Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Geometric characterization of the generalized Lommel–Wright function in the open unit disc

Hanaa M. Zayed[1*] and Teodor Bulboacă[2]

*Correspondence:
hanaa@science.menofia.edu.eg
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin Elkom 32511, Egypt
Full list of author information is available at the end of the article

Abstract

The present investigation aims to examine the geometric properties of the normalized form of the combination of generalized Lommel–Wright function $\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z) := \Gamma^m(\lambda+1)\Gamma(\lambda+\mu+1)2^{2\lambda+\mu}z^{1-(\nu/2)-\lambda}\mathcal{J}_{\lambda,\mu}^{\nu,m}(\sqrt{z})$, where the function $\mathcal{J}_{\lambda,\mu}^{\nu,m}$ satisfies the differential equation $\mathcal{J}_{\lambda,\mu}^{\nu,m}(z) := (1-2\lambda-\nu)J_{\lambda,\mu}^{\nu,m}(z) + z(J_{\lambda,\mu}^{\nu,m}(z))'$ with

$$J_{\nu,\lambda}^{\mu,m}(z) = \left(\frac{z}{2}\right)^{2\lambda+\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma^m(k+\lambda+1)\Gamma(k\mu+\nu+\lambda+1)}\left(\frac{z}{2}\right)^{2k}$$

for $\lambda \in \mathbb{C}\setminus\mathbb{Z}^-$, $\mathbb{Z}^- := \{-1,-2,-3,\dots\}$, $m \in \mathbb{N}$, $\nu \in \mathbb{C}$, and $\mu \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. In particular, we employ a new procedure using mathematical induction, as well as an estimate for the upper and lower bounds for the gamma function inspired by Li and Chen (J. Inequal. Pure Appl. Math. 8(1):28, 2007), to evaluate the starlikeness and convexity of order α, $0 \le \alpha < 1$. Ultimately, we discuss the starlikeness and convexity of order zero for $\mathfrak{J}_{\lambda,\mu}^{\nu,m}$, and it turns out that they are useful to extend the range of validity for the parameter λ to $\lambda \ge 0$ where the main concept of the proofs comes from some technical manipulations given by Mocanu (Libertas Math. 13:27–40, 1993). Our results improve, complement, and generalize some well-known (nonsharp) estimates.

Mathematics Subject Classification: 30C45; 30C50

Keywords: Analytic; Univalent; Starlike; Convex; Convolution; Pochhammer symbol; Generalized Lommel–Wright function

1 Introduction and mathematical preliminaries

It is widely known that many functions could be called "special". These include certain elementary functions like the exponential, trigonometric, hyperbolic functions and their inverses, logarithmic functions and poly-logarithms, but the class also expands into transcendental functions like Bessel, Lamé, and Mathieu functions. Some of them play a supplemental role, while others, such as the Bessel and Legendre functions, are of primary importance. These functions appear as solutions of the differential equations and systems used as mathematical models of scientific and other phenomena, particularly those systems that change with time and space. We will restrict our present study to the generalized Lommel–Wright function, which could be of particular interest in concrete problems in mechanics, physics, astronomy, and engineering.

Geometric function theory is an area of complex analysis that investigates the geometric properties of analytic functions. It is a mathematical field characterized by a combination of geometry and complex analysis, and its origin began in the nineteenth century. More recently, geometric function theory has become highly significant as the way of algebraic geometry; in addition, the function theory on compact Riemann surfaces has found some results by creating a finite-gap solutions to nonlinear integrable systems, which can be an area of mathematics with a link to mathematical physics.

The theory of univalent functions is one of the greatest and most interesting fields in geometric function theory. Its origin starts from 1851, when the well-known mapping theorem was constructed by Riemann in his doctoral thesis, and which can be regarded as one of the most useful theorems in classical complex analysis. From a planar topology perspective, it is well known that there exist simply connected domains with rough boundaries, and for these domains, there are no clear homeomorphisms between them. However, the Riemann mapping theorem states that such simply connected domains are not only homeomorphic but are also biholomorphic. The Riemann mapping theorem states that if D is a nonempty domain that is a simply connected open subset in the complex plane $\mathbb{C}$, then there exists an injective and holomorphic mapping f that maps D onto the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. This function is known as the Riemann mapping. Nevertheless, his proof was incomplete, while the proof was given completely in 1912 by Carathéodory using the Riemann surfaces. It was simplified by Koebe after two years in a way that did not require these (see, for example, [1, 11, 13, 20, 23]).

In more recent years, significant efforts have been made to study the geometric properties of certain (normalized) special functions such as close-to-convexity, starlikeness, and convexity mostly within $\mathbb{U}$. For additional details, we refer, for example, to [18, 24–26] for hypergeometric function, to [8, 9] for Bessel function, to [21, 30, 33] for generalized Struve function, to [29] for Lommel function, to [32, 34] for generalized Lommel–Wright function, and to [17] for Fox–Wright function. In addition, some radii problems for the Bessel, q-Bessel, Struve, and Lommel functions of the first kind were investigated in [2–7] and in the references therein. It was shown that these radii are actually solutions of some transcendental equations. These results could be important to deduce some of the geometric properties of complex functions.

The contents of the present paper are summarized as follows. We outline first various well-known mathematical facts that will be used in the subsequent sections. Moreover, we examine the geometric properties of $\mathfrak{J}^{\nu,m}_{\lambda,\mu}$, including the starlikeness and convexity of order $\alpha, 0 \leq \alpha < 1$, using the mathematical induction, as well as an estimate for the upper and lower bounds for the gamma function inspired by [16]. In addition, we discuss the starlikeness and convexity of order zero for $\mathfrak{J}^{\nu,m}_{\lambda,\mu}$, and it turns out that they are useful to extend the range of validity for the parameter λ where the leading concept of the proofs comes from some technical manipulations by [19]. Our results improve, complement, and generalize some well-known (nonsharp) estimates.

An analytic function f is called univalent (or schlicht) in a domain $D \subset \mathbb{C}$, which is a subset of the complex plane, if it is injective in D. Without loss of generality, we will assume that f is normalized by the conditions $f(0) = f'(0) - 1 = 0$ and is defined on $\mathbb{U}$, that is, an analytic function having the Maclaurin series expansion of the form

$$f(z)=\sum_{k=1}^{\infty}A_k z^k,\quad z\in\mathbb{U},\text{with } A_1=1. \tag{1.1}$$

This class of functions is denoted by $\mathcal{A}$, while the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ is denoted by $\mathcal{S}$. The most classic example of a function in $\mathcal{S}$ is the Koebe function, that is,

$$k(z)=\frac{z}{(1-z)^2}=\frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2-1\right]=\sum_{k=1}^{\infty}kz^k.$$

It maps the unit disc $\mathbb{U}$ onto $\mathbb{C}$ slit along the negative real axis from $-1/4$ to $-\infty$, i.e., $k(\mathbb{U})=\mathbb{C}\setminus(-\infty,-1/4]$. It is well known that this function plays the extremal role in many problems in the univalent function theory.

Besides, if $g\in\mathcal{A}$ has the form $g(z)=\sum_{k=1}^{\infty}B_k z^k$, $z\in\mathbb{U}$, with $B_1=1$, then the *convolution* of two power series f and g is given by $(f*g)(z):=\sum_{k=1}^{\infty}A_kB_kz^k$, $z\in\mathbb{U}$. The aforementioned definition of the convolution arises from the integration (see [11])

$$(f*g)\left(r^2e^{i\theta}\right)=\frac{1}{2\pi}\int_0^{2\pi}f\left(re^{i(\theta-t)}\right)g\left(re^{it}\right)dt,\quad r<1.$$

We are now in a position to recall the most important subclasses of the class of analytic functions, which can be regarded as the cornerstone in the theory of univalent functions, that is, the subclasses of starlike and convex functions (these classes were introduced by Robertson in 1936).

If $f(\mathbb{U})$ is a starlike domain with respect to the origin, then $f\in\mathcal{S}$ is called *starlike with respect to the origin (or briefly, starlike)*, denoted by $\mathcal{S}^*$. We shall recall that the domain $D\subset\mathbb{C}$ is *starlike with respect to an interior point* $z_0\in D$ if the line segment that joins z_0 to any other point of D lies entirely in D. In particular, if $z_0=0$, then the domain D is called *starlike domain*. A function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}^*$ if and only if $\operatorname{Re}(zf'(z)/f(z))>0$, $z\in\mathbb{U}$. The Koebe function and its rotations are an example of starlike functions, and this function is extremal for the class $\mathcal{S}^*$.

Moreover, if $f(\mathbb{U})$ is a convex domain, then $f\in\mathcal{S}$ is called *convex*, denoted by $\mathcal{K}$. It is well known that the domain $D\subset\mathbb{C}$ is *convex* if the line segment joining any two points of D lies entirely in D. Analytically, the convex functions $f\in\mathcal{A}$ can be represented as $\operatorname{Re}(zf''(z)/f'(z))+1>0$, $z\in\mathbb{U}$. The main branch of the function $f(z)=-\log(1-z)\in\mathcal{K}$ since $1+\operatorname{Re}(zf''(z)/f'(z))=1+\operatorname{Re}(z/(1-z))>1/2>0$ for all $z\in\mathbb{U}$.

Additionally, a function $f\in\mathcal{A}$ is *starlike of order* α, $0\leq\alpha<1$, if and only if $\operatorname{Re}(zf'(z)/f(z))>\alpha$, $z\in\mathbb{U}$, and it belongs to the class of *convex functions of order* α, denoted by $\mathcal{K}(\alpha)$, if and only if $\operatorname{Re}(zf''(z)/f'(z))+1>\alpha$, $z\in\mathbb{U}$. As is well known, $\mathcal{S}^*(\alpha)\subset\mathcal{S}^*(0)=:\mathcal{S}^*$, $\mathcal{K}(\alpha)\subset\mathcal{K}(0)=:\mathcal{K}$ and $\mathcal{K}\subset\mathcal{S}^*\subset\mathcal{S}$.

Oteiza et al. in [10] introduced the *generalized Lommel–Wright function* $J^{\nu,m}_{\lambda,\mu}$ as

$$J^{\nu,m}_{\lambda,\mu}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k\mu+\lambda+\nu+1)\Gamma^m(k+\lambda+1)} \left(\frac{z}{2}\right)^{2k+2\lambda+\nu}$$
$$= \left(\frac{z}{2}\right)^{2\lambda+\nu} {}_1\Psi_{m+1}\left[\begin{matrix}(1,1)\\(\lambda+1,1),\ldots,(\lambda+1,1),(\lambda+\nu+1,\mu)\end{matrix}\middle| -\frac{z^2}{4}\right] \tag{1.2}$$

for $\lambda,\nu \in \mathbb{C}$, $m \in \mathbb{N} := \{1,2,\ldots\}$, and $\mu > 0$. Here, ${}_p\Psi_q$ stands for the *Fox–Wright function* defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix}\middle| z\right] := {}_p\Psi_q\left[\begin{matrix}(\boldsymbol{\alpha}_p,\mathbf{A}_p)\\(\boldsymbol{\beta}_q,\mathbf{B}_q)\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \psi_k \frac{z^k}{k!}, \tag{1.3}$$

where

$$\psi_k := \frac{\Gamma(\alpha_1+A_1k)\cdots\Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k)\cdots\Gamma(\beta_q+B_qk)}$$

with $A_i,B_j \in \mathbb{R}^+$ $(i=1,\ldots,p, j=1,\ldots,q)$ and $\alpha_i,\beta_j \in \mathbb{C}$. It is observable that (1.3) is absolutely convergent in the entire complex z–plane when $\Omega := \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1$, while if $\Omega = -1$, it converges absolutely for $|z| < \rho$ and $|z| = \rho$ under the condition $\operatorname{Re}(\sigma) > 1/2$, where

$$\rho := \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{-B_j}\right), \qquad \sigma := \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i + \frac{p-q}{2}.$$

We refer for additional information regarding the Fox–Wright functions to [15] and the references therein.

