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};\varkappa)-\mathfrak{z}(\varkappa)\big| \le M(\delta_{\mathfrak{i}})^{\frac{\beta}{2}},$$

which proves the theorem. □

So, Theorem 4.1 and Theorem 4.2 give us the rate of convergence of operators $D_{\mathfrak{i}}^{[\nu]}$ to $\mathfrak{z}$.

5 Statistical convergence-weighted

This section focuses on studying the properties of the weighted approximation of $D_{\mathfrak{i}}^{[\nu]}$ using the weighted Korovkin-type theorem proposed by Gadjiev in [19]. The aim is to obtain approximation properties on infinite intervals. In the context of this study, the following notations are used for $\rho(\varkappa) = 1+\varkappa^2$.

Let B_ρ denote the set of all functions $\mathfrak{z}$ defined on $[0,\infty)$ that satisfy the condition $|\mathfrak{z}(\varkappa)| \le M_{\mathfrak{z}}\rho(\varkappa)$, where $M_{\mathfrak{z}}$ is a constant associated with each $\mathfrak{z}$. Consider C_ρ as the subspace of continuous functions in the space B_ρ. Additionally, let C_ρ^* be the subspace of functions $\mathfrak{z}\in C_\rho$ for which the finite limit of $\lim_{\varkappa\to\infty}\frac{\mathfrak{z}(\varkappa)}{\rho(\varkappa)}$ exists.

The space C_ρ^* can be regarded as a linear normed space with the norm defined as:

$$\|\mathfrak{z}\|_\rho = \sup_{\varkappa\ge 0}\frac{|\mathfrak{z}(\varkappa)|}{\rho(\varkappa)}. \tag{5.1}$$

In this section, the norm defined in (5.1) is used.

Now, for convergence in weighted statistical sense, we recollect Gadjiev's stated theorem in [19] as follows:

Theorem 5.1 *Let $B_{\mathfrak{i}}$ be a positive linear operators sequence from $C_\rho \to C_\rho$ (or B_ρ), which satisfies the following conditions*:

$$\lim_{\mathfrak{i}\to\infty}\big\|B_{\mathfrak{i}}(e_r;.)-e_r\big\|_\rho = 0; \quad r = 0,1,2,$$

then

$$\lim_{\mathfrak{i}\to\infty}\big\|B_{\mathfrak{i}}(\mathfrak{z};.)-\mathfrak{z}\big\|_\rho = 0$$

for any function $\mathfrak{z}\in C_\rho^0$, where $e_r = \varkappa^r$.

Lemma 5.2 ([19]) *For a positive linear operators sequence, we have* $B_{\mathfrak{i}} : C_\rho \to B_\rho$ *if and only if*

$$\|B_{\mathfrak{i}}(\rho;.)\| \leq M_\rho,$$

where M_ρ *is a constant that depends only on* ρ.

So, for the operators $D_{\mathfrak{i}}^{[\nu]}$ defined by (2.8), we obtain the main result.

Theorem 5.3 *Let* $D_{\mathfrak{i}}^{[\nu]}$ *be the sequence of positive linear operators defined by* (2.8), *then for all* $\mathfrak{z} \in C_\rho^{\circ}$, *we have*

$$\lim_{\mathfrak{i}\to\infty} \|D_{\mathfrak{i}}^{[\nu]}(\mathfrak{z};.) - \mathfrak{z}\| = 0,$$

where $\rho(\varkappa) = 1 + \varkappa^2$.

Proof To show this, we can prove the condition of Theorem 5.1.

First, we need to show that $D_{\mathfrak{i}}^{[\nu]} : C_\rho \to B_\rho$.

Since $\rho(\varkappa) = 1 + \varkappa^2$, so by using Lemma 3.1

$$\begin{aligned}\|D_{\mathfrak{i}}^{[\nu]}(\rho;.)\|_\rho &\leq \|D_{\mathfrak{i}}^{[\nu]}(1;.)\|_\rho + \|D_{\mathfrak{i}}^{[\nu]}(\varkappa^2;.)\|_\rho \\ &= \sup_{\varkappa\in[0,\infty)} \left(\frac{|1 + \varkappa^2 + (\nu + \frac{2}{\mathfrak{i}})\varkappa + \frac{1}{3\mathfrak{i}^2}|}{1 + \varkappa^2} \right) \\ &\leq 1.\end{aligned}$$

So, there exists a positive constant M such that $M < 1$. Hence,

$$\|D_{\mathfrak{i}}^{[\nu]}(\rho;.)\| \leq M.$$

Thus, by using Lemma 5.2, we have $D_{\mathfrak{i}}^{[\nu]} : C_\rho \to B_\rho$ follows.

