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The discrete analogue of high-order differential operator and its application to finding coefficients of optimal quadrature formulas

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Abstract

The discrete analog of the differential operator plays a significant role in constructing interpolation, quadrature, and cubature formulas. In this work, we consider a discrete analog $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 1$ designed specifically for even natural numbers m. The operator's effectiveness in constructing an optimal quadrature formula in the $L_2^{(2,0)}(0,1)$ space is demonstrated. The errors of the optimal quadrature formula in the $W_2^{(2,1)}(0,1)$ space and in the $L_2^{(2,0)}(0,1)$ space are compared numerically. The numerical results indicate that the optimal quadrature formula constructed in this work has a smaller error than the one constructed in the $W_2^{(2,1)}(0,1)$ space.

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Keywords: Differential operator; Discrete analogue; Hilbert space; Discrete argument functions; Optimal quadrature formula

1 Introduction

Quadrature formulas are widely used in various branches of mathematics and their applications. When obtaining approximations of integrals, a vital role is played by the general requirement that quadrature formulas approximate the given definite integrals as best as possible. Such quadrature formulas can be obtained, for example, using variational principles. Therefore, the problem of constructing optimal quadrature formulas for classes of differentiable functions using variational methods is one of the urgent problems of computational mathematics. When it comes to optimizing numerical integration formulas in the variational approach, the main task is to find the minimum norm of the error functional ℓ on a specified function space. This problem can be solved by utilizing the nodes and coefficients of a quadrature formula. The process of finding the minimum value of the error functional norm through the coefficients while keeping the nodes fixed is referred to as the Sard problem. You can find more information on this topic in [1]. The resulting formula is known as the optimal quadrature formula in the sense of Sard. One of the techniques for constructing optimal quadrature formulas is the Sobolev method.

The Sobolev method involves using a discrete analogue of a linear differential operator. By utilizing this approach, we can discover analytical shapes of the coefficients of the optimal quadrature, cubature and interpolation formulas. In [1, 2], S.L. Sobolev studied the problem of finding the minimum of the norm of the error functional of the cubature formulas in $L_2^{(m)}$ spaces, in which he obtained a system of linear equations of the Wiener–Hopf type with respect to the coefficients. The uniqueness and existence of this system's solution were proven, along with an algorithm provided for discovering the analytical coefficients of optimal cubature formulas. For this, Sobolev determined and studied the discrete analog $D_{hH}^{(m)}[\beta]$ of the polyharmonic operator Δ^m. Constructing the discrete operator $D_{hH}^{(m)}[\beta]$ for $n \geq 2$ variables is a highly complex issue. The one-dimensional discrete operator $D_h^{(m)}[\beta]$ was constructed by Z.Z. Zhamalov [3] and K.M. Shadimetov [4].

Using the discrete analog $D_h^{(m)}[\beta]$ of the differential operator d^{2m}/dx^{2m} in the space $L_2^{(m)}(0,1)$, the following results were obtained:

– Optimal quadrature formulas were constructed;

– The weighted optimal quadrature formula was obtained [5, 6];

– The Euler–Maclaurin type optimal quadrature formulas were constructed in the work [7, 8];

– Hermitian-type optimal quadrature and interpolation formulas were constructed in the work [9, 10];

– The problem of constructing D^m- splines were solved in the work [11];

– Optimal quadrature formulas were derived in [12] for approximating Fourier coefficients;

– In the space $L_2^{(m)}(0,1)$, the optimal quadrature and interpolation formulas, splines, which are exact for any algebraic polynomial of degree $(m-1)$, were constructed.

The paper [13] introduced a discrete analogue, $D_{m,W}(h\beta)$, of the differential operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ in the Hilbert space $W_2^{(m,m-1)}$. The construction was used to obtain the following results:

–A set of mathematical formulas, which includes the optimal quadrature formula and interpolation spline, was constructed and discussed in the following works [14, 15], and [16].

– The optimal quadrature formulas for the Fourier coefficients were obtained [17].

– The optimal quadrature formulas and splines in the space $W_2^{(m,m-1)}$ are exact for any polynomial of degree $(m-2)$ and exponential function e^{-x}.

A discrete analog $D_{m,K}[\beta]$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$ in the Hilbert space $K_2(P_m)$ was constructed in the work [18], and the following results were obtained using it:

– Construct optimal quadrature formulas and interpolation splines that minimize a certain semi-norm in a given space (see, [19, 20]);

– Constructed formulas are exact for any algebraic polynomial of degree $(m-3)$ and trigonometric functions $\sin(\omega x)$ and $\cos(\omega x)$.

A discrete analog of the differential operator $\frac{d^{2m}}{dx^{2m}} - 1$ was constructed in the Hilbert space $W_2^{(m,0)}$ in [21]. This work used the analog to obtain the following result:

– the optimal quadrature and interpolation formulas, which are exact to the exponential-trigonometric functions, were constructed in [22, 23];

– In addition, a natural spline function that gives a minimum to the semi-norm in the corresponding space was found.

Further, in the work [24] discrete analogs of differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$ (for m-even) and their properties were studied.

As can be seen from the above results, the optimal quadrature and interpolation formulas were constructed using the discrete analogs of the differential operators constructed in various Hilbert spaces. As a result, analytical forms of optimal coefficients were found. Therefore, in this paper, we construct a discrete analog to the differential operator to find the analytical expressions for the coefficients that give a minimum to the norm of the error functional in $L_2^{(m,0)}(0,1)$ space.

Here, we use functions with a discrete argument and the corresponding operations on them ([1], Chapter VII). For completeness, we present some of the definitions.

Let $\beta \in \mathbb{Z}$, $h = \frac{1}{N}$, and $N = 1, 2, \ldots$. Suppose that $\varphi(x)$ and $\psi(x)$ are real-valued functions defined on the real line $\mathbb{R}$.

Definition 1 The function $\varphi(h\beta)$ is *a function of discrete argument* if it is defined on a set of integer values of β.