We are now in a position to deduce certain special cases of the *generalized Lommel–Wright function*. If we set $m = 1$ in (1.2), we get the *Bessel–Maitland function* introduced by Pathak [22], which has the form

$$J^{\nu}_{\lambda,\mu}(z) := J^{\nu,1}_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k\mu+\lambda+\nu+1)\Gamma(k+\lambda+1)} \left(\frac{z}{2}\right)^{2k+2\lambda+\nu}$$

for $\mu > 0$ and $\lambda,\nu \in \mathbb{C}$. Putting $\lambda = 1/2$ and $m = \mu = 1$ in (1.2), we have the *Struve function* defined by

$$H_\nu(z) := J^{\nu,1}_{1/2,\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+3/2)\Gamma(k+3/2)} \left(\frac{z}{2}\right)^{\nu+2k+1}, \quad \nu \in \mathbb{C}.$$

For $\lambda = 0$ and $m = \mu = 1$ in (1.2), we get the *Bessel function* that has the power series expansion

$$J_\nu(z) := J^{\nu,1}_{0,1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k},$$

where $z \in \mathbb{C} \setminus \{0\}$, $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > -1$.

In the following part of the paper, we need the next definition.

Definition 1.1 The normalized form *of the combination of generalized Lommel–Wright function* is defined by

$$\mathfrak{J}^{\nu,m}_{\lambda,\mu}(z) := \Gamma^m(\lambda+1)\Gamma(\lambda+\mu+1)2^{2\lambda+\mu}z^{1-(\nu/2)-\lambda}\mathcal{J}^{\nu,m}_{\lambda,\mu}(\sqrt{z}),$$

where the function $\mathcal{J}^{\nu,m}_{\lambda,\mu}$ satisfies the differential equation $\mathcal{J}^{\nu,m}_{\lambda,\mu}(z) := (1-2\lambda-\nu)J^{\nu,m}_{\lambda,\mu}(z) + z(J^{\nu,m}_{\lambda,\mu}(z))'$ for $\lambda \in \mathbb{C}\setminus\mathbb{Z}^-$, $\mathbb{Z}^- := \{-1,-2,-3,\ldots\}$, $m \in \mathbb{N}$, $\nu \in \mathbb{C}$, and $\mu \in \mathbb{N}_0$. Clearly, $\mathfrak{J}^{\nu,m}_{\lambda,\mu}$ can be written as

$$\mathfrak{J}^{\nu,m}_{\lambda,\mu}(z) = z + \sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}z^{k+1}, \tag{1.4}$$

where $(r)_k$ denotes the *Pochhammer symbol* given by

$$(r)_k := \begin{cases} 1, & \text{if } k=0,\\ r(r+1)(r+2)\cdots(r+k-1), & \text{if } k\in\mathbb{N}. \end{cases}$$

Remark 1.1 1. First, we will determine sufficient conditions such that $\mathfrak{J}^{\nu,m}_{\lambda,\mu}$ given by (1.4) is well defined.

2. According to the definition of Pochhammer symbol, we should assume that $k\mu \in \mathbb{N}$ for all $k \in \mathbb{N}$, that it is equivalent to $\mu \in \mathbb{N}$.

3. We should assume that the denominator of the above definition formula is not vanishing for any $k \in \mathbb{N}$, which is equivalent to

$$\lambda+1 \notin \mathbb{Z}^-_0, \qquad \lambda+\nu+1 \notin \mathbb{Z}^-_0 \quad \Leftrightarrow \quad \lambda \notin \mathbb{Z}^-, \qquad \lambda+\nu \notin \mathbb{Z}^-. \tag{1.5}$$

4. Moreover, we should prove that the power series of (1.4) converges in the whole open unit disc $\mathbb{U}$. The radius of convergence of this power series is

$$\begin{aligned} R &:= \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{2k+1}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}\cdot\frac{4^{k+1}[(\lambda+1)_{k+1}]^m(\lambda+\nu+1)_{(k+1)\mu}}{2(k+1)+1}\right|\\ &= \lim_{k\to\infty}\left|\frac{4(2k+1)}{2(k+1)+1}\right|\cdot\lim_{k\to\infty}\left|\left[\frac{(\lambda+1)(\lambda+2)\cdots(\lambda+k+1)}{(\lambda+1)(\lambda+2)\cdots(\lambda+k)}\right]^m\right|\\ &\quad\cdot\lim_{k\to\infty}\left|\frac{(\lambda+\nu+1)\cdots(\lambda+\nu+k\mu)(\lambda+\nu+k\mu+1)\cdots(\lambda+\nu+k\mu+\mu)}{(\lambda+\mu+1)\cdots(\lambda+\nu+k\nu)}\right|\\ &= 4\cdot 1\cdot\lim_{k\to\infty}\left|(\lambda+k+1)\right|^m\cdot\lim_{k\to\infty}\left|(\lambda+\nu+k\mu+1)\cdots(\lambda+\nu+k\mu+\mu)\right|. \end{aligned}$$

Using assumptions (1.5) and the fact that $\mu \in \mathbb{N}$, the second of the above limits is $+\infty$, while the first one is

$$\lim_{n\to\infty}\left|(\lambda+k+1)\right|^m = \begin{cases} 0, & \text{if } m<0,\\ 1, & \text{if } m=0,\\ +\infty, & \text{if } m>0. \end{cases}$$

Therefore, $R \geq 1$ if and only if $m \geq 0$ and (1.5) is satisfied. Concluding, the power series defined by (1.4) is correctly defined and converges in $\mathbb{U}$ only if we make the following assumptions:

$$\mu \in \mathbb{N}, \qquad \lambda \notin \mathbb{Z}^-, \qquad \lambda + \nu \notin \mathbb{Z}^-, \qquad m \geq 0. \tag{1.6}$$

2 Sufficient conditions for starlikeness and convexity of order α

This section aims to investigate a fascinating aspect regarding the geometric properties of the function defined by (1.4), such as starlikeness and convexity of order α, $0 \leq \alpha < 1$, inside the open unit disc using mathematical induction, as well as an estimation for the upper and lower bounds for the gamma function inspired by [16]. Our results improve, complement, and generalize some well-known (nonsharp) estimates given in the literature.

Theorem 2.1 *Let* $\mu \in \mathbb{N}$, $\lambda \in \mathbb{N}$, $\nu \geq 0$, *and* $m \in \mathbb{N}$. *If*

$$0 \leq \alpha < 1 - \frac{1}{(\lambda+1)^m(\lambda+\nu+1)_\mu - 1} =: \alpha_s \tag{2.1}$$

or, equivalently,

$$(\lambda+1)^m(\lambda+\nu+1)_\mu > \frac{2-\alpha}{1-\alpha}, \tag{2.2}$$

then $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$.

Proof To prove that $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$, it is sufficient to show that

$$\left| \frac{z(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z))'}{\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U}.$$

By making use of the maximum modulus principle of an analytic function as well as the triangle inequality, together with assumptions (1.6) and the fact that $\Gamma(\zeta+1) = \zeta\Gamma(\zeta)$, $\operatorname{Re}\zeta > 0$, we get

$$\begin{aligned}
&\left| \left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)' - \frac{\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)}{z} \right| \\
&\quad = \left| \sum_{k=1}^{\infty} \frac{(-1)^k k(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} z^k \right| \\
&\quad < \sup_{\theta\in[0,2\pi]} \left| \sum_{k=1}^{\infty} \frac{(-1)^k k(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} e^{ik\theta} \right| < \sum_{k=1}^{\infty} \frac{k(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} \\
&\quad = \Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1) \cdot \sum_{k=1}^{\infty} \frac{k(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)}, \quad z \in \mathbb{U},
\end{aligned} \tag{2.3}$$

since $\lambda \in \mathbb{N}$ and $\nu \geq 0$. Define $F : [1, +\infty) \to \mathbb{R}$ by

$$F(t) := \frac{t(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)}, \tag{2.4}$$

where we assumed in addition, and according to the above assumptions, that $\lambda > 0$ and $\lambda + \nu > 0$. Differentiating logarithmically both sides of (2.4) to get

$$\frac{F'(t)}{F(t)} = \frac{1}{t} + \frac{2}{2t+1} - m\psi(\lambda + t + 1) - \mu\psi(\lambda + \nu + t\mu + 1) =: G(\lambda), \tag{2.5}$$

where ψ is the well-known digamma function defined by $\psi(z) := \Gamma'(z)/\Gamma(z)$. Assume in addition that $\lambda \in \mathbb{N}$ to use in the further proof the induction method.

Now, to prove that $G(\lambda) < 0$ for all $\lambda \in \mathbb{N}$, we will use the mathematical induction. For $\lambda = 1$, we have

$$G(1) = \frac{1}{t} + \frac{2}{2t+1} - m\psi(t+2) - \mu\psi(\nu + t\mu + 2).$$

Using the fact that $\nu \geq 0$ and since we already know that $t \geq 1$, $\mu \geq 1$, we have $\nu + t\mu + 2 \geq t\mu + 2 \geq t + 2 \geq 3$, and using the fact that the digamma function ψ is a strictly increasing function on $(0, +\infty)$, it follows that $\psi(t+2) \geq \psi(3)$, $\psi(\nu + t\mu + 2) \geq \psi(3)$. Thus, because $t \geq 1$, $\nu \geq 0$, $\mu \geq 1$, and $m \in \mathbb{N}$, we obtain that

$$G(1) \leq 1 + \frac{2}{3} - m\psi(3) - \mu\psi(3) = \frac{5}{3} - (m + \mu)\psi(3),$$

and we will prove that $(5/3) - (m + \mu)\psi(3) < 0$. For this purpose, using the relation

$$\psi(\zeta + 1) = \frac{1}{\zeta} + \psi(\zeta), \quad \operatorname{Re}\zeta > 0, \tag{2.6}$$

and the fact that $\psi(1) = -\gamma$, where γ is the Euler–Mascheroni constant given by

$$\gamma := \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n} - \ln k\right) = 0.57721566490\ldots, \tag{2.7}$$

we obtain

$$\frac{5}{3} - (m + \mu)\psi(3) = \frac{5}{3} - (m + \mu)\left(\frac{3}{2} - \gamma\right) \leq \frac{5}{3} - (m + 1)\left(\frac{3}{2} - \gamma\right) < 0$$

for $\mu \geq 1$ and $m \geq (1/3)(1 + 6\gamma)/(3 - 2\gamma) = 0.8061280444\ldots$. Therefore, $G(1) < 0$, whenever $t \geq 1$, $\nu \geq 0$, $\mu \geq 1$, and $m \geq (1/3)(1 + 6\gamma)/(3 - 2\gamma) = 0.8061280444\ldots$.

