Agwu et al. *Journal of Inequalities and Applications* (2024) 2024:58
https://doi.org/10.1186/s13660-024-03120-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

# Fixed point results involving a finite family of enriched strictly pseudocontractive and pseudononspreading mappings

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**Abstract**

In this study, we introduce a method for finding common fixed points of a finite family of $(\eta_i, k_i)$-enriched strictly pseudocontractive (ESPC) maps and $(\eta_i, \beta_i)$-enriched strictly pseudononspreading (ESPN) maps in the setting of real Hilbert spaces. Further, we prove the strong convergence theorem of the proposed method under mild conditions on the control parameters. Our main results are also applied in proving strong convergence theorems for $\eta_i$-enriched nonexpansive, strongly inverse monotone, and strictly pseudononspreading maps. Some nontrivial examples are given, and the results obtained extend, improve, and generalize several well-known results in the current literature.

**Mathematics Subject Classification:** 47H09; 47H10; 47J05; 65J15

**Keywords:** Variational inequality; Enriched nonlinear map; Pseudocontractive; Quasi-nonexpansive maps; Hilbert space

## 1 Introduction

The process of finding solutions to real-life problems has continually defied human intellect. To develop an approach for solving nonlinear problems, fixed point algorithmic technique has emerged as one of the indispensable tools. Consequently, it has drawn the attention of well-established mathematicians all over the world (see, for instance, [1–9] and the references therein), considering the vast applications of results obtained through this means in diverse fields, from pure mathematics to engineering to applied mathematics.

In this paper, we assume that $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, is a real Hilbert space, and $\emptyset \neq \Omega \subset \mathcal{H}$ is closed and convex; $\mathcal{N}$ and $\mathcal{R}$ will represent the set of all positive integers and the set of real numbers, respectively. If $\Im : \Omega \longrightarrow \Omega$ is a nonlinear map, then $F(\Im) = \{\wp \in \Omega : \Im\wp = \wp\}$ will denote the set of fixed point of $\Im$.

**Definition 1.1** Recall that the map $\Im : \Omega \longrightarrow \Omega$

1. is known as nonexpansive if

$$\|\Im\psi - \Im\wp\| \leq \|\psi - \wp\|, \quad \forall \psi, \phi \in \Omega; \tag{1.1}$$

2. is called quasi-nonexpansive if $F(\Im) \neq \emptyset$ and $\forall(\psi, \vartheta) \in \Omega \times F(\Im)$, we have

$$\|\Im\psi - \vartheta\| \leq \|\psi - \vartheta\|; \tag{1.2}$$

3. is called nonspreading [10] if $\forall \psi, \wp \in \Omega$, we have

$$2\|\Im\psi - \Im\wp\|^2 \leq \|\Im\psi - \wp\|^2 + \|\Im\wp - \psi\|^2; \tag{1.3}$$

4. is called $\beta$-strictly pseudononspreading [11] if there exists $\beta \in [0, 1)$ such that $\forall \psi, \wp \in \Omega$,

$$\|\Im\psi - \Im\wp\|^2 \leq \|\psi - \wp\|^2 + \beta\|\psi - \Im\psi - (\wp - \Im\wp) + 2\langle \wp - \Im\wp, \psi - \Im\rangle. \tag{1.4}$$

It is not difficult to show that (1.3) is equivalent to

$$\|\Im\psi - \Im\wp\|^2 \leq \|\psi - \wp\|^2 + 2\langle \wp - \Im\wp, \psi - \Im\rangle. \tag{1.5}$$

*Remark* 1.1 It is easy to see from Definition 1.1 [(3)and(4)] that

(a) if (1.3) holds and $F(\Im) \neq \emptyset$, then (1.2) surfaces immediately for all $\vartheta \in F(\Im)$;
(b) if (1.3) holds, then (1.4) holds with $\beta = 0$. However, the opposite is not true, as shown in the following example.

*Example* 1.1 Let $\Im : \mathcal{R} \longrightarrow \mathcal{R}$ be defined by

$$\Im\psi = \begin{cases} \psi, & \psi \in (-\infty, 0], \\ -2\psi, & \psi \in [0, \infty), \end{cases}$$

with the usual norm. Then, $\Im$ satisfies (1.4), but not (1.3). Thus, the class of maps satisfying (1.4) is more general than that of (1.3).

In 2011, Osilike and Isiogugu [11] initiated the concept of $\beta$-strictly pseudononspreading (SPN) maps and established weak convergence result of Bailion-type similar to that obtained in [10] and [12]. In addition, using the notion of mean convergence, they obtained strong convergence results similar to the those established in [10] and thus resolved an open problem posed by Kurokawa and Takahashi [10] for the case where the map $\Im$ is averaged.

A map $\Im : \mathcal{K} \longrightarrow \mathcal{H}$ is called $\vartheta$-inverse strongly monotone if there exists a positive number $\vartheta$ such that

$$\langle \psi - \Im\wp, \psi - \wp\rangle \geq \vartheta\|\Im\psi - \Im\wp\|^2, \quad \forall \psi, \wp \in \mathcal{K}. \tag{1.6}$$

Finding fixed points of nonexpansive, nonspreading, strictly pseudononspreading, and strictly pseudocontractive maps is an important topic in fixed point theory, and they have far-reaching applications in applied areas such as signal processing [13], split feasibility [14], and convex feasibility problems [15]. Subsequently, as an important generation of the above-mentioned maps, the notion of enriched nonlinear maps was first introduced by Berinde [16] (see also [17] and [5]) in the setup of a real Hilbert space. This concept was later extended to the more general Banach space by Saleem [18].

**Definition 1.2** A map $\Im : \Omega \longrightarrow \Omega$ is called $\Psi_{\Im}$-enriched Lipschitizian (or $(\eta, \Psi_{\Im})$-enriched Lipshitizian) (see [18]) if for all $\psi, \phi \in \Omega$, there exist $\eta \in [0, +\infty)$ and a continuous nondecreasing function $\Psi_{\Im} : R^{+} \longrightarrow R^{+}$, with $\Psi_{\Im}(0) = 0$, such that

$$\left\| \eta(\psi - \phi) + \Im\psi - \Im\phi \right\| \leq (\eta + 1)\Psi_{\Im}\big(\|\psi - \phi\|\big). \tag{1.7}$$

The following special cases emanating from inequality (1.7) are worth mentioning:

(i) if $\eta = 0$, inequality (1.7) reduces to a class of maps known as $\Psi_{\Im}$-Lipschitizian;

(ii) if $\eta = 0$ and $\Psi(t) = Lt$, for $L > 0$, then (1.7) reduces to a class of maps called $L$-Lipschitizian with $L$ as the Lipschitz constant. In a more special case where $\eta = 0$, $\Psi_{\eta}(t) = Lt$ and $L = 1$, $\Psi_{\Im}$-enriched Lipschitizian map immediately reduces to the class of nonexpansive maps on $\Omega$;

(iii) if $\Psi_{\Im}(s) = 1$, then inequality (1.7) becomes

$$\left\| \eta(\psi - \phi) + \Im\psi - \Im\phi \right\| \leq (\eta + 1)\|\psi - \phi\|, \tag{1.8}$$

which known as an $\eta$-enriched nonexpansive map. This class of maps was first studied by Berinde [5, 17] as a generalization of a well-known class of maps called nonexpansive.

Note that if $\Psi_{\Im}$ is not necessarily nondecreasing and satisfies the condition

$$\Psi_{\Im}(t) < t, \quad t > 0,$$

then we have the class of $\eta$-enriched contraction maps.

**Definition 1.3** A map $\Im$ is known as $(\eta, k)$-ESPC (see [18]) if for all $\psi, \phi \in \Omega$, there exist $\eta \in [0, +\infty)$ and $j(\psi - \phi) \in J(\psi - \phi)$ such that

$$\begin{aligned}&\big\langle \eta(\psi - \phi) + \Im\psi - \Im\phi, j\big((\eta + 1)(\psi - \phi)\big)\big\rangle \\ &\quad \leq (\eta + 1)^2\|\psi - \phi\|^2 - k\left\|\psi - \phi - (\Im\psi - \Im\phi)\right\|^2,\end{aligned} \tag{1.9}$$

where $k = \frac{1}{2}(1 - \lambda)$ for some $\lambda \in [0, 1)$.

In the setup of a real Hilbert space, inequality (1.9) is equivalent to the following:

$$\left\| \eta(\psi - \phi) + \Im\psi - \Im\phi \right\|^2 \leq (\eta + 1)^2\|\psi - \phi\|^2 + \lambda\left\|\psi - \phi - (\Im\psi - \Im\phi)\right\|^2, \tag{1.10}$$

where $\lambda = 1 - 2k$.

In [18], Saleem et al. established that if $\Omega$ is a bounded close and convex subset of a real Banach space and $\Im : \Omega \longrightarrow \Omega$ is a finite family of $(\eta, k)$-ESPC maps, then $\Im$ has a fixed point in $\Omega$.

In 2009, Takahashi and Shimoji [19] initiated the concept of $W$-map developed from $\Im_1, \Im_2, \dots, \Im_t$ and $\alpha_1, \alpha_2, \dots, \alpha_t$ in the following way:

**Definition 1.4** Let $\mathcal{E}$ be a Banach space and $\mathcal{C} \subset \mathcal{E}$ be convex. Let $N \in \mathcal{N}$, $\{\Im_i\}_{i=1}^{N} : \mathcal{C} \longrightarrow \mathcal{C}$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ be real numbers such that $0 \leq \alpha_k \leq 1$ for very $k = 1, 2, \dots, N$. Then, the map $W$ is defined as follows:

$$G_1 = \alpha_1\Im_1 + (1-\alpha)_1 I$$

$$G_2 = \alpha_2\Im_2 G_1 + (1-\alpha)_2 I$$

$$G_3 = \alpha_2\Im_3 G_2 + (1-\alpha)_3 I$$

$$\vdots\underline{a}$$

$$G_{N-1} = \alpha_{N-1}\Im_{N-1}G_{N-2} + (1-\alpha_{N-1})I$$

$$W = G_N = \alpha_N\Im_N G_{N-1} + (1-\alpha_N)I.$$

This is known as $W$-map developed from $\Im_1, \Im_2, \dots, \Im_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

The results obtained using $W$-map in [19, 20] were generalized in [20, 21] through the instrument of $K$-map, while, in [22, 23], the notion of $S$-map was studied and applied in generalizing the main results of [20, 21].

More recently, Ke and Ma [24] introduced and studied the following nonlinear maps:

**Definition 1.5** Let $\mathcal{E}$ be a real Banach space and $\emptyset \neq \mathcal{K} \subset \mathcal{E}$. Let $\{\Im_i\}_{i=1}^N$ be a finite family of maps of $\mathcal{K}$ into itself. For $i = 1, 2, \dots, \mathcal{N}$, let $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0,1]$ and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. The map $D$ is defined as follows:

$$G_0 = I$$

$$G_1 = \alpha_1\Im_1^2 G_0 + \beta_1\Im_1 G_0 + \gamma_1 G_0 + \delta_1 I$$

$$G_2 = \alpha_2\Im_2^2 G_1 + \beta_2\Im_1 G_1 + \gamma_2 G_1 + \delta_2 I$$

$$G_3 = \alpha_3\Im_3^2 G_2 + \beta_3\Im_3 G_2 + \gamma_3 G_2 + \delta_3 I$$

$$\vdots$$

$$G_{N-1} = \alpha_{N-1}\Im_{,N-1}^2 G_{N-2} + \beta_{N-1}\Im_{N-1}G_{N-2} + \gamma_{N-1}G_{N-2} + \delta_{N-1}I$$

$$D = G_N = \alpha_N\Im_N^2 G_{N-1} + \beta_N\Im_N G_{N-1} + \gamma_N G_{N-1} + \delta_N I.$$

This is known as a $G$-map developed from $\Im_1, \Im_2, \dots, \Im_N$ and $\tau_1, \tau_2, \dots, \tau_N$.

Using this map, they obtained the following main results as a generalization of the results in [22, 23].

**Lemma 1.1** [24] *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be $k_i$-strictly pseudocontractive (SPC) maps with $\bigcap_{i=1}^N F(\Im_i) \neq \emptyset$, and let $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0,1]$ with $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. Let $D$ be the $G$-map developed from the sequences $\{\Im_i\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$. If the following conditions are satisfied:*

(a) $\alpha_1 \leq \beta_1 < 1 - k_1$ *and* $(k_1 + \beta_1)\alpha_1 < \beta_1(1 - \alpha_1 - \beta_1)$;

(b) $\beta_i \geq k_i$, $k_i < \gamma_i < 1$ *and* $k_i\alpha_i \leq \beta_i\gamma_i - \beta_i k_i$, *for* $i = 1, 2, \dots, N$,

*then $F(G) = \bigcap_{i=1}^N F(\Im_i)$, and $D$ is nonexpansive.*

**Theorem 1.2** [24] *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be $k_i$-SPC maps and $S : \mathcal{K} \longrightarrow \mathcal{K}$ be a $\beta$-SPN map for $\beta \in [0,1)$. Let $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, for $i = 1,2,\dots,N$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0,1]$ with $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$ such that*

(a) $\alpha_1 \leq \beta_1 < 1 - k_1$ *and* $(k_1 + \beta_1)\alpha_1 < \beta_1(1 - \alpha_1 - \beta_1)$;

(b) $\beta_i \geq k_i$, $k_i < \gamma_i < 1$ *and* $k_i\alpha_i \leq \beta_i\gamma_i - \beta_i k_i$, *for* $i = 1,2,\dots,N$.

*Let $D$ be the G-map generated by the sequences $\{\Im_{\omega,i}\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$, where $\Im_{\omega,i} = (1-\omega)I + \omega\Im_i$. Suppose $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(\Im_i) \neq \emptyset$. Let $\{\wp_n\}$ be a sequence developed from arbitrary $u, \wp_0 \in \mathcal{K}$ by*

$$\begin{cases} \hbar_n = (1-\pi_n)\wp_n + \pi_n P_{\mathcal{K}}(1 - \lambda_n(I-S))\wp_n, \\ w_n = (1-\sigma_n)\wp_n + \sigma_n P_{\mathcal{K}}(1 - \lambda_n(I-S))\hbar_n, \\ \wp_{n+1} = a_n u + b_n w_n + c_n D w_n, \end{cases} \tag{1.11}$$

*where $\{\pi_n\}, \{\sigma_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0, 1-\beta)$ satisfying the conditions*:

(i) $a_n + b_n + c_n = 1$;

(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^\infty a_n = \infty$;

(iii) $\liminf_{n\to\infty} b_n > 0$ *and* $\liminf_{n\to\infty} c_n > 0$;

(iv) $\sum_{n=0}^\infty \lambda_n < \infty$;

(v) $\sum_{n=0}^\infty |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^\infty |\pi_{n+1} - \pi_n| < \infty$, $\sum_{n=0}^\infty |\sigma_{n+1} - \sigma_n| < \infty$, $\sum_{n=0}^\infty |a_{n+1} - a_n| < \infty$, $\sum_{n=0}^\infty |b_{n+1} - b_n| < \infty$, $\sum_{n=0}^\infty |c_{n+1} - c_n| < \infty$.