For simplicity, discrete argument functions are sometimes called discrete functions. The domain of the definition of a discrete function $\varphi(h\beta)$ is the set of all integer points, and the functions $\varphi(h\beta)$ themselves are considered as real-valued.

Definition 2 The formula for the inner product of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by:

$$\left[\varphi(h\beta), \psi(h\beta)\right] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right-hand side of the last equality converges absolutely.

Definition 3 *The convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the following inner product

$$\varphi(h\beta) * \psi(h\beta) = \left[\varphi(h\gamma), \psi(h\beta - h\gamma)\right] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Definition 4 The function $\overset{\sqcap}{\varphi}(x) = \sum_{\beta=-\infty}^{+\infty} \varphi(h\beta)\delta(x - h\beta)$ is called *harrow-shaped function* corresponding to the discrete argument function $\varphi(h\beta)$, where $\delta(x)$ is the Dirac delta-function.

We want to locate a discrete function, denoted by $D_m(h\beta)$, which fulfils the following equation

$$D_m(h\beta) * G_m(h\beta) = \delta_d(h\beta), \tag{1}$$

where

$$G_m(h\beta) = -\frac{\mathrm{sign}(h\beta)}{2m}\sum_{k=1}^{m} e^{h\beta\cdot\cos\frac{(2k-1)\pi}{2m}}\cdot\cos\left(h\beta\cdot\sin\frac{(2k-1)\pi}{2m}+\frac{(2k-1)\pi}{2m}\right), \tag{2}$$

and $\delta_d(h\beta)$ is the discrete delta function defined by a function that returns the value 1 only if the input is zero and returns the value 0 for all other inputs.

The discrete function $D_m(h\beta)$ plays a vital role in calculating the coefficients of an optimal quadrature in the space $L_2^{(m,0)}(0,1)$. Here $L_2^{(m,0)}(0,1)$ is the class of functions φ defined on the interval $[0,1]$, which have an absolutely continuous $(m-1)$th derivative on $[0,1]$, and the mth generalized derivative is in $L_2(0,1)$. Note that the equation (1) is a discrete analog of the equation

$$\left(\frac{d^{2m}}{dx^{2m}}+1\right)G_m(x) = \delta(x), \tag{3}$$

where m is an even natural number,

$$G_m(x) = -\frac{\mathrm{sign}(x)}{2m}\sum_{k=1}^{m} e^{x\cdot\cos\frac{(2k-1)\pi}{2m}}\cdot\cos\left(x\cdot\sin\frac{(2k-1)\pi}{2m}+\frac{(2k-1)\pi}{2m}\right)$$

and $\delta(x)$ is the Dirac's delta-function.

Furthermore, the discrete function $D_m(h\beta)$ has properties similar to the differential operator $\frac{d^{2m}}{dx^{2m}}+1$. The zeros of the discrete operator $D_m(h\beta)$ coincide with the discrete functions corresponding to the zeros of the operator $\frac{d^{2m}}{dx^{2m}}+1$.

In this paper, we present the following structure. Firstly, in Sect. 2, we introduce some well-known formulas and auxiliary results that are necessary for constructing the discrete function $D_m(h\beta)$. After that, in Sect. 3, we focus on constructing a discrete analog of $D_m(h\beta)$ for the differential operator $\frac{d^{2m}}{dx^{2m}}+1$. In Sect. 4, we use the discrete function $D_2(h\beta)$ to construct optimal quadrature formulas in the sense of Sard in $L_2^{(2,0)}$ space. In Sect. 5, we provide numerical findings to support our analysis.

2 Known formulas and auxiliary results

In this section, we present some well-known formulas (see, for example, [1, 25]) and auxiliary results that we use in constructing a discrete analog $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}+1$.

The Fourier transforms of the function φ are defined

$$F\big[\varphi(x)\big](p) = \int_{-\infty}^{+\infty}\varphi(x)e^{2\pi ipx}\,dx, \qquad F^{-1}\big[\varphi(p)\big](x) = \int_{-\infty}^{+\infty}\varphi(x)e^{-2\pi ipx}\,dp.$$

For the Fourier transform of the product and convolution of the functions φ and ψ, the following hold:

$$F[\varphi*\psi] = F[\varphi]\cdot F[\psi], \tag{4}$$

$$F[\varphi\cdot\psi] = F[\varphi]*F[\psi], \tag{5}$$

where $*$ is the convolution operator. The convolution of two functions φ and ψ defined as follows

$$(\varphi * \psi)(x) = \int_{-\infty}^{+\infty} \varphi(x-y)\psi(y)\,dy = \int_{-\infty}^{+\infty} \varphi(y)\psi(x-y)\,dy.$$

The Fourier transform of the delta function and its derivatives follows the following rules:

$$F\big[\delta(x)\big] = 1, \qquad F\big[\delta^{(\alpha)}(x)\big] = (-2\pi ip)^{\alpha}. \tag{6}$$

We also use the following well-known properties of the delta function

$$\delta(hx) = h^{-1}\delta(x), \tag{7}$$

$$\delta(x-a)\cdot f(x) = \delta(x-a)\cdot f(a), \tag{8}$$

$$\delta^{(\alpha)}(x) * f(x) = f^{(\alpha)}(x), \tag{9}$$

$$\phi_0(x) = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta), \qquad \sum_{\beta=-\infty}^{+\infty} e^{2\pi ix\beta} = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta). \tag{10}$$

To obtain the main results, we must calculate the series given below

$$S = \sum_{\beta=-\infty}^{+\infty} \frac{1}{(\beta - h(p+\frac{s_1 i}{2\pi}))(\beta - h(p+\frac{s_2 i}{2\pi}))\cdot\ldots\cdot(\beta - h(p+\frac{s_{2m} i}{2\pi}))}, \tag{11}$$

where $s_k = \cos\frac{(2k-1)\pi}{2m} + i\sin\frac{(2k-1)\pi}{2m}$, $k = 1,2,\ldots,2m$. The sum of the series given in (11) is shown in the following lemma.