Further, assuming that $G(\lambda_0) < 0$ for some $\lambda_0 \in \mathbb{N}$ and using (2.6), we have

$$\begin{aligned}
&G(\lambda_0 + 1) - G(\lambda_0)\\
&\quad = -m\big[\psi(\lambda_0 + t + 2) - \psi(\lambda_0 + t + 1)\big] - \mu\big[\psi(\lambda_0 + \nu + t\mu + 2) - \psi(\lambda_0 + \nu + t\mu + 1)\big]\\
&\quad = -\frac{m}{\lambda_0 + t + 1} - \frac{\mu}{\lambda_0 + \nu + t\mu + 1} < 0,
\end{aligned}$$

where $t \geq 1$, $\nu \geq 0$, $\mu \geq 1$, and $m \geq 0$. It follows that $G(\lambda_0 + 1) < G(\lambda_0) < 0$, therefore $G(\lambda) < 0$ for all $\lambda \in \mathbb{N}$. The well-known relation

$$\Gamma(\zeta) = \lim_{k\to+\infty}\frac{k!\,k^{\zeta}}{\zeta(\zeta+1)\cdots(\zeta+k)}, \quad \operatorname{Re}\zeta > 0, \tag{2.8}$$

leads to $F(t) > 0$ for all $t \in [1, +\infty)$.

Finally, using that $F(t) > 0$ for all $t \geq 1$ and the fact that $G(\lambda) < 0$ for all $\lambda \in \mathbb{N}$, from (2.5) it follows that $F'(t) < 0$, $t \in [1, +\infty)$, hence the function F is strictly decreasing on $[1, +\infty)$.

Consequently, the right-hand side of (2.3) implies that

$$\begin{aligned}&\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\sum_{k=1}^{\infty}\frac{k(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)}\\&\quad<\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)},\end{aligned}$$

and so

$$\left|\left(\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)\right)'-\frac{\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)}{z}\right|<\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)},\quad z\in\mathbb{U}. \tag{2.9}$$

On the other hand, from the maximum modulus theorem of an analytic function, it finds

$$\begin{aligned}\left|\frac{\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)}{z}\right|&=\left|1+\sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}z^k\right|\\&>1-\sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}e^{ik\theta}\right|\\&\geq1-\sum_{k=1}^{\infty}\frac{2k+1}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}},\quad z\in\mathbb{U},\end{aligned}$$

and the above inequality could be rewritten as

$$\left|\frac{\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)}{z}\right|>1-\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\cdot\sum_{k=1}^{\infty}\frac{2k+1}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)},\quad z\in\mathbb{U}.$$

If we define $G:[1,+\infty)\to\mathbb{R}$ by $G(t):=F(t)/t$ where F is defined by (2.4), since we already proved that F is a strictly decreasing function on $[1,+\infty)$, it follows that G is also a strictly decreasing on the same interval. Therefore, the above inequality leads to

$$\begin{aligned}\left|\frac{\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)}{z}\right|&>1-3\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\cdot\sum_{k=1}^{\infty}\frac{1}{4^k\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}\\&=1-\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}>0,\quad z\in\mathbb{U}.\end{aligned} \tag{2.10}$$

From inequalities (2.9) and (2.10) we deduce that

$$\left|\frac{z\left(\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)\right)'}{\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)}-1\right|<\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)-\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)},\quad z\in\mathbb{U},$$

and a simple computation shows that the right-hand side of the above inequality is less or equal than $1-\alpha$ if and only if (2.1) holds, or equivalently (2.2). □

Theorem 2.2 *Let $\mu \in \mathbb{N}$, $\lambda \in \mathbb{N}$, $\nu \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$. If*

$$0 \leq \alpha < 1 - \frac{2}{(\lambda+1)^m(\lambda+\nu+1)_\mu - 2} =: \alpha_c \tag{2.11}$$

or, equivalently,

$$(\lambda+1)^m(\lambda+\nu+1)_\mu > \frac{2(2-\alpha)}{1-\alpha}, \tag{2.12}$$

then $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}(\alpha)$, $0 \leq \alpha < 1$.

Proof We would like to find sufficient conditions such that $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}(\alpha)$, $0 \leq \alpha < 1$, with $\mathfrak{J}_{\lambda,\mu}^{\nu,m}$ given by (1.4) under the assumptions of Theorem 2.1 that are $\lambda, \mu, m \in \mathbb{N}$ and $\nu \geq 0$. To prove the required result, it is sufficient to show that

$$\left|\frac{z(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z))''}{(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z))'}\right| < 1-\alpha, \quad z \in \mathbb{U}.$$

Using the maximum modulus principle of the analytic functions and the triangle inequality, we obtain

$$\begin{aligned}
\left|z\big(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\big)''\right| &= \left|\sum_{k=1}^{\infty} \frac{(-1)^k k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} z^k\right| \\
&< \sup_{\theta\in[0,2\pi]} \left|\sum_{k=1}^{\infty} \frac{(-1)^k k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} e^{ik\theta}\right| \\
&< \sum_{k=1}^{\infty} \frac{k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m} \\
&= \Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1) \\
&\quad \cdot \sum_{k=1}^{\infty} \frac{k(k+1)(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)}, \quad z \in \mathbb{U}.
\end{aligned} \tag{2.13}$$

Consider the function $H : [1,+\infty) \to \mathbb{R}$ defined by

$$H(t) := \frac{t(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)}. \tag{2.14}$$

Differentiating logarithmically both sides of (2.14), we get

$$\frac{H'(t)}{H(t)} = \frac{1}{t} + \frac{1}{t+1} + \frac{2}{2t+1} - m\psi(\lambda+t+1) - \mu\psi(\lambda+\nu+t\mu+1) =: L(\lambda), \tag{2.15}$$

where ψ stands for the digamma function.

Now, using the mathematical induction, we will prove that $L(\lambda) < 0$ for all $\lambda \in \mathbb{N}$. For $\lambda = 1$, we have

$$L(1) = \frac{1}{t} + \frac{1}{t+1} + \frac{2}{2t+1} - m\psi(t+2) - \mu\psi(\nu+t\mu+2).$$

According to the assumptions that $\nu \geq 0$, $t \geq 1$, and $\mu \in \mathbb{N}$, we get $\nu + t\mu + 2 \geq t\mu + 2 \geq t + 2 \geq 3$, and using again the fact that ψ is a strictly increasing function on $(0, +\infty)$, we deduce that $\psi(t+2) \geq \psi(3)$, $\psi(\nu + t\mu + 2) \geq \psi(3)$, hence

$$L(1) \leq 1 + \frac{1}{2} + \frac{2}{3} - m\psi(3) - \mu\psi(3) = \frac{13}{6} - (m+\mu)\psi(3).$$

Moreover, $(13/6) - (m+\mu)\psi(3) < 0$ because by using (2.6) and the fact that $\psi(1) = -\gamma$ we get

$$\frac{13}{6} - (m+\mu)\psi(3) = \frac{13}{6} - (m+\mu)\left(\frac{3}{2} - \gamma\right) \leq \frac{13}{6} - (m+1)\left(\frac{3}{2} - \gamma\right) < 0,$$

and this last inequality holds under our assumptions $\mu \geq 1$ and $m \geq (2/3)(2+3\gamma)/(3-2\gamma) = 1.347966458\ldots$. Consequently, $L(1) < 0$, whenever $t \geq 1$, $\nu \geq 1$, $\mu \geq 1$, and $m \geq (2/3)(2+3\gamma)/(3-2\gamma) = 1.347966458\ldots$.

Assuming that $L(\lambda_0) < 0$ for some $\lambda_0 \in \mathbb{N}$, we have

$$\begin{aligned} &L(\lambda_0+1) - L(\lambda_0) \\ &\quad = -m\left[\psi(\lambda_0+t+2) - \psi(\lambda_0+t+1)\right] - \mu\left[\psi(\lambda_0+\nu+t\mu+2) - \psi(\lambda_0+\nu+t\mu+1)\right] \\ &\quad = -\frac{m}{\lambda_0+t+1} - \frac{\mu}{\lambda_0+\nu+t\mu+1} < 0, \end{aligned}$$

where $t \geq 1$, $\nu \geq 0$, $\mu \geq 1$, and $m \geq 0$. It follows that $L(\lambda_0+1) < L(\lambda_0) < 0$, therefore $L(\lambda) < 0$ for all $\lambda \in \mathbb{N}$. Using again relation (2.8), we get that $H(t) > 0$ for all $x \geq 1$, and from relation (2.15) and the fact that $L(\lambda) < 0$ for all $\lambda \in \mathbb{N}$, we conclude that $H'(t) < 0$, $t \in [1, +\infty)$. Therefore, the function H is strictly decreasing on $[1, +\infty)$, hence from (2.13) we have

$$\begin{aligned} &\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\sum_{k=1}^{\infty}\frac{k(k+1)(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)} \\ &\quad < \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}, \end{aligned}$$

and (2.13) leads to the inequality

$$\left|z\left(\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)\right)''\right| < \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}, \quad z \in \mathbb{U}. \tag{2.16}$$

From the maximum modulus principle of an analytic function, we get

$$\begin{aligned} \left|\left(\mathfrak{I}_{\lambda,\mu}^{\nu,m}(z)\right)'\right| &\geq 1 - \left|\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}z^k\right| \\ &> 1 - \sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}e^{ik\theta}\right| \\ &\geq 1 - \sum_{k=1}^{\infty}\frac{(k+1)(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}}, \quad z \in \mathbb{U}, \end{aligned}$$

and the above inequality could be rewritten as

$$\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)'\right| > 1-\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\cdot\sum_{k=1}^{\infty}\frac{(k+1)(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)},\quad z\in\mathbb{U}.$$

If we define the function $L:[1,+\infty)\to\mathbb{R}$ by $L(t):=H(t)/t$, where H is given by (2.14), since we already proved that H is a strictly decreasing function on $[1,+\infty)$, it follows that L is also strictly decreasing on the same interval. Therefore, the above inequality implies that

$$\begin{aligned}\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)'\right| &> 1-6\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\cdot\sum_{k=1}^{\infty}\frac{1}{4^k\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}\\ &= 1-\frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}>0,\quad z\in\mathbb{U}.\end{aligned}\tag{2.17}$$

From inequalities (2.16) and (2.17) we conclude that

$$\left|\frac{z(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z))''}{(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z))'}\right| < \frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)-2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)},\quad z\in\mathbb{U},$$

and a simple computation shows that the right-hand side of the above inequality is less or equal to $1-\alpha$ if and only if (2.11) holds, which is equivalent to (2.12). □

Example 2.1 If we take in Theorems 2.1 and 2.2 the particular values $\lambda=1$, $\mu=3$, $m=2$, and $\nu=0.5$, we get that

$$\begin{aligned}\mathfrak{J}_{1,3}^{0.5,2}(z) &= z-0.004761904762z^2+0.0000008222230443z^3-2.121318485\cdot10^{-11}z^4\\ &\quad+1.405428395\cdot10^{-16}z^5-3.216716321\cdot10^{-22}z^6+3.072295397\cdot10^{-28}z^7\\ &\quad-1.396355734\cdot10^{-34}z^8+3.326860555\cdot10^{-41}z^9+\cdots\in\mathcal{S}^*(\alpha_s)\cap\mathcal{K}(\alpha_c),\end{aligned}$$

