(2024) 2024:28

# RESEARCH

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# Some existence results for a differential equation and an inclusion of fractional order via (convex) *F*-contraction mapping

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# Abstract

The existence of solutions for a class of  $\mu$ -Caputo fractional differential equations and an inclusion problem equipped with nonlocal  $\mu$ -integral boundary conditions are investigated. We use *F*-contraction, convex *F*-contraction, and some consequences to achieve the desired goals. Finally, some examples are provided to illustrate the results.

Keywords: Convex F-contraction; Fixed point; Fractional differential inclusion

# **1** Introduction

The subject of fractional differential equations (FDEs) is gaining much importance and significance. Due to the influence memory function of a fractional derivative (FD), FDEs have been widely used to describe many physical phenomena such as seepage flow in porous media. The existence of solutions to such equations has also been investigated by many scientists ([4, 11–17, 20, 26, 38]). The authors in [37] studied the problem of a FDE involving nonlocal fractional integral (FI) conditions as

$$\begin{cases} {}^{RL}D_{0^+}^{\pi^*}w^*(\varsigma) = \mathfrak{h}(\varsigma, w^*(\varsigma)), \quad \varsigma \in [0, T]; \\ w^*(0) = 0, \\ w^*(T) = \sum_{i=1}^n \beta_i^{\mathcal{H}}I_{0^+}^{p_i}w^*(\zeta_i), \end{cases}$$

where  $1 < \pi^* \leq 2$ ,  ${}^{RL}D_{0^+}^{\pi^*}$ , and  ${}^{\mathcal{H}}I_{0^+}^{p_i}$  denote, respectively, the Riemann–Liouville (R–L) FD of order  $\pi^*$  and the Hadamard FI of order  $p_i > 0$ ,  $\zeta_i \in (0, T)$ ,  $\mathfrak{h} : [0, T] \times \mathfrak{R} \to \mathfrak{R}$  and  $\beta_i \in \mathfrak{R}$ ,  $i \in \{1, 2, ..., n\}$  with  $\sum_{i=1}^n \frac{\beta_i \zeta_i \pi^* - 1}{(\pi^* - 1)^{p_i}} \neq T^{\pi^* - 1}$ .

Ntouyas et al. ([29]) investigated the problem

$$\begin{cases} {}^{RL}D_{0^+}^{\pi^*}w^*(\varsigma) = \mathfrak{H}(\varsigma, w^*(\varsigma)), \quad \varsigma \in [0, T]; \\ w^*(0) = 0, \\ w^*(T) = \sum_{i=1}^n \beta_i^{\mathcal{H}}I_{0^+}^{p_i}w^*(\zeta_i), \end{cases}$$

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where  $1 < \pi^* \le 2$ ,  ${}^{RL}D_{0^+}^{\pi^*}$  and  ${}^{\mathcal{H}}I_{0^+}^{p_i}$  indicate, respectively, the R–L FD of order  $\pi^*$  and the Hadamard FI of order  $p_i > 0$ ,  $\zeta_i \in (0, T)$ ,  $\mathfrak{h} : [0, T] \times \mathfrak{R} \to \mathbb{P}(\mathfrak{R})$  and  $\beta_i \in \mathfrak{R}$ , i = 1, 2, ..., n with  $\sum_{i=1}^{n} \frac{\beta_i \zeta_i \pi^* - 1}{(\pi^* - 1) \mathcal{V}_i} \neq T^{\pi^* - 1}$ .

Almeida in [7], obtained a generalization of the classical Caputo operator to  $\mu$ -Caputo operator. To see some results in this area, refer to [1–3, 5, 6, 9, 10, 18, 19, 25, 28, 31–34, 36, 39].