*Then, $\{\wp_n\}$ converges strongly to $\bar{\wp} = P_{\mathcal{F}}u$.*

Considering the results of Ke and Ma [24] and other results in the reviewed works, the following question arises:

**Question 1.1** Could there be a nonlinear map that contains the $G$-map for which we would obtain the results in [24] as special cases?

Ke and Ma [24] considered the $G$-map and proved Lemma 1.1 and Theorem 1.2 as their main results in [24]. The results they extended and generalized were consistent with those from [22, 23]. In this paper, we first introduce a new class of nonlinear maps called $\eta$-enriched $D$-maps and give some nontrivial examples to demonstrate its existence. Further, we modify the iterative method studied in [24] and after that give an affirmative answer to Question 1.1.

## 2 Preliminaries

Further, in the course of establishing our main results, the following well-known results should help us. Let $\mathcal{H}$ be a real Hilbert space, and let $\{\psi_n\} \subset \mathcal{H}$. We shall represent weak convergence of $\{\psi_n\}$ to $\psi \in \mathcal{H}$ by $\psi_n \rightharpoonup \psi$ and the strong convergence of $\{\psi_n\}$ to $\psi \in \mathcal{H}$ by $\psi_n \to \psi$ as $n \to \infty$, respectively.

**Lemma 2.1** ([11, 24]) *Let $\mathcal{H}$ be a real Hilbert space. Then, the following results are valid*:

(i)

$$\|\wp + \hbar\|^2 = \|\wp\|^2 + 2\langle \wp, \hbar\rangle + \|\hbar\|^2, \quad \forall \hbar, \wp \in \mathcal{H};$$

(ii)

$$\|\hbar+\wp\|^2 \le \|\hbar\|^2 + 2\langle \wp, \hbar+\wp\rangle, \quad \forall \hbar, \wp \in \mathcal{H};$$

(iii)

$$\left\| \sum_{i=0}^{m} \beta_i \wp_i \right\|^2 = \sum_{i=0}^{m} \beta_i \|\wp_i\|^2 - \sum_{i=0}^{m} \beta_i\beta_j \|\wp_i - \wp_j\|^2;$$

(iv) *if* $\{\psi_n\}$ *is a sequence in* $\mathcal{H}$ *such that* $\psi_n \rightharpoonup \wp \in \mathcal{H}$, *then*

$$\limsup_{n\to\infty} \|\psi_n - \hbar\|^2 = \limsup_{n\to\infty} \|\psi_n - \wp\|^2 + \|\wp - \hbar\|^2.$$

**Definition 2.1** Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. The nearest point projection $P_{\mathcal{K}} : \mathcal{H} \longrightarrow \mathcal{K}$ defined from $\mathcal{H}$ onto $\mathcal{K}$ is a operator that assigns to each $\psi \in \mathcal{H}$ its nearest point represented with $P_{\mathcal{K}}\psi$ in $\Omega$. Thus, $P_{\mathcal{K}}$ is the unique point in $\Omega$ such that

$$\|\psi - P_{\mathcal{K}}\psi\| \le \|\psi - \hbar\|, \quad \forall \hbar \in \mathcal{K}.$$

**Lemma 2.2** [24] *Let* $\mathcal{H}$ *be a real Hilbert space,* $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ *be closed and convex and* $P_{\mathcal{K}} : \mathcal{H} \longrightarrow \mathcal{K}$ *be a metric projection. Then,*

(i)

$$\|P_{\mathcal{K}}\wp - P_{\mathcal{K}}\hbar\| \le \langle \wp - \hbar, P_{\mathcal{K}}\wp - P_{\mathcal{K}}\hbar\rangle, \quad \forall \wp, \hbar \in \mathcal{H};$$

(ii) $P_{\mathcal{K}}$ *is a nonexpansive map, that is,* $\|P_{\mathcal{K}}\wp - P_{\mathcal{K}}\hbar\| \le \|\wp - \hbar\|$;

(iii)

$$\langle \wp - P_{\mathcal{K}}\wp, \hbar - P_{\mathcal{K}}\wp\rangle \le 0, \quad \forall \wp, \hbar \in \mathcal{K}.$$

**Lemma 2.3** ([25]) *Let* $\mathcal{H}$ *be a real Hilbert space and* $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ *be closed and convex. Let* $P_{\mathcal{K}} : \mathcal{H} \longrightarrow \Omega$ *be the metric projection of* $\mathcal{H}$ *onto* $\mathcal{K}$. *Let* $\{\psi_n\}$ *be a sequence in* $\mathcal{K}$ *and*

$$\|\psi_{n+1} - \vartheta\| \le \|\psi_n - \vartheta\|, \quad \forall \vartheta \in \mathcal{K}.$$

*Then,* $\{P_{\Omega}\psi_n\}$ *converges strongly.*

**Lemma 2.4** *Let* $\mathcal{H}$ *be a real Hilbert space,* $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ *be closed and convex and* $\Im : \mathcal{K} \longrightarrow \mathcal{K}$ *be a* $\beta$*-ESPN map such that* $F(\Im) \neq \emptyset$. *Let* $\Im_\xi = \xi I + (1-\xi)\Im$, $\xi \in [\beta, 1)$. *Then, the following conclusions hold*:

1. $F(\Im) = F(\Im_\xi)$;
2. $I - \Im_\xi$ *is demiclosed at zero*;
3. $\|\Im\psi - \Im\psi\|^2 \le \|\psi - \phi\|^2 + \frac{2}{1-\xi}\langle \psi - \Im\psi, \phi - \Im\phi\rangle$;
4. $\Im_\xi$ *is quasi-nonexpansive map.*

**Lemma 2.5** ([26, 27]) *Let $\{\nu_n\}$ be a sequence of nonnegative real numbers satisfying*

$$\nu_{n+1} \leq (1-\pi_n)\nu_n + \mu_n,$$

*where $\{\pi_n\}$ and $\{\mu_n\}$ are real sequences such that*
(i) $\{\pi_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \pi_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\mu_n}{\pi_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\mu_n| < \infty$.
*Then*, $\lim_{n\to\infty} \nu_n = 0$.

**Lemma 2.6** ([26, 27]) *Let $\{\nu_n\} \subset [0,+\infty)$ be satisfying*

$$\nu_{n+1} \leq (1-\pi_n)\nu_n + \pi_n\mu_n,$$

*where $\{\pi_n\}$ and $\{\mu_n\}$ are real sequences such that*
(i) $\{\pi_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \pi_n = \infty$;
(ii) $\limsup_{n\to\infty} \mu_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\mu_n| < \infty$.
*Then*, $\lim_{n\to\infty} \nu_n = 0$.

Let $\mathcal{X}$ be a real Banach space. A map $\Im : \mathcal{D}(\Im) \longrightarrow \mathcal{R}(\Im)$, with domain $D(\Im)$ and range $R(\Im)$ in $\mathcal{X}$, is called demiclosed at a point $\vartheta$ (see, for instance, [28]) if whenever $\{\psi_n\}$ is a sequence in $D(\Im)$ such that $\psi_n \rightharpoonup \psi \in D(\Im)$ and $\{\Im\psi_n\}$ converges strongly to $\wp$, then $\Im\psi = \wp$.

**Lemma 2.7** [22] *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex and $\Im : \mathcal{K} \longrightarrow \mathcal{H}$ be a nonexpansive map. Then, the map $I - \Im$ is demiclosed at zero.*

**Lemma 2.8** (*Opial property* [29]) *Let $\mathcal{H}$ be a real Hilbert space. Suppose $\wp_n \rightharpoonup \omega$, then*

$$\liminf_{n\to\infty} \|\wp_n - \hbar\| > \liminf_{n\to\infty} \|\wp_n - \omega\|, \quad \forall \hbar \in \mathcal{H}, \hbar \neq \omega.$$

## 3 Results and discussion

Further, we state the following definition.

**Definition 3.1** Let $\mathcal{H}$ be a real Hilbert space. A map $\Im$ with domain $\mathcal{D}(\Im)$ and range $\mathcal{R}(\Im)$ in $\mathcal{H}$ is known as $(\eta,\beta)$-ESPN in the sense of Browder and Petryshyn [30] if there exist $\eta \in [0,\infty)$ and $\beta \in [0,1)$ such that for all $(\psi,\phi) \in \mathcal{D}(\Im)$,

$$\begin{aligned} &\bigl\|\eta(\psi-\phi) + \Im\psi - \Im\phi\bigr\|^2 \\ &\quad \leq (\eta+1)^2\|\psi-\phi\|^2 + \beta\bigl\|\psi - \Im\psi - (\phi - \Im\phi)\bigr\|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle. \end{aligned} \tag{3.1}$$

Let $\omega = \frac{1}{\eta+1}$, then it is clear that $\omega \in (0,1]$. In this case, inequality (3.1) becomes

$$\begin{aligned} &\left\|\frac{(1-\omega)}{\omega}(\psi-\phi) + \Im\psi - \Im\phi\right\|^2 \\ &\quad \leq \frac{1}{\omega^2}\|\psi-\phi\|^2 + \beta\bigl\|\psi - \Im\psi - (\phi - \Im\phi)\bigr\|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle, \end{aligned}$$

which, on simplification, yields

$$\|\Im_\omega\psi - \Im_\omega\phi\|^2 \leq \|\psi - \phi\|^2 + \beta\big\|\psi - \Im_\omega\psi - (\phi - \Im_\omega\phi)\big\|^2 + 2\langle\psi - \Im_\omega\psi, \phi - \Im_\omega\phi\rangle. \tag{3.2}$$

Inequality (3.2) is equivalently written as

$$\big\langle(I - \Im_\omega)\psi - (I - \Im_\omega)\phi, \psi - \phi\big\rangle \geq \lambda\big\|\psi - \Im_\omega\psi - (\phi - \Im_\omega\phi)\big\|^2 - \langle\psi - \Im_\omega\psi, \phi - \Im_\omega\phi\rangle, \tag{3.3}$$

where $\Im_\omega = (1-\omega)I + \omega\Im$, $\lambda = \frac{1}{2}(1-\beta)$, and $I$ denotes the identity map $\Omega$. Here, it is not difficult to see from (3.2) that the average operator $\Im_\omega$ is $\beta$-SPN.

The following example shows that the class of $(\eta, \beta)$-ESPN maps is larger than the class of $\beta$-SPN maps.

*Example* 3.1 Let $\Im : [-2, 2] \longrightarrow [-2, 2]$ be defined by

$$\Im\psi = -\frac{5}{3}\psi, \quad \psi \in [-2, 2].$$

Then, we have

$$\begin{aligned}
&\big|\eta(\psi - \phi) + \Im\psi - \Im\phi\big|^2 = \left(\eta - \frac{5}{3}\right)|\psi - \phi|^2,\\
&\frac{1}{4}\big|\psi - \Im\psi - (\phi - \Im\phi)\big|^2 = \frac{1}{4}\left|\psi + \frac{5}{3}\psi - \left(\phi + \frac{5}{3}\phi\right)\right|^2 = \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi - \phi|^2,\\
&2\langle\psi - \Im\psi, \phi - \Im\phi\rangle = 2\left\langle\psi + \frac{5}{3}\psi, \phi + \frac{5}{3}\phi\right\rangle = \frac{128}{9}\psi\phi.
\end{aligned}$$

Thus, for $\eta = \frac{5}{3}$, $\beta = \frac{1}{4}$ and $\Phi(\psi, \phi) = (\eta + 1)^2|\psi - \phi|^2 + \frac{1}{4}|\psi - \Im\psi - (\phi - \Im\phi)|^2 + 2\langle\psi - \Im\psi, \phi - \Im\phi\rangle$, we get

$$\begin{aligned}
\Phi(\psi, \phi) &= \frac{64}{9}|\psi - \phi|^2 + \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi - \phi|^2 + \frac{128}{9}\psi\phi\\
&= \frac{64}{9}\big[\psi^2 - 2\psi\phi + \phi^2\big] + \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi - \phi|^2 + \frac{128}{9}\psi\phi\\
&= \frac{64}{9}\big[\psi^2 + \phi^2\big] + \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi - \phi|^2\\
&> 0\\
&= \left|\frac{5}{3}(\psi - \phi) - \frac{5}{3}(\psi - \phi)\right|^2\\
&= \big|\eta(\psi - \phi) + \Im\psi - \Im\phi\big|^2.
\end{aligned}$$

Hence, $\Im$ is $(\frac{5}{3}, \frac{1}{4})$-ESPN map, but $\Im$ is not $\beta$-SPN since for $\psi = \frac{3}{2}$ and $\phi = -\frac{3}{2}$, we obtain

$$\|\Im\psi - \Im\phi\|^2 = \left|\Im\left(\frac{3}{2}\right) - \Im\left(-\frac{3}{2}\right)\right|^2 = \left|-\frac{5}{3}\left(\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right|^2 = \left|-\frac{10}{2}\right|^2 = 25,$$

$$
\begin{aligned}
&|\psi-\phi|^2=\left|\frac{3}{2}-\left(-\frac{3}{2}\right)\right|^2=9,\\
&\beta\left|(I-\Im)\left(\frac{3}{2}\right)-(I-\Im)\left(-\frac{3}{2}\right)\right|^2\\
&\quad=\frac{1}{4}\left|\frac{3}{2}+\frac{5}{3}\left(\frac{3}{2}\right)-\left(\left(-\frac{3}{2}\right)+\frac{5}{3}\left(-\frac{3}{2}\right)\right)\right|^2=\frac{1}{4}|8|^2=16,
\end{aligned}
$$

and

$$
2\left\langle(I-\Im)\left(\frac{3}{2}\right),(I-\Im)\left(-\frac{3}{2}\right)\right\rangle=2\left\langle\frac{3}{2}+\frac{5}{3}\left(\frac{3}{2}\right),\left(-\frac{3}{2}\right)+\frac{5}{3}\left(-\frac{3}{2}\right)\right\rangle=2(4)(-4)=-32.
$$

Therefore,

$$
\begin{aligned}
\|\Im\psi-\Im\phi\|^2&=25>9+16-32\\
&=\|\psi-\phi\|^2+\beta\left\|\psi-\Im\psi-(\phi-\Im\phi)\right\|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle,
\end{aligned}
$$

for $\beta=\frac{1}{4}$.

The examples below demonstrate the conclusion that the class of $(\eta,\lambda)$-ESPC maps and the class of maps studied in this paper are independent.

*Example* 3.2 Let $\Im:\mathcal{R}\longrightarrow\mathcal{R}$ be defined, for each $\psi\in\mathcal{R}$, by

$$
\Im\psi=\begin{cases}0, & \text{if } \psi\in(-\infty,2]\\ 1, & \text{if } \psi\in(2,\infty),\end{cases}
$$

where $\mathcal{R}$ denotes the reals with the usual norm. Then, for all $\psi,\phi\in(-\infty,2]$ and for all $\beta\in[0,1)$, $\Im$ is $(\eta,\beta)$-ESPN map with $\eta=0$ (see [11] for details). However, $\Im$ is not $(\eta,\lambda)$-ESPC map since every $(\eta,\lambda)$-ESPC map satisfies the Lipschitz condition (see, Proposition 3.3 below).