Now, since

$$D_{\mathfrak{i}}^{[\nu]}(e_0;\varkappa) = 1,$$

we can clearly say that

$$\mathfrak{st} - \lim_{\mathfrak{i}\to\infty} \|D_{\mathfrak{i}}^{[\nu]}(e_0;.) - e_0\|_\rho = 0.$$

Next, by using Lemma 3.1, we have

$$\begin{aligned}\|D_{\mathfrak{i}}^{[\nu]}(e_1;.) - e_1\|_\rho &= \left(\sup_{\varkappa\in[0,\infty)} \frac{|D_{\mathfrak{i}}^{[\nu]}(e_1;\varkappa) - \varkappa|}{\varkappa^2 + 1} \right) \\ &\leq \varkappa + \frac{1}{2\mathfrak{i}} - \varkappa \\ &= \frac{1}{2\mathfrak{i}}.\end{aligned}$$

Clearly, from (3.2)

$$\mathfrak{st}-\lim_{\mathrm{i}}\left(\frac{1}{2\mathrm{i}}\right)=0.$$

So,

$$\lim_{\mathrm{i}\to\infty}\left\|D_{\mathrm{i}}^{[\nu]}(e_1;.)-e_1\right\|_{\rho}=0.$$

Lastly, again from Lemma 3.1,

$$\begin{aligned}\left\|D_{\mathrm{i}}^{[\nu]}(e_2;.)-e_2\right\|_{\rho}&=\sup_{\varkappa\in[0,\infty)}\left(\frac{|D_{\mathrm{i}}^{[\nu]}(e_2;.)-\varkappa^2|}{1+\varkappa^2}\right)\\&\leq\left|D_{\mathrm{i}}^{[\nu]}(e_2;.)-\varkappa^2\right|\\&=\left(\nu+\frac{2}{\mathrm{i}}\right)\varkappa+\frac{1}{3\mathrm{i}^2}.\end{aligned}$$

Again, by using (3.2), we get

$$\mathfrak{st}-\lim_{\mathrm{i}\to\infty}\left[\left(\nu+\frac{2}{\mathrm{i}}\right)\varkappa+\frac{1}{3\mathrm{i}^2}\right]=0.$$

So,

$$\mathfrak{st}-\lim_{\mathrm{i}\to\infty}\left\|D_{\mathrm{i}}^{[\nu]}(e_2;.)-e_2\right\|_{\rho}=0.$$

Hence, by Theorem 5.1, our proof completes. □

6 Applications of modified Kantorovich operators

In this section, we have reviewed various applications in previously published papers on the applications of Kantorovich form of modified Szász–Mirakyan operators to demonstrate the significance and potential uses of the operators that have been developed and used in this research work. Our results can potentially be used for applications in the areas with analogous trends, which are discussed below.

6.1 Applications in the area of convergence in sustainability

In the cited work, [31] Turturean et al. explain the convergence in the long-term viability of the economies of the EU's constituent nations. The sustainability and economic policy factors have been examined in terms of both beta and sigma convergence. In order to estimate the beta equation, conditional beta convergence takes into account both absolute convergence and the factors that influence economic growth. Baumol established a methodology for the analyzing beta convergence in 1986 [32], and Sala-i-Martin proposed the idea of sigma convergence for the very first time in his PhD dissertation in 1990 [33]. This was also the very first time that it was employed. The phrase "sigma convergence" refers to the gradual shrinking of the difference between the mean of a set of countries or regions and their means over time. The phrase "sigma convergence" refers to the gradual reduction over time of the gap between the mean of a collection of countries or regions and their means.

However, as discussed in [34], it is not possible to calculate or measure limits or statistical limits with absolute precision. Various mathematical methodologies, like fuzzy set theory, fuzzy logic, interval analysis, set-valued analysis, etc., have been created to reflect and describe this imprecision. Among these methods is the neoclassical analysis. Fuzzy concepts, such as fuzzy limits, fuzzy continuity, and fuzzy derivatives, are applied to study various ordinary analysis structures, including functions, sequences, series, and operators. In neoclassical analysis, for instance, the set of fuzzy continuous functions encompasses the set of continuous functions studied in classical analysis. The techniques of traditional calculus are extended by neoclassical analysis to account for uncertainties that exist in computations and observations.