Lemma 1 *For the series in* (11), *the following is true*:

$$S = -\frac{h\lambda}{m}\cdot\left(-\frac{2\pi i}{h}\right)^{2m}\cdot\sum_{k=1}^{\frac{m}{2}} \frac{a_{1,k}\cdot\lambda^2 + a_{2,k}\cdot\lambda + a_{1,k}}{\lambda^4 + b_{1,k}\lambda^3 + b_{2,k}\lambda^2 + b_{1,k}\lambda + 1},$$

where

$$\begin{aligned}
a_{1,k} &= e^{h\cos\frac{(2k-1)\pi}{2m}}\cos\left(h\sin\frac{(2k-1)\pi}{2m} + \frac{(2k-1)\pi}{2m}\right)\\
&\quad - e^{-h\cos\frac{(2k-1)\pi}{2m}}\cos\left(h\sin\frac{(2k-1)\pi}{2m} - \frac{(2k-1)\pi}{2m}\right),\\
a_{2,k} &= 2\left(\sin\frac{(2k-1)\pi}{2m}\sin\left(2h\sin\frac{(2k-1)\pi}{2m}\right)\right.\\
&\quad \left. - \cos\frac{(2k-1)\pi}{2m}\cdot\sinh\left(2h\cos\frac{(2k-1)\pi}{2m}\right)\right),\\
b_{1,k} &= -4\cosh\left(h\cos\frac{(2k-1)\pi}{2m}\right)\cdot\cos\left(h\cdot\sin\frac{(2k-1)\pi}{2m}\right),\\
b_{2,k} &= 2\left[1 + \cosh\left(2h\cdot\cos\frac{(2k-1)\pi}{2m}\right) + \cos\left(2h\sin\frac{(2k-1)\pi}{2m}\right)\right],\\
&k = 1,2,\ldots,m-1.
\end{aligned}$$

Proof To compute the infinite series S in (11), we utilize a well-known formula from the theory of residues (see [26]).

$$\sum_{\beta=-\infty}^{+\infty} f(\beta) = -\sum_{z_1,z_2,\dots,z_n} \operatorname{res}\big(\pi\cot(\pi z) f(z)\big). \tag{12}$$

The given statement mentions that $z_1, z_2, \dots, z_n$ refer to the poles of the function $f(z)$.

Let us denote

$$f(z) = \frac{1}{(z-z_1)\cdot(z-z_2)\dots(z-z_k)}.$$

Here $z_k = h[p + \frac{is_k}{2\pi}]$, $(k = 1, 2, \dots, 2m)$ are poles of the first order. Then, for $z = z_k$, $(k = 1, 2, \dots, 2m)$ taking into account (12) from (11), we obtain

$$\begin{aligned}\operatorname{res}_{z=z_k}\big(\pi\cot(\pi z) f(z)\big) &= \lim_{z\to z_k}\big(\pi\cot(\pi z)\cdot f(z)\cdot(z-z_k)\big)\\ &= \lim_{z\to z_k}\frac{\pi\cot(\pi z)}{(z-z_1)(z-z_2)\dots(z-z_{k-1})(z-z_{k+1})\dots(z-z_{2m})}\\ &= \frac{\pi\cot(\pi hp + \frac{s_k hi}{2})}{(\frac{hi}{2\pi})^{2m-1}\prod_{i=1,i\neq k}^{2m}(s_k-s_i)}.\end{aligned} \tag{13}$$

Now let us simplify the denominator of the expression (13). We consider

$$s^{2m}+1 = \prod_{i=1}^{2m}(s-s_i).$$

From here, dividing the left and right sides of the last equality by $s - s_k$, we get

$$\frac{s^{2m}+1}{s-s_k} = \prod_{i=1 i\neq k}^{2m}(s-s_i).$$

We calculate the value of the above expression at $s = s_k$

$$\prod_{i=1 i\neq k}^{2m}(s_k-s_i) = \lim_{s\to s_k}\frac{s^{2m}+1}{s-s_k}.$$

It can be seen that the limit is undetermined of the form $\frac{0}{0}$, so let us calculate this limit using L'Hôpital's rule

$$\lim_{s\to s_k}\frac{s^{2m}+1}{s-s_k} = \lim_{s\to s_k}\frac{2ms^{2m-1}}{1} = 2ms_k^{2m-1} = 2m\frac{s_k^{2m}}{s_k} = -\frac{2m}{s_k}.$$

Using the result obtained above, we write the expression (13) in the following form

$$\begin{aligned}\operatorname{res}_{z=z_k}\big(\pi\cot(\pi z) f(z)\big) &= \frac{\pi\cot(\pi hp + \frac{s_k hi}{2})}{(\frac{hi}{2\pi})^{2m-1}\prod_{i=1 i\neq k}^{2m}(s_k-s_i)}\\ &= -\left(\frac{-2\pi i}{h}\right)^{2m-1}\frac{\pi s_k}{2m}\cot\left(\pi hp + \frac{s_k hi}{2}\right).\end{aligned}$$

To calculate the sum in formula (12), let us simplify it by grouping those whose multiplier is s_k $(k = 1, 2, \dots, m)$ and whose multiplier is s_{2m-k+1} $(k = 1, 2, \dots, m)$.