Here, we first consider the problem

$$\begin{cases} {}^{C}D_{0}^{\pi^{*,\mu}}w^{*}(\varsigma) = \mathfrak{h}(\varsigma, w^{*}(\varsigma)), \quad \varsigma \in \mathbb{I} = [s_{0}, T]; \\ w^{*}(s_{0}) = 0, \\ w^{*}(T) = \sum_{i=1}^{n} \beta_{i}^{RL} I_{0}^{p_{i},\mu} w^{*}(\zeta_{i}), \end{cases}$$
(1)

in which  ${}^{C}D_{0}^{\pi^{*},\mu}$  and  ${}^{RL}I_{0}^{p_{i},\mu}$  are, respectively, the  $\mu$ -Caputo FD of order  $\pi^{*}$ ,  $1 < \pi^{*} < 2$  and the R–L  $\mu$ -FI of order  $p_{i} > 0$ ,  $s_{0} > 0$ ,  $\zeta_{i} \in (s_{0}, T)$ ,  $\mathfrak{h} : \mathbb{I} \times \mathfrak{R} \to \mathfrak{R}$  and  $\beta_{i} \in \mathfrak{R}$  i = 1, 2, ..., n. Also, the existence of solutions of the problem

$$\begin{cases} {}^{\mathcal{C}}D_{0}^{\pi^{*,\mu}}w^{*}(\varsigma) \in \mathfrak{g}(\varsigma, w^{*}(\varsigma)), \quad \varsigma \in \mathbb{I}; \\ w^{*}(s_{0}) = 0, \\ w^{*}(T) = \sum_{i=1}^{n} \beta_{i}^{RL}I_{0}^{p_{i},\mu}w^{*}(\zeta_{i}), \end{cases}$$
(2)

will be investigated in which  $\mathfrak{g}: \mathbb{I} \times \mathfrak{R} \to \mathbb{P}(\mathfrak{R})$  is a set-valued compact map.

This study aims to provide a different approach to examining the existence of solutions of (1) and (2). We utilize new techniques based on the application of some F-contraction mappings that are defined in appropriate cones of positive functions.

Here, weaker conditions have been applied compared to other works including the Banach contraction. In the contraction that we apply, the used functions may not even be continuous, while in the Banach contraction, the functions are uniformly continuous, and therefore the number of functions that apply to our contraction is much more than the number of functions that apply to other contractions. Therefore, the number of problems that we can discuss with this contraction will be far more than in similar cases.

# 2 Requisites preliminaries

We present some basic and auxiliary concepts in this section. The following definitions were given in references [35, 40], and [26].

For function  $w^* : [0, +\infty) \to \Re$ , we recall the R–L FI of order  $\pi^* > 0$  as

$${}^{RL}\mathcal{I}_{s_0}^{\pi^*}w^*(\varsigma) = \int_{s_0}^{\varsigma} \frac{(\varsigma-q)^{\pi^*-1}}{\Gamma(\pi^*)} w^*(q) \,\mathrm{d}q.$$
(3)

Here, we assume  $n - 1 < \pi^* < n$ , so that  $n = [\pi^*] + 1$ . For a continuous function  $w^*$ :  $[0, +\infty) \rightarrow \Re$ , the R–L FD of order  $\pi^*$  is given by

$${}^{RL}D_{s_0}^{\pi^*}w^*(\varsigma) = \left(\frac{\mathrm{d}}{\mathrm{d}\varsigma}\right)^n \int_{s_0}^{\varsigma} \frac{(\varsigma-q)^{n-\pi^*-1}}{\Gamma(n-\pi^*)} w^*(q) \,\mathrm{d}q.$$
(4)

Let  $w^* \in \mathcal{AC}^{(n)}_{\mathfrak{R}}([0, +\infty))$  (absolutely continuous mappings). The Caputo FD is defined by:

$${}^{\mathcal{C}}D_{s_0}^{\pi^*}w^*(\varsigma) = \int_{s_0}^{\varsigma} \frac{(\varsigma-q)^{n-\pi^*-1}}{\Gamma(n-\pi^*)} w^{*(n)}(q) \,\mathrm{d}q.$$
(5)

Here, consider an increasing function  $\mu \in C^n(\mathbb{I})$  with  $\mu'(\varsigma) > 0$  for every  $s_0 \leq \varsigma \leq M$ . Then, the integral in the sense of  $\mu$ -R–L of  $w^* : \mathbb{I} \to \mathfrak{R}$  of order  $\pi^*$  depending on  $\mu$  is introduced as

$${}^{RL}\mathcal{I}_{s_0}^{\pi^*;\mu}w^*(\varsigma) = \frac{1}{\Gamma(\pi^*)} \int_{s_0}^{\varsigma} \mu'(q) \big(\mu(\varsigma) - \mu(q)\big)^{\pi^* - 1} w^*(q) \,\mathrm{d}q.$$
(6)