*Example* 3.3 Let $\Im:\mathcal{R}\longrightarrow\mathcal{R}$ be defined, for each $\psi\in\mathcal{R}$, by

$$
\Im\psi=-3\psi, \tag{3.4}
$$

where $\mathcal{R}$ denotes the reals with the usual norm. It is shown in [11] that $\Im$ is $(\eta,\lambda)$-ESPC map with $\eta=0$. Nevertheless, it is not difficult to see that $\Im$ is not $(\eta,\beta)$-ESPN map. Indeed, for $\eta=0$, if $\psi=\frac{1}{2}$ and $\phi=-\frac{1}{2}$, then

$$
\begin{aligned}
\left|\eta(\psi-\phi)+\Im\psi-\Im\phi\right|^2&=9(\eta+1)\\
&=(\eta+1)|\psi-\phi|^2+\left|\psi-\Im\psi-(\phi-\Im\phi)\right|^2\\
&\quad+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle\\
&>(\eta+1)|\psi-\phi|^2+\beta\left|\psi-\Im\psi-(\phi-\Im\phi)\right|^2\\
&\quad+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle,
\end{aligned}
$$

for all $\beta\in[0,1)$.

**Proposition 3.1** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\Im : \mathcal{K} \longrightarrow \mathcal{K}$ be an $(\eta, \beta)$-ESPN map with $F(\Im) \neq \emptyset$. Then,*

$$\left\|\eta\left(\wp_n - \wp^\star\right) + \Im\wp_n - \wp^\star\right\| \leq \frac{(\eta+1)(1+\beta)}{1-\beta}\left\|\wp_n - \wp^\star\right\|, \quad \forall\left(\wp_n, \wp^\star\right) \in \mathcal{K} \times F(\Im).$$

*Proof* Since $\Im$ is an $(\eta, \beta)$-ESPN, we get

$$\begin{aligned}
\left\|\eta\left(\wp_n - \wp^\star\right) + \Im\wp_n - \wp^\star\right\|^2 &\leq (\eta+1)^2\left\|\wp_n - \wp^\star\right\|^2 + \beta\left\|\wp_n - \wp^\star - \left(\Im\wp_n - \Im\wp^\star\right)\right\|^2 \\
&\quad + \left\langle\wp_n - \Im\wp_n, \wp^\star - \Im\wp^\star\right\rangle \\
&= (\eta+1)^2\left\|\wp_n - \wp^\star\right\|^2 \\
&\quad + \beta\left\|(\eta+1)\left(\wp_n - \wp^\star\right) - \left[\eta\left(\wp_n - \wp^\star\right) + \left(\Im\wp_n - \Im\wp^\star\right)\right]\right\|^2 \\
&\quad + \left\langle\wp_n - \Im\wp_n, \wp^\star - \Im\wp^\star\right\rangle \\
&= (\eta+1)^2\left\|\wp_n - \wp^\star\right\|^2 + \beta\left[(\eta+1)^2\left\|\wp_n - \wp^\star\right\|^2\right. \\
&\quad + \left\|\eta\left(\wp_n - \wp^\star\right) + \Im\wp_n - \Im\wp^\star\right\|^2 \\
&\quad \left. - 2(\eta+1)\left\langle\wp_n - \wp^\star, \Im\wp_n - \wp^\star\right\rangle\right],
\end{aligned}$$

from which

$$\begin{aligned}
(1-\beta)\left\|\eta\left(\wp_n - \wp^\star\right) + \Im\wp_n - \wp^\star\right\|^2 &\leq (\eta+1)^2(1+\beta)\left\|\wp_n - \wp^\star\right\|^2 \\
&\quad + 2\beta(\eta+1)\left\|\wp_n - \wp^\star\right\|\left\|\Im\wp_n - \wp^\star\right\|.
\end{aligned} \tag{3.5}$$

Set $C = \|\eta(\wp_n - \wp^\star) + \Im\wp_n - \wp^\star\|$ and $D = \|\wp_n - \wp^\star\|$ in (3.5) so that

$$\begin{aligned}
0 &\geq (1-\beta)C^2 - (\eta+1)^2(1+\beta)D^2 - 2\beta(\eta+1)CD \\
&= (1-\beta)C^2 - (\eta+1)^2(1+\beta)D^2 - \left[(\eta+1)^2(1+\beta)D^2 + \beta(\eta+1)CD\right] \\
&= (1-\beta)C^2 - (\eta+1)^2(1+\beta)D^2 + (\eta+1)CD - \left[(\eta+1)^2(1+\beta)D^2\right. \\
&\quad \left. + \beta(\eta+1)CD + (\eta+1)CD\right] \\
&= (\eta+1)(1-\beta)\left(\frac{C^2}{\eta+1} + CD\right) - \left[(\eta+1)^2(1+\beta)\left(D^2 + \frac{CD}{\eta+1}\right)\right] \\
&= (\eta+1)(1-\beta)C\left(\frac{C}{\eta+1} + D\right) - \left[(\eta+1)^2(1+\beta)D\left(D + \frac{C}{\eta+1}\right)\right].
\end{aligned}$$

The last inequality implies that

$$C \leq \frac{(\eta+1)(1+\beta)}{1-\beta}D, \tag{3.6}$$

and this completes the proof. □

**Proposition 3.2** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\Im : \mathcal{K} \longrightarrow \mathcal{K}$ be an $(\eta, \beta)$-ESPN map with $F(\Im) \neq \emptyset$. Then, $F(\Im) = VI(\mathcal{K}, (I - \Im))$.*

*Proof* It is not difficult to see that $F(\Im) \subseteq VI(\mathcal{K},(I-\Im))$. Let $A = I - \Im$, $\wp \in VI(\mathcal{K},(I-\Im))$ and $\wp^\star \in F(\Im)$. Since $\wp \in VI(\mathcal{K},A)$, it follows that

$$\langle \hbar - \wp, A\wp \rangle \geq 0, \quad \forall \hbar \in \mathcal{K}. \tag{3.7}$$

Since $\Im$ is an $(\eta,\beta)$-ESPN map with $F(\Im) \neq \emptyset$, it follows that

$$\begin{aligned}
\left\| \eta\left(\wp - \wp^\star\right) + \Im\wp_n - \wp^\star \right\|^2 &= \left\| \eta\left(\wp - \wp^\star\right) + (I-A)\wp_n - (I-A)\wp^\star \right\|^2 \\
&= \left\| (\eta+1)\left(\wp - \wp^\star\right) - \left(A\wp_n - A\wp^\star\right) \right\|^2 \\
&= (\eta+1)^2 \left\| \wp - \wp^\star \right\| + \|A\wp_n - A\wp^\star)\|^2 \\
&\quad - 2(\eta+1)\left\langle \wp - \wp^\star, A\wp - A\wp^\star \right\rangle \\
&= (\eta+1)^2 \left\| \wp - \wp^\star \right\| + \|A\wp_n\|^2 \\
&\quad - 2(\eta+1)\left\langle \wp - \wp^\star, A\wp \right\rangle \\
&\leq (\eta+1)^2 \left\| \wp - \wp^\star \right\| + \beta \left\| (I-\Im)\wp - (I-\Im)\wp^\star \right\|^2 \\
&\quad + 2\left\langle \wp - \Im\wp, \wp^\star - \Im\wp^\star \right\rangle \\
&= (\eta+1)^2 \left\| \wp - \wp^\star \right\| + \beta \left\| (I-\Im)\wp \right\|^2,
\end{aligned}$$

from which

$$\begin{aligned}
(1-\beta)\|\wp - \Im\wp\|^2 &\leq 2(\eta+1)\left\langle \wp - \wp^\star, \wp - \Im\wp \right\rangle \\
&= -2(\eta+1)\left\langle \wp^\star - \wp, \wp - \Im\wp \right\rangle \\
&\leq 0 \quad (\text{by } (3.7)).
\end{aligned}$$

Consequently, $\wp \in F(\Im)$ and $VI(\mathcal{K},A) \subseteq F(\Im)$. Hence, $VI(\mathcal{K},A) = F(\Im)$. □

*Remark* 3.1 From Lemma 2.2 and (3.7), we obtain

$$F(\Im) = F\left(P_{\mathcal{K}}\left(I - \lambda(I-\Im)\right)\right), \quad \forall \lambda > 0.$$

**Proposition 3.3** ([31]) *Let $\mathcal{E}$ be a normed space and $\Im : D(\Im) \subseteq \mathcal{E} \longrightarrow \mathcal{E}$ be an $(\eta,k)$-SPC map. Then, $\Im$ is an L-Lipschitizian map.*

*Proof* Since $\Im$ is an $(\eta,k)$-SPC map, $\exists k \in [0,1)$ such that $\forall \psi, \phi \in d(\Im)$,

$$\left\| \eta(\psi - \phi) + \Im\psi - \Im\phi \right\|^2 \leq (\eta+1)^2 \|\psi - \phi\|^2 + k \left\| \psi - \Im\psi - (\phi - \Im\phi) \right\|^2.$$

From the above inequality, we obtain

$$\begin{aligned}
&\left\| \eta(\psi - \phi) + \Im\psi - \Im\phi \right\|^2 \\
&\quad \leq (\eta+1)^2 \|\psi - \phi\|^2 + k \left\| \psi - \Im\psi - (\phi - \Im\phi) \right\|^2 \\
&\quad \leq \left[ (\eta+1)\|\psi - \phi\|^2 + \sqrt{k} \left\| \psi - \Im\psi - (\phi - \Im\phi) \right\| \right]^2
\end{aligned}$$

$$\leq (\eta+1)\|\psi-\phi\|^2+\sqrt{k}\big\|(\eta+1)(\psi-\phi)-\big[\eta(\psi-\phi)+\Im\psi-\Im\phi\big]\big\|$$
$$\leq (\eta+1)\|\psi-\phi\|^2+\sqrt{k}(\eta+1)\|\psi-\phi\|^2+\sqrt{k}\big\|\eta(\psi-\phi)+\Im\psi-\Im\phi\big\|.$$

Therefore,

$$\big\|\eta(\psi-\phi)+\Im\psi-\Im\phi\big\|^2\leq L\|\psi-\phi\|,$$

with $L=\frac{(\eta+1)(1+\sqrt{k})}{1-\sqrt{k}}$. □

**Proposition 3.4** ([31]) *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset\neq\Omega\subset\mathcal{H}$ and $\Im:\Omega\longrightarrow\Omega$ be an $(\eta,\beta)$-ESPN map. Then, $(I-\Im)$ is demiclosed at* 0.

*Proof* Let $\{\psi_n\}$ be a sequence in $\Omega$, which converges weakly to $\vartheta$ and $\{\psi_n-\Im\psi_n\}$ converges strongly to 0. We want to show that $\vartheta\in F(\Im)$. Now, since $\{\psi_n\}$ converges weakly, it is bounded.

For each $\psi\in\mathcal{H}$, define $f:\mathcal{H}\longrightarrow[0,\infty)$ by

$$f(\psi)=\limsup_{n\to\infty}\|\psi_n-\psi\|^2.$$

Then, using Lemma 2.1(iii), we get

$$f(\psi)=\limsup_{n\to\infty}\|\psi_n-\vartheta\|^2+\|\vartheta-\psi\|^2,\quad\forall\psi\in\mathcal{H}.$$

Consequently,

$$f(\psi)=f(\vartheta)+\|\vartheta-\psi\|^2,\quad\forall\psi\in\mathcal{H},$$

and

$$f(\Im_\omega)=f(\vartheta)+\|\vartheta-\Im_\omega\vartheta\|^2=f(\vartheta)+\frac{1}{(\eta+1)^2}\|\vartheta-\Im\vartheta\|^2,\quad\forall\psi\in\mathcal{H}.\tag{3.8}$$

Observe that

$$\begin{aligned}
f(\Im_\omega) &= \limsup_{n\to\infty}\|\psi_n-\Im_\omega\vartheta\|^2\\
&= \limsup_{n\to\infty}\|\psi_n-\Im_\omega\psi_n+\Im_\omega\psi_n-\Im_\omega\vartheta\|^2\\
&= \limsup_{n\to\infty}\big\|\psi_n-\big[(1-\omega)\psi_n+\omega\Im\psi_n\big]+(1-\omega)\psi_n+\omega\Im\psi_n-\big[(1-\omega)\vartheta+\omega\Im\vartheta\big]\big\|^2\\
&= \limsup_{n\to\infty}\big\|\omega(\psi_n-\Im\psi_n)+(1-\omega)(\psi_n-\vartheta)+\omega(\Im\psi_n-\Im\vartheta)\big\|^2\\
&= \limsup_{n\to\infty}\left\|\frac{\eta}{\eta+1}(\psi_n-\vartheta)+\frac{1}{\eta+1}(\Im\psi_n-\Im\vartheta)\right\|^2\\
&= \frac{1}{(\eta+1)^2}\limsup_{n\to\infty}\big\|\eta(\psi_n-\vartheta)+\Im\psi_n-\Im\vartheta\big\|^2\\
&\leq \frac{1}{(\eta+1)^2}\limsup_{n\to\infty}\big[(\eta+1)^2\|\psi_n-\vartheta\|^2+\beta\|\vartheta-\Im\vartheta\|^2\big]
\end{aligned}$$

$$= f(\vartheta) + \frac{\beta}{(\eta+1)^2}\|\vartheta - \Im\vartheta\|^2. \tag{3.9}$$

Then, (3.8) and (3.9) give that

$$(1-\beta)\|\vartheta - \Im\vartheta\| \le 0$$

so that $\vartheta \in F(\Im)$ as required. □

**Proposition 3.5** ([31]) *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \Omega \subset \mathcal{H}$ and $\Im : \Omega \longrightarrow \Omega$ be an $(\eta,\beta)$-ESPN map. Then, $F(\Im)$ is closed and convex.*

*Proof* Let $\{\psi_n\}$ be a sequence in $\Omega$, which converges to $\psi$. We want to show that $\psi \in F(\Im)$. Since

$$\begin{aligned}
\|\Im_\omega\psi - \psi\| &= \omega\|\Im\psi - \psi\| \\
&\le \omega\|\Im\psi - \Im\psi_n\| + \omega\|\psi_n - \psi\| \le \omega\|\psi - \psi_n\| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which follows that $\psi = \Im\psi$. Hence, $\psi \in F(\Im)$.