If $k = 1$:

$$S_1^* = s_1 \cot\left(\pi hp + \frac{s_1 hi}{2}\right) + s_{2m} \cot\left(\pi hp + \frac{s_{2m} hi}{2}\right)$$
$$= \frac{2\cos\frac{\pi}{2m}\sin(2\pi hp + hi\cos\frac{\pi}{2m}) + 2i\sin\frac{\pi}{2m}\sin(h\sin\frac{\pi}{2m})}{\cos(h\sin\frac{\pi}{2m}) - \cos(2\pi hp + hi\cos\frac{\pi}{2m})},$$
$$\vdots$$

when $k = m$:

$$S_m^* = s_m \cot\left(\pi hp + \frac{s_m hi}{2}\right) + s_{m+1} \cot\left(\pi hp + \frac{s_{m+1} hi}{2}\right)$$
$$= \frac{2\cos\frac{(2m-1)\pi}{2m}\sin(2\pi hp + hi\cos\frac{(2m-1)\pi}{2m}) + 2i\sin\frac{(2m-1)\pi}{2m}\sin(h\sin\frac{(2m-1)\pi}{2m})}{\cos(h\sin\frac{(2m-1)\pi}{2m}) - \cos(2\pi hp + hi\cos\frac{(2m-1)\pi}{2m})}.$$

By introducing the notation $\lambda = e^{2\pi iph}$, using the last m equalities and taking into account the following well-known formulas $\cos(z) = \frac{e^{zi}+e^{-zi}}{2}$, $\sin(z) = \frac{e^{zi}-e^{-zi}}{2i}$, $\cosh(z) = \frac{e^z+e^{-z}}{2}$, $\sinh(z) = \frac{e^z-e^{-z}}{2}$, after some simplifications, we obtain the following expression

$$S = -\frac{h\lambda}{m} \cdot \left(-\frac{2\pi i}{h}\right)^{2m} \cdot \sum_{k=1}^{\frac{m}{2}} \frac{a_{1,k} \cdot \lambda^2 + a_{2,k} \cdot \lambda + a_{1,k}}{\lambda^4 + b_{1,k}\lambda^3 + b_{2,k}\lambda^2 + b_{1,k}\lambda + 1}.$$

Lemma 1 is proven. □

3 Construction of a discrete operator

In this section, we construct the function $D_m(h\beta)$ for an even natural number m, which is a discrete analog of the differential operator $\frac{d^{2m}}{dx^{2m}} + 1$. We also obtain some of its properties.

Theorem 1 *The discrete analog of the differential operator* $\frac{d^{2m}}{dx^{2m}} + 1$, *satisfying equality* (1) *when m is even, takes the form*:

$$D_m(h\beta) = -\frac{m}{K} \cdot \begin{cases} M_1 - K_1 + \sum_{k=1}^{m-1} \frac{A_k}{\lambda_k}, & \beta = 0, \\ 1 + \sum_{k=1}^{m-1} A_k, & |\beta| = 1, \\ \sum_{k=1}^{m-1} A_k \cdot \lambda_k^{|\beta|-1}, & |\beta| \geq 2. \end{cases} \tag{14}$$

Here

$$K = \sum_{k=1}^{m/2} a_{1,k}, \qquad K_1 = \frac{\sum_{k=1}^{m/2}[a_{2,k} + a_{1,k} \cdot \sum_{k\neq j, j=1}^{m/2} b_{1,j}]}{K},$$
$$K_2 = \frac{\sum_{k=1}^{m/2}[a_{1,k} + a_{2,k}\sum_{k\neq j,j=1}^{m/2} b_{1,j} + a_{1,k}\sum_{k\neq j,j=1}^{m/2} b_{2,j}]}{K},$$
$$K_3 = \frac{\sum_{k=1}^{m/2}[2a_{1,k}\sum_{k\neq j,j=1}^{m/2} b_{1,j} + a_{2,k}\sum_{k\neq j,j=1}^{m/2} b_{2,j}]}{K},$$

$$M_1 = \sum_{k=1}^{m/2} b_{1,k}, \qquad A_k = \frac{B_{2m}(\lambda_k)}{\lambda_k \cdot Q'_{2m-2}(\lambda_k)},$$

$$M_2 = \prod_{k=1}^{m/2} b_{1,k} \sum_{j=1}^{m/2} \frac{1}{b_{1,j}} + \sum_{k=1}^{m/2} b_{2,k},$$

$$Q_{2m-2}(\lambda) = \lambda^{2m-2} + K_1\lambda^{2m-3} + K_2\lambda^{2m-4} + K_3\lambda^{2m-5} + \cdots + K_3\lambda^3 + K_2\lambda^2 + K_1\lambda + 1,$$

$$A_{2m}(\lambda) = \prod_{k=1}^{m/2}\left[\lambda^4 + b_{1,k}\lambda^3 + b_{2,k}\lambda^2 + b_{1,k}\lambda + 1\right] = \lambda^{2m} + M_1\lambda^{2m-1} + M_2\lambda^{2m-2} + \cdots + M_2\lambda^2 + M_1\lambda + 1,$$

λ_k *represents the roots of the polynomial* $Q_{2m-2}(\lambda)$ *with an absolute value less than* 1, *i.e.*, $|\lambda_k| < 1$.

Theorem 2 *The discrete analog* $D_m(h\beta)$ *of the differential operator* $\frac{d^{2m}}{dx^{2m}} + 1$ *satisfies the equalities*

(1) $D_m(h\beta) * e^{h\beta\cos\frac{(2k-1)\pi}{2m}} \cos(h\beta \sin\frac{(2k-1)\pi}{2m}) = 0$, $k = 1, 2, \ldots, m$,

(2) $D_m(h\beta) * e^{h\beta\cos\frac{(2k-1)\pi}{2m}} \sin(h\beta \sin\frac{(2k-1)\pi}{2m}) = 0$, $k = 1, 2, \ldots, m$.

Here $G_m(h\beta)$ is defined by equality (2).

Proof It is more convenient to perform operations on harrow-shaped functions instead of discrete argument functions. The harrow-shaped function corresponding to the discrete argument function $D_m(h\beta)$ has a specific form

$$\overset{\sqcap}{D}_m(x) = \sum_{\beta=-\infty}^{+\infty} D_m(h\beta)\delta(x - h\beta).$$

Equation (1) in the class of harrow-shaped functions becomes the equation

$$\overset{\sqcap}{D}_m(x) * \overset{\sqcap}{G}_m(x) = \delta(x), \tag{15}$$

where $\overset{\sqcap}{G}_m(x) = \sum_{\beta=-\infty}^{+\infty} G_m(h\beta) \cdot \delta(x - h\beta)$.