6.2 Applications in the area of medical diagnosis

Costarelli and Vinti [35] utilized sampling Kantorovich operators in enhancing the diagnosis of certain vascular apparatus disorders in the medical field. A concrete example is provided by processing a section of a CT (computerized tomography) image representing theaorta artery. The family of bivariate sampling Kantorovich operators permits picture reconstruction and enhancement. A precise diagnosis of vascular apparatus pathology can be made using augmented biomedical imaging. The region of interest is the vessel's lumen, which is essential from a medical standpoint since it helps doctors identify thrombotic zones(areas of blood clot)from the vessel's lumen and correct diagnoses of diseases. The aforementioned problem could be solved by using a contrast medium, which is used to improve images of the inside of the body in CT investigations; however, we can make a more accurate diagnosis from the original CT pictures taken without contrast media, as contrast medium is too invasive to utilize. The primary objective of image processing is to highlight the lumen in the vessel (Fig. 2), which is delineated by the red square (240240 pixels) on the CT image (Fig. 1). Figure 3 depicts the augmented image produced by operators. The increase of the final image relative to the original image indicates that the final image has been built with twice the resolution (960960 pixels). Figure 3 is generally more detailed than the image in Fig. 2. The image reconstructed using Kantorovich algorithm depicts the lumen of the blood artery more accurately. Enhancing images with sampling Kantorovich operators is extremely beneficial from a medical standpoint, enabling doctors to make more accurate diagnoses.

7 Findings and implications

The convergence in statistical sense of Kantorovich form of modified Szász–Mirakyan operators has been obtained in classical and weighted space. The finding is important in medical applications and traditional mathematics; one way to get a close approximation of the Riemann integrable functions is through the use of the Kantorovich modification of positive linear operators. According to the discussion of Sect. 6, the use of Kantorovich operators is tremendously helpful from a medical standpoint; in this work, the rate of convergence has been approximated through modulus of continuity. The obtained results in this study can be compared with other approximation results that were obtained using different tools for the same operators.

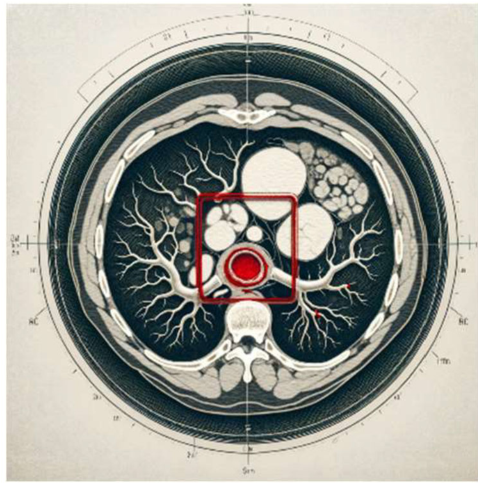

Figure 1 Depicting a CT scan of the abdomen without contrast medium, with a red square highlighting the aorta artery

Figure 2 Focusing on the ROI of the CT image, specifically on the aorta artery, enhanced to better visualize the area

8 Conclusion

We examined the convergence in statistical sense of the modified Szász–Mirakyan operators in the Kantorovich form. The rate of convergence of the operators is determined for those functions that are continuous and bounded on the interval $[0,\infty)$ as well as those that belong to the Lipschitz class. Functional analysis is significantly aided by the contributions made by the theory and practice of summability. Therefore, it would be worthwhile to investigate various convergence in statistical sense and summability approaches for the operators. Additionally, we also consider the topic of convergence in statistical sense of the operators in a weighted space.

The potential future applications of this study might include looking at the convergence in statistical sense rate of a Bézier variation of the operators that have been established, as well as their respective blended forms in weighted space. With the assistance of functions

Figure 3 Showing an enhanced version of Fig. 2, Kantorovich operators is used to enhance Fig. 2 with more clarity and detail

that belong to the Lipschitz class $\mathrm{Lip}_M()$. The discovery has implications in both traditional mathematics and medical applications; the Kantorovich modification of positive linear operators is one way to closely approximate the Riemann integrable functions. The finding is useful in both medical applications and traditional mathematics.

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