Next, let $\vartheta_1, \vartheta_2 \in F(\Im)$. We prove that $\lambda\vartheta_1 + (1-\lambda)\vartheta_2 \in F(\Im)$. Set $\wp = \lambda\vartheta_1 + (1-\lambda)\vartheta_2$. Then, $\vartheta_1 - \wp = (1-\lambda)(\vartheta_1 - \vartheta_2)$ and $\vartheta_2 - \wp = \lambda(\vartheta_2 - \vartheta_1)$. Since

$$\begin{aligned}
\omega^2\|\Im\wp - \wp\|^2 &= \|\wp - \Im_\omega\wp\|^2 \\
&= \left\|\lambda\vartheta_1 + (1-\lambda)\vartheta_2 - \Im_\omega\wp\right\|^2 \\
&= \left\|\lambda(\vartheta_1 - \Im_\omega\wp) + (1-\lambda)(\vartheta_2 - \Im_\omega\wp)\right\|^2 \\
&= \lambda\|\vartheta_1 - \Im_\omega\wp\|^2 + (1-\lambda)\|\vartheta_2 - \Im_\omega\wp\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda\left\|(1-\omega)\vartheta_1 + \omega\Im\vartheta_1 - \left[(1-\omega)\wp + \omega\Im\wp\right]\right\|^2 \\
&\quad + (1-\lambda)\left\|(1-\omega)\vartheta_2 + \omega\Im\vartheta_2 - \left[(1-\omega)\wp + \omega\Im\wp\right]\right\|^2 \\
&\quad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \left\|(1-\omega)(\vartheta_1 - \wp) + \omega(\Im\vartheta_1 - \Im\wp)\right\|^2 \\
&\quad + (1-\lambda)\left\|(1-\omega)(\vartheta_2 - \wp) + \omega(\Im\vartheta_2 - \Im\wp)\right\|^2 \\
&\quad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \frac{\lambda}{(\eta+1)^2}\left\|\eta(\vartheta_1 - \wp) + \Im\vartheta_1 - \Im\wp\right\|^2 \\
&\quad + \frac{1-\lambda}{(\eta+1)^2}\left\|\eta(\vartheta_2 - \wp) + \Im\vartheta_2 - \Im\wp\right\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\le \frac{\lambda}{(\eta+1)^2}\left[(\eta+1)^2\|\vartheta_1 - \wp\|^2 + \beta\|\wp - \Im\wp\|^2\right] \\
&\quad + \frac{1-\lambda}{(\eta+1)^2}\left[(\eta+1)^2\|\vartheta_2 - \wp\|^2 + \beta\|\wp - \Im\wp\|^2\right] \\
&\quad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda(1-\lambda)^2\|\vartheta_1 - \vartheta_2\|^2 + \frac{\beta}{(\eta+1)^2}\|\wp - \Im\wp\|^2
\end{aligned}$$

$$+ (1-\lambda)\lambda^2\|\vartheta_2-\vartheta_1\|^2 - \lambda(1-\lambda)\|\vartheta_1-\vartheta_2\|^2$$
$$= \lambda(1-\lambda)[1-\lambda+\lambda]\|\vartheta_1-\vartheta_2\|^2 + \frac{\beta}{(\eta+1)^2}\|\wp - \Im\wp\|^2$$
$$- \lambda(1-\lambda)\|\vartheta_1-\vartheta_2\|^2,$$

it follows that $(1-\beta)\|\wp - \Im\wp\| \leq 0$. Therefore, $\wp = \Im\wp$ and $\wp \in F(\Im)$ as required. □

**Definition 3.2** Let $\mathcal{E}$ be a real Banach space and $\emptyset \neq \mathcal{K} \subset \mathcal{E}$. Let $\{\Im_i\}_{i=1}^N$ be a finite family of maps of $\mathcal{K}$ into itself. For $i = 1,2,\dots,\mathcal{N}$, let $\tau_i = (\alpha_i,\beta_i,\gamma_i,\delta_i)$, where $\alpha_i,\beta_i,\gamma_i,\delta_i \in [0,1]$ with $\alpha_i+\beta_i+\gamma_i+\delta_i = 1$. The map $D:\mathcal{K} \longrightarrow \mathcal{K}$ is defined as follows:

$$\begin{aligned}
&G_0 = I\\
&G_1 = \alpha_1\Im_{\omega,1}^2 G_0 + \beta_1\Im_{\omega,1}G_0 + \gamma_1 G_0 + \delta_1 I\\
&G_2 = \alpha_2\Im_{\omega,2}^2 G_1 + \beta_2\Im_{\omega,1}G_1 + \gamma_2 G_1 + \delta_2 I\\
&G_3 = \alpha_3\Im_{\omega,3}^2 G_2 + \beta_3\Im_{\omega,3}G_2 + \gamma_3 G_2 + \delta_3 I\\
&\vdots\\
&G_{N-1} = \alpha_{N-1}\Im_{\omega,N-1}^2 G_{N-2} + \beta_{N-1}\Im_{\omega,N-1}G_{N-2} + \gamma_{N-1}G_{N-2} + \delta_{N-1}I\\
&D = G_N = \alpha_N\Im_{\omega,N}^2 G_{N-1} + \beta_N\Im_{\omega,N}G_{N-1} + \gamma_N G_{N-1} + \delta_N I,
\end{aligned}$$

where $\Im_{\omega,i} = (1-\omega)I + \omega\Im_i$, $i = 1,2,\dots,N$. This is known as an $\eta$-enriched $D$-map developed from $\Im_{\omega,1},\Im_{\omega,2},\dots,\Im_{\omega,N}$ and $\tau_1,\tau_2,\dots,\tau_N$.

*Remark* 3.2 If $\eta = 0$, then $\omega = 1$ and $\eta$-enriched $D$-map become $G$-map; if $\eta = 0$ and for every $i = 1,2,\dots,N$, $\alpha_i = 0$, then $\eta$-enriched $D$-map becomes $S$-map; if $\eta = 0$ and for every $i = 1,2,\dots,N$, $\alpha_i = 0$ and $\gamma_i = 0$, then $\eta$-enriched $D$-map becomes $W$-map, and if $\eta = 0$ and for every $i = 1,2,\dots,N$, $\alpha_i = 0$ and $\delta_i = 0$, then $\eta$-enriched $D$-map becomes $K$-map.

**Lemma 3.6** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be $(\eta_i,\beta_i)$-ESPN maps with $\bigcap_{i=1}^N F(\Im_i) \neq \emptyset$, and let $\tau_i = (\alpha_i,\beta_i,\gamma_i,\delta_i)$, where $\alpha_i,\beta_i,\gamma_i,\delta_i \in [0,1]$ and $\alpha_i+\beta_i+\gamma_i+\delta_i = 1$. Let $D$ be an $\eta$-enriched $D$-map developed from the sequences $\{\Im_i\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$. If the following conditions are satisfied*:

(a) $\alpha_1 \leq \beta_1 < 1-k_1$ *and* $\beta_1\gamma_1 < \beta_1(1-\alpha_1-\beta_1)$;

(b) $\beta_i \geq k_i$, $k_i < \gamma_i < 1$ *and* $\eta_i \leq \sqrt{\frac{\beta_i\gamma_i-(\alpha_i+\beta_i)k_i}{\alpha_i}}$, *for* $i = 1,2,\dots,N$.

*Then,* $F(D) = \bigcap_{i=1}^N F(\Im_i)$ *and $D$ is nonexpansive.*

*Proof* It is not difficult to see that $\bigcap_{i=1}^N F(\Im_i) \subseteq F(D)$. So, it suffices for us to show that $F(D) \subseteq \bigcap_{i=1}^N F(\Im_i)$. Let $\wp_0 \in F(D)$ and $\wp^\star \in \bigcap_{i=1}^N F(\Im_i)$ so that

$$\begin{aligned}
\|\wp_0 - \wp^\star\|^2 &= \|D\wp_0 - \wp^\star\|^2\\
&= \big\|\alpha_N\big(\Im_{\omega,N}^2 G_{N-1}\wp_0 - \wp^\star\big) + \beta_N\big(\Im_{\omega,N}G_{N-1}\wp_0 - \wp^\star\big) + \gamma_N\big(G_{N-1}\wp_0 - \wp^\star\big)\\
&\quad + \delta_N\big(\wp_0 - \wp^\star\big)\big\|^2\\
&= \alpha_N\big\|\Im_{\omega,N}^2 G_{N-1}\wp_0 - \wp^\star\big\|^2 + \beta_N\big\|\Im_{\omega,N}G_{N-1}\wp_0 - \wp^\star\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma_N \left\| G_{N-1}\wp_0 - \wp^\star \right\|^2 \\
&+ \delta_N \left\| \wp_0 - \wp^\star \right\|^2 - \alpha_N\beta_N \left\| \Im^2_{\omega,N} G_{n-}\wp_0 - \Im_{\omega,N} G_{N-1}\wp_0 \right\|^2 \\
&- \alpha_N\gamma_N \left\| \Im^2_{\omega,N} G_{n-}\wp_0 - G_{N-1}\wp_0 \right\|^2 - \alpha_N\delta_N \left\| \Im^2 G_{n-}\wp_0 - \wp_0 \right\|^2 \\
&- \beta_N\gamma_N \| \Im_{\omega,N} G_{n-}\wp_0 - G_{N-1}\wp_0 \|^2 - \beta_N\delta_N \| \Im_{\omega,N} G_{n-}\wp_0 - \wp_0 \|^2 \\
&- \gamma_N\delta_N \| G_{N-1}\wp_0 - \wp_0 \|^2 \\
\le{}& \alpha_N \left\| \Im^2_{\omega,N} G_{N-1}\wp_0 - \wp^\star \right\|^2 + \beta_N \left\| \Im_{\omega,N} G_{N-1}\wp_0 - \wp^\star \right\|^2 \\
&+ \gamma_N \left\| G_{N-1}\wp_0 - \wp^\star \right\|^2 \\
&+ \delta_N \left\| \wp_0 - \wp^\star \right\|^2 - \alpha_N\beta_N \left\| \Im^2_{\omega,N} G_{n-}\wp_0 - \Im_{\omega,N} G_{N-1}\wp_0 \right\|^2 \\
&- \beta_N\gamma_N \| \Im_{\omega,N} G_{n-}\wp_0 - G_{N-1}\wp_0 \|^2.
\end{aligned}
\tag{3.10}
$$

Since

$$
\begin{aligned}
\left\| \Im^2_{\omega,N} G_{N-1}\wp_0 - \wp^\star \right\|^2 &= \left\| (1-\omega)\Im_N G_{N-1}\wp_0 + \omega \Im^2_N G_{N-1}\wp_0 - \left[ (1-\omega)\wp^\star + \omega \Im_N \wp_0^\star \right] \right\|^2 \\
&= \frac{1}{(\eta_N+1)^2} \left\| \eta_N \left( \Im_N G_{N-1}\wp_0 - \wp^\star \right) + \Im^2_N G_{N-1}\wp_0 - \Im_N \wp_0^\star \right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| \Im_{\omega,N} G_{N-1}\wp_0 - \wp^\star \right\|^2 &= \left\| (1-\omega) G_{N-1}\wp_0 + \omega \Im_N G_{N-1}\wp_0 - \left[ (1-\omega)\wp^\star + \omega \Im_N \wp_0^\star \right] \right\|^2 \\
&= \frac{1}{(\eta_N+1)^2} \left\| \eta_N \left( G_{N-1}\wp_0 - \wp^\star \right) + \Im_N G_{N-1}\wp_0 - \Im_N \wp_0^\star \right\|^2,
\end{aligned}
$$