Note that in the case of  $\mu(\varsigma) = \varsigma$ , the R–L  $\mu$ -FI (6) reduces to the classical R–L FI (3). The  $\pi^*$  ordered R–L  $\mu$ -FD of the continuous function  $w^* : [0, +\infty) \to \Re$  is illustrated as (see [24, 30, 35])

$${}^{RL}D_{s_0}^{\pi^*;\mu}w^*(\varsigma) = \frac{1}{\Gamma(n-\pi^*)} \left(\frac{1}{\mu'(\varsigma)} \frac{\mathrm{d}}{\mathrm{d}\varsigma}\right)^n \int_{s_0}^{\varsigma} \mu'(q) \left(\mu(\varsigma) - \mu(q)\right)^{n-\pi^*-1} w^*(q) \,\mathrm{d}q.$$
(7)

Similarly, in the case of  $\mu(\varsigma) = a$ , the R–L  $\mu$ -FD (7) becomes the classical R–L FD (4). Almeida has presented the following derivative ([7])

$${}^{\mathcal{C}}D_{s_0}^{\pi^*;\mu}w^*(\varsigma) = \frac{1}{\Gamma(n-\pi^*)} \int_{s_0}^{\varsigma} \mu'(q) \big(\mu(\varsigma) - \mu(q)\big)^{n-\pi^*-1} \bigg(\frac{1}{\mu'(q)}\frac{\mathrm{d}}{\mathrm{d}q}\bigg)^n w^*(q) \,\mathrm{d}q.$$
(8)

Note again that in the case of  $\mu(\varsigma) = \varsigma$ , the  $\mu$ -Caputo FD (8) reduces to the classical Caputo derivative (5). Some properties of the mentioned operators are given in the following lemmas.

**Lemma 2.1** ([7, 24, 30, 35]) Let  $\pi^*$ ,  $\varrho^*$ , and  $\beta^*$  be positive and  $\mu \in C^n(\mathbb{I})$  be a mapping with  $\mu'(\varsigma) > 0$ . Then,

(i1)  ${}^{RL}\mathcal{I}_{s_0}^{\pi^*;\mu}({}^{RL}\mathcal{I}_{s_0}^{\varrho^*;\mu}w^*)(\varsigma) = ({}^{RL}\mathcal{I}_{s_0}^{\pi^*+\varrho^*;\mu}w^*)(\varsigma);$ (i2)  ${}^{RL}\mathcal{I}_{s_0}^{\pi^*;\mu}(\mu(\varsigma) - \mu(s_0))^{\beta^*} = \frac{\Gamma(\beta^*+1)}{\Gamma(\pi^*+\beta^*+1)}(\mu(\varsigma) - \mu(s_0))^{\pi^*+\beta^*};$ (i3)  ${}^{C}D_{s_0}^{\pi^*;\mu}(\mu(\varsigma) - \mu(s_0))^{\beta^*} = \frac{\Gamma(\beta^*+1)}{\Gamma(\beta^*-\pi^*+1)}(\mu(\varsigma) - \mu(s_0))^{\beta^*-\pi^*}, (\beta^* > \pi^*);$ (i4)  ${}^{RL}D_{s_0}^{\pi^*;\mu}({}^{RL}\mathcal{I}_{s_0}^{\varrho^*;\mu}w^*)(\varsigma) = ({}^{RL}\mathcal{I}_{s_0}^{\varrho^*-\pi^*;\mu}w^*)(\varsigma), (\pi^* < \varrho^*).$ 

**Lemma 2.2** ([7]) Let  $\pi^* \in (n-1, n)$  and  $\mu \in C^n(\mathbb{I})$  be a function with  $\mu'(\varsigma) > 0$  for every  $\varsigma \in \mathbb{I}$ . Then, for any  $w^* \in C^{n-1}(\mathbb{I})$ ,

$${}^{RL}\mathcal{I}_{s_0}^{\pi^*;\mu} \Big({}^{\mathcal{C}}D_{s_0}^{\pi^*;\mu}w^*\Big)(\varsigma) = w^*(\varsigma) - \sum_{j=0}^{n-1} \frac{(\delta_{\mu})^j w^*(s_0)}{j!} \Big(\mu(\varsigma) - \mu(s_0)\Big)^j, \quad \left(\delta_{\mu} = \frac{1}{\mu'(\varsigma)} \frac{\mathrm{d}}{\mathrm{d}\varsigma}\right).$$