It is known that the class of harrow-shaped functions and the class of functions with a discrete argument are isomorphic [1]. Therefore, instead of the discrete argument function $D_m(h\beta)$, it is sufficient to study the function $\overset{\sqcap}{D}_m(x)$. After applying the Fourier transform to both sides of the equation (15) while taking into account (4) and (6), we can obtain:

$$F\left[\overset{\sqcap}{D}_m(x)\right] = \frac{1}{F[\overset{\sqcap}{G}_m(x)]}. \tag{16}$$

It is necessary to use the Fourier transform $F[\overset{\sqcap}{G}_m(x)]$ in this case. Taking into account (8), (10), and also using the formulas (5) and (7), we have

$$F\left[\overset{\sqcap}{G}_m(x)\right] = F\left[G_m(x)\right] * \phi_0(hp). \tag{17}$$

To calculate the Fourier transform $F[G_m(x)]$ of a function $G_m(x)$, we use equalities (3) and (9). After considering the equalities (4) and (6), we can conclude that

$$F\big[G_m(x)\big] = \frac{1}{F[\delta^{(2m)}(x) + \delta(x)]} = \frac{1}{(-2\pi ip)^{2m} + 1} = \frac{1}{(2\pi ip)^{2m} + 1}.$$

We can use the last equality to simplify the expression in (17)

$$\begin{aligned}
F\big[\overset{\sqcap}{G_m}(x)\big] &= F\left[G_m(x) \cdot \sum_{\beta=-\infty}^{+\infty} \delta(x - h\beta)\right] \\
&= F\big[G_m(x)\big] * F\left[\sum_{\beta=-\infty}^{+\infty} \delta(x - h\beta)\right] \\
&= \frac{1}{(2\pi ip)^{2m} + 1} * \sum_{\beta=-\infty}^{+\infty} \delta(hp - \beta) \\
&= \sum_{\beta=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\delta(hy - \beta)}{(2\pi i(p - y))^{2m} + 1}\, dy \\
&= h^{-1} \sum_{\beta=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\delta(y - h^{-1}\beta)}{(2\pi i(p - y))^{2m} + 1}\, dy \\
&= h^{-1} \cdot \sum_{\beta=-\infty}^{+\infty} \frac{1}{(2\pi i(p - h^{-1}\beta))^{2m} + 1} \\
&= h^{-1}\left(-\frac{h}{2\pi i}\right)^{2m} \sum_{\beta=-\infty}^{+\infty} \frac{1}{\prod_{k=1}^{2m}(\beta - h[p + \frac{s_k i}{2\pi}])}
\end{aligned} \tag{18}$$

where $s_k = \cos\frac{(2k-1)\pi}{2m} + i\sin\frac{(2k-1)\pi}{2m}$, $k = 1, 2, \ldots, 2m$.

Now, expanding the right side of (18) into elementary fractions and taking into account (16), we have

$$F\big[\overset{\sqcap}{D_m}(x)\big] = \left[\frac{h^{2m-1}}{(2\pi i)^{2m}} \cdot \sum_{\beta=-\infty}^{+\infty} \frac{1}{\prod_{k=1}^{2m}(\beta - h[p + \frac{s_k i}{2\pi}])}\right]^{-1}.$$

Let us consider a function $F[\overset{\sqcap}{D_m}(x)](p)$ and assume that its Fourier series has a particular form

$$F[\overset{\sqcap}{D_m}](p) = \sum_{\beta=-\infty}^{+\infty} \widetilde{D}_m(h\beta) \cdot e^{2\pi iph\beta}, \tag{19}$$

where $\widetilde{D}_m(h\beta)$ are the Fourier coefficients of the function $F[\overset{\sqcap}{D_m}](p)$, i.e.,

$$\widetilde{D}_m(h\beta) = \int_0^{h^{-1}} F[\overset{\sqcap}{D_m}](p) e^{-2\pi iph\beta}\, dp. \tag{20}$$

Applying the inverse Fourier transform to equality (19), we obtain the harrow-shaped function

$$\overset{\sqcap}{D}_m(x) = \sum_{\beta=-\infty}^{+\infty} \widetilde{D}_m(h\beta)\delta(x-h\beta).$$

From here, based on the definition of harrow-shaped functions, we conclude that the discrete function $\widetilde{D}_m(h\beta)$ is the desired function of the discrete argument $D_m(h\beta)$. Here, to find the function $\widetilde{D}_m(h\beta)$, we do not use the formula (20), but find it as follows. Using Lemma 1, from (20) we obtain

$$F[\overset{\sqcap}{D}_m](p) = -\frac{m}{K}\cdot\frac{B_{2m}(\lambda)}{\lambda Q_{2m-2}(\lambda)}. \tag{21}$$

In order to find the explicit form of the discrete operator $D_m(h\beta)$, we decompose the expression (21) into elementary fractions. Since the polynomial $Q_{2m-2}(\lambda)$ from Theorem 2 has $2m-2$ roots and $\lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_{2m-3}\cdot\lambda_{2m-2}=1$, where $\lambda_j\cdot\lambda_{2m-1-j}=1$, $j=\overline{1,m-1}$, then for the right-hand side (21) we have

$$-\frac{m}{K}\cdot\frac{B_{2m}(\lambda)}{\lambda\prod_{k=1}^{2m-2}(\lambda-\lambda_k)} = -\frac{m}{K}\left[\lambda+M_1-K_1+\frac{A_0}{\lambda}+\sum_{k=1}^{2m-2}\frac{A_k}{\lambda-\lambda_k}\right]. \tag{22}$$

To find the unknowns $A_0, A_1, A_2, \ldots, A_{2m-3}, A_{2m-2}$ of equation (22), we multiply both sides of equation (22) by $\lambda\prod_{i=1}^{2m-2}(\lambda-\lambda_i)$, to get for $\lambda=0$

$$A_0 = 1,$$

and for $\lambda=\lambda_k$

$$A_k = \frac{B_{2m}(\lambda_k)}{\lambda_k Q'_{2m-2}(\lambda_k)}, \quad k=1,2,\ldots,2m-2. \tag{23}$$