it follows from (3.10) that

$$
\begin{aligned}
\left\| \wp_0 - \wp^\star \right\|^2 \le{}& \frac{\alpha_N}{(\eta_N+1)^2} \left\| \eta_N \left( \Im_N G_{N-1}\wp_0 - \wp^\star \right) + \Im^2_N G_{N-1}\wp_0 - \Im_N \wp_0^\star \right\|^2 \\
&+ \frac{\beta_N}{(\eta_N+1)^2} \left\| \eta_N \left( G_{N-1}\wp_0 - \wp^\star \right) + \Im_N G_{N-1}\wp_0 - \Im_N \wp_0^\star \right\|^2 \\
&+ \gamma_N \left\| G_{n-1}\wp_0 - \wp^\star \right\|^2 + \delta_N \left\| \wp_0 - \wp^\star \right\|^2 \\
&- \frac{\alpha_N\beta_N}{(\eta_N+1)^2} \left\| \Im_N (\Im_N G_{N-1}\wp_0) - \Im_N G_{N-1}\wp_0 \right\|^2 \\
&- \frac{\beta_N\gamma_N}{(\eta_N+1)^2} \| G_{N-1}\wp_0 - \Im_N G_{N-1}\wp_0 \|^2 \\
\le{}& \frac{\alpha_N}{(\eta_N+1)^2} \left[ (\eta_N+1)^2 \left\| \Im_N G_{N-1}\wp_0 - \wp^\star \right\|^2 + k_N \left\| (I - \Im_N) \Im_N G_{n-1}\wp_0 \right\|^2 \right] \\
&+ \frac{\beta_N}{(\eta_N+1)^2} \left\| \eta_N \left( G_{N-1}\wp_0 - \wp^\star \right) + \Im_N G_{N-1}\wp_0 - \Im_N \wp_0^\star \right\|^2 \\
&+ \gamma_N \left\| G_{n-1}\wp_0 - \wp^\star \right\|^2 + \delta_N \left\| \wp_0 - \wp^\star \right\|^2 \\
&- \frac{\alpha_N\beta_N}{(\eta_N+1)^2} \left\| \Im_N (\Im_N G_{N-1}\wp_0) - \Im_N G_{N-1}\wp_0 \right\|^2 \\
&- \frac{\beta_N\gamma_N}{(\eta_N+1)^2} \| G_{N-1}\wp_0 - \Im_N G_{N-1}\wp_0 \|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\alpha_N}{(\eta_N+1)^2}\big\|\eta_N\big(G_{N-1}\wp_0-\wp^\star\big)+\Im_N G_{N-1}\wp_0-\Im_N\wp_0^\star \\
&\quad -\eta_N(G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0)\big\|^2+\frac{\alpha_N k_N}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{n-1}\wp_0\big\|^2 \\
&\quad +\frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N\big(G_{N-1}\wp_0-\wp^\star\big)+\Im_N G_{N-1}\wp_0-\Im_N\wp_0^\star\big\|^2 \\
&\quad +\gamma_N\big\|G_{n-1}\wp_0-\wp^\star\big\|^2+\delta_N\big\|\wp_0-\wp^\star\big\|^2 \\
&\quad -\frac{\alpha_N\beta_N}{(\eta_N+1)^2}\big\|\Im_N(\Im_N G_{N-1}\wp_0)-\Im_N G_{N-1}\wp_0\big\|^2 \\
&\quad -\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2 \\
&\leq \frac{\alpha_N}{(\eta_N+1)^2}\big\|\eta_N\big(G_{N-1}\wp_0-\wp^\star\big)+\Im_N G_{N-1}\wp_0-\Im_N\wp_0^\star\big\|^2 \\
&\quad +\frac{\alpha_N\eta_N^2}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0)\|^2 \\
&\quad +\frac{\alpha_N k_N}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{n-1}\wp_0\big\|^2 \\
&\quad +\frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N\big(G_{N-1}\wp_0-\wp^\star\big)+\Im_N G_{N-1}\wp_0-\Im_N\wp_0^\star\big\|^2 \\
&\quad +\gamma_N\big\|G_{n-1}\wp_0-\wp^\star\big\|^2+\delta_N\big\|\wp_0-\wp^\star\big\|^2 \\
&\quad -\frac{\alpha_N\beta_N}{(\eta_N+1)^2}\big\|\Im_N(\Im_N G_{N-1}\wp_0)-\Im_N G_{N-1}\wp_0\big\|^2 \\
&\quad -\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2 n \\
&= \frac{\alpha_N+\beta_N}{(\eta_N+1)^2}\big\|\eta_N\big(G_{N-1}\wp_0-\wp^\star\big)+\Im_N G_{N-1}\wp_0-\Im_N\wp_0^\star\big\|^2 \\
&\quad +\frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{n-1}\wp_0\big\|^2 \\
&\quad +\frac{\alpha_N\eta_N^2-\beta_N\gamma_N}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2 \\
&\quad +\gamma_N\big\|G_{n-1}\wp_0-\wp^\star\big\|^2+\delta_N\big\|\wp_0-\wp^\star\big\|^2 \\
&\leq \frac{\alpha_N+\beta_N}{(\eta_N+1)^2}\big[(\eta_N+1)^2\big\|G_{N-1}\wp_0-\wp^\star\big\|^2+k_N\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2\big] \\
&\quad +\frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{n-1}\wp_0\big\|^2 \\
&\quad +\frac{\alpha_N\eta_N^2-\beta_N\gamma_N}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2 \\
&\quad +\gamma_N\big\|G_{n-1}\wp_0-\wp^\star\big\|^2+\delta_N\big\|\wp_0-\wp^\star\big\|^2 \\
&= (1-\delta_N)\big\|G_{N-1}\wp_0-\wp^\star\big\|^2+\big(1-(1-\delta_N)\big)\big\|\wp_0-\wp^\star\big\|^2 \\
&\quad +\frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{N-1}\wp_0\big\|^2 \\
&\quad +\frac{(\alpha_N+\beta_N)k_N+\alpha_N\eta_N^2-\beta_N\gamma_N}{(\eta_N+1)^2}\|G_{N-1}\wp_0-\Im_N G_{N-1}\wp_0\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le (1-\delta_N)\left\|G_{N-1}\wp_0-\wp^\star\right\|^2+\left(1-(1-\delta_N)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\vdots\\
&\le (1-\delta_N)\left[(1-\delta_{N-1})\left\|G_{N-1}\wp_0-\wp^\star\right\|^2+\left(1-(1-\delta_{N-1})\right)\left\|\wp_0-\wp^\star\right\|^2\right]\\
&\quad+\left(1-(1-\delta_N)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&=(1-\delta_N)(1-\delta_{N-1})\left\|G_{N-1}\wp_0-\wp^\star\right\|^2\\
&\quad+\left(1-(1-\delta_N)(1-\delta_{N-1})\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\vdots\\
&\le \prod_{k=3}^{N}(1-\delta_k)\left\|G_2\wp_0-\wp^\star\right\|^2+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\le \prod_{k=3}^{N}(1-\delta_k)\Bigg[(1-\delta_2)\left\|G_1\wp_0-\wp^\star\right\|^2+\delta_2\left\|\wp_0-\wp^\star\right\|^2\\
&\quad+\frac{\alpha_2(k_2-\beta_2)}{(\eta_2+1)^2}\left\|(I-\Im_2)\Im_2G_1\wp_0\right\|^2\\
&\quad+\frac{(\alpha_2+\beta_2)k_2+\alpha_2\eta_2^2-\beta_2\gamma_2}{(\eta_2+1)^2}\|G_1\wp_0-\Im_2G_{N-1}\wp_0\|^2\Bigg]\\
&\quad+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2 \qquad (3.11)\\
&\le \prod_{k=2}^{N}(1-\delta_k)\left\|G_1\wp_0-\wp^\star\right\|^2+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\le \prod_{k=2}^{N}(1-\delta_k)\Big\{\alpha_1\left\|\Im_{\omega,1}^2\wp_0-\wp^\star\right\|^2\\
&\quad+\beta_1\left\|\Im_{\omega,1}\wp_0-\wp^\star\right\|^2+(1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2\\
&\quad-\alpha_1\beta_1\left\|\Im_{\omega,1}^2\wp_0-\Im_{\omega,1}\wp_0\right\|^2-\alpha_1(1-\alpha_1-\beta_1)\left\|\Im_{\omega,1}^2\wp_0-\wp_0\right\|^2\\
&\quad-\beta_1(1-\alpha_1-\beta_1)\|\Im_{\omega,1}\wp_0-\wp_0\|^2\Big\}+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\le \prod_{k=2}^{N}(1-\delta_k)\Big\{\alpha_1\left\|\Im_{\omega,1}^2\wp_0-\wp^\star\right\|^2+\beta_1\left\|\Im_{\omega,1}\wp_0-\wp^\star\right\|^2\\
&\quad+(1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2\\
&\quad-\alpha_1\beta_1\left\|\Im_{\omega,1}^2\wp_0-\Im_{\omega,1}\wp_0\right\|^2-\beta_1(1-\alpha_1-\beta_1)\|\Im_{\omega,1}\wp_0-\wp_0\|^2\Big\}\\
&\quad+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&=\prod_{k=2}^{N}(1-\delta_k)\Big\{\frac{\alpha_1}{(\eta_1+1)^2}\left\|\eta_1\left(\Im_1\wp_0-\wp^\star\right)+\Im_1^2\wp_0-\wp^\star\right\|^2\\
&\quad+\frac{\beta_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2+(1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\alpha_1 \beta_1}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\} \\
& + \left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2 \\
\leq{} & \prod_{k=2}^{N}(1-\delta_k)\left\{\frac{\alpha_1}{(\eta_1+1)^2}\left[(\eta_1+1)^2\left\|\Im_1\wp_0-\wp^\star\right\|^2 + k_1\left\|(I-\Im_1)\Im_1\wp_0\right\|^2\right]\right. \\
& + \frac{\beta_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2 + (1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2 \\
& \left. - \frac{\alpha_1 \beta_1}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\right\} \\
& + \left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2 \\
={} & \prod_{k=2}^{N}(1-\delta_k)\left\{\frac{\alpha_1}{(\eta_1+1)^2}\|\left[\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star-\eta_1(\wp_0-\Im_1\wp_0)\|^2\right]\right. \\
& + \frac{\alpha_1 k_1}{(\eta_1+1)^2}(\eta_1+1)^2\left\|(I-\Im_1)\Im_1\wp_0\right\|^2] \\
& + \frac{\beta_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2 + (1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2 \\
& \left. - \frac{\alpha_1 \beta_1}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\right\} \\
& + \left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2 \\
\leq{} & \prod_{k=2}^{N}(1-\delta_k)\left\{\frac{\alpha_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2\right. \\
& + \frac{\alpha_1 \eta_1^2}{(\eta_1+1)^2}\|\wp_0-\Im_1\wp_0\|^2 \\
& + \frac{\alpha_1 k_1}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 \\
& + \frac{\beta_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2 + (1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2 \\
& \left. - \frac{\alpha_1 \beta_1}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\right\} \\
& + \left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2 \\
={} & \prod_{k=2}^{N}(1-\delta_k)\left\{\frac{\alpha_1+\beta_1}{(\eta_1+1)^2}\left\|\eta_1\left(\wp_0-\wp^\star\right)+\Im_1\wp_0-\wp^\star\right\|^2\right. \\
& + \frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2 \\
& \left. + \frac{\alpha_1\eta_1^2-\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2 + (1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&\le \prod_{k=2}^{N}(1-\delta_k)\{\frac{\alpha_1+\beta_1}{(\eta_1+1)^2}\left[(\eta_1+1)^2\left\|\wp_0-\wp^\star\right\|^2+k_1\left\|(I-\Im_1)\wp_0\right\|^2\right]\\
&\quad+\frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2+\frac{\alpha_1\eta_1^2-\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\\
&\quad+(1-\alpha_1-\beta_1)\left\|\wp_0-\wp^\star\right\|^2\}+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2\\
&=\prod_{k=2}^{N}(1-\delta_k)\left\{\left\|\wp_0-\wp^\star\right\|^2+\frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2\right.\\
&\quad\left.+\frac{\alpha_1\eta_1^2+(\alpha_1+\beta_1)k_1-\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2\right\}\\
&\quad+\left(1-\prod_{k=3}^{N}(1-\delta_k)\right)\left\|\wp_0-\wp^\star\right\|^2.
\end{aligned}
\tag{3.12}
$$

If we set

$$
\begin{aligned}
\mathcal{U}_1={}&\frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\Im_1\wp_0\right\|^2\\
&+\frac{\alpha_1\eta_1^2+(\alpha_1+\beta_1)k_1-\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\|\Im_1\wp_0-\wp_0\|^2,
\end{aligned}
$$

then by conditions [(a) and (b)], we obtain

$$
\mathcal{U}_1\le 0.
\tag{3.13}
$$

Using (3.12) and the fact that $\delta_k<1$ for $k=1,2,\dots,N$, it follows that

$$
\mathcal{U}_1\ge 0.
\tag{3.14}
$$

(3.13) and (3.14) imply that

$$
\|\Im_1\wp_0-\wp_0\|=0.
\tag{3.15}
$$

Consequently, $\Im_1\wp_0=\wp_0$, that is, $\wp_0\in F(\Im_1)=F(\Im_{\omega,1})$. Using the definition $G_1$ and the fact that $\Im_{\omega,1}=(1-\omega)I+\omega\Im_1$, we obtain

$$
\begin{aligned}
G_1\wp_0&=\alpha_1\Im_{\omega,1}^2G_0\wp_0+\beta_1\Im_{\omega,1}G_0\wp_0+\gamma_1G_0\wp_0+\delta_1\wp_0\\
&=\alpha_1\Im_{\omega,1}^2\wp_0+\beta_1\Im_{\omega,1}\wp_0+\gamma_1\wp_0+\delta_1\wp_0\\
&=\alpha_1\Im_{\omega,1}\wp_0+\beta_1\wp_0+\gamma_1\wp_0+\delta_1\wp_0\\
&=\alpha_1\wp_0+\beta_1\wp_0+\gamma_1\wp_0+\delta_1\wp_0=\wp_0.
\end{aligned}
\tag{3.16}
$$

Again, set

$$\begin{aligned}\mathcal{U}_2 &= \frac{\alpha_2(k_2-\beta_2)}{(\eta_2+1)^2}\big\|(I-\Im_2)\Im_2 G_1\wp_0\big\|^2 \\ &\quad + \frac{(\alpha_2+\beta_2)k_2+\alpha_2\eta_2^2-\beta_2\gamma_2}{(\eta_2+1)^2}\|G_1\wp_0-\Im_2 G_1\wp_0\|^2,\end{aligned}$$

then by (3.12), (3.16), and the fact that $\delta_k < 1$ for $k = 3, 4, \dots, N$, we obtain

$$\begin{aligned}\mathcal{U}_2 &= \frac{\alpha_2(k_2-\beta_2)}{(\eta_2+1)^2}\big\|(I-\Im_2)\Im_2\big\|^2 + \frac{(\alpha_2+\beta_2)k_2+\alpha_2\eta_2^2-\beta_2\gamma_2}{(\eta_2+1)^2}\|\wp_0-\Im_2\wp_0\|^2, \\ &\geq 0 \end{aligned} \tag{3.17}$$

Using condition [(a) and (b)], it follows that

$$\|\wp_0-\Im_2\wp_0\| = 0.$$

Hence, $\Im_2\wp_0 = \wp_0$; that is, $\wp_0 \in F(\Im_2) = F(\Im_{\omega,2})$. By the definition $G_2$, we obtain

$$G_2\wp_0 = \wp_0. \tag{3.18}$$

Continuing in this manner, we obtain

$$\wp_0 \in F(\Im_i) = F(\Im_{\omega,i}). \tag{3.19}$$

Consequently,

$$F(D) \subseteq \bigcap_{i=1}^{N} F(\Im_i) = \bigcap_{i=1}^{N} F(\Im_{\omega,i}).$$

Next, we show that $D$ is nonexpansive. Indeed, for any $\wp, \hbar \in \mathcal{K}$, we have

$$\begin{aligned}\|D\wp-D\hbar\|^2 &= \big\|\alpha_N\big(\Im_{\omega,N}^2 G_{N-1}\wp-\Im_{\omega,N}^2 G_{N-1}\hbar\big)+\beta_N(\Im_{\omega,N}G_{N-1}\wp-\Im_{\omega,N}G_{N-1}\hbar) \\ &\quad + \gamma_N(G_{N-1}\wp-G_{N-1}\hbar)+\delta_N(\wp-\hbar)\big\|^2 \\ &= \alpha_N\big\|\Im_{\omega,N}^2 G_{N-1}\wp-\Im_{\omega,N}^2 G_{N-1}\hbar\big\|^2+\beta_N\|\Im_{\omega,N}G_{N-1}\wp-\Im_{\omega,N}G_{N-1}\hbar\|^2 \\ &\quad + \gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2 \\ &\quad - \alpha_N\beta_N\big\|(I-\Im_{\omega,N})\Im_{\omega,N}G_{N-1}\wp-(I-\Im_{\omega,N})\Im_{\omega,N}G_{N-1}\hbar\big\|^2 \\ &\quad - \beta_N\gamma_N\big\|(I-\Im_{\omega,N})G_{N-1}\wp-(I-\Im_{\omega,N})G_{N-1}\hbar\big\|^2 \\ &= \frac{\alpha_N}{(\eta_N+1)^2}\big\|\eta_N(\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar)+\Im_N^2 G_{N-1}\wp-\Im_N^2 G_{N-1}\hbar\big\|^2 \\ &\quad + \frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\big\|^2 \\ &\quad + \gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2 \\ &\quad - \alpha_N\beta_N\big\|\omega(1-\omega)\big[(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big]\end{aligned}$$