From the previous lemma, the authors in [7] have considered the series solution of the homogeneous equation  $({}^{\mathcal{C}}D_{s_0}^{\pi^*;\mu}w^*)(\varsigma) = 0$  as

$$w^{*}(\varsigma) = \sum_{j=0}^{n-1} \tilde{k}_{j}^{*} (\mu(\varsigma) - \mu(s_{0}))^{j}$$
  
=  $\tilde{k}_{0}^{*} + \tilde{k}_{1}^{*} (\mu(\varsigma) - \mu(s_{0})) + \tilde{k}_{2}^{*} (\mu(\varsigma) - \mu(s_{0}))^{2} + \dots + \tilde{k}_{n-1}^{*} (\mu(\varsigma) - \mu(s_{0}))^{n-1},$ 

where  $n - 1 < \pi^* < n$  and  $\tilde{k}_0^*, \tilde{k}_1^*, \dots, \tilde{k}_{n-1}^* \in \mathfrak{R}$ .

Let  $(\mathfrak{D}, \|\cdot\|)$  be a normed space. The classes  $\mathbb{P}_{\mathbb{CL}}(\mathfrak{D})$ ,  $\mathbb{P}_{\mathbb{BN}}(\mathfrak{D})$ ,  $\mathbb{P}_{\mathbb{CP}}(\mathfrak{D})$ , and  $\mathbb{P}_{\mathbb{CV}}(\mathfrak{D})$  are, respectively, closed, bounded, compact, and convex subsets of  $\mathfrak{D}$ .

**Definition 2.3** ([21, 23]) Let (X, d, s) be a *b*-metric space and  $T : X \to X$ . Then, *T* is said to be a convex *F*-contraction if there exists  $F : (0, \infty) \to \Re$  such that

- (i) *F* is strictly increasing on  $(0, \infty)$ ;
- (ii) for each sequence  $\{\alpha_n\}$  of positive numbers, if  $\lim_{n\to\infty} F(\alpha_n) = -\infty$ , then  $\lim_{n\to\infty} \alpha_n = 0$ ;
- (iii) there exists  $k \in (0, \frac{1}{1 + \log_2 s})$  with property  $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$ ;
- (iv) there exists  $\tau > 0, \lambda \in [0, 1)$  with

$$\tau + F(d_n) \le F(\lambda d_n + (1 - \lambda)d_{n-1}),\tag{9}$$

for all  $d_n > 0$ , where  $n \in \mathbb{N}$ .

To obtain the desired results, the following theorems are crucial.

**Theorem 2.4** ([21]) Suppose that (X, d, s) is a b-complete b-metric space and T is a continuous convex F-contraction on X. Then, T has a fixed point (FP) in X.

**Theorem 2.5** ([41]) Let (X, d, s > 1) be a complete b-metric space and  $\mathfrak{F} : X \to CB(X)$ . Assume that there exists a strictly increasing function  $\mathbb{F} : (0, \infty) \to (-\infty, +\infty)$  and  $\tau > 0$  such that

$$2\tau + \mathbb{F}\big(s.H(\mathfrak{F}\iota,\mathfrak{F}\nu)\big) \le \mathbb{F}\big(d(\iota,\nu)\big),\tag{10}$$

for all  $\iota, \nu \in X$  with  $\mathfrak{F}\iota \neq \mathfrak{F}\nu$ . Then,  $\mathfrak{F}$  has a FP.

# 3 Main results

Here, we derive some conditions for the existence of at least one solution to problems (1) and (2). In respect of achieving the goals, let  $\mathfrak{D} = \{w^*(\varsigma) : w^*(\varsigma) \in C(\mathbb{I}, \mathfrak{R})\}$  and  $d : \mathfrak{D} \times \mathfrak{D} \rightarrow [0, \infty)$  be given by

$$d(w^*, \rho^*) = ||w^* - \rho^*||^2 = \sup_{\varsigma \in \mathbb{I}} |w^*(\varsigma) - \rho^*(\varsigma)|^2.$$

Evidently,  $(\mathfrak{D}, \|\cdot\|)$  is a complete *b*-metric space with  $\upsilon = 2$  but is not a metric space.

