Rasham et al. *Journal of Inequalities and Applications* (2024) 2024:57
https://doi.org/10.1186/s13660-024-03115-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Novel results for separate families of fuzzy-dominated mappings satisfying advanced locally contractions in b-multiplicative metric spaces with applications

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Abstract

The objective of this research is to present new fixed point theorems for two separate families of fuzzy-dominated mappings. These mappings must satisfy a unique locally contraction in a complete b-multiplicative metric space. Also, we have obtained novel results for families of fuzzy-dominated mappings on a closed ball that meet the requirements of a generalized locally contraction. This research introduces new and challenging fixed-point problems for families of ordered fuzzy-dominated mappings in ordered complete b-multiplicative metric spaces. Moreover, we demonstrate a new concept for families of fuzzy graph-dominated mappings on a closed ball in these spaces. Additionally, we present novel findings for graphic contraction endowed with graphic structure. These findings are groundbreaking and provide a strong foundation for future research in this field. To demonstrate the uniqueness of our novel findings, we provide evidence of their applicability in obtaining the common solution of integral and fractional differential equations. Our findings have resulted in modifications to several contemporary and classical results in the research literature. This provides further evidence of the originality and impact of our work.

Mathematics Subject Classification: 46S40; 54H25; 47H10; 26A33

Keywords: Fixed point; b-multiplicative metric space; Advanced locally contraction; Families of fuzzy-dominated mappings; Graph contraction; Ordered families of fuzzy mappings; Integral equation; Fractional differential equation

1 Introduction and preliminaries

In a broad range of mathematical, computing, economic, and engineering problems, the existence of a theoretical or practical solution can be compared to the occurrence of a fixed point (abbreviated as F$\mathcal{P}$). F$\mathcal{P}$ theory is a mixture of several branches of mathematics, such as mathematical analysis, general topology, and functional analysis. These branches provide numerous applications in pure and practical mathematics, including computer science, engineering, fuzzy theory, and game theory. Scientists widely use F$\mathcal{P}$

theory techniques to demonstrate the presence of solutions to problems in science that involve integral equations or differential equations. In 1922, Banach [10] proved a significant result in metric F$\mathscr{P}$ theory, which is now famously known as the Banach contraction principle. Due to its profound significance, well-known authors have presented various formulations and interpretations of his seminal result.

One of most important generalization of metric space is the concept of multiplicative metric spaces established by Ozavsar and Cevikel [27]. They proved some new F$\mathscr{P}$ results fulfilling contractive mappings in such spaces and a few related topological settings. In 2017, Ali et al. [3] introduced the conception of b-multiplicative metric space (*bMMS*) and achieved some new F$\mathscr{P}$ results with graph structure and also use their main theorem to achieve unique solution of Fredholm type multiplicative nonlinear integral equations. Recently, Rasham et al. [34] introduced novel F$\mathscr{P}$ theorems on *bMMS* with applications to non-linear integral and functional equations.

Wardowski [43] established an extension of Banach's contraction result and named F-contraction. After this, Acar et al. [1], Aydi et al. [8], Hussain et al. [14], Karapinar et al. [18], Padcharoen et al. [28] and Piri et al. [29] introduced novel extensions of F-contraction with significant applications. Weiss [44] and Butnariu [11] introduced the notion of fuzzy mappings and proved many important results in the field theory of F$\mathscr{P}$.

Heilpern [13] demonstrated a significant F$\mathscr{P}$ hypothesis for fuzzy contractions that become broader then Nadler's result [24]. Motivated from Heilpern's outcomes, F$\mathscr{P}$ theory for fuzzy contractive mappings by using the Hausdorff distance spaces has become more significant in different fields by a large number of authors [30, 35, 40]. In addition, Rasham et al. [32] established F$\mathscr{P}$ problems for families of fuzzy-dominated local contractions in b-metric-like spaces and use their main hypothesis to obtain the solution of integral equations. Further results in this direction can be found in [6, 16, 36, 37].

This article presents several generalized F$\mathscr{P}$ theorems for two separate families of fuzzy-dominated maps that satisfy a novel generalized locally contraction on a closed ball in complete *bMMS*. Additionally, we demonstrate new F$\mathscr{P}$ theorems for families of fuzzy-dominated mappings on a closed ball in ordered complete *bMMS*. Furthermore, we introduce a new concept for two families of fuzzy graph-dominated mappings on a closed ball in these spaces and present novel findings for graphic contraction endowed with graphic structure. Finally, we provide evidence of the originality and impact of our new findings by demonstrating applications to obtain the common solution of integral and fractional differential equations.

Let us begin by presenting our main result.

Definition 1.1 [3] Let $\mathscr{K}$ be a nonempty set and suppose that $\mathfrak{m} \geq 1$. If the following properties are satisfied, a function $d : \mathscr{K} \times \mathscr{K} \to [1, \infty)$ is referred to as a *bMMS* with coefficient $\mathfrak{m}$:

i. $d(e, f) > 1$ for all $e, f \in \mathscr{K}$ with $e \neq f$ and $d(e, f) = 1$, iff $e = f$;
ii. $d(e, f) = d(f, e)$ for all $e, f \in \mathscr{K}$;
iii. $d(e, z) \leq [d(e, f) \cdot d(f, z)]^{\mathfrak{m}}$ for all $e, f, z \in \mathscr{K}$;

The triplet $(\mathscr{K}, d, \mathfrak{m})$ is called a b-multiplicative metric space shortly as *bMMS*. Let $c \in \mathscr{K}$ and $r > 0$, $\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0, r)} = \{g \in \mathscr{K} : d(e, z) \leq r\}$ be a closed ball in *bMMS*.

Example 1.2 [3] Let $\mathscr{K} = [0,\infty)$ and a function $d : \mathscr{K} \times \mathscr{K} \to [1,\infty)$.

$$d(s,t) = a^{(s-t)^2},$$

where $a > 1$. Then, d is a *bMMS* on $\mathscr{K}$ with $\mathfrak{m} = 2$. It is noted here that d is certainly not a multiplicative metric on $\mathscr{K}$. Taking $a = 3$, $r = 3^4$ and $t = 1$, then $\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0, r)} = [0,3]$ is the closed ball in $\mathscr{K}$.

Definition 1.3 [3] Let $(\mathscr{K}, d)$ be a *bMMS*.

i. A sequence $\{s_n\}$ in $\mathscr{K}$ is convergent if a point $s \in \mathscr{K}$ exist such that $d(s_n, s) \to 1$ as $n \to +\infty$.
ii. A sequence $\{s_n\}$ is said b-multiplicative Cauchy iff $d(s_m, s_n) \to 1$ as $m, n \to +\infty$.
iii. Every multiplicative Cauchy sequence in $\mathscr{K}$ converges to some $s \in \mathscr{K}$, then $(\mathscr{K}, d)$ is said to be complete.

Definition 1.4 [34] Let $\mathscr{B}$ be a non-empty subset of $\mathscr{K}$ and $a \in \mathscr{K}$. Then, $f_0 \in \mathscr{B}$ is a best approximation in $\mathscr{B}$ if

$$d(a,\mathscr{B}) = d(a,f_0), \quad \text{where } d(a,\mathscr{B}) = \inf_{f_0 \in \mathscr{B}} d(a,f).$$

Here $\mathscr{P}(\mathscr{K})$ represents the set of all compact subsets of $\mathscr{K}$.

Definition 1.5 [34] Suppose $H_d : \mathscr{P}(\mathscr{K}) \times \mathscr{P}(\mathscr{K}) \to R^+$ be a function, defined by

$$H_d(\mathscr{L}, \mathscr{Z}) = \max\left\{\sup_{\delta \in \mathscr{L}} d(\delta, \mathscr{Z}), \sup_{\hbar \in \mathscr{Z}} d(\mathscr{L}, \hbar)\right\}.$$

Then, H_d is said as Hausdorff b-multiplicative metric on $\mathscr{P}(\mathscr{K})$.

Definition 1.6 [35] Let $\mathscr{K} \neq \{\phi\}$ and take $\mathcal{Q} : \mathscr{K} \to \mathscr{P}(\mathscr{K})$ as a mapping, $\mathscr{T} \subseteq \mathscr{K}$ and $\varphi : \mathscr{K} \times \mathscr{K} \to [0,+\infty)$, and $\mathcal{Q}$ is considered as φ_*-admissible on $\mathscr{T}$, if

$$\varphi_*(\mathcal{Q}x, \mathcal{Q}y) = \inf\left\{\varphi(x,y) : x \in \mathcal{Q}x, y \in \mathcal{Q}y\right\} \geq 1.$$

Definition 1.7 [35] Suppose $\mathscr{K} \neq \{\phi\}$ and $\xi : \mathscr{K} \to \mathscr{P}(\mathscr{K})$ is a mapping, $\mathscr{N} \subseteq A$, and $\varphi : \mathscr{K} \times \mathscr{K} \to [0,+\infty)$ is a function. If for each $\mathfrak{e} \in \mathscr{O}$, then ξ is referred as φ_*-dominated mapping on $\mathscr{N}$ if satisfies

$$\varphi_*(\mathfrak{e}, \xi\mathfrak{e}) = \inf\left\{\varphi(\mathfrak{e}, m) : m \in \xi\mathfrak{e}\right\} \geq 1.$$

Definition 1.8 [43] Let $(\mathscr{K}, d)$ be a metric space. A map $L : \mathscr{K} \to \mathscr{K}$ is known as an $\mathscr{F}$-contraction if there exists $\tau > 0$ so that for all $d, e \in \mathscr{K}$ with $d(Ld, Le) > 0$, satisfying the inequality given as:

$$\tau + \mathscr{F}\left(d(Ld, Le)\right) \leq \mathscr{F}\left(d(d, e)\right),$$

where $\mathscr{F} : \mathbb{R}_+ \to \mathbb{R}$ fulfills the given assumptions:

i. $\mathscr{F}$ is strictly increasing function;
ii. For each sequence $\{\gamma_n\}$ in $\mathbb{R}_+$, so that $\lim_{n\to+\infty}\gamma_n = 0$, iff $\lim_{n\to+\infty}\gamma_n = -\infty$;
iii. $\lim_{n\to+\infty}\gamma^{\mathscr{b}}\mathscr{F}(\gamma) = 0$, if there exists $\mathscr{b} \in (0,1)$.