$$
\begin{aligned}
&+\omega\big[(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big]\big\|^2\\
&-\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\\
\le{}&\frac{\alpha_N}{(\eta_N+1)^2}\big\|\eta_N(\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar)+\Im_N^2G_{N-1}\wp-\Im_N^2G_{N-1}\hbar\big\|^2\\
&+\frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\big\|^2\\
&+\gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2\\
&-\frac{\alpha_N\beta_N}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\\
&-\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\\
\le{}&\frac{\alpha_N}{(\eta_N+1)^2}\big[(\eta_N+1)^2\|\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\|^2\\
&+k_N\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\big]\\
&+\frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\big\|^2\\
&+\gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2\\
&-\frac{\alpha_N\beta_N}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\\
&-\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\\
={}&\frac{\alpha_N}{(\eta_N+1)^2}\big[\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\\
&-\eta_N\big[(G_{N-1}\wp-G_{N-1}\hbar)-\Im_N G_{N-1}\wp+\Im_N G_{N-1}\hbar\big]\big\|^2 \qquad (3.20)\\
&+k_N\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\big]\\
&+\frac{\beta_N}{(\eta_N+1)^2}\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\big\|^2\\
&+\gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2\\
&-\frac{\alpha_N\beta_N}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\\
&-\frac{\beta_N\gamma_N}{(\eta_N+1)^2}\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\\
={}&\frac{\alpha_N+\beta_N}{(\eta_N+1)^2}\big\|\eta_N(G_{N-1}\wp-G_{N-1}\hbar)+\Im_N G_{N-1}\wp-\Im_N G_{N-1}\hbar\big\|^2\\
&+\frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\big\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\big\|^2\\
&+\gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2\\
&+\frac{\alpha_N\eta_N^2-\beta_N\gamma_N}{(\eta_N+1)^2}\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\\
\le{}&\frac{\alpha_N+\beta_N}{(\eta_N+1)^2}\big[(\eta_N+1)^2\|G_{N-1}\wp-G_{N-1}\hbar\|^2\\
&+k_N\big\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\big\|^2\big\|^2\big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\left\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\right\|^2\\
&+ \gamma_N\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\delta_N\|\wp-\hbar\|^2\\
&+ \frac{\alpha_N\eta_N^2-\beta_N\gamma_N}{(\eta_N+1)^2}\left\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\right\|^2\\
=\ &(1-\delta_N)\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\big(1-(1-\delta_N)\big)\|\wp-\hbar\|^2\\
&+ \frac{\alpha_N(k_N-\beta_N)}{(\eta_N+1)^2}\left\|(I-\Im_N)\Im_N G_{N-1}\wp-(I-\Im_N)\Im_N G_{N-1}\hbar\right\|^2\\
&+ \frac{\alpha_N\eta_N^2+(\alpha_N+\beta_N)k_N-\beta_N\gamma_N}{(\eta_N+1)^2}\left\|(I-\Im_N)G_{N-1}\wp-(I-\Im_N)G_{N-1}\hbar\right\|^2\\
=\ &(1-\delta_N)\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\big(1-(1-\delta_N)\big)\|\wp-\hbar\|^2\\
&\vdots\\
\le\ &(1-\delta_N)\big[(1-\delta_{N-1})\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\big(1-(1-\delta_{N-1})\big)\|\wp-\hbar\|^2\big]\\
&+ \big(1-(1-\delta_N)\big)\|\wp-\hbar\|^2\\
=\ &(1-\delta_N)(1-\delta_{N-1})\|G_{N-1}\wp-G_{N-1}\hbar\|^2\\
&+ \big(1-(1-\delta_N)(1-\delta_{N-1})\big)\|\wp-\hbar\|^2\\
\le\ &\prod_{k=2}^{N}(1-\delta_k)\|G_{N-1}\wp-G_{N-1}\hbar\|^2+\left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2\\
=\ &\prod_{k=2}^{N}(1-\delta_k)\left\|\alpha_1\big(\Im_{\omega,1}^2\wp-\Im_{\omega,1}^2\hbar\big)+\beta_1(\Im_{\omega,1}\wp-\Im_{\omega,1}\hbar)\right.\\
&\left.+ (1-\alpha_1-\beta_1)(\wp-\hbar)\right\|^2\\
&+ \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2\\
=\ &\prod_{k=2}^{N}(1-\delta_k)\big[\alpha_1\left\|\Im_{\omega,1}^2\wp-\Im_{\omega,1}^2\hbar\right\|^2+\beta_1\|\Im_{\omega,1}\wp-\Im_{\omega,1}\hbar\|^2\\
&+ (1-\alpha_1-\beta_1)\|\wp-\hbar\|^2\\
&- \alpha_1\beta_1\left\|(I-\Im_{\omega,1})\Im_{\omega,1}\wp-(I-\Im_{\omega,1})\Im_{\omega,1}\hbar\right\|^2\\
&- \beta_1(1-\alpha_1-\beta_1)\left\|(I-\Im_{\omega,1})\wp-(I-\Im_{\omega,1})\hbar\right\|^2\big]\\
&+ \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2\\
=\ &\prod_{k=2}^{N}(1-\delta_k)[\frac{\alpha_1}{(\eta_1+1)^2}\left\|\eta_1(\Im_1\wp-\Im_1\hbar)+\Im_1^2\wp-\Im_1^2\hbar\right\|^2\\
&+ \frac{\beta_1}{(\eta_1+1)^2}\left\|\eta_1(\wp-\hbar)+\Im_1\wp-\Im_1\hbar\right\|^2+(1-\alpha_1-\beta_1)\|\wp-\hbar\|^2\\
&- \alpha_1\beta_1\left\|(I-\Im_{\omega,1})\Im_{\omega,1}\wp-(I-\Im_{\omega,1})\Im_{\omega,1}\hbar\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\big] \\
&\quad + \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2 \\
&= \prod_{k=2}^{N}(1-\delta_k)\bigg[\frac{\alpha_1}{(\eta_1+1)^2}\big[(\eta_1+1)^2\|\Im_1\wp-\Im_1\hbar\|^2 \\
&\quad + k_1\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\big] \\
&\quad + \frac{\beta_1}{(\eta_1+1)^2}\big\|\eta_1(\wp-\hbar)+\Im_1\wp-\Im_1\hbar\big\|^2+(1-\alpha_1-\beta_1)\|\wp-\hbar\|^2 \\
&\quad - \frac{\alpha_1\beta_1}{(\eta_1+1)^2}\big\|(I-\Im_1)\Im_1\wp-(I-\Im_{\omega,1})\Im_{\omega,1}\hbar\big\|^2 \\
&\quad - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\bigg] \\
&\quad + \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2 \\
&= \prod_{k=2}^{N}(1-\delta_k)\bigg[\frac{\alpha_1}{(\eta_1+1)^2}\big[\big\|\eta_1(\wp-\hbar)+\Im_1\wp \\
&\quad - \Im_1\hbar-\eta_1\big[(\wp-\hbar)-\Im_1\wp+\Im_1\hbar\big]\big\|^2 \\
&\quad + k_1\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\big] \\
&\quad + \frac{\beta_1}{(\eta_1+1)^2}\big\|\eta_1(\wp-\hbar)+\Im_1\wp-\Im_1\hbar\big\|^2+(1-\alpha_1-\beta_1)\|\wp-\hbar\|^2 \\
&\quad - \frac{\alpha_1\beta_1}{(\eta_1+1)^2}\big\|(I-\Im_1)\Im_1\wp-(I-\Im_{\omega,1})\Im_{\omega,1}\hbar\big\|^2 \\
&\quad - \frac{\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\bigg] \\
&\quad + \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2 \\
&\leq \prod_{k=2}^{N}(1-\delta_k)\bigg[\frac{\alpha_1+\beta_1}{(\eta_1+1)^2}\big\|\eta_1(\wp-\hbar)+\Im_1\wp-\Im_1\hbar\big\|^2 \\
&\quad + \frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2+(1-\alpha_1-\beta_1)\|\wp-\hbar\|^2 \\
&\quad + \frac{\alpha_1\eta_1^2-\beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\bigg] \\
&\quad + \left(1-\prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2 \\
&\leq \prod_{k=2}^{N}(1-\delta_k)\bigg[\frac{\alpha_1+\beta_1}{(\eta_1+1)^2}\big[(\eta_1+1)^2\|\wp-\hbar\|^2+k_1\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2\big] \\
&\quad + \frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\big\|(I-\Im_1)\wp-(I-\Im_1)\hbar\big\|^2+(1-\alpha_1-\beta_1)\|\wp-\hbar\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{\alpha_1\eta_1^2 - \beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\wp - (I-\Im_1)\hbar\right\|^2\Bigg] \\
&\quad + \left(1 - \prod_{k=2}^{N}(1-\delta_k)\right)\|\wp - \hbar\|^2 \\
&= \prod_{k=2}^{N}(1-\delta_k)\Bigg[\|\wp-\hbar\|^2 + \frac{\alpha_1(k_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\wp - (I-\Im_1)\hbar\right\|^2 \\
&\quad + \frac{\alpha_1\eta_1^2 + (\alpha_1\beta_1)k_1 - \beta_1(1-\alpha_1-\beta_1)}{(\eta_1+1)^2}\left\|(I-\Im_1)\wp - (I-\Im_1)\hbar\right\|^2\Bigg] \\
&\quad + \left(1 - \prod_{k=2}^{N}(1-\delta_k)\right)\|\wp - \hbar\|^2 \\
&\leq \prod_{k=2}^{N}(1-\delta_k)\|\wp-\hbar\|^2 + \left(1 - \prod_{k=2}^{N}(1-\delta_k)\right)\|\wp-\hbar\|^2 = \|\wp-\hbar\|^2. \qquad (3.21)
\end{aligned}
$$

□

*Remark* 3.3 Since $F(D) = \bigcap_i^N F(\Im_i) \neq \emptyset$, it follows that the map $D$ is quasi-nonexpansive, that is,

$$\left\|D\wp - \wp^\star\right\| \leq \left\|\wp - \wp^\star\right\|, \quad \forall\left(\wp, \wp^\star\right) \in \mathcal{K} \times F(D). \qquad (3.22)$$

*Example* 3.4 Let $\Im_1, \Im_2 : \mathcal{R} \longrightarrow \mathcal{R}$ be defined as follows:

$$\Im_1\wp = \begin{cases} \wp, & \wp \in (-\infty, 0], \\ -\frac{3}{2}\wp, & \wp \in [0, +\infty), \end{cases}$$

and

$$\Im_2\wp = \begin{cases} -2\wp, & \wp \in (-\infty, 0], \\ \wp, & \wp \in [0, +\infty). \end{cases}$$

Then, $F(\Im_1) = (-\infty, 0]$ and $F(\Im_2) = [0, +\infty)$. Consequently, $F(\Im_1) \cap F(\Im_2) = \{0\}$. Also, it is shown in [24] that $\Im_1$ is a $(0, k_1)$-ESPC map (with $k_1 = \frac{1}{5}$) and $\Im_2$ is a $(0, k_2)$-ESPC map (with $k_2 = \frac{1}{3}$). Further, if we set $\tau_1 = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5})$, which satisfies condition (a) of Lemma 3.1, then it follows from

$$\Im_1^2\wp = \begin{cases} \wp, & \wp \in (-\infty, 0], \\ -\frac{3}{2}\wp, & \wp \in [0, +\infty), \end{cases}$$

that

$$G_1\wp = \frac{1}{5}\Im_{\omega,1}^2\wp + \frac{1}{5}\Im_{\omega,1}\wp + \frac{2}{5}\wp + \frac{1}{5}\wp.$$

Since $\Im_1$ is $(0, k_1)$-ESPC, it follows that $\eta_1 = 0$ so that $\omega = \frac{1}{\eta_1+1} = 1$. Thus,

$$\Im_{\omega,1}^2\wp = \Im_{1,1}^2\wp = (1-\omega)\wp + \omega\Im_1^2\wp = \Im_1^2\wp,$$

and

$$\Im_{\omega,1}\wp = \Im_{1,1}^2\wp = (1-\omega)\wp + \omega\Im_1\wp = \Im_1\wp.$$

Hence,

$$G_1\wp = \frac{1}{5}\Im_1^2\wp + \frac{1}{5}\Im_1\wp + \frac{2}{5}\wp + \frac{1}{5}\wp = \begin{cases} \wp, & \wp \in (-\infty, 0], \\ 0, & \wp \in [0, +\infty). \end{cases}$$

Again, if we set $\tau_2 = (\frac{1}{7}, \frac{1}{3}, \frac{1}{2}, \frac{1}{42})$, which satisfies condition (b) of Lemma 3.1, then it follows from

$$\Im_2^2\wp = \begin{cases} -2\wp, & \wp \in (-\infty, 0], \\ 0, & \wp \in [0, +\infty), \end{cases}$$

that

$$D\wp = G_1\wp = \frac{1}{7}\Im_{\omega,1}^2 G_1\wp + \frac{1}{3}\Im_{\omega,1} G_1\wp + \frac{1}{2} g_1\wp + \frac{1}{42}\wp.$$

Since $\Im_2$ is $(0, k_2)$-ESPC, it follows that $\eta_2 = 0$ so that $\omega = \frac{1}{\eta_2+1} = 1$. Thus,

$$\Im_{\omega,2}^2\wp = \Im_{1,2}^2\wp = (1-\omega)\wp + \omega\Im_2^2\wp = \Im_2^2\wp,$$

and

$$\Im_{\omega,2}\wp = \Im_{1,2}^2\wp = (1-\omega)\wp + \omega\Im_2\wp = \Im_2\wp.$$

Hence,

$$\begin{aligned} D\wp = G_2\wp &= \frac{1}{7}\Im_2^2 G_1\wp + \frac{1}{3}\Im_2 G_1\wp + \frac{1}{2} G_1\wp + \frac{1}{42}\wp \\ &= \frac{1}{7}\Im_2^2\wp + \frac{1}{3}\Im_2\wp + \frac{1}{2}\wp + \frac{1}{42}\wp = \begin{cases} -\frac{3}{7}\wp, & \wp \in (-\infty, 0], \\ \frac{1}{42}\wp, & \wp \in [0, +\infty). \end{cases} \end{aligned}$$

Using the above information, it is not difficult to see that $F(D) = \{0\} = F(\Im_1) \cap F(\Im_2)$. It has also been demonstrated in [24] that $D$ is nonexpansive.

**Theorem 3.7** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be $(\eta_i, k_i)$-ESPC maps and $S : \mathcal{K} \longrightarrow \mathcal{K}$ be an $(\eta, \beta)$-ESPN map for $\beta \in [0, 1)$. Let $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, for $i = 1, 2, \dots, N$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ with $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$ such that*

(a) *$\alpha_i \leq \beta_i < 1 - k_i$; $\beta_i\gamma_i < \beta_i(1 - \alpha_i - \beta_i)$ for $i = 1, 2, \dots, N$;*

(b) *$\beta_i \geq k_i$, $k_i < \gamma_i < 1$ and $\eta_i \leq \sqrt{\frac{\beta_i\gamma_i - (\alpha_i+\beta_i)k_i}{\alpha_i}}$, for $i = 1, 2, \dots, N$.*

*Let $D$ be the D-map generated by the sequences $\{\Im_{\omega,i}\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$, where $\Im_{\omega,i} = (1-\omega)I + \omega\Im_i$. Suppose $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(\Im_i) \neq \emptyset$. Let $\{\wp_n\}$ be a sequence as defined in (1.11) with the conditions $(i)$ – $(v)$. Then, $\{\wp_n\}_{n=0}^\infty$ converges strongly to $\bar{\wp} = P_{\mathcal{F}} u$.*