**Lemma 3.1** ([22]) Let  $\Phi^* \in C(\mathbb{I}, \mathfrak{R})$ . Then,  $w^*(\varsigma) \in \mathfrak{D}$  is a solution for the boundary value problem (BVP)

$$\begin{cases} {}^{\mathcal{C}}D_{0}^{\pi^{*,\mu}}w^{*}(\varsigma) = \Phi^{*}(\varsigma), \quad \varsigma \in \mathbb{I}; \\ w^{*}(s_{0}) = 0, \\ w^{*}(T) = \sum_{i=1}^{n} \beta_{i}^{RL} I_{0}^{p_{i,\mu}}w^{*}(\zeta_{i}) \end{cases}$$
(11)

*if and only if*  $w^*(\varsigma)$  *is a solution for* 

$$w^{*}(\varsigma) = {}^{RL} I_{0}^{\pi^{*},\mu} \Phi^{*}(\varsigma) - \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \left[ {}^{RL} I_{0}^{\pi^{*},\mu} \Phi^{*}(T) - \sum_{i=1}^{n} \beta_{i}^{RL} I_{0}^{p_{i}+\pi^{*},\mu} \Phi^{*}(\zeta_{i}) \right], \quad (12)$$

where  $\Lambda = (\mu(T) - \mu(s_0)) - \sum_{i=1}^n \beta_i \frac{\xi^{p_i+1}}{\Gamma(p_i+2)} \neq 0.$ 

Throughout this work, we apply

$${}^{RL}I_{0}^{\pi^{*},\mu}\mathfrak{h}(\varsigma,w^{*}(\varsigma)) = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big(\mu(\varsigma) - \mu(q)\big)^{\pi^{*}-1}\mathfrak{h}(q,w^{*}(q)) \,\mathrm{d}q \tag{13}$$

and

$${}^{RL}I_{0}^{p_{i}+\pi^{*},\mu}\mathfrak{h}(\zeta_{i},w^{*}(\zeta_{i})) = \frac{1}{\Gamma(p_{i}+\pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) \big(\mu(\zeta_{i})-\mu(q)\big)^{p_{i}+\pi^{*}-1}\mathfrak{h}(q,w^{*}(q)) \,\mathrm{d}q, \quad (14)$$

where  $\zeta_i \in \mathbb{I}$ ,  $i = 1, 2, \ldots, n$ .

Let  $\mathfrak{D} = C(\mathbb{I}, \mathfrak{R})$ . Define the operator  $\mathfrak{T} : \mathfrak{D} \to \mathfrak{D}$  by

$$\mathfrak{T}w^{*}(\varsigma) = {}^{RL} I_{0}^{\pi^{*},\mu} \mathfrak{h}(\varsigma, w^{*}(\varsigma)) - \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \left[ {}^{RL} I_{0}^{\pi^{*},\mu} \mathfrak{h}(T, w^{*}(T)) - \sum_{i=1}^{n} \beta_{i}^{RL} I_{0}^{p_{i}+\pi^{*},\mu} \mathfrak{h}(\zeta_{i}, w^{*}(\zeta_{i})) \right].$$
(15)

The BVP (11) has a solution if  $\mathfrak{T}$  has a FP.

Note that in the following theorem we prove that  $\Omega_1$  is represented as

$$\Omega_{1} = \frac{(|\mu(T) - \mu(s_{0})|)^{\pi^{*}}}{\Gamma(\pi^{*} + 1)} + \frac{(|(\mu(T) - \mu(s_{0}))|)^{\pi^{*} + 1}}{|\Lambda|\Gamma(\pi^{*} + 1)} + \sum_{i=1}^{n} \beta_{i} \frac{n(|(\mu(T) - \mu(s_{0}))|)^{p_{i} + \pi^{*} + 1}}{|\Lambda|\Gamma(p_{i} + \pi^{*} + 1)}.$$
(16)

**Theorem 3.2** Let  $\mathfrak{h}: \mathbb{I} \times \mathfrak{D} \to \mathfrak{D}$  be a continuous function. Suppose also that

$$\left|\mathfrak{h}(\varsigma, w_1^*(\varsigma)) - \mathfrak{h}(\varsigma, w_2^*(\varsigma))\right| \le rac{\lambda^*}{\sqrt{2 heta}} \left|w_1^* - w_2^*\right|,$$

for all  $w_1^*, w_2^* \in \mathfrak{D}$ ,  $\lambda^* = \frac{1}{\Omega_1}$ , and  $\theta > 0$ . Then, at least one solution of (1) exists.