Example 1.9 [35] The function $\varphi_* : A \times A \to [0,\infty)$ is given by

$$\varphi_*(p,q) = \begin{cases} 1 & \text{if } p > q, \\ \frac{1}{4} & \text{if } p \le q. \end{cases}$$

Consider the mappings $\mathrm{G}, R : A \to \mathscr{P}(A)$ defined by

$$\mathrm{G}s = [-4+s, -3+s] \quad \text{and} \quad Rm = [-2+m, -1+m],$$

respectively. Then G and R are φ_*-dominated, but they are not φ_*-admissible.

Lemma 1.10 [34] *Let* $(\mathscr{K},\mathscr{d})$ *be a bMMS and* $(\mathscr{P}(\mathscr{K}), H_{\mathscr{d}})$ *a dislocated Hausdorff bMMS on* $\mathscr{K}$. *For each* $\mathrm{E},\Gamma \in \mathscr{P}(\mathscr{K})$ *for all* $\mathscr{g} \in \mathrm{E}$, *and* $h_{\mathscr{g}} \in \Gamma$ *such that* $\mathscr{d}(\mathscr{g},\Gamma) = \mathscr{d}(\mathscr{g},h_{\mathscr{g}})$, *then,* $H_{\mathscr{d}}(\mathrm{E},\Gamma) \ge \mathscr{d}(\mathscr{g},h_{\mathscr{g}})$ *holds.*

Definition 1.11 [45] $F(\mathscr{K})$ represents the class of all fuzzy sets in $\mathscr{K}$ and a fuzzy set H is the function from $\mathscr{K}$ to $[0,1]$. If $f \in \mathscr{K}$, then $H(f)$ is called the grade of membership of the element f in H. Then, $[H]_\gamma$ denotes the γ-level set of H and is given as

$$[H]_\gamma = \{f : H(f) \ge \gamma\} \quad \text{where } 0 < \gamma \le 1,$$
$$[H]_0 = \overline{\{f : H(f) > 0\}}.$$

Now, we chose a subset from the family $F(\mathscr{K})$ of all fuzzy sets, a sub family with finer restrictions, i.e., the class of subfamily of the approximate quantities, signified by $W(\mathscr{K})$.

Definition 1.12 [41] A subset H of fuzzy set $\mathscr{K}$ is an approximate quantity iff its $\gamma-$ level set is convex subset of $\mathscr{K}$ for each $\gamma \in [0,1]$ and $\sup_{f\in\mathscr{K}} H(f) = 1$.

Definition 1.13 [41] Let $\mathscr{K}$ be a metric space and V an arbitrary set. A mapping from V to $W(\mathscr{K})$ is said to be a fuzzy map. Then, the mapping $Z : V \to W(\mathscr{K})$ is a fuzzy subset of $V \times \mathscr{K}, Z : V \times \mathscr{K} \to [0,1]$, in reference to $Z(e,f) = Z(e)(f)$.

Definition 1.14 [41] Let $Z : \mathscr{K} \to W(\mathscr{K})$ be a fuzzy mapping. A point $c \in \mathscr{K}$ is called a fuzzy $\mathrm{F}\mathscr{P}$ of Z if there exists $0 < \gamma \le 1$ such that $c \in [Zc]_\gamma$.

Example 1.15 [32] Let $\mathscr{K} = \{0,1,2\}$. A fuzzy mapping $Z : \mathscr{K} \to W(\mathscr{K})$ defined as

$$Z(0)(c) = Z(1)(c) = Z(2)(c) = \begin{cases} \frac{2}{3}, & \text{if } c = 2, \\ 0, & \text{if } c = 0,1. \end{cases}$$

For $\gamma \in (0,\frac{2}{3}]$, we have $[Zc]_\gamma = \{2\}$ for all $c \in \mathscr{K}$. Here, $2 \in \mathscr{K}$ is a fuzzy $\mathrm{F}\mathscr{P}$ of the mapping Z.

Definition 1.16 [35] Let B be a non- empty set, $R : A \to W(A)$ a fuzzy mapping, $V \subseteq B$ and $\alpha : B \times B \to [0, +\infty)$. Then R is called α_*-dominated fuzzy mapping on V, if for each $e \in V$ and $0 < \gamma \leq 1$,

$$\alpha_*\big(e, [Re]_\gamma\big) = \inf\big\{\alpha(e, l) : l \in [Re]_\gamma\big\} \geq 1.$$

We are now starting to present our main findings.

2 Main results

Let $(\mathscr{K}, \mathscr{d})$ be a *bMMS*, $\mathfrak{m}_0 \in \mathscr{K}$ and $\{\mathscr{P}_o : o \in \mathbb{N}^o\}$ and $\{\mathscr{Q}_e : e \in \mathbb{N}^e\}$ are two families of fuzzy mappings on $W(\mathscr{K})$. Moreover, let $\gamma, \delta : \mathscr{K} \to [0, 1]$ are two real functions. Let $\mathfrak{m}_1 \in [\mathscr{P}_1\mathfrak{m}_0]_{\gamma(\mathfrak{m}_0)}$ be an element such that $\mathscr{d}(\mathfrak{m}_0, [\mathscr{P}_1\mathfrak{m}_0]_{\gamma(\mathfrak{m}_0)}) = \mathscr{d}(\mathfrak{m}_0, \mathfrak{m}_1)$. Let $\mathfrak{m}_2 \in [\mathscr{Q}_2\mathfrak{m}_1]_{\delta(\mathfrak{m}_1)}$ be such that $\mathscr{d}(\mathfrak{m}_1, [\mathscr{Q}_2\mathfrak{m}_1]_{\delta(\mathfrak{m}_1)}) = \mathscr{d}(\mathfrak{m}_1, \mathfrak{m}_2)$, where $1 \in \mathbb{N}^o$ and $2 \in \mathbb{N}^e$. Let $\mathfrak{m}_3 \in [\mathscr{P}_3\mathfrak{m}_2]_{\gamma(\mathfrak{m}_2)}$ be such that $\mathscr{d}(\mathfrak{m}_2, [\mathscr{P}_3\mathfrak{m}_2]_{\gamma(\mathfrak{m}_2)}) = \mathscr{d}(\mathfrak{m}_2, \mathfrak{m}_3)$ and $\mathfrak{m}_4 \in [\mathscr{Q}_4\mathfrak{m}_3]_{\delta(\mathfrak{m}_1)}$ be such that $\mathscr{d}(\mathfrak{m}_3, [\mathscr{Q}_4\mathfrak{m}_4]_{\delta(\mathfrak{m}_4)}) = \mathscr{d}(\mathfrak{m}_3, \mathfrak{m}_4)$, where $3 \in \mathbb{N}^o$ and $4 \in \mathbb{N}^e$. Proceeding with this cycle, we find a sequence $\{\mathfrak{m}_n\}$ of points in $\mathscr{K}$ such that

$$\begin{aligned} &\mathfrak{m}_{2n+1} \in [\mathscr{P}_i\mathfrak{m}_{2n}]_{\gamma(\mathfrak{m}_{2n})} \quad \text{and} \quad \mathfrak{m}_{2n+2} \in [\mathscr{Q}_j\mathfrak{m}_{2n+1}]_{\delta(\mathfrak{m}_{2n+1})}, \\ &\quad i \in \mathbb{N}^o \text{ (odd natuarals)}, j \in \mathbb{N}^e \text{ (even naturals) for } n = 0, 1, 2, \dots. \end{aligned}$$

Also,

$$\begin{aligned} &\mathscr{d}\big(\mathfrak{m}_{2n}, [\mathscr{P}_i\mathfrak{m}_{2n}]_{\gamma(\mathfrak{m}_{2n})}\big) = \mathscr{d}(\mathfrak{m}_{2n}, \mathfrak{m}_{2n+1}), \\ &\mathscr{d}\big(\mathfrak{m}_{2n+1}, [\mathscr{Q}_j\mathfrak{m}_{2n+1}]_{\delta(\mathfrak{m}_{2n+1})}\big) = \mathscr{d}(\mathfrak{m}_{2n+1}, \mathfrak{m}_{2n+2}). \end{aligned}$$

$\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$ be the representation of above sequence. We say that $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$ is a sequence in $\mathscr{K}$ generated by $\mathfrak{m}_0$. We define $\mathscr{D}(u, v)$ as

$$\mathscr{D}_{(o,e)}(u, v) = \max\left\{\begin{matrix} \mathscr{d}(u, v), \mathscr{d}\big(u, [\mathscr{P}_o u]_{\gamma(u)}\big), \mathscr{d}\big(v, [\mathscr{Q}_e v]_{\delta(v)}\big) \\ \mathscr{d}\big(u, [\mathscr{Q}_e v]_{\delta(v)}\big), \mathscr{d}\big(v, [\mathscr{P}_o u]_{\gamma(u)}\big) \end{matrix}\right\}^{\hbar}.$$