Hence, taking into account that $\lambda_j\cdot\lambda_{2m-1-j}=1$, $j=1,2,\ldots,m-1$, we have

$$A_{2m-2} = -\frac{1}{\lambda_1^2}A_1, \qquad A_{2m-3} = -\frac{1}{\lambda_2^2}A_2, \ldots, A_{m-1} = -\frac{1}{\lambda_m^2}A_m. \tag{24}$$

Finally, taking into account the equalities (23)–(24) and using the formula for the sum of an infinitely killing geometric progression from (22), we obtain

$$\begin{aligned} F\big[\overset{\sqcap}{D}_m(x)\big] &= -\frac{m}{K}\cdot\left(\lambda+M_1-K_1+\frac{1}{\lambda}+\sum_{\gamma=0}^{+\infty}\sum_{j=1}^{m-1}\left[\frac{A_j}{\lambda}\left(\frac{\lambda_j}{\lambda}\right)^{\gamma}+\frac{A_j}{\lambda_j}(\lambda_j\lambda)^{\gamma}\right]\right) \\ &= \sum_{\gamma=-\infty}^{+\infty} D_m(h\gamma)\lambda^{\gamma}. \end{aligned}$$

From here, bearing in mind that $\lambda = e^{2\pi i p h}$, we obtain the explicit form (14) of the discrete function $D_m(h\beta)$. Theorem 1 is proven. □

The proof of Theorem 2 is obtained using the definition of convolution of discrete functions and directly calculating the left sides of equalities (a) and (b).

In (14) note that the function $D_m(h\beta)$ is even, i.e., $D_m(-h\beta) = D_m(h\beta)$.

4 Optimal quadrature formula in the space $L_2^{(2,0)}(0,1)$

This section is dedicated to the use of the discrete operator $D_2(h\beta)$ to construct an optimal quadrature formula.

In this section, we focus on Sard's problem of constructing the optimal quadrature formula in the Hilbert space $L_2^{(2,0)}$. Here, $L_2^{(2,0)}$ represents the set of functions φ defined on the interval $[0,1]$, which have an absolutely continuous first derivative on $[0,1]$ and a second derivative that belongs to $L_2(0,1)$. The class $L_2^{(2,0)}$ with the inner-product

$$\langle \varphi, \psi \rangle = \int_0^1 \bigl(\varphi''(x)\psi''(x) + \varphi(x)\psi(x)\bigr)\,dx$$

is a Hilbert space equipped with the norm

$$\|\varphi\|_{L_2^{(2,0)}} = \left\{ \int_0^1 \bigl[\bigl(\varphi''(x)\bigr)^2 + \bigl(\varphi(x)\bigr)^2\bigr]\,dx \right\}^{1/2}.$$

For a function φ from the space $L_2^{(2,0)}$, consider a quadrature formula of the form

$$\int_0^1 \varphi(x)\,dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi(h\beta), \tag{25}$$

where $C[\beta]$ and $[\beta] = h\beta$ are the coefficients and nodes of the formula (25), respectively, φ is an element of the Hilbert space $L_2^{(2,0)}(0,1)$.

The following difference between the integral and the quadrature sum

$$(\ell, \varphi) = \int_0^1 \varphi(x)\,dx - \sum_{\beta=0}^{N} C[\beta]\varphi(h\beta) \tag{26}$$

is called the error of the quadrature formula (25) and here

$$(\ell, \varphi) = \int_{-\infty}^{+\infty} \ell(x)\varphi(x)\,dx.$$

This difference corresponds to the error functional ℓ, which has the form:

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C[\beta]\delta(x - h\beta). \tag{27}$$

Here $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$.

According to the Cauchy–Schwartz inequality, the absolute value of the error (26) can be estimated using the norm

$$\bigl|(\ell, \varphi)\bigr| \le \|\ell\|_{L_2^{(2,0)*}} \|\varphi\|_{L_2^{(2,0)}}, \tag{28}$$

of the error functional ℓ as follows

$$\|\ell\|_{L_2^{(2,0)*}} = \sup_{\|\varphi\|_{L_2^{(2,0)}}=1} \left|(\ell,\varphi)\right| \tag{29}$$

where $L_2^{(2,0)*}$ is the dual space to the space $L_2^{(2,0)}$.

To construct the optimal quadrature formula in the space $L_2^{(2,0)}(0,1)$, we need to calculate a specific quantity

$$\|\mathring{\ell}\|_{L_2^{(2,0)*}} = \inf_{C[\beta]} \|\ell\|_{L_2^{(2,0)*}}, \tag{30}$$

i.e., in finding the minimum value of the norm (28) for the error functional ℓ by the coefficients $C[\beta]$.

To calculate (30) we need a discrete analog $D_2(h\beta)$ of the operator $\frac{d^4}{dx^4}+1$. For the case $m=2$, we obtain the following results from Theorem 1 and 2:

Corollary 1 *Discrete analog of the differential operator* $\frac{d^4}{dx^4}+1$, *satisfying the equation* $D_2(h\beta)*G_2(h\beta)=\delta_d(h\beta)$, *has the form*

$$D_2(h\beta) = \frac{\sqrt{2}}{K}\cdot\begin{cases} M_1 - K_1 + \frac{A_1}{\lambda_1}, & \beta=0,\\ 1+A_1, & |\beta|=1,\\ A_1\cdot\lambda_1^{|\beta|-1}, & |\beta|\geq 2,\end{cases}$$

where $K, M_1, K_1, A_1, \lambda_1$ *are defined in Theorem* 1 *in* [27].