*Proof* First, we show that $(I-\lambda_n A)\wp_n$, where $A = I - S$, is nonexpansive. Now, since $S$ is $(\eta,\beta)$-ESPN map, it follows from Definition 3.1 that

$$\begin{aligned}\big\langle \eta(\wp-\hbar)+S\wp-S\hbar, \eta(\wp-\hbar)+S\wp-S\hbar\big\rangle \le{}& \big\langle(\eta+1)(\wp-\hbar),(\eta+1)(\wp-\hbar)\big\rangle\\ &+\beta\big\|\wp-S\wp-(\hbar-S\hbar)\big\|^2\\ &+2\langle\wp-S\wp,\hbar-S\hbar\rangle,\end{aligned}$$

or equivalently

$$\begin{aligned}\big\langle\wp-S\wp-(\hbar-S\hbar),\wp-S\wp-(\hbar-S\hbar)\big\rangle \le{}& \beta\big\|\wp-S\wp-(\hbar-S\hbar)\big\|^2\\ &+2\langle\wp-S\wp,\hbar-S\hbar\rangle\end{aligned}$$

so that

$$\frac{1-\beta}{2}\beta\big\|\wp-S\wp-(\hbar-S\hbar)\big\|^2\le\langle\wp-S\wp,\hbar-S\hbar\rangle,$$

which, when $A = I - S$, yields

$$\frac{1-\beta}{2}\|A\wp-A\hbar\|^2\le\langle A\wp,A\hbar\rangle. \tag{3.23}$$

Also, since

$$\begin{aligned}\langle A\wp,A\hbar\rangle &= \big\langle\wp-\hbar+A\wp-(\wp-\hbar),-(A\wp-A\hbar)+A\wp\big\rangle\\ &= -\langle\wp-\hbar+A\wp,A\wp-A\hbar\rangle+\langle\wp-\hbar,A\wp-A\hbar\rangle\\ &\quad-\big\langle\wp-\hbar-\big[(\wp-\hbar)-A\wp\big],A\wp\big\rangle\\ &= -\|\wp-\hbar+A\wp\|\|A\wp-A\hbar\|+\langle\wp-\hbar,A\wp-A\hbar\rangle\\ &\quad-\|\wp-\hbar-[(\wp-\hbar)-A\wp\|\|A\wp\|\\ &= -\|\wp-\hbar+A\wp\|\|A\wp-A\hbar\|+\langle\wp-\hbar,A\wp-A\hbar\rangle-\|A\wp\|^2,\end{aligned}$$

it follows from (3.23) that

$$\begin{aligned}\frac{1-\beta}{2}\|A\wp-A\hbar\|^2 &\le -\|\wp-\hbar+A\wp\|\|A\wp-A\hbar\|+\langle\wp-\hbar,A\wp-A\hbar\rangle-\|A\wp\|^2\\ &\le\langle\wp-\hbar,A\wp-A\hbar\rangle.\end{aligned} \tag{3.24}$$

Thus, if $S$ is an $(\eta,\beta)$-ESPN, then $A = I - S$ is $\frac{1-\beta}{2}$-inverse strongly monotone map. From (3.24), we obtain, for any $\hbar = \bar{\wp}^\star \in F(S)$, that

$$\begin{aligned}\big\|(I-\lambda_nA)\wp_n-(I-\lambda_nA)\wp^\star\big\|^2 &= \big\|\big(\wp_n-\wp^\star\big)-\lambda_n\big(A\wp_n-A\wp^\star\big)\big\|^2\\ &= \big\|\wp_n-\wp^\star\big\|^2-\lambda_n\big\langle\wp_n\wp^\star,A\wp_n\big\rangle+\lambda_n^2\|A\wp_n\|^2\\ &\le\big\|\wp_n-\wp^\star\big\|^2-\lambda_n\big((1-\beta)-\lambda_n\big)\|A\wp_n\|^2\\ &\le\big\|\wp_n-\wp^\star\big\|^2.\end{aligned} \tag{3.25}$$

Next, we set

$$Q = \max\{\|u\|, \|\wp_n\|, \|w_n\|, \|Dw_n\|, \|P_{\mathcal{K}}(1-\lambda_n(I-S))\wp_n\|, \|P_{\mathcal{K}}(1-\lambda_n(I-S))\hbar_n, \\ \|(I-\Im)\wp_n-(I-\Im)\wp_{n-1}\|, \|(I-\Im)\hbar_n-(I-\Im)\hbar_{n-1}\|, \|(I-\Im)\wp_n\|, \|(I-\Im)\hbar_n\|\}$$

and then show that $Q$ is bounded. To do this, let $\wp^\star \in \mathcal{F}(\Im)$ be arbitrarily chosen. Then, using (1.11), we get

$$\begin{aligned} \|\wp_{n+1}-\wp^\star\| &= \|a_n u + b_n w_n + c_n Dw_n - \wp^\star\| \\ &= \|a_n(u-\wp^\star) + b_n(w_n-\wp^\star) + c_n(Dw_n-\wp^\star)\| \\ &\leq a_n\|u-\wp^\star\| + b_n\|w_n-\wp^\star\| + c_n\|Dw_n-\wp^\star\| \\ &\leq a_n\|u-\wp^\star\| + b_n\|w_n-\wp^\star\| + c_n\|w_n-\wp^\star\| \\ &= a_n\|u-\wp^\star\| + (1-a_n)\|w_n-\wp^\star\|. \end{aligned} \tag{3.26}$$

Using Lemma 2.2(ii) and (3.25), we also obtain from (1.11) that

$$\begin{aligned} \|w_n-\wp^\star\| &= \|(1-\sigma_n)\wp_n + \sigma_n P_{\mathcal{K}}(1-\lambda_n(I-S))\hbar_n - \wp^\star\| \\ &\leq (1-\sigma_n)\|\wp_n-\wp^\star\| + \sigma_n\|P_{\mathcal{K}}(1-\lambda_n(I-S))\hbar_n-\wp^\star\| \\ &\leq (1-\sigma_n)\|\wp_n-\wp^\star\| + \sigma_n\|(1-\lambda_n(I-S))\hbar_n-\wp^\star\| \\ &\leq (1-\sigma_n)\|\wp_n-\wp^\star\| + \sigma_n\|\hbar_n-\wp^\star\|, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \|\hbar_n-\wp^\star\| &= \|(1-\pi_n)\wp_n + \pi_n P_{\mathcal{K}}(1-\lambda_n(I-S))\wp_n - \wp^\star\| \\ &\leq (1-\pi_n)\|\wp_n-\wp^\star\| + \pi_n\|P_{\mathcal{K}}(1-\lambda_n(I-S))\wp_n-\wp^\star\| \\ &\leq (1-\pi_n)\|\wp_n-\wp^\star\| + \pi_n\|(1-\lambda_n(I-S))\wp_n-\wp^\star\| \\ &\leq (1-\pi_n)\|\wp_n-\wp^\star\| + \pi_n\|\wp_n-\wp^\star\| \\ &= \|\wp_n-\wp^\star\|. \end{aligned} \tag{3.28}$$

Thus, (3.26), (3.27), and (3.28) imply that

$$\|\wp_{n+1}-\wp^\star\| \leq a_n\|u-\wp^\star\| + (1-a_n)\|\wp_n-\wp^\star\|. \tag{3.29}$$

It is now easy to see using (3.29) and inductional hypothesis that

$$\|\wp_{n+1}-\wp^\star\| \leq \max\{\|u-\wp^\star\|, \|\wp_n-\wp^\star\|\}, \tag{3.30}$$

which consequently implies that $\{\wp_n\}$ is bounded. In addition, $\{\hbar_n\}$, $\{w_n\}$, and $\{Dw_n\}$ are also bounded. Now, from Remark 3.1, we have $\wp^\star \in F(P_{\mathcal{K}}(I-\lambda(I-\Im)))$, and from the nonexpansiveness of $P_{\mathcal{K}}$, we obtain

$$\|P_{\mathcal{K}}(I-\lambda(I-\Im))\wp_n-\wp^\star\|^2 = \|P_{\mathcal{K}}(I-\lambda(I-\Im))\wp_n - P_{\mathcal{K}}(I-\lambda(I-\Im))\wp^\star\|^2$$

$$\begin{aligned}&\leq \big\|\big(I-\lambda(I-\Im)\big)\wp_n-\big(I-\lambda(I-\Im)\big)\wp^\star\big\|^2\\&\leq \big\|\wp_n-\wp^\star\big\|^2(\text{by }(3.25)).\end{aligned}$$

Thus, using the boundedness of $\{\wp_n\}$ and $\{\hbar_n\}$, it will not be difficult to see that $\{P_{\mathcal{K}}(I-\lambda(I-\Im))\wp_n\}$ and $\{P_{\mathcal{K}}(I-\lambda(I-\Im))\hbar_n\}$ are bounded. From Proposition 3.1, we also obtain that $\{(I-\Im)\wp_n-(I-\Im)\wp_{n-1}\}$ and $\{(I-\Im)\hbar_n-(I-\Im)\hbar_{n-1}\}$ are bounded. Hence, $Q$ is bounded.

Next, we show that $\lim_{n\to\infty}\|\wp_{n+1}-\wp_n\|=0$. Using (1.11), we obtain the following estimates

$$\begin{aligned}&\|\wp_{n+1}-\wp_n\|\\&\quad=\big\|a_nu+b_nw_n+c_nDw_n-(a_{n-1}u+b_{n-1}w_{n-1}+c_{n-1}Dw_{n-1})\big\|\\&\quad=\big\|(a_n-a_{n-1})u+b_n(w_n-w_{n-1})+(b_n-b_{n-1})w_{n-1}+c_n(Dw_n-Dw_{n-1})\\&\qquad+(c_n-c_{n-1})Dw_{n-1}\big\|\\&\quad=|a_n-a_{n-1}|\|u\|+b_n\|w_n-w_{n-1}\|+|b_n-b_{n-1}|\|w_{n-1}\|+c_n\|Dw_n-Dw_{n-1}\|\\&\qquad+|c_n-c_{n-1}|\|Dw_{n-1}\|\\&\quad\leq|a_n-a_{n-1}|Q+b_n\|w_n-w_{n-1}\|+|b_n-b_{n-1}|Q+c_n\|w_n-w_{n-1}\|\\&\qquad+|c_n-c_{n-1}|Q\\&\quad=(1-a_n)\|w_n-w_{n-1}\|+|a_n-a_{n-1}|Q+|b_n-b_{n-1}|Q\\&\qquad+|c_n-c_{n-1}|Q,\end{aligned}\tag{3.31}$$

$$\begin{aligned}&\|w_{n+1}-w_n\|\\&\quad=\big\|(1-\sigma_n)\wp_n+\sigma_nP_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\hbar_n-\big((1-\sigma_{n-1})\wp_{n-1}\\&\qquad+\sigma_{n-1}P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\hbar_{n-1}\big)\big\|\\&\quad\leq\big\|(1-\sigma_n)\wp_n-(1-\sigma_{n-1})\wp_{n-1}\big\|+\big\|\sigma_nP_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\hbar_n\\&\qquad-\sigma_{n-1}P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\hbar_{n-1}\big\|\\&\quad\leq(1-\sigma_n)\|\wp_n-\wp_{n-1}\|+|\sigma_n-\sigma_{n-1}|\|\wp_{n-1}\|+\sigma_n\big\|P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\hbar_n\\&\qquad-P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\hbar_{n-1}\big\|+|\sigma_n-\sigma_{n-1}|\}P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\hbar_{n-1}\|\\&\quad\leq(1-\sigma_n)\|\wp_n-\wp_{n-1}\|+|\sigma_n-\sigma_{n-1}|Q+\sigma_n\big\|\big(1-\lambda_n(I-S)\big)\hbar_n\\&\qquad-\big(1-\lambda_n(I-S)\big)\hbar_{n-1}\big\|+|\sigma_n-\sigma_{n-1}|Q\\&\quad\leq(1-\sigma_n)\|\wp_n-\wp_{n-1}\|+2|\sigma_n-\sigma_{n-1}|Q+\sigma_n\|\hbar_n-\hbar_{n-}\|\\&\qquad+\sigma_n\|\lambda_n(I-S)\hbar_n-\lambda_n(I-S))\hbar_{n-1}+\lambda_n(I-S))\hbar_{n-1}-\lambda_{n-1}(I-S))\hbar_{n-1}\|\\&\quad\leq(1-\sigma_n)\|\wp_n-\wp_{n-1}\|+2|\sigma_n-\sigma_{n-1}|Q+\sigma_n\|\hbar_n-\hbar_{n-}\|\\&\qquad+\sigma_n\lambda_n\|(I-S)\hbar_n-(I-S))\hbar_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-S))\hbar_{n-1}\|\\&\quad\leq(1-\sigma_n)\|\wp_n-\wp_{n-1}\|+2|\sigma_n-\sigma_{n-1}|Q+\sigma_n\|\hbar_n-\hbar_{n-}\|\\&\qquad+\sigma_n\lambda_nQ+|\lambda_n-\lambda_{n-1}|Q,\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}
\|\hbar_{n+1} - \hbar_n\| &= \big\|(1-\pi_n)\wp_n + \pi_n P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_n - \big((1-\pi_{n-1})\wp_{n-1} \\
&\quad + \pi_{n-1} P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\wp_{n-1}\big)\big\| \\
&\le \big\|(1-\pi_n)\wp_n - (1-\pi_{n-1})\wp_{n-1}\big\| + \big\|\pi_n P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_n \\
&\quad - \pi_{n-1} P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\wp_{n-1}\big\| \\
&\le (1-\pi_n)\|\wp_n - \wp_{n-1}\| + |\pi_n - \pi_{n-1}|\|\wp_{n-1}\| \\
&\quad + \pi_n \big\|P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_n - P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_{n-1}\big\| \\
&\quad + |\pi_n - \pi_{n-1}|\big\|P_{\mathcal{K}}\big(1-\lambda_{n-1}(I-S)\big)\wp_{n-1}\big\| \\
&\le (1-\pi_n)\|\wp_n - \wp_{n-1}\| + |\pi_n - \pi_{n-1}|Q \\
&\quad + \pi_n \big\|P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_n - P_{\mathcal{K}}\big(1-\lambda_n(I-S)\big)\wp_{n-1}\big\| \\
&\quad + |\pi_n - \pi_{n-1}|Q \\
&\le (1-\pi_n)\|\wp_n - \wp_{n-1}\| + 2|\pi_n - \pi_{n-1}|Q \\
&\quad + \pi_n\|\wp_n - \wp_{n-}\| + \pi_n\big\|\lambda_n(I-S)\wp_n - \lambda_n(I-S)\wp_{n-1} \\
&\quad + \lambda_n(I-S)\wp_{n-1} - \lambda_{n-1}(I-S)\wp_{n-1}\big\| \\
&\le (1-\pi_n)\|\wp_n - \wp_{n-1}\| + 2|\pi_n - \pi_{n-1}|Q \\
&\quad + \pi_n\|\wp_n - \wp_{n-}\| + \pi_n\lambda_n\big\|(I-S)\wp_n - (I-S)\wp_{n-1}\big\| \\
&\quad + \pi_n|\lambda_n - \lambda_{n-1}|\big\|(I-S)\wp_{n-1}\big\| \\
&\le \|\wp_n - \wp_{n-1}\| + 2|\pi_n - \pi_{n-1}|Q + \pi_n\lambda_n Q \\
&\quad + \pi_N|\lambda_n - \lambda_{n-1}|Q.
\end{aligned} \tag{3.33}$$