Theorem 2.1 *Let $(\mathscr{K}, \mathscr{d})$ be a complete bMMS. Suppose there exists a function $\varphi : \mathscr{K} \times \mathscr{K} \to [0, +\infty)$. Let $\mathfrak{m}_0 \in \overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)} \subseteq \mathscr{K}$, $r > 0$, and $\{\mathscr{P}_o : o \in \mathbb{N}^o\}$ and $\{\mathscr{Q}_e : e \in \mathbb{N}^e\}$ be two families φ_*-dominated fuzzy mappings from $\mathscr{K}$ to $W(\mathscr{K})$ on $\overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)}$. Suppose $\tau > 0$ there exists $\hbar \in (0, \frac{1}{s})$ with $s > 1$ and $\mathscr{F}$ is a strictly increasing function satisfying*:

$$\tau + \big(H_{\mathscr{d}}\big([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e v]_{\delta(v)}\big)\big) \leq \mathscr{F}\big(\mathscr{D}_{(o,e)}(u, v)\big), \tag{2.1}$$

where $\gamma(u), \delta(v) \in (0, 1]$ for each, $u, v \in \overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)} \cap \{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$, $\varphi(u, v) \geq 1$, $H_{\mathscr{d}}([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e v]_{\delta(v)}) > 0$, such that

$$\mathscr{d}\big(\mathfrak{m}_0, [\mathscr{P}_o u]_{\gamma(u)}\big) \leq r^{\frac{1-sh}{s}}. \tag{2.2}$$

Then, $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$ is a sequence in $\overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)}$, $\varphi(m_\mathfrak{n}, m_{\mathfrak{n}+1}) \geq 1$ for all $\mathfrak{n} \in \mathbb{N} \cup \{0\}$ and $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\} \to \mathfrak{m}^* \in \overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)}$. Again, if inequality (2.1) holds for $\mathfrak{m}^*$ in $\overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)}$ with $\varphi(\mathfrak{m}_\mathfrak{n}, \mathfrak{m}^*) \geq 1$ or $\varphi(\mathfrak{m}^*, \mathfrak{m}_\mathfrak{n}) \geq 1$ for all naturals. Then, $\mathscr{P}_o$ and $\mathscr{Q}_e$ have common $F\mathcal{P}$ $\mathfrak{m}^*$ in $\overline{B_{\mathscr{d}\mathfrak{m}}(\mathfrak{m}_0, r)}$ for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

Proof Consider the sequence $\{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_n)\}$. From (2.2), we obtain

$$d(\mathfrak{m}_o,\mathfrak{m}_1)=d\big(\mathfrak{m}_{0,}[\mathcal{P}_1\mathfrak{m}_0]_{\gamma(\mathfrak{m}_0)}\big)\leq r^{\frac{1-sh}{s}}<r.$$

It follows that,

$$\mathfrak{m}_1\in\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}.$$

Let $\mathfrak{m}_2\cdots\mathfrak{m}_j\in\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}$ for some $j\in\mathbb{N}$. Suppose $j=2\ell+1$, where $\ell=1,2,\dots j-\frac{1}{2}$. Since $\{\mathcal{P}_o:o\in\mathbb{N}^o\}$ and $\{\mathcal{Q}_e:e\in\mathbb{N}^e\}$ are two families φ_*-dominated fuzzy mapping. So, $\varphi_*(\mathfrak{m}_{2\ell},[\mathcal{P}_o\mathfrak{m}_{2\ell}]_{\gamma(b_{2l})})\geq 1$ and $\varphi_*(\mathfrak{m}_{2\ell+1},[\mathcal{Q}_e\mathfrak{m}_{2\ell+1}]_{\delta(b_{2\ell+1})})\geq 1$. As $\varphi_*(\mathfrak{m}_{2\ell},[\mathcal{P}_o\mathfrak{m}_{2\ell}]_{\gamma(b_{2\ell})})\geq 1$, for all $o\in\mathbb{N}^o$ and $e\in\mathbb{N}^e$ this implies that $\inf\{\varphi(\mathfrak{m}_{2\ell},b):b\in[\mathcal{P}_o\mathfrak{m}_{2\ell}]_{\gamma_{(b_{2\ell})}}\}\geq 1$.

Also, $m_{2\ell+1}\in[\mathcal{P}_c\mathfrak{m}_{2\ell}]_{\gamma_{(b_{2\ell})}}$ for some $c\in\mathbb{N}^o$, so $\varphi(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+1})\geq 1$ and $m_{2\ell+2}\in[\mathcal{Q}_v\mathfrak{m}_{2\ell+1}]_{\delta(b_{2\ell+1})}$ for some $v\in\mathbb{N}^e$. Now, by using Lemma 1.10 and inequality (2.1), we have

$$\begin{aligned}
&\tau+\mathcal{F}\big(d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})\big)\\
&\quad\leq\tau+\mathcal{F}\big(H_d\big([\mathcal{P}_c\mathfrak{m}_{2\ell}]_{\gamma_{(b_{2\ell})}},[\mathcal{Q}_v\mathfrak{m}_{2\ell+1}]_{\delta_{(b_{2\ell+1})}}\big)\big)\leq\mathcal{F}\big(\mathcal{D}(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+1})\big)\\
&\quad\leq\mathcal{F}\left(\max\left\{\begin{matrix}d(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+1}),d(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+1}),d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})\\ d(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+2})^{\frac{1}{2s}},d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+1})\end{matrix}\right\}\right)^{k}\\
&\quad\leq\mathcal{F}\big(\max\big\{d(\mathfrak{m}_{2\ell,}\mathfrak{m}_{2\ell+1}),d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})\big\}\big)^{k}.
\end{aligned}$$

Thus,

$$\tau+\mathcal{F}d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})\leq\mathcal{F}\big(d(\mathfrak{m}_{2\ell},\mathfrak{m}_{2\ell+1})\big)^{\mu},$$

for each $\ell\in\mathbb{N}$, where $\mu=\frac{1-sk}{s}$. As $\mathcal{F}$ is a strictly increasing function, then

$$d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})<d(m_{2\ell},m_{2\ell+1})^{\mu}.\tag{2.3}$$

Similarly, if j is even, we have

$$d(\mathfrak{m}_{2\ell+2},\mathfrak{m}_{2\ell+3})<d(\mathfrak{m}_{2\ell+1},\mathfrak{m}_{2\ell+2})^{\mu}.\tag{2.4}$$

We have,

$$(\mathfrak{m}_j,\mathfrak{m}_{j+1})<d(\mathfrak{m}_{j-1},\mathfrak{m}_j)^{\mu}\quad\text{for all } j\in N.\tag{2.5}$$

Therefore,

$$d(\mathfrak{m}_j,\mathfrak{m}_{j+1})<d(\mathfrak{m}_{j-1},\mathfrak{m}_j)^{\mu}<d(\mathfrak{m}_{j-2},\mathfrak{m}_{j-1})^{\mu^2}<\cdots<d(\mathfrak{m}_0,\mathfrak{m}_1)^{j}.\tag{2.6}$$

Now,

$$\begin{aligned}
d(\mathfrak{m}_0,\mathfrak{m}_{j+1})&\leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{s}.d(\mathfrak{m}_1,\mathfrak{m}_2)^{s^2}.d(\mathfrak{m}_1,\mathfrak{m}_2)^{s^3}.\cdots.d(\mathfrak{m}_0,\mathfrak{m}_1)^{s^{j+1}}\\
&\leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{s}.d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu s^2}.d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^2s^3}.d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^3s^4}
\end{aligned}$$

$$\begin{aligned} & d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^4 s^5}.\cdots .d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^{\mathrm{j}} s^{\mathrm{j}+1}} \\ & \leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{s(\mu^0+s\mu^1+s^2\mu^2+s^2\mu^2+s^3\mu^3+\cdots+s^{\mathrm{j}}\mu^{\mathrm{j}})} \\ & \leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{s(\frac{1}{1-s\mu})}. \end{aligned}$$

Then, we have

$$d(\mathfrak{m}_0,\mathfrak{m}_{\mathrm{j}+1}) \leq r^{\frac{(1-s(\mu))\times s}{s\times(1-s(\mu))}} \leq r.$$

This shows that $\mathfrak{m}_{\mathrm{j}+1} \in \overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}$. It follows that for all $\mathfrak{n} \in \mathbb{N}$, by induction, $\mathfrak{m}_{\mathfrak{n}} \in \overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}$. Also, $\varphi(\mathfrak{m}_{\mathfrak{n}},\mathfrak{m}_{\mathfrak{n}+1}) \geq 1$, for any $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. Now, we deduce

$$d(\mathfrak{m}_{\mathfrak{n}},\mathfrak{m}_{\mathfrak{n}+1}) < d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^{\mathfrak{n}}}, \quad \text{for all } \mathfrak{n} \in \mathbb{N} \tag{2.7}$$

Now, $e, g (g > e)$ be the positive integer, then

$$\begin{aligned} d(\mathfrak{m}_e,\mathfrak{m}_g) & \leq d(\mathfrak{m}_e,\mathfrak{m}_{e+1})^s.d(\mathfrak{m}_{e+1},\mathfrak{m}_{e+2})^{s^2}.\cdots .d(\mathfrak{m}_{g-1},\mathfrak{m}_g)^{s^g\mu^{g-1}} \\ & \leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{s\mu^e}.d(\mathfrak{m}_0,\mathfrak{m}_1)^{\mu^{e+1}s^2}.\cdots .d(\mathfrak{m}_0,\mathfrak{m}_1)^{s^z\mu^{g-1}} \quad (\text{by } (2.7)) \\ & \leq d(\mathfrak{m}_0,\mathfrak{m}_1)^{(s\mu^e+s^2\mu^{e+1}+s^3\mu^{e+2}+s^4\mu^{e+3}+\cdots+s^g\mu^{g-1})} \\ & < d(\mathfrak{m}_0,\mathfrak{m}_1)^{(s\mu^e+s^2\mu^{e+1}+s^3\mu^{e+2}+\cdots)}, \\ d(m_e,m_g) & < d(\mathfrak{m}_0,\mathfrak{m}_1)^{(\frac{s\mu^e}{1-s\mu})}. \end{aligned}$$