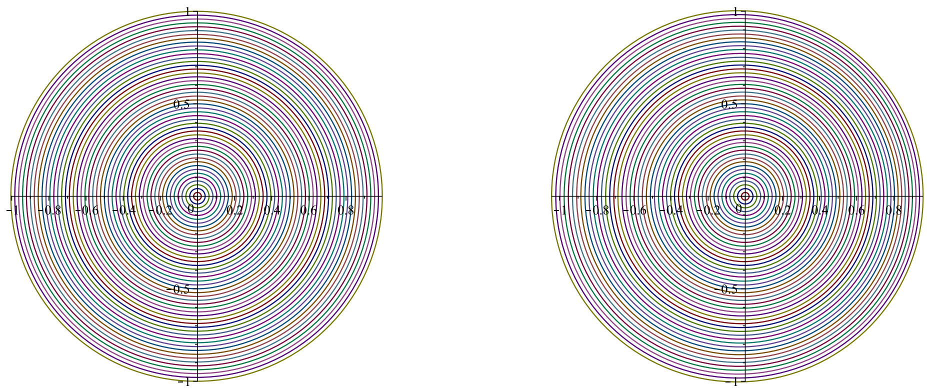
where

$$\alpha_s=0.9936102236\ldots\quad\text{and}\quad\alpha_c=0.9871382637\ldots,$$

and the image of the open unit disc $\mathbb{U}$ by $\mathfrak{J}_{1,3}^{0.5,2}$ is shown in Fig. 1(a).

Example 2.2 For the special case $\lambda=\mu=1$, $m=3$, and $\nu=0.2$, Theorems 2.1 and 2.2 lead to

$$\begin{aligned}\mathfrak{J}_{1,1}^{0.2,3}(z) &= z-0.04261363636z^2+0.0002055055766z^3-0.0000002675853863z^4\\ &\quad+0.0000000001323224437z^5-3.019105272\cdot10^{-14}z^6\\ &\quad+3.611954874\cdot10^{-18}z^7-2.481683640\cdot10^{-22}z^8\\ &\quad+1.048404154\cdot10^{-26}z^9\cdots\in\mathcal{S}^*(\alpha_s)\cap\mathcal{K}(\alpha_c)\end{aligned}$$



(A) The image of $\mathfrak{J}_{1,3}^{0.5,2}(\mathbb{U})$ (B) The image of $\mathfrak{J}_{1,1}^{0.2,3}(\mathbb{U})$

Figure 1 Figures for Examples 2.1 and 2.2

with

$$\alpha_s = 0.9397590361\ldots \quad \text{and} \quad \alpha_c = 0.8717948718\ldots,$$

while the image of $\mathfrak{J}_{1,1}^{0.2,3}(\mathbb{U})$ is presented in Fig. 1(B).

Using Theorem 1 of [27], we obtain in the following result a sufficient condition for the parameters $\lambda \geq 1$, $\mu, m \in \mathbb{N}$, and $\nu \geq 0$ such that $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{S}^*(\alpha)$, which extends Theorem 2.1, where we assumed that $\lambda \in \mathbb{N}$.

Theorem 2.3 *Suppose that* $\mu \in \mathbb{N}$, $\lambda \geq 1$, $\nu \geq 0$, *and* $m \in \mathbb{N}$. *If*

$$0 \leq \alpha < 1 - \frac{1}{(\lambda+1)^m(\lambda+\nu+1)_\mu - 1} = \alpha_s$$

or, equivalently,

$$(\lambda+1)^m(\lambda+\nu+1)_\mu > \frac{2-\alpha}{1-\alpha}, \tag{2.18}$$

then $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{S}^*(\alpha)$.

Proof As is well known from [27, Theorem 1], if f is of the form (1.1) and satisfies $\sum_{k=2}^{\infty}(k-\alpha)|A_k| \leq 1-\alpha$, then $f \in \mathcal{S}^*(\alpha)$. Thus, according to (1.4), it is enough to prove that

$$\mathrm{H}_1 := \sum_{k=2}^{\infty}(k-\alpha)\left|\frac{(-1)^{k-1}(2k-1)}{4^{k-1}[(\lambda+1)_{k-1}]^m(\lambda+\nu+1)_{(k-1)\mu}}\right| \leq 1-\alpha.$$

Since $\lambda > -1$ and $\mu \in \mathbb{N}$, we have

$$\mathrm{H}_1 = \sum_{k=2}^{\infty}\frac{(k-\alpha)(2k-1)}{4^{k-1}[(\lambda+1)_{k-1}]^m(\lambda+\nu+1)_{(k-1)\mu}}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{k(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}} + (1-\alpha)\sum_{k=1}^{\infty} \frac{2k+1}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}} \\
&= \Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\left(\sum_{k=1}^{\infty} \frac{k(2k+1)}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)}\right. \\
&\quad \left. + (1-\alpha)\sum_{k=1}^{\infty} \frac{2k+1}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)}\right). \qquad (2.19)
\end{aligned}$$

Consider the function $U:[1,+\infty)\to\mathbb{R}$ defined by

$$U(t) := \frac{t(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)}.$$

Now, we would like to show that $U(t+1) < U(t)$ for $t \geq 1$, hence we will give a negative upper bound for

$$\begin{aligned}
&U(t+1) - U(t) \\
&= \frac{(t+1)(2t+3)}{\Gamma(\lambda+\nu+t\mu+\mu+1)\Gamma^m(\lambda+t+2)} - \frac{t(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)} \\
&= \frac{1}{\Gamma^m(\lambda+t+2)}\left(\frac{(t+1)(2t+3)}{\Gamma(\lambda+\nu+t\mu+\mu+1)} - \frac{t(2t+1)(\lambda+t+1)^m}{\Gamma(\lambda+\nu+t\mu+1)}\right). \qquad (2.20)
\end{aligned}$$

In Theorem 1 of [16], it was proved that

$$\frac{t^{t-\gamma}}{e^{t-1}} < \Gamma(t) < \frac{t^{t-1/2}}{e^{t-1}}, \quad t > 1,$$

but

$$\left.\frac{t^{t-\gamma}}{e^{t-1}}\right|_{t=1} = 1 = \Gamma(1) = \left.\frac{t^{t-1/2}}{e^{t-1}}\right|_{t=1} = 1,$$

therefore, we conclude that

$$\frac{t^{t-\gamma}}{e^{t-1}} \leq \Gamma(t) \leq \frac{t^{t-1/2}}{e^{t-1}}, \quad t \geq 1, \qquad (2.21)$$

where γ is the Euler–Mascheroni constant given by (2.7), relation (2.20) becomes

$$\begin{aligned}
&U(t+1) - U(t) \\
&\leq \frac{1}{\Gamma^m(\lambda+t+2)} \cdot \left[\frac{(t+1)(2t+3)e^{\lambda+\nu+t\mu+\mu}}{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}} - \frac{t(2t+1)(\lambda+t+1)^m e^{\lambda+\nu+t\mu}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}}\right] \\
&= \frac{e^{\lambda+\nu+t\mu+\mu}}{\Gamma^m(\lambda+t+2)} \cdot \frac{U_1(t)}{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}}, \qquad (2.22)
\end{aligned}$$

where

$$U_1(t) := (t+1)(2t+3)(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}$$

$$-t(2t+1)(\lambda+t+1)^m(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}e^{-\mu},$$

and could be written as

$$\begin{aligned}U_1(t)=(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}\bigg[&(t+1)(2t+3)\\&-t(2t+1)(\lambda+t+1)^m\frac{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}e^{\mu}}\bigg].\end{aligned}$$

Since $\lambda\geq 1$, $\mu\in\mathbb{N}$, $m\geq 1$, and $\nu\geq 0$, we have

$$\begin{aligned}U_1(t)&\leq(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}\bigg[(t+1)(2t+3)\\&\quad-t(2t+1)(t+2)\frac{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}}\cdot\frac{1}{e^{\mu}}\bigg]\\&=(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}\big[(t+1)(2t+3)-t(2t+1)(t+2)A(y)\big],\end{aligned}\tag{2.23}$$

where, using the same above assumptions, we have

$$A(y):=\frac{(y+\mu)^{y+\mu-\gamma}}{y^{y-\frac{1}{2}}}\cdot\frac{1}{e^{\mu}}\quad\text{with } y=\lambda+\nu+t\mu+1\geq\mu+2 \text{ for } t\geq 1.$$

It is well known that

$$\left(1+\frac{1}{x}\right)^x<e<\left(1+\frac{1}{x}\right)^{x+1},\quad x>0,$$

and replacing $t:=y/\mu$, since $\mu\geq 1$, we get

$$\left(1+\frac{\mu}{y}\right)^y<e^{\mu}<\left(1+\frac{\mu}{y}\right)^{y+\mu},\quad y>0.$$

From the assumption $\mu\geq 1$, using the second of the above inequalities, it follows

$$A(y):=\frac{(y+\mu)^{y+\mu-\gamma}}{y^{y-\frac{1}{2}}}\cdot\frac{1}{e^{\mu}}>\frac{(y+\mu)^{y+\mu-\gamma}}{y^{y-\frac{1}{2}}}\cdot\frac{y^{y+\mu}}{(y+\mu)^{y+\mu}}=\frac{y^{\mu+\frac{1}{2}-\gamma}}{(1+\frac{\mu}{y})^{\gamma}}=:B(y),\quad y\geq\mu+2,$$

and since the function B is strictly increasing on $[\mu+2,+\infty)$, we have

$$B(y)\geq B(\mu+2)=\frac{(\mu+2)^{\mu+\frac{1}{2}-\gamma}}{(1+\frac{\mu}{\mu+2})^{\gamma}}=\frac{(\mu+2)^{\mu+\frac{1}{2}-\gamma}}{2^{\gamma}(\mu+1)^{\gamma}}=:\Phi(\mu),\quad y\geq\mu+2.$$

Using the MAPLE™ computer software code "`minimize(`$\Phi,\mu\geq 1$`)`", we obtain that

$$B(y)\geq\min\big\{\Phi(\mu):\mu\geq 1\big\}=\Phi(1)=\frac{(\mu+2)^{\frac{3}{2}-\gamma}}{2^{\gamma}(\mu+1)^{\gamma}}=1.238116644\ldots,\quad y\geq\mu+2,$$

which leads to $A(y)>1.2381$, $y\geq\mu+2$. Therefore, using (2.23) we deduce

$$U_1(t)<(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}\big[(t+1)(2t+3)-t(2t+1)(t+2)\cdot 1.2381\big],\quad t\geq 1,$$

or

$$U_1(t) < (\lambda + \nu + t\mu + 1)^{\lambda+\nu+t\mu+1/2} U_2(t), \quad t \geq 1, \tag{2.24}$$

where

$$U_2(t) := (t+1)(2t+3) - 1.2381t(2t+1)(t+2), \quad t \geq 1.$$

A simple computation shows that

$$U_2'(t) = -7.4286t^2 - 8.3810t + 2.5238 < 0, \quad t \geq 1,$$

hence U_2 is a strictly decreasing function on $[1, +\infty)$ that implies

$$U_2(t) \leq U_2(1) = -13.2858\ldots < 0, \quad t \geq 1,$$

and according to (2.24) this inequality implies $U_1(t) < 0$ for all $t \geq 1$. Therefore, taking into the account inequality (2.22), we obtain that $U(t+1) < U(t)$ for $t \geq 1$. Consequently, since $U(k+1) < U(k)$ for all $k \in \mathbb{N}$, for the first term of the sum (2.19), we deduce that

$$\sum_{k=1}^{\infty} \frac{k(2k+1)}{4^k \Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)} < \frac{1}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}. \tag{2.25}$$