Using (3.32) and (3.33) in (3.31), we infer

$$\begin{aligned}
\|\wp_{n+1} - \wp_n\| &\le (1-a_n)\big[(1-\sigma_n)\|\wp_n - \wp_{n-1}\| + 2|\sigma_n - \sigma_{n-1}|Q \\
&\quad + \sigma_n\big(\|\wp_n - \wp_{n-1}\| + 2|\pi_n - \pi_{n-1}|Q + \pi_n\lambda_n Q \\
&\quad + \pi_n|\lambda_n - \lambda_{n-1}|Q\big) + \sigma_n\lambda_n Q + |\lambda_n - \lambda_{n-1}|Q\big] + |a_n - a_{n-1}|Q \\
&\quad + |b_n - b_{n-1}|Q + |c_n - c_{n-1}|Q \\
&= (1-a_n)\big[(1-\sigma_n)\|\wp_n - \wp_{n-1}\| + 2|\sigma_n - \sigma_{n-1}|Q \\
&\quad + \sigma_n\|\wp_n - \wp_{n-1}| + 2\sigma_n|\pi_n - \pi_{n-1}|Q + \sigma_n\pi_n\lambda_n Q \\
&\quad + \sigma_n\pi_n|\lambda_n - \lambda_{n-1}|Q + \sigma_n\lambda_n Q + |\lambda_n - \lambda_{n-1}|Q\big] + |a_n - a_{n-1}|Q \\
&\quad + |b_n - b_{n-1}|Q + |c_n - c_{n-1}|Q \\
&= (1-a_n)\|\wp_n - \wp_{n-1}\| + 2(1-a_n)|\sigma_n - \sigma_{n-1}|Q \\
&\quad + 2(1-a_n)\sigma_n|\pi_n - \pi_{n-1}|Q + \sigma_{n(1-a_n)}\pi_n\lambda_n Q \\
&\quad + (1-a_n)\sigma_n\pi_n|\lambda_n - \lambda_{n-1}|Q + (1-a_n)\sigma_n\lambda_n Q + (1-a_n)|\lambda_n - \lambda_{n-1}|Q
\end{aligned}$$

$$\begin{aligned}&+|a_n-a_{n-1}|Q+|b_n-b_{n-1}|Q+|c_n-c_{n-1}|Q\\&=(1-a_n)\|\wp_n-\wp_{n-1}\|+\ell_n,\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}\ell_n&=2(1-a_n)|\sigma_n-\sigma_{n-1}|Q+2(1-a_n)\sigma_n|\pi_n-\pi_{n-1}|Q+\sigma_{n(1-a_n)}\pi_n\lambda_nQ\\&\quad+(1-a_n)\sigma_n\pi_n|\lambda_n-\lambda_{n-1}|Q+(1-a_n)\sigma_n\lambda_nQ+(1-a_n)|\lambda_n-\lambda_{n-1}|Q\\&\quad+|a_n-a_{n-1}|Q+|b_n-b_{n-1}|Q+|c_n-c_{n-1}|Q.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\ell_n<\infty$ (by conditions [(iv) and (v)]), it follows from Lemma 2.5 and (3.34) that

$$\lim_{n\to\infty}\|\wp_{n+1}-\wp_n\|=0. \tag{3.35}$$

Next, we show that $\lim_{n\to\infty}\|Dw_n-w_n\|=0$ and $\lim_{n\to\infty}\|\wp_n-w_n\|=0$. Since, from (1.11), (3.32), (3.33) and Lemma 2.1,

$$\begin{aligned}\|\wp_{n+1}-\wp^\star\|^2&=\|a_nu+b_nw_n+c_nDw_n-\wp^\star\|^2\\&=\|a_n(u-\wp^\star)+b_n(w_n-\wp^\star)+c_n(Dw_n-\wp^\star)\|^2\\&=a_n\|u-\wp^\star\|^2+b_n\|w_n-\wp^\star\|^2+c_n\|Dw_n-\wp^\star\|^2-a_nb_n\|u-w_n\|^2\\&\quad-a_nc_n\|u-Dw_n\|^2-b_nc_n\|Dw_n-w_n\|^2\\&\le a_n\|u-\wp^\star\|^2+b_n\|w_n-\wp^\star\|^2+c_n\|w_n-\wp^\star\|^2-b_nc_n\|Dw_n-w_n\|^2\\&\le a_n\|u-\wp^\star\|^2+(1-a_n)\|\wp_n-\wp^\star\|^2-b_nc_n\|Dw_n-w_n\|^2.\\&\le a_n\|u-\wp^\star\|^2+\|\wp_n-\wp^\star\|^2-b_nc_n\|Dw_n-w_n\|^2\end{aligned}$$

and

$$\begin{aligned}&\|\wp_n-\wp^\star\|^2-\|\wp_{n+1}-\wp^\star\|^2\\&=\|\wp_n-\wp_{n+1}\|^2+2\|\wp_n-\wp_{n+1}\|\|\wp_{n+1}-\wp^\star\|\\&=(\|\wp_n-\wp_{n+1}\|+2\|\wp^\star-\wp_{n+1}\|)\|\wp_n-\wp_{n+1}\|\\&\le(\|\wp_n-\wp^\star\|+\|\wp^\star-\wp_{n+1}\|+2\|\wp_{n+1}-\wp^\star\|)\|\wp_n-\wp_{n+1}\|\\&\le(\|\wp_n-\wp^\star\|+3\|\wp_{n+1}-\wp^\star\|)\|\wp_n-\wp_{n+1}\|,\end{aligned}$$

it follows that

$$\begin{aligned}b_nc_n\|Dw_n-w_n\|^2&\le a_n\|u-\wp^\star\|^2+\|\wp_n-\wp^\star\|^2-\|\wp_{n+1}-\wp^\star\|^2\\&=a_n\|u-\wp^\star\|^2+(\|\wp_n-\wp^\star\|+3\|\wp_{n+1}-\wp^\star\|)\\&\quad\times\|\wp_n-\wp_{n+1}\|.\end{aligned} \tag{3.36}$$

Using conditions [(ii) and (iii)], the boundedness of $\|u-\wp^\star\|$ and $\{\wp_n\}$ and the fact that $\lim_{n\to\infty}\|\wp_{n+1}-\wp_n\|=0$, we obtain from (3.36) that

$$\lim_{n\to\infty}\|Dw_n-w_n\|=0. \tag{3.37}$$

Also, since

$$\begin{aligned}\|\wp_n - w_n\| &\leq \|\wp_n - \wp_{n+1}\| + \|\wp_{n+1} - w_n\| \\ &= \|\wp_n - \wp_{n+1}\| + \|a_n u + b_n w_n + c_n D w_n - w_n\| \\ &\leq \|\wp_n - \wp_{n+1}\| + a_n\|u - w_n\| + +c_n\|Dw_n - w_n\|,\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \|\wp_n - w_n\| = 0. \tag{3.38}$$

Next, let $\bar{\wp} = P_{\mathcal{F}}u$. Then, we show that $\limsup_{n\to\infty}\langle u - \bar{\wp}, \wp_n - \bar{\wp}\rangle \leq 0$. Consider a subsequence $\{\wp_{n_k}\}$ of $\{\wp_n\}$ such that

$$\limsup_{n\to\infty}\langle u - \bar{\wp}, \wp_n - \bar{\wp}\rangle = \lim_{n\to\infty}\langle u - \bar{\wp}, \wp_{n_k} - \bar{\wp}\rangle. \tag{3.39}$$

From the boundedness of $\{\wp_n\}$, we can find a subsequence $\{\wp_{n_k}\}$ that converges weakly to a point of $\mathcal{K}$. Without loss of generality, we may consider that $\wp_{n_k} \rightharpoonup \wp^\star$. Hence, from (3.38), we get $w_{n_k} \rightharpoonup \wp^\star$. Using Lemma 2.7 and (3.37), we also obtain $D\wp^\star = \wp^\star$; or equivalently, $\wp^\star \in F(D)$. Since $w_{n_k} \rightharpoonup \wp^\star$, it follows that $\wp^\star \in F(S)$. Now, suppose otherwise and consider $\wp^\star \in F(S)$. Then,

$$\big(I - \lambda_{n_k}(I - S)\big)\wp^\star \neq \wp^\star$$

and by Lemma 2.8, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\big\|\wp_{n_k} - \wp^\star\big\| &< \liminf_{n\to\infty}\big\|\wp_{n_k} - \big(I - \lambda_n(I - S)\big)\wp^\star\big\| \\ &\leq \liminf_{n\to\infty}\big(\big\|\wp_{n_k} - \wp^\star\big\| + \lambda_{n_k}\big\|(I - S)\wp^\star\big\|\big) \\ &\leq \liminf_{n\to\infty}\big\|\wp_{n_k} - \wp^\star\big\|,\end{aligned}$$

which is a contradiction. Consequently,

$$\wp^\star \in F(S) \cap \bigcap_{i=1}^{N} F(\Im_i). \tag{3.40}$$

From (3.40) in combination with the property of metric projection, we get

$$\limsup_{n\to\infty}\langle u - \bar{\wp}, \wp_n - \bar{\wp}\rangle = \lim_{n\to\infty}\langle u - \bar{\wp}, \wp_{n_k} - \bar{\wp}\rangle = \big\langle u - \bar{\wp}, \wp^\star - \bar{\wp}\big\rangle \leq 0. \tag{3.41}$$

Last, we shall prove that $\wp_n \to \wp^\star$ as $n \to \infty$. Now, from (1.11)

$$\begin{aligned}\|\wp_{n+1} - \bar{\wp}\|^2 &= \langle a_n u + b_n w_n + c_n D w_n - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle \\ &= \big\langle a_n(u - \bar{\wp}) + b_n(w_n - \bar{\wp}) + c_n(Dw_n - \bar{\wp}), \wp_{n+1} - \bar{\wp}\big\rangle \\ &\leq a_n\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle + b_n\|w_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\|\end{aligned}$$

$$
\begin{aligned}
&+ c_n\|Dw_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\| \\
&\le a_n\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle + b_n\|w_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\| \\
&\quad + c_n\|w_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\| \\
&\le a_n\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle + \frac{b_n}{2}\big(2\|\wp_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\|\big) \\
&\quad + \frac{c_n}{2}\big(2\|\wp_n - \bar{\wp}\|\|\wp_{n+1} - \bar{\wp}\|\big) \\
&\le a_n\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle + \frac{1 - a_n}{2}\big(\|\wp_n - \bar{\wp}\|^2 + \|\wp_{n+1} - \bar{\wp}\|^2\big),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\|\wp_{n+1} - \bar{\wp}\| &\le \left(\frac{1 - a_n}{1 + a_n}\right)\|\wp_n - \bar{\wp}\|^2 + \frac{a_n}{1 + a_n}\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle \\
&= \left(1 - \frac{2a_n}{1 + a_n}\right)\|\wp_n - \bar{\wp}\|^2 + \frac{a_n}{1 + a_n}\langle u - \bar{\wp}, \wp_{n+1} - \bar{\wp}\rangle. \qquad (3.42)
\end{aligned}
$$

It is not difficult to see that $\sum_{n=0}^{\infty} \frac{2a_n}{1+a_n} = \infty$. Using this fact, together with (3.41), (3.42) and Lemma 2.6, we get that $\wp_n \to \bar{\wp}$ as $n \to \infty$. The proof is completed. □

## 4 Application

The following theorems can easily be obtained from Theorem 3.7.

**Theorem 4.1** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be $\eta_i$-enriched nonexpansive maps and $S : \mathcal{K} \longrightarrow \mathcal{K}$ be an $(\eta, \beta)$-ESPN map for $\beta \in [0, 1)$. Let $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, for $i = 1, 2, \dots, N$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ with $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$ and the conditions* (a) *and* (b) *in Theorem 3.7.*

*Let $D$ be the D-map generated by the sequences $\{\Im_{\omega,i}\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$, where $\Im_{\omega,i} = (1 - \omega)I + \omega\Im_i$. Suppose $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(\Im_i) \neq \emptyset$. Let $\{\wp_n\}$ be a sequence as defined in* (1.11) *with the conditions* (i)–(v) *. Then, $\{\wp_n\}_{n=0}^{\infty}$ converges strongly to $\bar{\wp} = P_{\mathcal{F}}u$.*

**Lemma 4.2** [24] *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\Im : \mathcal{K} \longrightarrow \mathcal{H}$ be $\vartheta$-inverse strongly monotone map. Then, for all $\wp, \hbar \in \mathcal{K}$ and $\nu > 0$,*

$$
\begin{aligned}
\big\|(I - \nu\Im)\wp - (I - \nu\Im)\hbar\big\|^2 &= \big\|\wp - \hbar - \nu(\Im\wp - \Im\hbar)\big\|^2 \\
&= \|\wp - \hbar\|^2 - 2\nu\langle\Im\wp - \Im\hbar, \wp - \hbar\rangle + \nu^2\|\Im\wp - \Im\hbar\|^2 \\
&\le \|\wp - \hbar\|^2 - 2\nu(\nu - 2\vartheta)\langle\Im\wp - \Im\hbar, \wp - \hbar\|\Im\wp - \Im\hbar\rangle. \qquad (4.1)
\end{aligned}
$$

*Thus, if $0 < \nu \le 2\vartheta$, then $I - \nu\Im$ is a nonexpansive map.*

Using (4.1) and Lemma 4.2, the following result emerges as an immediate consequence of Theorem 4.1.

**Theorem 4.3** *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{K} \subset \mathcal{H}$ be closed and convex. Let $\{\mathcal{B}_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{H}$ be an $\vartheta_i$-inverse strongly monotone maps and let $S : \mathcal{K} \longrightarrow \mathcal{K}$ be an $(\eta, \beta)$-ESPN map for $\beta \in [0, 1)$. Let $\{\Im_i\}_{i=1}^N : \mathcal{K} \longrightarrow \mathcal{K}$ be defined $\Im_i\wp = P_{\mathcal{K}}(I - \nu_i\mathcal{B}_i)\wp$ for every*

$\wp \in \mathcal{K}$ *and* $\nu_i \in (0, 2\vartheta)$, *and let* $\tau_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, *for* $i = 1, 2, \ldots, N$, *where* $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0,1]$ *with* $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$ *and the conditions* (a) *and* (b) *in Theorem* 3.7.

*Let D be the D-map generated by the sequences* $\{\Im_{\omega,i}\}_{i=1}^N$ *and* $\{\tau_i\}_{i=1}^N$, *where* $\Im_{\omega,i} = (1-\omega)I + \omega\Im_i$. *Suppose* $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(\Im_i) \neq \emptyset$. *Let* $\{\wp_n\}$ *be a sequence as defined in* (1.11) *with the conditions* (i)–(v) . *Then*, $\{\wp_n\}$ *converges strongly to* $\bar{\wp} = P_{\mathcal{F}}u$.

## 5 Conclusion

In this paper, a method for finding common fixed points of a finite family of $(\eta_i, k_i)$-ESPC maps and $(\eta_i, \beta_i)$-ESPN maps have been introduced in the setup of a real Hilbert space. Further, strong convergence theorems of the proposed method under mild conditions on the control parameters have been established. The main results have been applied in proving strong convergence theorems for $\eta_i$-enriched nonexpansive, strongly inverse monotone, and strictly pseudononspreading maps. Some nontrivial examples have also been constructed to demonstrate the effectiveness of the proposed method.

**Author contributions**

I.K.A. made conceptualization, methodology and writing draft preparation. H.I. performed the formal analysis, writing-review and editing. D.I.I. made investigation, review and validation. All authors read and approved the final version.

**Funding**

This work does not receive any external funding.

**Data availability**

Not applicable.

## Declarations

**Ethics approval and consent to participate**

This article does not contain any studies with human participants or animals performed by any of the authors.

**Competing interests**

The authors declare no competing interests.

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Received: 13 June 2023 Accepted: 11 March 2024 Published online: 16 April 2024

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