As $e, g \to +\infty$, then $d(m_e,m_g) \to 1$. Therefore, $\{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_g)\}$ is a Cauchy sequence in $\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}$. So, there is a $\mathfrak{m}^* \in \overline{B_{d\mathfrak{m}}(\mathfrak{m}_0,r)}$ and $\{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_g)\} \to \mathfrak{m}^*$ such that $g \to +\infty$. Then,

$$\lim_{g\to+\infty}\left(\mathfrak{m}_g,\mathfrak{m}^*\right) = 1. \tag{2.8}$$

Now, by using the inequality, we have

$$d\left(\mathfrak{m}^*,\left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\delta(\mathfrak{m}^*)}\right) \leq d\left(\mathfrak{m}^*,\mathfrak{m}_{2z+1}\right)^s.d\left(\mathfrak{m}_{2z+1},\left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\delta(\mathfrak{m}^*)}\right)^s.$$

Using Lemma 1.10 and inequality (2.1), we obtain

$$d\left(\mathfrak{m}^*,\left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\gamma(\mathfrak{m}^*)}\right) \leq d\left(\mathfrak{m}^*,\mathfrak{m}_{2g+1}\right)^s.H_d\left([\mathcal{P}_o\mathfrak{m}_{2g}]_{\gamma(2g)},\left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\delta(\mathfrak{m}^*)}\right)^s. \tag{2.9}$$

By supposition $\varphi(\mathfrak{m}_g,\mathfrak{m}^*) \geq 1$. Suppose that $d(\mathfrak{m}^*,[\mathcal{Q}_e\mathfrak{m}^*]_{\delta(\mathfrak{m}^*)}) > 0$ and p is a non-negative integer that exists, such that $d(\mathfrak{m}_{\mathfrak{n}},[\mathcal{Q}_e\mathfrak{m}^*]_{\delta(\mathfrak{m}^*)}) > 0$, for all $g \geq p$, we have

$$\begin{aligned} & d\left(\mathfrak{m}^*,[\mathcal{Q}_e\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}\right) \\ & \quad < d(\mathfrak{m}^*,\mathfrak{m}_{2g+1})^s.\left(\max\left\{\begin{array}{c} d(\mathfrak{m}_{2g},\mathfrak{m}^*), d(\mathfrak{m}_{2g},[\mathcal{Q}_e\mathfrak{m}^*]_{\delta(\mathfrak{m}^*)}), d(\mathfrak{m}_{2g+1},[\mathcal{Q}_e\mathfrak{m}^*]_{\delta(\mathfrak{m}^*)}) \\ d(\mathfrak{m}_{2z},\mathfrak{m}_{2g+1})^{\frac{1}{2s}}, d(\mathfrak{m}_{2g+1},\mathfrak{m}_{2g+2}) \end{array}\right\}^p\right)^s. \end{aligned} \tag{2.10}$$

By taking lim $\mathfrak{n} \to +\infty$ and inequality (2.8) from both sides of (2.9), we have a result that is not generally true, $d(\mathfrak{m}^*, [\mathcal{Q}_e\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}) < d(\mathfrak{m}^*, [\mathcal{Q}_e\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)})^{ps}$. Our assumption is not true because $ps < 1$. Therefore, $d(\mathfrak{m}^*, [\mathcal{Q}_e\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}) = 1$ or $\mathfrak{m}^* \in [\mathcal{Q}_e\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}$. Similarly, using Lemma 1.10 and inequality (2.8), we can obtain either $d(\mathfrak{m}^*, [\mathcal{P}_o\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}) = 1$ or $\mathfrak{m}^* \in [\mathcal{P}_o\mathfrak{m}^*]_{\gamma(\mathfrak{m}^*)}$. Therefore, in $\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0, r)}$, $\mathcal{P}_o$ and $\mathcal{Q}_e$ admit a fuzzy F$\mathcal{P}$ that is $\mathfrak{m}^*$. Now, using the above multiplicative triangular inequality, we get

$$d\left(\mathfrak{m}^*, \mathfrak{m}^*\right) \leq \left[d\left(\mathfrak{m}^*, \left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\delta(\mathfrak{m}^*)}\right).d\left(\left[\mathcal{Q}_e\mathfrak{m}^*\right]_{\delta(\mathfrak{m}^*)}, \mathfrak{m}^*\right)\right]^s.$$

This indicate that $d(\mathfrak{m}^*, \mathfrak{m}^*) = 1$. □

Ran and Reuring [31] established the conception of ordered metric spaces and achieved well-known F$\mathcal{P}$ hypothesis in these spaces. Lateral, Arshad et al. [6], Rasham et al. [35], Shatanawi et al. [39] and Nieto et al. [26] showed F$\mathcal{P}$ results for the setting of ordered complete distance spaces.

Definition 2.2 [35] Let $\mathcal{L}$ be a non-empty set, $\preccurlyeq$ be a partial order on $B \subseteq \mathcal{K}$. We say that $a \preccurlyeq B$ whenever for all $b \in B$, we have $a \preccurlyeq b$. A family of mapping $\{\mathcal{P}_o : o \in \mathbb{N}^o\}$ from $\mathcal{K}$ to $W(\mathcal{K})$ is said to be fuzzy $\preccurlyeq$-dominated on B, if $a \preccurlyeq [\mathcal{P}_o a]_{\gamma(a)}$ for each $a \in \mathcal{K}$ and $\gamma \in (0, 1]$.

Now we prove upcoming theorem for fuzzy $\preccurlyeq$-dominated maps on $\{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_n)\}$ in an ordered complete *bMMS*.

Theorem 2.3 *Let $(\mathcal{K}, \preccurlyeq, d)$ be an ordered complete bMMS. Assume that there exists a function $\varphi : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$. Let $\mathfrak{m}_0 \in \overline{B_{d\mathfrak{m}}(\mathfrak{m}_0, r)} \subseteq \mathcal{K}$, $r > 0$, and $\{\mathcal{P}_o : o \in \mathbb{N}^o\}$ and $\{\mathcal{Q}_e : o \in \mathbb{N}^e\}$ be two families of φ_*-dominated fuzzy maps from $\mathcal{K}$ to $W(\mathcal{K})$ on $\overline{B_{d\mathfrak{m}}(\mathfrak{m}_0, r)}$. Suppose there exist $\tau > 0, \gamma(u), \delta(v) \in (0, 1]$ and $\mathcal{F}$ is a function of strictly increasing such that the following holds*:

$$\tau + \left(H_d\left([\mathcal{P}_o u]_{\gamma(u)}, [\mathcal{Q}_e v]_{\delta(v)}\right)\right) \leq \mathcal{F}\left(\max\left\{\begin{array}{c} d(u, v), d\left(u, [\mathcal{P}_o u]_{\gamma(u)}\right), d\left(v, [\mathcal{Q}_e v]_{\delta(v)}\right) \\ d\left(u, [\mathcal{Q}_e v]_{\delta(v)}\right), d\left(v, [\mathcal{P}_o u]_{\gamma(u)}\right) \end{array}\right\}\right)^{k}, \tag{2.11}$$

whenever , $v \in \{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_n)\}$, with either $u \preccurlyeq v$ or $v \preccurlyeq u$, and $H_d([\mathcal{P}_o u]_{\gamma(u)}, [\mathcal{Q}_e v]_{\delta(v)}) > 0$. Then $\{\mathcal{Q}_e\mathcal{P}_o(\mathfrak{m}_n)\} \to \mathfrak{m}^* \in \mathcal{K}$. Also, if (2.11) holds for $\mathfrak{m}^*$, $u_n \preccurlyeq \mathfrak{m}^*$ and $\mathfrak{m}^* \preccurlyeq u_n$ for all $n \in \{0, 1, 2, \dots\}$, then $\mathfrak{m}^*$ belongs to both $[\mathcal{Q}_e v]_{\delta(v)}$ and $[[\mathcal{P}_o u]_{\gamma(u)}$ for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

Proof Let $\varphi : \mathcal{K} \times \mathcal{K} \to [0, +\infty)$ be a mapping defined by $\varphi(u, v) = 1$ for all $u \in \mathcal{K}$ with $\preccurlyeq v$, and $\varphi(u, v) = 0$ for all elements $u, v \in \mathcal{K}$. As $\mathcal{P}_o$ and $\mathcal{Q}_e$ are the fuzzy-dominated mappings on $\mathcal{K}$, so $u \preccurlyeq [\mathcal{P}_o u]_{\gamma(u)}$ and $u \preccurlyeq [\mathcal{Q}_e u]_{\delta(u)}$ for all $u \in \mathcal{K}$. This implies that $u \preccurlyeq b$ for all $b \in [\mathcal{P}_o u]_{\gamma(u)}$ and $u \preccurlyeq e$ for all $u \in [\mathcal{Q}_e u]_{\beta(u)}$. So, $\varphi(u, b) = 1$ for all $b \in [\mathcal{P}_o u]_{\gamma(u)}$ and $\alpha(u, e) = 1$ for each $u \in [\mathcal{Q}_e u]_{\delta(u)}$. This implies that

$$\inf\left\{\varphi(u, v) : v \in [\mathcal{P}_o u]_{\gamma(u)}\right\} = 1, \quad \text{and} \quad \inf\{\varphi(u, v) : v \in [\mathcal{Q}_e u]_{\delta(u)} = 1.$$