To evaluate the second term of the sum (2.19), we will define the function $V : [1, +\infty) \to \mathbb{R}$ by

$$V(t) := \frac{2t+1}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)}.$$

Since

$$V(t+1) - V(t) = \frac{U(t+1)}{t+1} - \frac{U(t)}{t} < \frac{1}{t}\big(U(t+1) - U(t)\big), \quad t \geq 1,$$

and because we already proved that $U(t+1) - U(t) < 0$ for all $t \geq 1$, it follows that $V(t+1) - V(t) < 0$, $t \geq 1$. Similarly, for the second term of the sum (2.19), we have

$$\sum_{k=1}^{\infty} \frac{2k+1}{4^k \Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)} < \frac{1}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}. \tag{2.26}$$

Using relation (2.19) combined with inequalities (2.25) and (2.26), it follows that

$$H_1 < (2-\alpha)\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma^m(\lambda+2)\Gamma(\lambda+\nu+1+\mu)} = \frac{2-\alpha}{(\lambda+1)^m(\lambda+\nu+1)_\mu},$$

and from assumption (2.18) it follows that $H_1 \leq 1 - \alpha$, thus $\mathfrak{I}_{\lambda,\mu}^{\nu,m} \in \mathcal{S}^*(\alpha)$. □

Using similar reasons, the next theorem gives us sufficient conditions such that $\mathfrak{I}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}(\alpha)$ for a much weaker assumption on λ, that is, only $\lambda \geq 1$.

Theorem 2.4 *Suppose that $\mu \in \mathbb{N}$, $\lambda \geq 1$, $\nu \geq 0$, and $m \in \mathbb{N}$. If*

$$0 \leq \alpha < 1 - \frac{2}{(\lambda+1)^m(\lambda+\nu+1)_\mu - 2} = \alpha_c$$

or, equivalently,

$$(\lambda+1)^m(\lambda+\nu+1)_\mu > \frac{2(2-\alpha)}{1-\alpha}, \tag{2.27}$$

then $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}(\alpha)$.

Proof As is well known, a function f of the form (1.1) belongs to the class $\mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.

Since

$$z\big(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\big)' = z + \sum_{k=1}^{\infty} B_k z^k, \quad z \in \mathbb{U},$$

with

$$B_k := \frac{(-1)^{k-1}k(2k-1)}{4^{k-1}[(\lambda+1)_{k-1}]^m(\lambda+\nu+1)_{(k-1)\mu}}, \quad k \geq 2,$$

according to [27, Theorem 1], to prove our result, we will show that $\sum_{k=2}^{\infty}(k-\alpha)|B_k| \leq 1-\alpha$. A simple computation leads to the fact that this inequality is equivalent to

$$\mathrm{H}_2 := \sum_{k=2}^{\infty}(k-\alpha)\left|\frac{(-1)^{k-1}k(2k-1)}{4^{k-1}[(\lambda+1)_{k-1}]^m(\lambda+\nu+1)_{(k-1)\mu}}\right| \leq 1-\alpha.$$

Using that $\lambda > -1$ and $\mu \in \mathbb{N}$, it follows

$$\begin{aligned}
\mathrm{H}_2 &= \sum_{k=2}^{\infty} \frac{(k-\alpha)k(2k-1)}{4^{k-1}[(\lambda+1)_{k-1}]^m(\lambda+\nu+1)_{(k-1)\mu}} \\
&= \sum_{k=1}^{\infty} \frac{k(k+1)(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}} + (1-\alpha)\sum_{k=1}^{\infty} \frac{(k+1)(2k+1)}{4^k[(\lambda+1)_k]^m(\lambda+\nu+1)_{k\mu}} \\
&= \Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\left(\sum_{k=1}^{\infty} \frac{k(k+1)(2k+1)}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)}\right. \\
&\quad \left. + (1-\alpha)\sum_{k=1}^{\infty} \frac{(k+1)(2k+1)}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)}\right).
\end{aligned} \tag{2.28}$$

Consider the functions $\widehat{U} : [1,+\infty) \to \mathbb{R}$ and $\widehat{V} : [1,+\infty) \to \mathbb{R}$ defined by

$$\widehat{U}(t) := \frac{t(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)}$$

and

$$\widehat{V}(t) := \frac{(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)},$$

respectively.

First, we would like to show that $\widehat{U}(t+1) < \widehat{U}(t)$ for $t \geq 1$, hence we will try to find a negative upper bound for the difference

$$\begin{aligned}
&\widehat{U}(t+1) - \widehat{U}(t) \\
&\quad = \frac{(t+1)(t+2)(2t+3)}{\Gamma(\lambda+\nu+t\mu+\mu+1)\Gamma^m(\lambda+t+2)} - \frac{t(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)} \\
&\quad = \frac{t+1}{\Gamma^m(\lambda+t+2)}\left(\frac{(t+2)(2t+3)}{\Gamma(\lambda+\nu+t\mu+\mu+1)} - \frac{t(2t+1)(\lambda+t+1)^m}{\Gamma(\lambda+\nu+t\mu+1)}\right).
\end{aligned} \tag{2.29}$$

Using again the double inequality (2.21), from relation (2.29) it follows that

$$\begin{aligned}
&\widehat{U}(t+1) - \widehat{U}(t) \\
&\quad \leq \frac{t+1}{\Gamma^m(\lambda+t+2)} \cdot \left(\frac{(t+2)(2t+3)e^{\lambda+\nu+t\mu+\mu}}{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}} - \frac{t(2t+1)(\lambda+t+1)^m e^{\lambda+\nu+t\mu}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}}\right) \\
&\quad = \frac{(t+1)e^{\lambda+\nu+t\mu+\mu}}{\Gamma^m(\lambda+t+2)} \cdot \frac{\widehat{U}_1(t)}{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}},
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
\widehat{U}_1(t) :=\ &(t+2)(2t+3)(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2} \\
&- t(2t+1)(\lambda+t+1)^m(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}e^{-\mu}
\end{aligned}$$

or

$$\begin{aligned}
\widehat{U}_1(t) = (\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}\Bigg[&(t+2)(2t+3) \\
&- t(2t+1)(\lambda+t+1)^m \frac{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}e^{\mu}}\Bigg].
\end{aligned}$$

Since $\lambda \geq 1$, $\mu \in \mathbb{N}$, $m \geq 1$, and $\nu \geq 0$, we get

$$\begin{aligned}
\widehat{U}_1(t) \leq\ &(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}(t+2)\Bigg[2t+3 \\
&- t(2t+1)\frac{(\lambda+\nu+t\mu+\mu+1)^{\lambda+\nu+t\mu+\mu+1-\gamma}}{(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}} \cdot \frac{1}{e^{\mu}}\Bigg] \\
=\ &(\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}(t+2)\big[2t+3-t(2t+1)\widehat{A}(y)\big],
\end{aligned} \tag{2.31}$$

where, using again the above mentioned assumptions, we deduce that

$$\widehat{A}(y) := \frac{(y+\mu)^{y+\mu-\gamma}}{y^{y-\frac{1}{2}}} \cdot \frac{1}{e^{\mu}} \quad \text{with } y = \lambda+\nu+t\mu+1 \geq \mu+2 \text{ for } t \geq 1.$$

Since $\widehat{A}(y)$ has the same form like in the proof of Theorem 2.3, using the MAPLE™ computer software, we obtain $\widehat{A}(y) > 1.2381$, $y \geq \mu+2$, and from inequality (2.31) we get

$$\widehat{U}_1(t) < (\lambda+\nu+t\mu+1)^{\lambda+\nu+t\mu+1/2}(t+2)\big[2t+3-t(2t+1)\cdot 1.2381\big], \quad t \geq 1,$$

or

$$\widehat{U}_1(t) < (\lambda + \nu + t\mu + 1)^{\lambda+\nu+t\mu+1/2}(t+2)\widehat{U}_3(t), \quad t \geq 1, \tag{2.32}$$

where

$$\widehat{U}_2(t) := 2t + 3 - 1.2381t(2t+1) = -2.4762t^2 + 0.7619t + 3, \quad t \geq 1.$$

Since $\widehat{U}_2(t) = 0$ if $t \in \{-0.9575518016, 1.265241003\}$, it follows that

$$\widehat{U}_2(t) = -2.4762t^2 + 0.7619t + 3, \quad t \geq 2,$$

and according to (2.32) and (2.30) it follows that $\widehat{U}(t+1) < \widehat{U}(t)$ for $t \geq 2$.

To show that the last inequality holds also for $t = 1$, we should prove that

$$\widehat{U}(2) - \widehat{U}(1) = \frac{2}{\Gamma^m(\lambda+3)}\left(\frac{15}{\Gamma(\lambda+\nu+2\mu+1)} - \frac{3(\lambda+2)^m}{\Gamma(\lambda+\nu+\mu+1)}\right) < 0$$

for $\lambda \geq 1$, $\mu \in \mathbb{N}$, $m \geq 1$, and $\nu \geq 0$ or, equivalently,

$$\begin{aligned}
&\frac{\Gamma(\lambda+\nu+\mu+1)}{\Gamma(\lambda+\nu+2\mu+1)} < \frac{(\lambda+2)^m}{5} \\
&\quad\Leftrightarrow\quad S(\lambda,\nu,\mu) := \frac{1}{(\lambda+\nu+2\mu)(\lambda+\nu+2\mu-1)\cdots(\lambda+\nu+\mu+1)} \\
&\qquad\qquad < \frac{(\lambda+2)^m}{5} =: T(\lambda,m).
\end{aligned}$$

Since $\lambda \geq 1$, $\mu \in \mathbb{N}$, $m \geq 1$, and $\nu \geq 0$, we have

$$T(\lambda,m) \geq T(1,1) = \frac{3}{5} \quad \text{and} \quad S(\lambda,\nu,\mu) \leq S(1,0,1) = \frac{1}{3},$$

therefore

$$S(\lambda,\nu,\mu) \leq \frac{1}{3} < \frac{3}{5} \leq T(\lambda,m).$$

Hence, the inequality $\widehat{U}(t+1) < \widehat{U}(t)$ holds also for $t = 1$, therefore $\widehat{U}(t+1) < \widehat{U}(t)$ for $t \in \{1\} \cup [2,+\infty)$.