Corollary 2 *The discrete operator* $D_2(h\beta)$ *has the following properties*:

(1) $D_2(h\beta)*e^{\frac{\sqrt{2}}{2}h\beta}\cos(\frac{\sqrt{2}}{2}h\beta)=0$,
(2) $D_2(h\beta)*e^{\frac{\sqrt{2}}{2}h\beta}\sin(\frac{\sqrt{2}}{2}h\beta)=0$,
(3) $D_2(h\beta)*e^{-\frac{\sqrt{2}}{2}h\beta}\cos(\frac{\sqrt{2}}{2}h\beta)=0$,
(4) $D_2(h\beta)*e^{-\frac{\sqrt{2}}{2}h\beta}\sin(\frac{\sqrt{2}}{2}h\beta)=0$.

These equalities are obtained in the work [27].

Using Corollaries 1 and 2, we get the following result.

Theorem 3 *The coefficients of optimal quadrature formula* (25) *in* $L_2^{(2,0)}$ *space have the next form*

$$C[\beta] = \frac{\sqrt{2}}{K}\begin{cases} T + f_2^-(h) + m_1(1+\lambda_1), & \beta=0,\\ T + m_1(\lambda_1^{\beta}+\lambda_1^{N-\beta}), & 0<\beta<N,\\ T + f_2^+(h) + m_1(\lambda_1^N+1), & \beta=N,\end{cases}$$

where

$$T = \frac{A_1}{\lambda_1}\cdot\frac{1+\lambda_1}{1-\lambda_1} + M_1 - K_1 + 2,$$

$$
\begin{aligned}
m_1 = A_1 \cdot \Bigg[& \left(\frac{1 + e^{-\frac{\sqrt{2}}{2}} \cos \frac{\sqrt{2}}{2}}{4} - d_1 \right) \cdot \frac{e^{-\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) - \lambda_1 \cdot e^{-\sqrt{2}h}}{\lambda_1^2 \cdot e^{-\sqrt{2}h} - 2\lambda_1 e^{-\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) + 1} \\
& + \left(d_2 - \frac{e^{-\frac{\sqrt{2}}{2}} \sin \frac{\sqrt{2}}{2}}{4} \right) \cdot \frac{e^{-\frac{\sqrt{2}h}{2}} \sin(\frac{\sqrt{2}h}{2})}{\lambda_1^2 \cdot e^{-\sqrt{2}h} - 2\lambda_1 e^{-\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) + 1} \\
& + \left(\frac{1 + e^{\frac{\sqrt{2}}{2}} \cos \frac{\sqrt{2}}{2}}{4} - d_3 \right) \cdot \frac{e^{\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) - \lambda_1 \cdot e^{\sqrt{2}h}}{\lambda_1^2 \cdot e^{\sqrt{2}h} - 2\lambda_1 e^{\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) + 1} \\
& + \left(d_4 - \frac{e^{\frac{\sqrt{2}}{2}} \sin \frac{\sqrt{2}}{2}}{4} \right) \cdot \frac{e^{\frac{\sqrt{2}h}{2}} \sin(\frac{\sqrt{2}h}{2})}{\lambda_1^2 \cdot e^{\sqrt{2}h} - 2\lambda_1 e^{\frac{\sqrt{2}h}{2}} \cos(\frac{\sqrt{2}h}{2}) + 1} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
f_2^-(h) = & -d_1 \cdot e^{\frac{-\sqrt{2}}{2}h} \cos\left(\frac{\sqrt{2}}{2}h\right) + d_2 \cdot e^{\frac{-\sqrt{2}}{2}h} \sin\left(\frac{\sqrt{2}}{2}h\right) \\
& - d_3 \cdot e^{\frac{\sqrt{2}}{2}h} \cos\left(\frac{\sqrt{2}}{2}h\right) + d_4 \cdot e^{\frac{\sqrt{2}}{2}h} \sin\left(\frac{\sqrt{2}}{2}h\right) - 1 \\
& + \frac{1}{2}\left[\cos\left(\frac{\sqrt{2}}{2}h\right) \cosh\left(\frac{\sqrt{2}}{2}h\right) + \cos\left(\frac{\sqrt{2}}{2}(1+h)\right) \cosh\left(\frac{\sqrt{2}}{2}(1+h)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
f_2^+(h) = & \, d_1 \cdot e^{\frac{\sqrt{2}}{2}(h+1)} \cos\left(\frac{\sqrt{2}}{2}(h+1)\right) + d_2 \cdot e^{\frac{\sqrt{2}}{2}(h+1)} \sin\left(\frac{\sqrt{2}}{2}(h+1)\right) \\
& + d_3 \cdot e^{-\frac{\sqrt{2}}{2}(h+1)} \cos\left(\frac{\sqrt{2}}{2}(h+1)\right) + d_4 \cdot e^{-\frac{\sqrt{2}}{2}(h+1)} \sin\left(\frac{\sqrt{2}}{2}(h+1)\right) - 1 \\
& + \frac{1}{2}\left[\cos\left(\frac{\sqrt{2}}{2}(h+1)\right) \cosh\left(\frac{\sqrt{2}}{2}(h+1)\right) + \cos\left(\frac{\sqrt{2}}{2}h\right) \cosh\left(\frac{\sqrt{2}}{2}h\right)\right],
\end{aligned}
$$

and d_k $(k = 1, 2, 3, 4)$ *are defined in Theorem* 2.4 *of* [28].