Hence, $\varphi_*(u,[\mathscr{P}_o u]_{\gamma(u)}) = 1$, $\varphi_*(u,[\mathscr{Q}_e u]_{\delta(u)}) = 1$ for all $u \in \mathscr{K}$. So, $\mathscr{P}_o, \mathscr{Q}_e : \mathscr{K} \to W(\mathscr{K})$ are two families of φ_*-dominated fuzzy mappings. Furthermore, (2.11) exists and it can be expressed as

$$\tau + \mathscr{F}\left(H_d\left([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e g]_{\delta(v)}\right)\right) \leq \mathscr{F}\left(\mathscr{D}_{(o,e)}(u,v)\right),$$

for all elements u, v in $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$ with either $\varphi(u,v) \geq 1$ or $\varphi(v,u) \geq 1$. Then, by Theorem 2.1, $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$ is a sequence in $\mathscr{K}$ and $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\} \to \mathfrak{m}^* \in \mathscr{K}$. Now, $\mathfrak{m}_n, \mathfrak{m}^* \in \mathscr{K}$ and either $\mathfrak{m}_n \preccurlyeq \mathfrak{m}^*$, or $\mathfrak{m}^* \preccurlyeq \mathfrak{m}_n$ implies that either $\varphi(\mathfrak{m}_n, \mathfrak{m}^*)$, or $\varphi(\mathfrak{m}^*, \mathfrak{m}_n) \geq 1$. So, all requirements of Theorem 2.5 hold. Hence, by Theorem 2.5, $\mathfrak{m}^*$ is the common fuzzy F$\mathcal{P}$ of both $\mathscr{P}_o$ and $\mathscr{Q}_e$ in $\mathscr{K}$ and $d(\mathfrak{m}^*, \mathfrak{m}^*) = 0$. □

Example 2.4 Take $\mathscr{K} = [0, +\infty)$ and the mapping $d : \mathscr{K} \times \mathscr{K} \to \mathscr{K}$ defined by

$$d(u,v) = e^{|u-v|^2} \quad \text{for all } u, v \in \mathscr{K}.$$

Now for $u, v \in \mathscr{K}, \eta, \mu \in [0,1]$. Define $\mathscr{P}_o, \mathscr{Q}_e : \mathscr{K} \to W(\mathscr{K})$ as two distinct families of fuzzy mappings for $\mathscr{P}_o, \mathscr{Q}_e \in W(\mathscr{K})$, defined as

$$(\mathscr{P}_o u)(t) = \begin{cases} \eta & \text{if } 0 \leq t < \frac{u}{2l}, \\ \frac{\eta}{2} & \text{if } \frac{u}{2l} \leq t \leq \frac{3u}{4l}, \\ \frac{\eta}{4} & \text{if } \frac{3u}{4l} < t \leq u, \\ 0 & \text{if } u < t \leq 1, \end{cases}$$

and

$$(\mathscr{Q}_e u)(t) = \begin{cases} \mu & \text{if } 0 \leq t < \frac{u}{3z}, \\ \frac{\mu}{4} & \text{if } \frac{u}{3z} \leq t \leq \frac{2u}{3z}, \\ \frac{\mu}{6} & \text{if } \frac{2u}{3z} < t \leq u, \\ 0 & \text{if } u < t \leq 1. \end{cases}$$

Now, we consider

$$(\mathscr{P}_o u)_{\frac{\eta}{2}} = \left[\frac{u}{2l}, \frac{3u}{4l}\right] \quad \text{and} \quad (\mathscr{Q}_e u)_{\frac{\mu}{4}} = \left[\frac{u}{3z}, \frac{2u}{3z}\right].$$

Consider, $u_0 = \frac{1}{2}$, $r = 36$. Thus, $\overline{B_{dm}(u_0, r)} = [0, 5.5]$. So, we have

$$d\left(u_0, [\mathscr{P}_1 u_0]_{\frac{\eta}{2}}\right) = d\left(\frac{1}{2}, \left[\mathscr{P}_1 \frac{1}{2}\right]_{\frac{\eta}{2}}\right) = d\left(\frac{1}{2}, \frac{1}{8}\right) = \text{So,} \qquad u_1 = \frac{1}{8},$$

$$d\left(u_1, [\mathscr{Q}_2 u_1]_{\frac{\mu}{4}}\right) = d\left(\frac{1}{8}, \left[\mathscr{Q}_2 \frac{1}{8}\right]_{\frac{\eta}{2}}\right) = d\left(\frac{1}{8}, \left(\frac{1}{3}\right)\left(\frac{1}{8}\right)\right) = d\left(\frac{1}{2}, \frac{1}{24}\right).$$

As,

$$u_2 = \frac{1}{24} d\big(u_2, [\mathscr{P}_3 u_2]_{\frac{\eta}{2}}\big) = d\left(\frac{1}{24}, \left[\mathscr{P}_3 \frac{1}{24}\right]_{\frac{\eta}{2}}\right)$$
$$= d\left(\frac{1}{24}, \left(\frac{1}{4}\right)\left(\frac{1}{24}\right)\right) = d\left(\frac{1}{24}, \frac{1}{96}\right).$$

So, $u_3 = \frac{1}{96}$.

So, we achieved a sequence of the form $\{\mathscr{Q}_e\mathscr{P}_o(u_n)\} = \{\frac{1}{2}, \frac{1}{8}, \frac{1}{24}, \frac{1}{96}, \ldots\}$ in $\mathscr{K}$ generated by u_0. Let the function $\varphi : \mathscr{K} \times \mathscr{K} \to [0,\infty)$ be defined by

$$\varphi(u,v) = \begin{cases} 1 & \text{if } u > v, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

So, $u, v \in \overline{B_{dm}(u_0, r)} \cap \{\mathscr{Q}_e\mathscr{P}_o(u_n)\}$ with $\varphi(u,v) \geq 1$, we have

$$\begin{aligned} &Hd\big([\mathscr{P}_o u]_{\frac{\eta}{2}}, [\mathscr{Q}_e v]_{\frac{\mu}{4}}\big) \\ &= \max\Big\{\sup_{c \in [\mathscr{P}_o u]_{\frac{\eta}{2}}} d\big(c, [\mathscr{Q}_e v]_{\frac{\mu}{4}}\big), \sup_{d \in [\mathscr{Q}e]_{\frac{\mu}{4}}} d\big([\mathscr{P}_o u]_{\frac{\eta}{2}}, d\big)\Big\} \\ &= \max\left\{d\left(\frac{3u}{4l}, \left[\frac{v}{3m}, \frac{2v}{3m}\right]\right), d\left(\left[\frac{u}{2l}, \frac{3u}{4l}\right], \frac{2v}{3m}\right)\right\} \\ &= \max\left\{d\left(\frac{3u}{4l}, \frac{v}{3m}\right), d\left(\frac{u}{2l}, \frac{2v}{3m}\right)\right\} \\ &= \max\left\{e^{|\frac{3u}{4l} - \frac{v}{3m}|^2}, e^{|\frac{u}{2l} - \frac{2v}{3m}|^2}\right\} \\ &< \max\begin{pmatrix} d(u,v), d(u, \frac{u}{2l}), d(v, \frac{v}{3m}) \\ d(u, \frac{v}{3m})^{\frac{1}{2s}}, d(v, \frac{u}{2l}) \end{pmatrix}^{\hbar} \\ &< \max\begin{pmatrix} e^{|u-v|^2}, e^{|u - \frac{u}{2l}|^2}, e^{|v - \frac{v}{3m}|^2} \\ e^{(|u - \frac{v}{3m}|^2)^{\frac{1}{2s}}}, e^{|v - \frac{u}{2l}|^2} \end{pmatrix}^{\hbar}. \end{aligned}$$

Thus,

$$Hd\big([\mathscr{P}_o u]_{\frac{\eta}{2}}, [\mathscr{Q}_e v]_{\frac{\mu}{4}}\big) < \mathscr{D}_{(o,e)}(u,v),$$

this implies that, for any $\tau \in (0, \frac{12}{95})$ and the logarithm function $\mathscr{F}(t) = \ln(t)$, that is strictly increasing, we obtain

$$\tau + \mathscr{F}\big(_{Hd}\big([\mathscr{P}_o u]_{\frac{\eta}{2}}, [\mathscr{Q}_e v]_{\frac{\mu}{4}}\big)\big) \leq \mathscr{F}\big(\mathscr{D}_{(o,e)}(u,v)\big).$$

Now, we take $6, 11 \in \mathscr{K}$, then $\varphi(6,11) \geq 1$. But we have

$$\tau + \mathscr{F}\big(_{Hd}\big([\mathscr{P}_{o6}]_{\frac{\eta}{2}}, [\mathscr{Q}_{e11}]_{\frac{\mu}{4}}\big)\big) \geq \mathscr{F}\big(\mathscr{D}_{(o,e)}(6,11)\big).$$

So, our assumption is not satisfied on $\mathscr{K}$. Thus, the consequences of Theorem 2.1 exist by maps $\mathscr{P}_o$ and $\mathscr{Q}_e$ for $u, v \in \overline{B_{dm}(u_0, r)} \cap \{\mathscr{Q}_e\mathscr{P}_o(u_n)\}$ with $\varphi(u, v) \geq 1$. As a result, $\mathscr{P}_o$ and $\mathscr{Q}_e$ admit a fuzzy F$\mathscr{P}$ for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

3 Application to graph theory

In 2007, Jachymski [17] appeared the notion of graph contraction and investigated some novel F$\mathscr{P}$ theorems. Afterward, Asl et al. [7], Hussain et al. [15], and Shazad et al. [41] discussed concerning F$\mathscr{P}$ results endowed with graph theory.