For the second term of sum (2.28), we see that for the previously defined function $\widehat{V}$ we have

$$\widehat{V}(t+1) - \widehat{V}(t) = \frac{\widehat{U}(t+1)}{t+1} - \frac{\widehat{U}(t)}{t} < \frac{1}{t}\bigl(\widehat{U}(t+1) - \widehat{U}(t)\bigr), \quad t \in \{1\} \cup [2,+\infty),$$

and because we already proved that $\widehat{U}(t+1) - \widehat{U}(t) < 0$ for all $t \in \{1\} \cup [2,+\infty)$, it follows that $\widehat{V}(t+1) - \widehat{V}(t) < 0$, $t \in \{1\} \cup [2,+\infty)$.

Consequently, since $\mathbb{N} \subset \{1\} \cup [2,+\infty)$, like in the proof of Theorem 2.3, the above results yield that

$$\widehat{U}(k) \leq \widehat{U}(1) = \frac{6}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}, \quad k \in \mathbb{N},$$

and

$$\widehat{V}(k) \le \widehat{V}(1) = \frac{6}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}, \quad k \in \mathbb{N}.$$

It follows that for both terms of the sums that appeared in (2.28) we have

$$\sum_{k=1}^{\infty} \frac{k(k+1)(2k+1)}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)} < \frac{2}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}, \tag{2.33}$$

and

$$\sum_{k=1}^{\infty} \frac{(k+1)(2k+1)}{4^k\Gamma^m(\lambda+k+1)\Gamma(\lambda+\nu+1+k\mu)} < \frac{2}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)}. \tag{2.34}$$

Finally, from relation (2.28) together with inequalities (2.33) and (2.34), we deduce that

$$H_2 < 2(2-\alpha)\frac{\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma^m(\lambda+2)\Gamma(\lambda+\nu+1+\mu)} = \frac{2(2-\alpha)}{(\lambda+1)^m(\lambda+\nu+1)_\mu},$$

and from assumption (2.27) it follows that $H_2 \le 1-\alpha$, therefore $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}(\alpha)$. □

Using Theorems 2.3 and 2.4, in the next two examples we find the order of starlikeness and convexity for the functions $\mathfrak{J}_{1.5,2.5}^{3,2}$ and $\mathfrak{J}_{0.2,3.5}^{2,2}$, and we emphasize that $\lambda \notin \mathbb{N}$, hence these results cannot be obtained from Theorems 2.1 and 2.2, respectively.

Example 2.3 Taking in Theorems 2.3 and 2.4 the values $\lambda = 2.5$, $\mu = 3$, $m = 2$, and $\nu = 1.5$, we get that

$$\begin{aligned}\mathfrak{J}_{2.5,3}^{1.5,2}(z) = z &- 0.00029154518 95z^2 + 0.000000008331766962z^3 \\ &- 5.617751131 \cdot 10^{-14}z^4 + 1.271982052 \cdot 10^{-19}z5 - 1.188430019 \cdot 10^{-25}z^6 \\ &+ 5.259621189 \cdot 10^{-32}z^7 - 1.218192959 \cdot 10^{-38}z^8 \\ &+ 1.592722130 \cdot 10^{-45}z^9 + \cdots \in \mathcal{S}^*(\alpha_s) \cap \mathcal{K}(\alpha_c),\end{aligned}$$

where

$$\alpha_s = 0.9996111219\ldots \quad \text{and} \quad \alpha_c = 0.9992219413\ldots,$$

and the image of the open unit disc $\mathbb{U}$ by $\mathfrak{J}_{2.5,3}^{1.5,2}$ is presented in Fig. 2 (A).

Example 2.4 Putting $\lambda = 3.5$, $\mu = m = 2$, and $\nu = 0.2$, Theorems 2.3 and 2.4 lead to

$$\begin{aligned}\mathfrak{J}_{3.5,2}^{0.2,2}(z) = z &- 0.001382494850z^2 + 0.0000003691147274z^3 - 3.623361937 \cdot 10^{-11}z^4 \\ &+ 1.653880131 \cdot 10^{-15}z^5 - 4.020055735 \cdot 10^{-20}z^6 + 5.702411482 \cdot 10^{-25}z^7 \\ &- 5.047520972 \cdot 10^{-30}z^8 + 2.935433795 \cdot 10^{-35}z^9 + \cdots \in \mathcal{S}^*(\alpha_s) \cap \mathcal{K}(\alpha_c)\end{aligned}$$

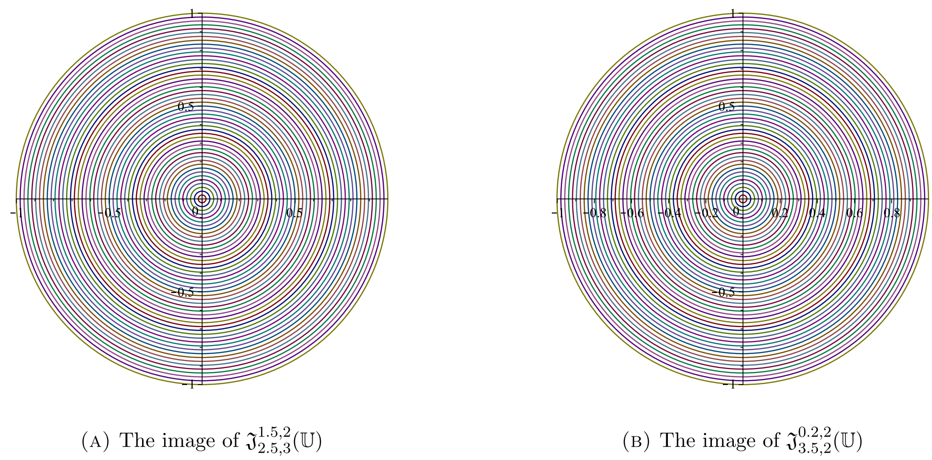


Figure 2 Figures for Examples 2.3 and 2.4

with

$$\alpha_s = 0.9981532694\ldots \quad \text{and} \quad \alpha_c = 0.9962997054\ldots,$$

and the image of $\mathfrak{J}^{0.2,2}_{3.5,2}(\mathbb{U})$ is shown in Fig. 2(B).

Remark 2.1 1. We could see that Theorems 2.3 and 2.4 are more general than Theorems 2.1 and 2.2, respectively. That is because in the first case we replace the assumption $\lambda \in \mathbb{N}$ with $\lambda \geq 1$, while in the second case the assumptions $\lambda \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{1\}$ were replaced by $\lambda \geq 1$ and $m \in \mathbb{N}$ only.

2. Remark that

$$\min \alpha_s = \frac{2}{3}$$

under the assumptions of Theorems 2.1 and 2.3, hence if the parameters satisfy the conditions of these two theorems, then $\mathfrak{J}^{\nu,m}_{\lambda,\mu} \in \mathcal{S}(2/3)$. Also,

$$\min \alpha_c = \frac{2}{3} \quad \text{and} \quad \min \alpha_c = 0$$

under the assumptions of Theorems 2.2 and 2.4, respectively. Thus, if the parameters satisfy the conditions of Theorem 2.2, then $\mathfrak{J}^{\nu,m}_{\lambda,\mu} \in \mathcal{K}(2/3)$, while if they satisfy the assumptions of Theorem 2.4, then $\mathfrak{J}^{\nu,m}_{\lambda,\mu} \in \mathcal{K}(0) =: \mathcal{K}$.

3 Sufficient conditions for starlikeness and convexity

In this section we give sufficient conditions for the starlikeness and convexity of $\mathfrak{J}^{\nu,m}_{\lambda,\mu}$ by using the next results, respectively.

Lemma 3.1 [19, *Corollary* 1.2] *If* $f(z) = z + a_{k+1}z^{k+1} + \cdots$, $k \geq 1$, *is analytic in* $\mathbb{U}$ *and*

$$\left|f'(z) - 1\right| < \frac{k+1}{\sqrt{(k+1)^2 + 1}}, \quad z \in \mathbb{U},$$

then $f \in \mathcal{S}^*$.

Remark that for $k=1$ the above result was previously obtained by [28, Theorem 3], and we will use it in our next first result.

Lemma 3.2 [19, *Theorem* 2] *If* $f(z)=z+a_{k+1}z^{k+1}+\cdots$, $k\geq 1$, *is analytic in* $\mathbb{U}$ *and*

$$\left|f''(z)\right|\leq\frac{k}{k+1},\quad z\in\mathbb{U},$$

then $|f''(z)/f'(z)|\leq 1$, $z\in\mathbb{U}$, *and hence* $f\in\mathcal{K}$. *This result is sharp.*

Theorem 3.1 *Let* $\mu\in\mathbb{N}$, $\lambda\geq 0$, $\nu\in\mathbb{R}$ *with* $\lambda+\nu+\mu\geq 1$, $\lambda+\nu\notin\mathbb{Z}^-$, *and* $m\in\mathbb{N}$. *If*

$$\begin{aligned}&\frac{7}{6}-m\left[\ln(\lambda+1)+\frac{1}{\lambda+3/2}-\frac{1}{\lambda+2}\right]\\&\quad-\mu\left[\ln(\lambda+\nu+\mu)+\frac{1}{\lambda+\nu+\mu+0.5}-\frac{1}{\lambda+\nu+\mu+1}\right]\leq 0,\end{aligned}\tag{3.1}$$

and

$$(\lambda+1)^m(\lambda+\nu+1)_\mu\geq\sqrt{5},\tag{3.2}$$

then $\mathfrak{J}^{\nu,m}_{\lambda,\mu}\in\mathcal{S}^*$.

Proof To prove the above result, we will use Lemma 3.1 for the particular case $k=1$. Thus, to obtain $\mathfrak{J}^{\nu,m}_{\lambda,\mu}\in\mathcal{S}^*$, it is sufficient to show that

$$\left|\left(\mathfrak{J}^{\nu,m}_{\lambda,\mu}(z)\right)'-1\right|<\frac{2}{\sqrt{5}},\quad z\in\mathbb{U}.\tag{3.3}$$

Like in the proof of Theorem 2.1, from the assumptions of the parameters we have that

$$\begin{aligned}&\left|\left(\mathfrak{J}^{\nu,m}_{\lambda,\mu}(z)\right)'-1\right|\\&\quad=\left|\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m}z^k\right|\\&\quad<\sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{(-1)^k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m}e^{in\theta}\right|<\sum_{k=1}^{\infty}\frac{(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m}\\&\quad=\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)\cdot\sum_{k=1}^{\infty}\frac{(k+1)(2k+1)}{4^k\Gamma(\lambda+\nu+k\mu+1)\Gamma^m(\lambda+k+1)},\quad z\in\mathbb{U}.\end{aligned}\tag{3.4}$$

If we define the function $K:[1,+\infty)\to\mathbb{R}$ by

$$K(t):=\frac{(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)},$$

then

$$\frac{K'(t)}{K(t)}=\frac{1}{t+1}+\frac{2}{2t+1}-m\psi(\lambda+t+1)-\mu\psi(\lambda+\nu+t\mu+1)=:\widehat{K}(t).\tag{3.5}$$