Now, we find the square of the norm of the error functional (27) for the optimal quadrature formula (25). The following result holds:

Theorem 4 *The square of the norm of the error functional* (27) *for the optimal quadrature formula* (25) *on the space* $L_2^{(2,0)}(0,1)$ *has the form*

$$
\begin{aligned}
\|\mathring{\ell}\|^2 = & \, 1 - \frac{\sqrt{2}}{2} \cdot \left(\sin \frac{\sqrt{2}}{2} \cdot \cosh \frac{\sqrt{2}}{2} + \cos \frac{\sqrt{2}}{2} \cdot \sinh \frac{\sqrt{2}}{2} \right) \\
& - C[0] - C[N] - \frac{\sqrt{2}}{K} \cdot (N-1) \cdot T - Q_1 - Q_2
\end{aligned}
$$

where

$$
Q_1 = \frac{2\sqrt{2}}{K} \cdot \frac{\lambda_1 - \lambda_1^N}{1 - \lambda_1} \cdot m_1,
$$

$$
\begin{aligned}
Q_2 = & \, d_1 \left[\frac{\sqrt{2}}{2} e^{\frac{\sqrt{2}}{2}} \left(\cos \frac{\sqrt{2}}{2} + \sin \frac{\sqrt{2}}{2} \right) + \frac{\sqrt{2}}{2} \right] \\
& + d_2 \left[\frac{\sqrt{2}}{2} e^{\frac{\sqrt{2}}{2}} \left(\sin \frac{\sqrt{2}}{2} - \cos \frac{\sqrt{2}}{2} \right) - \frac{\sqrt{2}}{2} \right]
\end{aligned}
$$

$$+ d_3\left[\frac{\sqrt{2}}{2}e^{-\frac{\sqrt{2}}{2}}\left(\sin\frac{\sqrt{2}}{2} - \cos\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right]$$
$$- d_4\left[\frac{\sqrt{2}}{2}e^{-\frac{\sqrt{2}}{2}}\left(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\right],$$

where λ_1 *is given in Theorem* 2 *and* $|\lambda_1| < 1$.

5 Discussion and numerical results

We numerically compare the above results with results in other spaces. For convenience, we represent the absolute value of the error (26) for the optimal quadrature formula (25). $|R_N(\varphi)| = |(\ell,\varphi)|$.

Then by the Cauchy–Schwartz inequality, we have

$$\left|R_N(\varphi)\right| \le \|\varphi\|_{L_2^{(m,0)}} \cdot \|\ell\|_{L_2^{(m,0)*}}. \tag{31}$$

In the space $L_2^{(2,0)}$, using Theorem 4 and inequality (31) for the error of optimal quadrature formula (25), we have

$$N = 10: \left|R_N(\varphi)\right| \le \|\varphi\|_{L_2^{(2,0)}} \cdot 0.77896 \cdot 10^{-2},$$
$$N = 50: \left|R_N(\varphi)\right| \le \|\varphi\|_{L_2^{(2,0)}} \cdot 0.15346 \cdot 10^{-4},$$
$$N = 100: \left|R_N(\varphi)\right| \le \|\varphi\|_{L_2^{(2,0)}} \cdot 0.37811 \cdot 10^{-5}.$$

Numerical results show that as the value of N increases, the error of the optimal quadrature formula in the space $L_2^{(2,0)}$ decreases. Let us compare the absolute errors of constructed optimal quadrature formulas in the spaces $L_2^{(2,0)}$ and $W_2^{(2,1)}$. Consider the following functions:

$$\varphi_1(x) = \sin\left(\frac{\sqrt{2}}{2}x\right), \qquad \varphi_2(x) = e^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right),$$

and

$$\varphi_3(x) = \cos\left(\frac{\sqrt{2}}{2}x\right)\sinh\left(\frac{\sqrt{2}}{2}x\right) - \sin\left(\frac{\sqrt{2}}{2}x\right)\cosh\left(\frac{\sqrt{2}}{2}x\right).$$

For convenience, we denote the absolute values of the error of optimal quadrature formulas in the spaces $L_2^{(2,0)}$ and $W_2^{(2,1)}$ by R_L and R_W, respectively.

Tables 1 and 2 provide clear evidence that the absolute error of $|R_L|$ in $L_2^{(2,0)}(0,1)$ space is much smaller than the absolute error $|R_W|$ in the space $W_2^{(2,1)}(0,1)$ for the functions $\varphi_1(x)$, $\varphi_2(x)$, and $\varphi_3(x)$.

Table 1 Error $|R_L(\varphi)|$ of the optimal quadrature formula in $L_2^{(2,0)}$ space

N	$\lvert R_L(\varphi_1)\rvert$	$\lvert R_L(\varphi_2)\rvert$	$\lvert R_L(\varphi_3)\rvert$
10	$2.52519 \cdot 10^{-4}$	$3.10892 \cdot 10^{-4}$	$3.126471 \cdot 10^{-45}$
100	$2.081351 \cdot 10^{-7}$	$2.567667 \cdot 10^{-7}$	$2.009139 \cdot 10^{-41}$
1000	$2.04889 \cdot 10^{-10}$	$2.527556 \cdot 10^{-10}$	$2.576504 \cdot 10^{-38}$

Table 2 Error $|R_W(\varphi)|$ of the optimal quadrature formula in $W_2^{(2,1)}$ space

N	$\|R_W(\varphi_1)\|$	$\|R_W(\varphi_2)\|$	$\|R_W(\varphi_3)\|$
10	$3.60277 \cdot 10^{-4}$	$1.97605 \cdot 10^{-3}$	$7.06542 \cdot 10^{-4}$
100	$3.96087 \cdot 10^{-6}$	$2.17169 \cdot 10^{-5}$	$7.75977 \cdot 10^{-6}$
1000	$3.996868 \cdot 10^{-8}$	$2.191418 \cdot 10^{-7}$	$7.830202 \cdot 10^{-8}$

6 Conclusion

Thus, in this work, using the Sobolev method, we constructed a discrete operator $D_m(h\beta)$ for even natural numbers m in the space $L_2^{(m,0)}(0,1)$. By applying this discrete operator in the case of $m = 2$, we obtained explicit expressions for the optimal coefficients $C[\beta]$ ($\beta = \overline{0,N}$) and using these coefficients we constructed an optimal quadrature formula of the form (25) in the space $L_2^{(2,0)}(0,1)$. At the conclusion of our work, we provided numerical results. The numerical results indicate that the absolute error of the quadrature formula in the $L_2^{(2,0)}(0,1)$ space is significantly smaller than that in the $W_2^{(2,1)}(0,1)$ space for several functions.

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