Definition 3.1 [44] Let $\mathscr{W} = (\mathscr{X}(\mathscr{W}), \mathscr{Y}(\mathscr{W}))$ be a graph such that $\mathscr{X}(\mathscr{W}) = \mathscr{Z}$, $\mathscr{H} \subseteq \mathscr{K}$, and let $\mathscr{K}$ be a not-empty set. It is referred to as fuzzy graph dominated on $\mathscr{H}$ for a family of fuzzy mapping $\{\mathscr{P}_o : o \in \mathbb{N}^o\} : \mathscr{K} \to \mathscr{M}(\mathscr{K})$. When it turns out that (u, v) is an edge that belongs to $\mathscr{Y}(\mathscr{W})$ for each u that belongs to $\mathscr{H}$ and each v that belongs to $\mathscr{P}_o u$.

Theorem 3.2 *Let $(\mathscr{K}, d)$ be a complete bMMS endowed with a graph $\mathscr{W}$. Let $r \geq 0$, $m_0 \in \overline{\mathscr{B}_{dm}(m_0, r)}$, and $\{\mathscr{P}_o : o \in \mathbb{N}^o\}$, $\{\mathscr{Q}_e : e \in \mathbb{N}^e\} : \mathscr{K} \to \mathscr{M}(\mathscr{K})$. Suppose that, for some $\gamma(u), \delta(v) \in (0, 1]$, the following criteria are fulfilled*:

i. *Let $\mathscr{P}_o$ and $\mathscr{Q}_e$ be fuzzy graph dominated on $\overline{\mathscr{B}_{dm}(m_0, r)} \cap \{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$.*
ii. *The contractive requirements are satisfied by a strictly increasing function $\mathscr{F}$ and $\tau > 0$,*

$$\tau + \left(\mathscr{H}_d\left([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e v]_{\delta(v)}\right)\right) \leq \mathscr{F}\left(\mathscr{D}_{(o,e)}(u, v)\right), \tag{3.1}$$

whenever $m_0 \in \overline{\mathscr{B}_{dm}(m_0, r)} \cap \{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\}$, $(u, v) \in \mathscr{Y}(\mathscr{W})$ and $\mathscr{H}_d([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e v]_{\delta(v)}) > 0$.

iii. $d(m_0, [\mathscr{P}_o u]_{\gamma(u)}) \leq r^{\frac{1-s\hbar}{s}}$, *where $\hbar \in (0, \frac{1}{s})$ with $s > 1$.*

Then $\{\mathscr{Q}_e\mathscr{P}_o(m_n)\}$ is a sequence in $\overline{\mathscr{B}_{dm}(m_0, r)}$, $(m_n, m_{n+1}) \in \mathscr{Y}(\mathscr{W})$ and $\{\mathscr{Q}_e\mathscr{P}_o(\mathfrak{m}_n)\} \to c^*$. Furthermore, we assume that inequality (3.1) holds for c^* and $(m_n, c^*) \in \mathscr{Y}(\mathscr{W})$ or $(c^*, m_n) \in \mathscr{Y}(\mathscr{W})$ for all $n \in \mathbb{N} \cup \{0\}$. Then, in $\overline{\mathscr{B}_{dm}(m_0, r)}$ both the maps $\mathscr{P}_o$ and $\mathscr{Q}_e$ have a fuzzy F$\mathscr{P}$ in c^* for all $o \in \mathbb{N}^o$ and $e \in \mathbb{N}^e$.

Proof Define $\gamma : \mathscr{K} \times \mathscr{K} \to [0, +\infty)$ by

$$\gamma(u, \hbar) = \begin{cases} 1, & \text{if } \hbar \in \overline{\mathscr{B}d_\ell(m_0, r)}, (u, \hbar) \in \mathscr{Y}(\mathscr{W}), \\ 0, & \text{otherwise.} \end{cases}$$

If (ii) gives the guarantees that $\mathscr{P}_o$ and $\mathscr{Q}_e$ are two separate families of fuzzy graph-dominated maps on $\overline{\mathscr{B}_{dm}(m_0, r)}$, then for $u \in \overline{\mathscr{B}_{dm}(m_0, r)}$, $(u, \hbar) \in \mathscr{Y}(\mathscr{W})$ for each $\hbar \in [\mathscr{P}_o u]_{\gamma(u)}$ and $(u, \hbar) \in \mathscr{Y}(\mathscr{W})$ for every $\hbar \in [\mathscr{Q}_e v]_{\delta(v)}$. So, $\gamma(u, \hbar) = 1$ for all $\hbar \in [\mathscr{P}_o u]_{\gamma(u)}$ and $\gamma(u, \hbar) = 1$ for each $\hbar \in [\mathscr{Q}_e v]_{\delta(v)}$. It follows that $\inf\{\gamma(u, \hbar) : \hbar \in [\mathscr{P}_o u]_{\gamma(u)}\} = 1$ and $\inf\{\gamma(u, \hbar) : \hbar \in [\mathscr{Q}_e v]_{\delta(v)}\} = 1$. Therefore, $\varphi_*(u, [\mathscr{P}_o u]_{\gamma(u)}) = 1$, and $\varphi_*(u, [\mathscr{Q}_e v]_{\delta(v)}) = 1$ for all $\hbar \in \overline{\mathscr{B}_{dm}(m_0, r)}$. So, $\mathscr{P}_o, \mathscr{Q}_e : \mathscr{K} \to \mathscr{W}(\mathscr{K})$ are semi φ_*-dominated, fuzzy-dominated mappings on $\overline{\mathscr{B}_{dm}(m_0, r)}$. Inequality (3.1) can also be written as:

$$\tau + \left(\mathscr{H}_d\left([\mathscr{P}_o u]_{\gamma(u)}, [\mathscr{Q}_e \hbar]_{\delta(\hbar)}\right)\right) \leq \mathscr{F}\left(\mathscr{D}_{(o,e)}(u, v)\right),$$

where $u, \hbar \in \overline{\mathcal{B}_{dm}(m_0, r)} \cap \{Q_e \mathcal{P}_o(\mathfrak{m}_n)\}$, $\gamma(u, \hbar) \geq 1$ and $\mathcal{H}_d([\mathcal{P}_o u]_{\gamma(u)}, [Q_e \hbar]_{\delta(\hbar)}) > 0$. In addition, condition (iii) allows Theorem 2.1 to state that $\{Q_\pi \mathcal{P}_\emptyset(\mathfrak{m}_n)\}$ is a sequence in $\overline{\mathcal{B}_{dm}(m_0, r)}$ and that $\{Q_e \mathcal{P}_o(\mathfrak{m}_n)\} \to c^* \in \overline{\mathcal{B}_{dm}(m_0, r)}$. Finally, $m_n, c^* \in \overline{\mathcal{B}_{dm}(m_0, r)}$ and either $(m_n, c^*) \in \mathcal{Y}(\mathcal{W})$ or $(c^*, m_n) \in \mathcal{Y}(\mathcal{W})$ indicates that either $\gamma(m_n, c^*) \geq 1$ or $\gamma(c^*, m_n) \geq 1$. Therefore, Theorem 2.1 verifies all conditions. Therefore, according to Theorem 3.2, $\mathcal{P}_o$ and Q_e share a fuzzy F$\mathcal{P}$ called c^* in $\overline{\mathcal{B}_{dm}(m_0, r)}$ with $d(c^*, c^*) = 0$. □

4 Application to the integral equations

In the setting of different abstract spaces using generalized contractions, a specified number of well-known authors observed sufficient and compulsory conditions for the solution of linear and nonlinear first and second type of both (Fredholm and Volterra) type integrals in the field of F$\mathcal{P}$ theory. Rasham et al., [37] showed some new F$\mathcal{P}$ results for a couple of multifunction, and they utilized their fundamental outcome to analyze the important circumstances for solving integral equations. More great recent results with fundamental integral applications can be found here [2, 4, 5, 8, 33, 38].

Theorem 4.1 *Let $(\mathcal{K}, d)$ be a bMMS and $\mathcal{P}_o, Q_e : \mathcal{K} \to \mathcal{K}$ be two mappings. Suppose that there are $\tau > 0$ and a strictly increasing function $\mathcal{F}$ satisfying the following*:

$$\tau + \mathcal{F}\left(d\left([\mathcal{P}_o u]_{\gamma(u)}, [Q_e v]_{\delta(v)}\right)\right) \leq \left(\mathcal{D}_{(o,e)}(u, v)\right), \tag{4.1}$$

for each $u, v \in \{\mathcal{O}_n\}$ and $d(\mathcal{P}_o u, Q_e v) > 0$. Then $\{\mathcal{O}_n\} \to q \in \mathcal{K}$. Also, inequality (4.1) holds for q, therefore $\mathcal{P}_o$, Q_e have a F$\mathcal{P}$ q in $\mathcal{K}$.

Proof The proof of Theorem 4.1 is same as of Theorem 2.1.

In this part, we prove an application of F$\mathcal{P}$ Theorem 4.1 to obtain unique solution of given Volterra-type integral equations presented as;

$$g(i) = \int_o^i \mathcal{M}\left(i, c, g(c)\right) dc, \tag{4.2}$$

$$o(i) = \int_o^i \mathcal{N}\left(\ell, c, o(c)\right) dc \tag{4.3}$$

for all $i \in [0,1]$ and $\mathcal{M}$, $\mathcal{N}$ are the mappings from $[0,1] \times [0,1] \times \hat{C}([0,1], R_+)$ to R_+. We investigate solution of (4.2) and (4.3). Let $\hat{C}([0,1], R_+)$ be the space of all continuous function on $[0,1]$. For $g \in \hat{C}([0,1], R_+)$, a norm defined as: $\|g\|_\tau^2 = \sup_{i\in[0,1]}\{e^{|g(i)|^2} e^{-\tau i}\}$, where $\tau > 0$. Then define

$$d_\tau(g, o) = \left[\sup_{i\in[0,1]} \left\{e^{|g(i)-o(i)|} e^{-\tau i}\right\}\right]^2 = e^{\|g-o\|_\tau^2},$$

for each $g, o \in \hat{C}([0,1], R_+)$, with these conditions $\hat{C}([0,1], R_+, d_\tau)$ turns into a complete *bMMS.* □

We now demonstrate an important hypothesis to examine the arrangement for solving two non-linear integral equations.