On the other hand, by using the left-hand side of the inequality (see [14, Lemma 1])

$$\ln t - \frac{1}{t} < \psi(t) < \ln t - \frac{1}{2t}, \quad t \in (0, +\infty), \tag{3.6}$$

followed by the left-hand side of (see [12, ineq. (13)])

$$\frac{1}{t + 0.5} < \ln\left(1 + \frac{1}{t}\right) < \frac{1}{t + 0.4}, \quad t \in [1, +\infty), \tag{3.7}$$

since $\lambda \geq 0$, $m \geq 0$, and $\mu \in \mathbb{N}$, it follows that

$$\begin{aligned}\widehat{K}(1) < \frac{7}{6} &- m\left[\ln(\lambda + 1) + \frac{1}{\lambda + 3/2} - \frac{1}{\lambda + 2}\right] \\ &- \mu\left[\ln(\lambda + \nu + \mu) + \frac{1}{\lambda + \nu + \mu + 0.5} - \frac{1}{\lambda + \nu + \mu + 1}\right] \leq 0\end{aligned}$$

under assumption (3.1). Since $\widehat{K}$ is a strictly decreasing function on $[1, +\infty)$, then $\widehat{K}(t) \leq \widehat{K}(1) < 0$, $t \geq 1$, and using that $K(t) > 0$, $t \geq 1$, relation (3.5) leads to $K'(t) < 0$, $t \geq 1$. This last inequality implies that K is also a strictly decreasing function on $[1, +\infty)$, and from (3.4) we get

$$\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)' - 1\right| < \frac{2\Gamma^m(\lambda + 1)\Gamma(\lambda + \nu + 1)}{\Gamma(\lambda + \nu + \mu + 1)\Gamma^m(\lambda + 2)}, \quad z \in \mathbb{U}.$$

A simple computation shows that under assumption (3.2) the right-hand side of the above inequality is less or equal than $2/\sqrt{5}$. Thus, according to (3.3), the required result follows. □

Theorem 3.2 *Let $\mu \in \mathbb{N}$, $\lambda \geq 0$, $\nu \in \mathbb{R}$ with $\lambda + \nu + \mu \geq 1$, $\lambda + \nu \notin \mathbb{Z}^-$, and $m \in \mathbb{N}$. If*

$$\begin{aligned}\frac{13}{6} &- m\left[\ln(\lambda + 1) + \frac{1}{\lambda + 3/2} - \frac{1}{\lambda + 2}\right] \\ &- \mu\left[\ln(\lambda + \nu + \mu) + \frac{1}{\lambda + \nu + \mu + 0.5} - \frac{1}{\lambda + \nu + \mu + 1}\right] \leq 0,\end{aligned} \tag{3.8}$$

and

$$(\lambda + 1)^m(\lambda + \nu + 1)_\mu \geq 4, \tag{3.9}$$

then $\mathfrak{J}_{\lambda,\mu}^{\nu,m} \in \mathcal{K}$.

Proof To use Lemma 3.2 for $k = 1$, we should prove that under our assumption we have

$$\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)''(z)\right| \leq \frac{1}{2}, \quad z \in \mathbb{U}. \tag{3.10}$$

Using similar computations like in the proof of Theorem 2.2, we get

$$\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)''\right| = \left|\sum_{k=1}^{\infty} \frac{(-1)^k k(k+1)(2k+1)}{4^k(\lambda + \nu + 1)_{k\mu}[(\lambda + 1)_k]^m} z^{k-1}\right|$$

$$< \sup_{\theta\in[0,2\pi]}\left|\sum_{n=1}^{\infty}\frac{(-1)^k k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m}e^{i(k-1)\theta}\right|$$
$$< \sum_{k=1}^{\infty}\frac{k(k+1)(2k+1)}{4^k(\lambda+\nu+1)_{k\mu}[(\lambda+1)_k]^m},\quad z\in\mathbb{U}. \tag{3.11}$$

If we define the function $Q:[1,+\infty)\to\mathbb{R}$ by

$$Q(t):=\frac{t(t+1)(2t+1)}{\Gamma(\lambda+\nu+t\mu+1)\Gamma^m(\lambda+t+1)},$$

then

$$\frac{Q'(t)}{Q(t)}=\frac{1}{t}+\frac{1}{t+1}+\frac{2}{2t+1}-m\psi(\lambda+t+1)-\mu\psi(\lambda+\nu+t\mu+1)=:\widehat{Q}(t). \tag{3.12}$$

Like in the proof of Theorem 3.1, using (3.6), (3.7) and the facts that $m,\mu\geq 0$, we obtain

$$\widehat{Q}(1)<\frac{13}{6}-m\left[\ln(\lambda+1)+\frac{1}{\lambda+3/2}-\frac{1}{\lambda+2}\right]$$
$$-\mu\left[\ln(\lambda+\nu+\mu)+\frac{1}{\lambda+\nu+\mu+0.5}-\frac{1}{\lambda+\nu+\mu+1}\right]\leq 0,$$

under assumption (3.8). Since $\widehat{Q}$ is a strictly decreasing function on $[1,+\infty)$, then $\widehat{Q}(t)\leq\widehat{Q}(1)<0$, $x\geq 1$, and using that $Q(t)>0$, $t\geq 1$, relation (3.12) yields that $Q'(t)<0$, $t\geq 1$. This last inequality implies that Q is also a strictly decreasing function on $[1,+\infty)$, and from (3.11) we get

$$\left|\left(\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)\right)''\right|<\frac{2\Gamma^m(\lambda+1)\Gamma(\lambda+\nu+1)}{\Gamma(\lambda+\nu+\mu+1)\Gamma^m(\lambda+2)},\quad z\in\mathbb{U}.$$

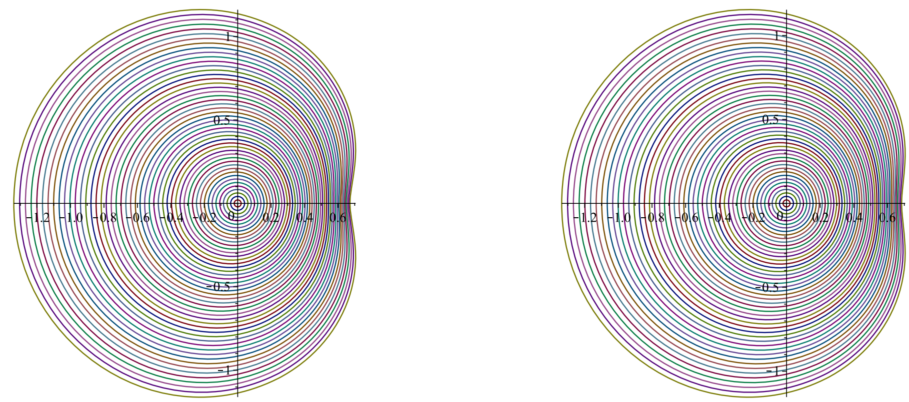
Consequently, assumption (3.9) implies that the right-hand side of the above inequality is less or equal than 1/2, and according to (3.10), we obtain our result. □

Example 3.1 In this example we will show that Theorem 3.1 is useful if $\alpha=0$ for the case $\lambda\geq 0$, which is not included in the assumptions of Theorems 2.1 or 2.3, where it was assumed that $\lambda\in\mathbb{N}$ or $\lambda\geq 1$, respectively. We mention that for $\alpha=0$ assumption (3.2) is stronger than (2.18), but with the additional condition (3.1) we could obtain the starlikeness of some functions like $\mathfrak{J}_{0.8,1}^{0.2,3}$.

Thus, if we put in Theorem 3.1 the values $\lambda=0.2$, $\mu=1$, $m=3$, and $\nu=0.1$, we get that

$$\mathfrak{J}_{0.2,1}^{0.1,3}(z)=z-0.3338675214z^2+0.005680244799z^3-0.00001838532122z^4$$
$$+0.00000001854980606z^5-7.605778219\cdot 10^{-12}z^6$$
$$+1.496644711\cdot 10^{-15}z^7-1.584478248\cdot 10^{-19}z^8$$
$$+9.809893741\cdot 10^{-24}z^9+\cdots\in\mathcal{S}^*,$$

and the image of $\mathbb{U}$ by $\mathfrak{J}_{0.2,1}^{0.1,3}$ is shown in Fig. 3(a).



(A) The image of $\mathfrak{J}^{0.1,3}_{0.2,1}(\mathbb{U})$ (B) The image of $\mathfrak{J}^{0.8,2}_{0.1,1}(\mathbb{U})$

Figure 3 Figures for Example 3.1

Taking in Theorem 3.1 the values $\lambda = 0.1$, $\mu = 1$, $m = 2$, and $\nu = 0.8$, we get that

$$\begin{aligned}\mathfrak{J}^{0.8,2}_{0.1,1}(z) = z &- 0.3262287951z^2 + 0.01062856085z^3 - 0.00009925548431z^4\\ &+ 0.0000003873247036z^5 - 0.0000000007712106488z^6\\ &+ 8.874725574 \cdot 10^{-13}z^7 - 6.428343262 \cdot 10^{-16}z^8\\ &+ 3.119153053 \cdot 10^{-19}z^9 + \cdots \in \mathcal{S}^*,\end{aligned}$$

and the image of $\mathfrak{J}^{0.8,2}_{0.1,1}(\mathbb{U})$ is shown in Fig. 3(B).

Example 3.2 Next we will show that Theorem 3.2 is useful for $\alpha = 0$ for the case $\lambda \geq 0$, which is not included in the assumptions of Theorems 2.2 or 2.4, where we assumed that $\lambda \in \mathbb{N}$ or $\lambda \geq 1$, respectively. We emphasize that for $\alpha = 0$ assumption (3.9) is stronger than (2.27), but adding condition (3.8) we could obtain the convexity of some functions like $\mathfrak{J}^{0.8,2}_{0.7,1}$.

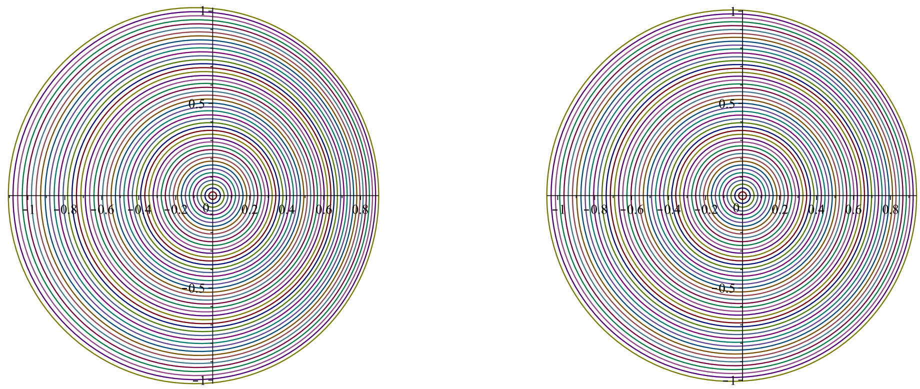
Taking in Theorem 3.2 the values $\lambda = 0.7$, $\mu = 1$, $m = 2$, and $\nu = 0.8$, we have

$$\begin{aligned}\mathfrak{J}^{0.8,2}_{0.7,1}(z) = z &- 0.1038062284z^2 + 0.001695183036z^3 - 0.000009630940061z^4\\ &+ 0.00000002547972595z^5 - 3.686564775 \cdot 10^{-11}z^6\\ &+ 3.235204040 \cdot 10^{-14}z^7 - 1.851779260 \cdot 10^{-17}z^8\\ &+ 7.296671191 \cdot 10^{-21}z^9 + \cdots \in \mathcal{K},\end{aligned}$$

and the image of $\mathbb{U}$ by $\mathfrak{J}^{0.8,2}_{0.7,1}$ is shown in Fig. 4(A).