Theorem *Suppose that (4.1) and (4.2) are satisfied*;

i. $\mathscr{M}, \mathscr{N} : [0,1] \times [0,1] \times \hat{C}([0,1], R_+) \to R$;
ii. *Define* $\mathscr{P}_o, \mathscr{Q}_e : \hat{C}([0,1], R_+) \to \hat{C}([0,1], R_+)$ *by*

$$(\mathscr{P}_o g)(\imath) = \int_o^{\imath} \mathscr{M}\big(\imath, c, g(c)\big)\, dc,$$
$$(\mathscr{Q}_e o)(\imath) = \int_o^{\imath} \mathscr{N}\big(\imath, c, o(c)\big)\, dc.$$

Assume that for $\tau > 0$ *such that*

$$e^{|\mathscr{M}(\imath,c,g(c)) - \mathscr{N}(\imath,c,o(c))|^2} \leq \frac{\tau \mathscr{D}_{(o,e)}(g, o)(e^{\mathscr{D}_{(o,e)}(g,o) \underline{e}_{-e}^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}})}{e^{\tau(1-\mathscr{b})}}$$

for each $\imath, c \in [0,1]$ *and* $g, o \in \hat{C}([0,1], R_+)$, *where*

$$\mathscr{D}_{(o,e)}(g, o) = \max \begin{Bmatrix} e^{|g(\imath) - o(\imath)|^2}, e^{|g(\imath) - (\mathscr{P}_o g(\imath))|^2}, e^{|g(\imath) - (\mathscr{Q}_e o(\imath))|^2} \\ \big(e^{|g(\imath) - (\mathscr{Q}_e o(\imath))|^2}\big)^{\frac{1}{2\jmath}}, e^{|o(\imath) - (\mathscr{P}_o g(\imath))|^2} \end{Bmatrix}^{k}.$$

Then, (4.2) *and* (4.3) *have a unique solution in* $\hat{C}([0,1], R_+)$.

Proof By condition (ii)

$$\begin{aligned} e^{|(\mathscr{P}_o g(\imath)) - (\mathscr{Q}_e o(\imath))|^2} &= \int_0^{\imath} e^{|\mathscr{M}(\imath,c,o(c)) - \mathscr{N}(\imath,c,o(c))|^2}\, d\mathscr{b} \\ &\leq \int_0^{\imath} \frac{\tau \mathscr{D}_{(o,e)}(g, o)(e^{\mathscr{D}_{(o,e)}(g,o) - e^{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}})}{e^{\tau(1-\mathscr{b})}}\, d\mathscr{b} \\ &\leq \frac{\tau \mathscr{D}_{(o,e)}(g, o)(e^{\mathscr{D}_{(o,e)}(g,o) - e^{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}})}{e^{\tau}} \int_0^{\imath} e^{\tau \mathscr{b}}\, d\mathscr{b} \\ &\leq \frac{\mathscr{D}_{(o,e)}(g, o)(e^{\mathscr{D}_{(o,e)}(g,o) \underline{e}_{-e}^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}})}{e^{\tau}} e^{\tau \imath} \end{aligned}$$

this implies,

$$\begin{aligned} &\frac{e^{|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)|^2} e^{-\tau \imath}}{\mathscr{D}_{(o,e)}(g, o)} \leq \frac{e^{\mathscr{D}_{(o,e)}(g,o) - e^{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}}}{e^{\tau}}, \\ &\frac{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}{\mathscr{D}_{(o,e)}(g, o)} \leq \frac{e^{\mathscr{D}_{(o,e)}(g,o) - e^{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}}}{e^{\tau}}, \\ &\frac{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}{e^{\mathscr{D}_{(o,e)}(g,o) - e^{e^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2}}}} \leq \frac{\mathscr{D}_{(o,e)}(g, o)}{e^{\tau}}. \end{aligned}$$

Taking ln of both sides, we get

$$\ln(e_{)+}^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2} \mathscr{D}_{(o,e)}(g, o) + e_{\leq}^{\|(\mathscr{P}_o g)(\imath) - (\mathscr{Q}_e o)(\imath)\|_\tau^2} \ln\big(\mathscr{D}_{(o,e)}(g, o)\big)_+ \ln e^{-\tau}.$$

This implies that

$$\tau + \ln\left(e^{\|(\mathscr{P}_o g)(\imath)-(\mathscr{Q}_e o)(\imath)\|_\tau^2}\right) + e^{\|(\mathscr{P}_o g)(\imath)-(\mathscr{Q}_e o)(\imath)\|_\tau^2} \leq \ln\left(\mathscr{D}_{(o,e)}(g, o)\right)_+ \mathscr{D}_{(o,e)}(g, o).$$

Hence, all conditions of Theorem 4.1 are true for $\mathscr{F}(o) = \ln(o^2 + o)$; $o > 0$, and $d_\tau(g, o) =_e^{\|g-o\|_\tau^2}$. So, (4.2) and (4.3) have common solution. □

5 Application to fractional differential equations

We apply our new findings to solve the fractional differential equations. Recently, many researchers used F$\mathscr{P}$ techniques to investigate the solution of fractional differential equations, considered in [5, 9, 25, 42].

Consider the space of continuous function $\hat{C}[0,1]$. Let $d(j, p) = e^{|j-p|^2}$ for all $j, p \in \hat{C}[0,1]$. The space $(\hat{C}[0,1], d)$is a complete *bMMS*. Let $\mathscr{P}_1, \mathscr{P}_2 : [0,1] \times \mathrm{R} \to \mathrm{R}$ be the continuous mappings. The equations of Caputo fractional derivatives of order μ will be examined.

$$\mathscr{D}^\mu r(q) = \mathscr{P}_1\left(q, r(q)\right) \tag{5.1}$$

with integral boundary condition $r(0) = 0$, $Ir(1) = r'(0)$, and

$$\mathscr{D}^\mu g(y) = \mathscr{P}_2\left(y, g(y)\right) \tag{5.2}$$

with integral boundary condition $g(0) = 0$, $Ig(1) = g'(0)$.

Then, Caputo derivative of fractional order μ can be written as follows

$$\mathscr{D}^\mu \mathscr{P}_1(q)) = \frac{1}{\Gamma(n-\mu)} \int_0^q (q-n)^{n-\mu-1} \mathscr{P}_1^n(q)\, dq$$

where $n - 1 < \mu < n$ and $n = [\mu] + 1$ and $I^\mu \mathscr{P}_1$ is given as

$$I^\mu \mathscr{P}_1(q) = \frac{1}{\Gamma(\mu)} \int_0^q (q-n)^{\mu-1} \mathscr{P}_1(q)\, dq \quad \text{with } \mu > 0.$$

The equation (5.1) can be written as

$$\begin{aligned} r(q) = {} & \frac{1}{\Gamma(\mu)} \int_0^q (q-n)^{\mu-1} \mathscr{P}_1\left(j, r(j)\right) dj \\ & + \frac{2q}{\Gamma(\mu)} \int_0^1 \int_0^j (n-y)^{\mu-1} \mathscr{P}_1\left(y, r(y)\right) dy\, dj \end{aligned}$$

and (5.2) can be modified as

$$\begin{aligned} g(\ell) = {} & \frac{1}{\Gamma(\mu)} \int_0^\ell (\ell-n)^{\mu-1} \mathscr{P}_2\left(j, r(j)\right) dj \\ & + \frac{2\ell}{\Gamma(\mu)} \int_0^1 \int_0^j (n-p)^{\mu-1} \mathscr{P}_2\left(y, r(y)\right) dp\, dj. \end{aligned}$$

Theorem 5.1 *Assume that*:

i. *For all* $c, g \in \hat{C}[0,1]$, *there exist* $\tau > 0$ *and* $e^{3\tau} < 1$,

$$\left|\mathscr{P}_1\big(y, c(y)\big)\,dy - \mathscr{P}_2\big(y, g(y)\big)\right| \leq \frac{\tau\Gamma(\mu+1)}{4}\left|c(y) - g(y)\right|;$$

ii. *There exist* $h, p \in \hat{C}[0,1]$ *and for all* $x, z \in \hat{C}[0,1]$,

$$h(x) = \frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_1\big(j, r(j)\big)\,dj + \frac{2q}{\Gamma(\mu)}\int_0^1\int_0^j (n-y)^{\mu-1}\mathscr{P}_1\big(y, r(y)\big)\,dy\,dj$$

and

$$p(z) = \frac{1}{\Gamma(\mu)}\int_0^\ell (\ell-n)^{\mu-1}\mathscr{P}_2\big(j, r(j)\big)\,dj + \frac{2\ell}{\Gamma(\mu)}\int_0^1\int_0^j (n-p)^{\mu-1}\mathscr{P}_2\big(y, r(y)\big)\,dp\,dj.$$

Then, equations (5.1) *and* (5.2) *have a unique solution in* $\hat{C}[0,1]$.

Proof The mappings $\mathscr{P}_o, \mathscr{Q}_e : \hat{C}[0,1] \to \hat{C}[0,1]$ defined by

$$\mathscr{P}_o\big(r(q)\big) = \left(\frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_1(j, r(j))\,dj + \frac{2q}{\Gamma(\mu)}\int_0^1\int_0^j (n-y)^{\mu-1}\mathscr{P}_1(y, r(y))\,dy\,dj\right)$$

and

$$\mathscr{N}\big(p(z)\big) = \left(\frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_2(j, r(j))\,dj + \frac{2q}{\Gamma(\mu)}\int_0^1\int_0^j (n-y)^{\mu-1}\mathscr{P}_2(y, r(y))\,dy\,dj\right).$$

By (ii), it can be shown that there are $h, p \in \hat{C}[0,1]$ so that $h_n = \mathscr{P}_{\text{on}}(h)$ and $p_n = \mathscr{Q}_{\text{en}}(p)$. The continuity of $\mathscr{P}_1$ and $\mathscr{P}_2$ leads the continuity of mappings $\mathscr{P}_o$ and $\mathscr{Q}_e$ on $\hat{C}[0,1]$. All the hypothesis of Theorem 5.1 are satisfied. For this, we have

$$e^{|\mathscr{P}_o(r(q)) - \mathscr{Q}_e(p(z))|^2} = e^{\left|\begin{array}{l}\frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_1(j, r(j))\,dj \\ \quad - \frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_2(j, r(j))\,dj \\ \quad + \frac{2q}{\Gamma(\mu)}\int_0^1\int_0^j (n-y)^{\mu-1}\mathscr{P}_1(y, r(y))\,dy\,dj \\ \quad - \frac{2q}{\Gamma(\mu)}\int_0^1\int_0^j (n-y)^{\mu-1}\mathscr{P}_2(y, r(y))\,dy\,dj\end{array}\right|^2}.$$

By using the result $e^x = e^y$ iff $x = y$, we deduce that

$$\left|\mathscr{P}_o\big(r(q)\big) - \mathscr{Q}_e\big(p(z)\big)\right|^2 = \left|\frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_1\big(j, r(j)\big)\,dj - \frac{1}{\Gamma(\mu)}\int_0^q (q-n)^{\mu-1}\mathscr{P}_2\big(j, r(j)\big)\,dj\right.$$

$$+\frac{2q}{\Gamma(\mu)}\int_0^1\int_0^{j}(n-y)^{\mu-1}\mathscr{P}_1\big(y,r(y)\big)\,dy\,dj$$
$$-\frac{2q}{\Gamma(\mu)}\int_0^1\int_0^{j}(n-y)^{\mu-1}\mathscr{P}_2\big(y,r(y)\big)\,dy\,dj\bigg|^2.$$

Taking square root of both sides, we have

$$\begin{aligned}\big|\mathscr{P}_o\big(r(q)\big)-\mathscr{Q}_e\big(p(z)\big)\big| = \bigg|&\frac{1}{\Gamma(\mu)}\int_0^{q}(q-n)^{\mu-1}\mathscr{P}_1\big(j,r(j)\big)\,dj\\&-\frac{1}{\Gamma(\mu)}\int_0^{q}(q-n)^{\mu-1}\mathscr{P}_2\big(j,r(j)\big)\,dj\\&+\frac{2q}{\Gamma(\mu)}\int_0^1\int_0^{j}(n-y)^{\mu-1}\mathscr{P}_1\big(y,r(y)\big)\,dy\,dj\\&-\frac{2q}{\Gamma(\mu)}\int_0^1\int_0^{j}(n-y)^{\mu-1}\mathscr{P}_2\big(y,r(y)\big)\,dy\,dj\bigg|.\end{aligned}$$

Implies that

$$\begin{aligned}&\leq\bigg|\int_0^{q}\bigg(\frac{1}{\Gamma(\mu)}(q-n)^{\mu-1}\mathscr{P}_1\big(j,r(j)\big)-\frac{1}{\Gamma(\mu)}(q-n)^{\mu-1}\mathscr{P}_2\big(j,r(j)\big)\bigg)\,dj\bigg|\\&\quad+\bigg|\int_0^1\int_0^{j}\bigg(\frac{2q}{\Gamma(\mu)}(n-y)^{\mu-1}\mathscr{P}_1\big(y,r(y)\big)-\frac{2q}{\Gamma(\mu)}(n-y)^{\mu-1}\mathscr{P}_2\big(y,r(y)\big)\bigg)\,dy\,dj\bigg|\\&\leq\frac{1}{\Gamma(\mu)}.\frac{\tau.\Gamma(\mu+1)}{4}\int_0^{q}(q-n)^{\mu-1}\big(h(z)-p(z)\big)\,dz\\&\quad+\frac{2}{\Gamma(\mu)}.\frac{\tau.\Gamma(\mu+1)}{4}\int_o^1\int_0^{j}(n-y)^{\mu-1}\big(h(y)-p(y)\big)\,dy\,dj\\&\leq\frac{1}{\Gamma(\mu)}.\frac{\tau.\Gamma(\mu+1)}{4}.|j-p|\int_0^{q}(q-n)^{\mu-1}\,dz\\&\quad+\frac{2}{\Gamma(\mu)}.\frac{\tau.\Gamma(\mu).\Gamma(\mu+1)}{4\Gamma(\mu).\Gamma(\mu+1)}.|j-p|\int_o^1\int_0^{j}(n-y)^{\mu-1}\,dy\,dj\\&\leq\frac{\tau.\Gamma(\mu).\Gamma(\mu+1)}{4\Gamma(\mu).\Gamma(\mu+1)}.|j-p|+2\tau\mathscr{B}(\mu+1,1)\frac{\Gamma(\mu).\Gamma(\mu+1)}{4\Gamma(\mu).\Gamma(\mu+1)}.|j-p|,\\&\big|\mathscr{P}_o\big(r(q)\big)-\mathscr{Q}_e\big(p(z)\big)\big|\leq\frac{\tau}{4}|j-p|+\frac{\tau}{2}|j-p|<\tau.|j-p|,\\&\big|\mathscr{P}_o\big(r(q)\big)-\mathscr{Q}_e\big(p(z)\big)\big|<\tau|j-p|,\end{aligned}$$

where $\mathscr{B}$ is a beta function. Again, using the inequality if $x<y$, then $e^x<e^y$, we have

$$e^{|\mathscr{P}_o(r(q))-\mathscr{Q}_e(p(z))|}<e^{\tau|j-p|}.$$

Squaring both sides, we get

$$e^{|\mathscr{P}_o(r(q))-\mathscr{Q}_e(p(z))|^2}<e^{2\tau}.e^{|j-p|^2}.$$

For any $\tau>0$ and $e^{3\tau}<1$, it implies that $e^{2\tau}<e^{-\tau}$, then we deduce that

$$e^{|\mathscr{M}_\emptyset(r(q))-\mathscr{Q}_e(p(z))|^2}<e^{-\tau}.e^{|j-p|^2}. \tag{5.3}$$

The expression for the inequality (5.3) can be written as

$$\begin{aligned} & d\left(\mathcal{P}_o\left(r(q)\right), \mathcal{N}_\pi\left(p(z)\right)\right) < e^{-\tau}\mathcal{D}_{(o,e)}(u,v), \\ & e^{\tau} d\left(\mathcal{P}_o\left(r(q)\right), \mathcal{Q}_e\left(p(z)\right)\right) < \mathcal{D}_{(o,e)}(u,v), \end{aligned} \tag{5.4}$$

for all $j, p \in \hat{C}[0,1]$ and define $\mathcal{F}(q) = \ln q$, then the inequality (5.4) can be expressed as,

$$\begin{aligned} & \ln\left(e^{\tau} d\left(\mathcal{P}_o\left(r(q)\right), \mathcal{Q}_e\left(p(z)\right)\right)\right) \le \ln\left(\mathcal{D}_{(o,e)}(u,v)\right), \\ & \ln e^{\tau} + \ln\left(d\left(\mathcal{P}_o\left(r(q)\right), \mathcal{Q}_e\left(p(z)\right)\right)\right) \le \ln\left(\mathcal{D}_{(o,e)}(u,v)\right), \\ & \tau + \mathcal{F}\left(d\left(\mathcal{P}_o\left(r(q)\right), \mathcal{Q}_e\left(p(z)\right)\right)\right) \le \mathcal{F}\left(\mathcal{D}_{(o,e)}(u,v)\right). \end{aligned}$$

All assumptions of Theorem 2.1 hold. So, the maps $\mathcal{P}_o$ and $\mathcal{Q}_e$ have admitted a common fuzzy F$\mathcal{P}$. Hence, (5.1) and (5.2) have a common solution. □

6 Conclusion

In this manuscript, we prove some new F$\mathcal{P}$ results for two distinct families of fuzzy-dominated mappings verifying a novel generalized locally contraction on a closed ball in complete *bMMS*. New F$\mathcal{P}$ results for families of ordered fuzzy-dominated mappings are also introduced for ordered complete *bMMS*. Also, a new concept for two families of fuzzy graph-dominated mappings is demonstrated on closed ball in such spaces and some novel findings for graphic contraction endowed with graphic structure presented. Finally, to show the originality of our new findings, we prove applications to obtain the common solution of integral and fractional differential equations. Furthermore, this article expands upon and enhances the findings of Rasham et al. [32, 34, 35], Karapinar et al. [18], Piri et al. [29], Moussaoui et al. [21–23] and other related research (click here for more details [2, 12, 19, 20, 26, 30, 37, 40, 41, 43]). Our work can be further improved in the future by investigating intuitionistic fuzzy maps, L-fuzzy maps, and bipolar fuzzy maps.

Acknowledgements
The authors would like to thank the editors and reviewers for their valuable comments and suggestions to refine and improve our paper. The third and fifth authors thank Prince Sultan University for supporting this paper.

Funding
This paper has been funded by Prince Sultan University through TAS lab.

Data availability
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Investigation, T. R; Methodology, T. R and F. H; Supervision, R. P. A and W. S.; Writing – original draft, T. R, R. Q. and F. H; Writing – review & editing, R. P. A and W. S.

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Received: 6 November 2023 Accepted: 11 March 2024 Published online: 16 April 2024

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