Putting in Theorem 3.1 the values $\lambda = 0.5$, $\mu = 1.5$, $m = 3$, and $\nu = 0.7$, we get that

$$\begin{aligned}\mathfrak{J}^{0.7,3}_{0.5,1.5}(z) = z &- 0.05870665083z^2 + 0.0002004168671z^3 - 0.0000001289274849z^4\\ &+ 2.486094240 \cdot 10^{-11}z^5 - 1.860843213 \cdot 10^{-15}z^6\\ &+ 6.383641038 \cdot 10^{-20}z^7 - 1.127257518 \cdot 10^{-24}z^8\end{aligned}$$



(A) The image of $\mathfrak{J}_{0.7,1}^{0.8,2}(\mathbb{U})$ (B) The image of $\mathfrak{J}_{0.5,1.5}^{0.7,3}(\mathbb{U})$

Figure 4 Figures for Example 3.2

$$+ 1.116507623 \cdot 10^{-29} z^9 + \cdots \in \mathcal{K},$$

while the image of $\mathfrak{J}_{0.5,1.5}^{0.7,3}(\mathbb{U})$ is shown in Fig. 4(B).

Note that all the figures of this article were made by using the MAPLE™computer software.

4 Concluding remarks and outlook

In the present section the highlights of the paper are listed below:

1. In Theorem 2.1 we have used the principle of mathematical induction to generate the starlikeness of order α_s, given by (2.1), for the function $\mathfrak{J}_{\lambda,\mu}^{\nu,m}(z)$ defined by (1.4) for $\mu \in \mathbb{N}$, $\lambda \in \mathbb{N}$, $\nu \geq 0$, and $m \in \mathbb{N}$. Further, in Theorem 2.2, the mathematical induction was also used to obtain the convexity of order α_c for $\mu \in \mathbb{N}$, $\lambda \in \mathbb{N}$, $\nu \geq 0$, and $m \in \mathbb{N} \setminus \{1\}$;

2. In Theorems 2.3 and 2.4, an estimate for the upper and lower bounds for the gamma function inspired by [16] has been used to evaluate the orders α_s for $\mu \in \mathbb{N}$, $\lambda \geq 1$, $\nu \geq 0$, $m \in \mathbb{N}$, and α_c for $\mu \in \mathbb{N}$, $\lambda \geq 1$, $\nu \geq 0$, and $m \in \mathbb{N}$;

3. It could be seen that in [31] and [34] the authors investigated the orders of starlikeness and convexity of order α_s and α_c, respectively, using some well-known estimation for gamma, digamma, and Fox–Wright functions. It is worth mentioning that our results in this paper slightly improve the results in [31] and [34];

4. Finally, the starlikeness and convexity of order zero for $\mathfrak{J}_{\lambda,\mu}^{\nu,m}$ are studied using some technical manipulations proved by [19] that if $f(z) = z + a_{n+1}z^{n+1} + \cdots$, $n \geq 1$ is analytic in $\mathbb{U}$ and $|f'(z) - 1| < (n+1)/\sqrt{(n+1)^2 + 1}$, $z \in \mathbb{U}$, and $|f''(z)| \leq n/(n+1)$, $z \in \mathbb{U}$, then $f \in \mathcal{S}^*$ and $f \in \mathcal{K}$, respectively. These results are useful to extend the range of validity for the parameter λ to $\lambda \geq 0$.

Acknowledgements
The authors are grateful to the referees for the valuable remarks, comments, and advices that helped us to improve the quality of the manuscript.

Funding
Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

Data availability
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Both authors contributed equally to the writing of this paper. They read and approved the final version of the paper.

Author details
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin Elkom 32511, Egypt. [2]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania.

Received: 2 September 2023 Accepted: 20 February 2024 Published online: 12 March 2024

References
1. Ahlfors, L.V.: Complex Analysis. McGraw-Hill, New York (1953)
2. Aktaş, I., Baricz, Á.: Bounds for radii of starlikeness of some q-Bessel functions. Results Math. **72**, 947–963 (2017). https://link.springer.com/article/10.1007/s00025-017-0668-6
3. Aktaş, I., Baricz, Á., Singh, S.: Bounds for radii of starlikeness and convexity of some special functions. Turk. J. Math. **42**(1), 211–226 (2018). https://journals.tubitak.gov.tr/cgi/viewcontent.cgi?article=1829&context=math
4. Aktaş, I., Baricz, Á., Singh, S.: Geometric and monotonic properties of hyper-Bessel functions. Ramanujan J. **51**, 275–295 (2020). https://link.springer.com/article/10.1007/s11139-018-0105-9
5. Aktaş, I., Baricz, Á., Yağmur, N.: Bounds for the radii of univalence of some special functions. Math. Inequal. Appl. **20**(3), 825–843 (2017). https://files.ele-math.com/articles/mia-20-52.pdf
6. Aktaş, I., Orhan, H.: Bounds for radii of convexity of some q-Bessel functions. Bull. Korean Math. Soc. **57**(2), 355–369 (2020). https://bkms.kms.or.kr/journal/view.html?uid=3087
7. Aktaş, I., Toklu, E., Orhan, H.: Radii of k-starlikeness of order α of Struve and Lommel functions. Miskolc Math. Notes **22**(1), 5–15 (2021). http://mat76.mat.uni-miskolc.hu/mnotes/contents/22/1
8. Baricz, Á.: Bessel transforms and Hardy space of generalized Bessel functions. Mathematica (Cluj) **48(71)**(2), 127–136 (2006). https://math.ubbcluj.ro/~mathjour/fulltext/2006-2/baricz.pdf
9. Baricz, Á., Ponnusamy, S.: Starlikeness and convexity of generalized Bessel functions. Integral Transforms Spec. Funct. **21**(9), 641–653 (2010). https://www.tandfonline.com/doi/abs/10.1080/10652460903516736
10. de Oteiza, M.B.M., Kalla, S., Conde, S.: Un estudio sobre la función Lommel-Maitland. Rev. Téc. Fac. Ing., Univ. Zulia **9**(2), 33–40 (1986)
11. Duren, P.L.: Univalent Functions. Springer, New York (1983)
12. Ghosh, A.: An asymptotic formula for the Chebyshev theta function. Notes Number Theory Discrete Math. **25**(4), 1–7 (2019). https://nntdm.net/volume-25-2019/number-4/1-7/
13. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Dekker, New York (2003)
14. Guo, B.-N., Qi, F.: Refinements of lower bounds for polygamma functions. Proc. Am. Math. Soc. **141**(3), 1007–1015 (2013). https://www.ams.org/journals/proc/2013-141-03/S0002-9939-2012-11387-5/S0002-9939-2012-11387-5.pdf
15. Kilbas, A.A., Saigo, M., Trujillo, J.J.: On the generalized Wright function. Fract. Calc. Appl. Anal. **5**(4), 437–460 (2002). https://www.diogenes.bg/fcaa/volume5/fcaa54/aKST_MAI.pdf
16. Li, X., Chen, C.-P.: Inequalities for the gamma function. J. Inequal. Pure Appl. Math. **8**(1), Article 28 (2007). https://www.emis.de/journals/JIPAM/images/264_06_JIPAM/264_06.pdf
17. Mehrez, K.: Some geometric properties of a class of functions related to the Fox-Wright functions. Banach J. Math. Anal. **14**, 1222–1240 (2020). https://link.springer.com/article/10.1007/s43037-020-00059-w
18. Miller, S.S., Mocanu, P.T.: Univalence of Gaussian and confluent hypergeometric functions. Proc. Am. Math. Soc. **110**(2), 333–342 (1990). https://www.ams.org/journals/proc/1990-110-02/S0002-9939-1990-1017006-8/S0002-9939-1990-1017006-8.pdf
19. Mocanu, P.T.: Some simple criteria for starlikeness and convexity. Libertas Math. **13**, 27–40 (1993). https://system.lm-ns.org/index.php/lm/article/view/627/499
20. Nehari, Z.: Conformal Mapping. McGraw-Hill, New York (1952)
21. Orhan, H., Yağmur, N.: Geometric properties of generalized Struve functions. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **63**(2), 229–244 (2017). https://www.math.uaic.ro/~annalsmath/pdf-uri_anale/F2(2017)/Orhan_Yagmur_pg229.pdf
22. Pathak, R.S.: Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. Proc. Natl. Acad. Sci., India **A-36**(1), 81–86 (1966)
23. Pommerenke, C.: Univalent Functions. Vandenhoeck & Ruprecht, Göttingen (1975)
24. Ponnusamy, S.: The Hardy space of hypergeometric functions. Complex Var. Theory Appl. **29**(1), 83–96 (1996). https://www.tandfonline.com/doi/abs/10.1080/17476939608814876
25. Ponnusamy, S., Rønning, F.: Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane. Integral Transforms Spec. Funct. **8**(1–2), 121–138 (1999). https://www.tandfonline.com/doi/abs/10.1080/10652469908819221
26. Ponnusamy, S., Vuorinen, M.: Univalence and convexity properties for confluent hypergeometric functions. Complex Var. Theory Appl. **36**(1), 73–97 (1998). https://www.tandfonline.com/doi/abs/10.1080/17476939808815101
27. Silverman, H.: Univalent functions with negative coefficients. Proc. Am. Math. Soc. **51**(1), 109–116 (1975). https://www.ams.org/journals/proc/1975-051-01/S0002-9939-1975-0369678-0/S0002-9939-1975-0369678-0.pdf
28. Singh, V.: Univalent functions with bounded derivative in the unit disc. Indian J. Pure Appl. Math. **8**(11), 1370–1377 (1977). https://archive.org/stream/in.ernet.dli.2015.205676/2015.205676.Indian-Journal_djvu.txt

29. Yağmur, N.: Hardy space of Lommel functions. Bull. Korean Math. Soc. **52**(3), 1035–1046 (2015). https://koreascience.kr/article/JAKO201517058945935.pdf
30. Yağmur, N., Orhan, H.: Hardy space of generalized Struve functions. Complex Var. Elliptic Equ. **59**(7), 929–936 (2014). https://www.tandfonline.com/doi/abs/10.1080/17476933.2013.799148
31. Zayed, H.M., Bulboacă, T.: On some geometric properties for the combination of generalized Lommel-Wright function. J. Inequal. Appl. **2021**, 158 (2021). https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-021-02690-z
32. Zayed, H.M., Bulboacă, T.: Normalized generalized Bessel function and its geometric properties. J. Inequal. Appl. **2022**, 158 (2022). https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-022-02891-0
33. Zayed, H.M., Bulboacă, T., Morais, J.: The geometric characterizations for a combination of generalized Struve functions. Comput. Methods Funct. Theory **22**, 699–714 (2022). https://link.springer.com/article/10.1007/s40315-021-00421-5
34. Zayed, H.M., Mehrez, K.: Generalized Lommel–Wright function and its geometric properties. J. Inequal. Appl. **2022**, 115 (2022). https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-022-02851-8

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.