*Proof* First, we show that  $\mathfrak{T}$  is continuous. Let  $\{w_n^*\} \in C(\mathbb{I}, \mathfrak{R})$  be a sequence that  $w_n^* \to w^* \in C(\mathbb{I}, \mathfrak{R})$ . For all  $\varsigma \in \mathbb{I}$ , we have

$$\begin{split} \mathfrak{T}w_{n}^{*}(\varsigma) &- \mathfrak{T}w^{*}(\varsigma) \Big| \\ &\leq \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big( \mu(\varsigma) - \mu(q) \big)^{\pi^{*}-1} \big| \mathfrak{h}(q, w_{n}^{*}(q)) - \mathfrak{h}(q, w^{*}(q)) \big| \, \mathrm{d}q \\ &+ \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) \big( \mu(T) - \mu(q) \big)^{\pi^{*}-1} \big| \mathfrak{h}(q, w_{n}^{*}(q)) - \mathfrak{h}(q, w^{*}(q)) \big| \, \mathrm{d}q \\ &+ \sum_{i=1}^{n} \beta_{i} \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(p_{i} + \pi^{*})} \\ &\times \int_{s_{0}}^{\zeta_{i}} \mu'(q) \big( \mu(\zeta_{i}) - \mu(q) \big)^{p_{i}+\pi^{*}-1} \big| \mathfrak{h}(q, w_{n}^{*}(q)) - \mathfrak{h}(q, w^{*}(q)) \big| \, \mathrm{d}q. \end{split}$$

Put,  $\mathfrak{h}(q, w_n^*(q)) = \mathfrak{h}_{w_n^*}(q)$  and  $\mathfrak{h}(q, w_n^*(q)) = \mathfrak{h}_{w^*}(q)$ . Then,

$$\left|\mathfrak{h}_{w_n^*}(q)-\mathfrak{h}_{w^*}(q)\right|\leq rac{\lambda^*}{\sqrt{2 heta}}\left|w_n^*-w^*\right|.$$

Now,  $w_n^* \to w^*$  as  $n \to \infty$  implies  $\mathfrak{h}_{w_n^*}(q) \to \mathfrak{h}_{w^*}(q)$ ,  $q \in \mathbb{I}$ . Let  $\aleph > 0$  such that for  $q \in \mathbb{I}$ , we have  $|\mathfrak{h}_{w_n^*}(q)| \leq \aleph$  and  $|\mathfrak{h}_{w^*}(q)| \leq \aleph$ . Thus,

$$\begin{split} \mu'(q) \big(\mu(j) - \mu(q)\big)^{\pi^* - 1} \big| \mathfrak{h}_{w_n^*}(q) - \mathfrak{h}_{w^*}(q) \big| &\leq \mu'(q) \big(\mu(j) - \mu(q)\big)^{\pi^* - 1} \big( \big| \mathfrak{h}_{w_n^*}(q) \big| + \big| \mathfrak{h}_{w^*}(q) \big| \big) \\ &\leq 2 \aleph \mu'(q) \big(\mu(j) - \mu(q)\big)^{\pi^* - 1}, \end{split}$$

where  $j = \zeta$ , *T*,  $\xi_i \in \mathbb{I}$ . For each  $j \in \mathbb{I}$ ,  $2 \aleph \mu'(q) (\mu(j) - \mu(q))^{\pi^* - 1}$  is integrable. Therefore, applying the Lebesgue dominated convergence theorem, we obtain  $|\mathfrak{T}w_n^*(\zeta) - \mathfrak{T}w^*(\zeta)| \to 0$  as  $n \to \infty$ . Hence, particularly  $\max_{\varsigma \in \mathbb{I}} |\mathfrak{T}w_n^*(\varsigma) - \mathfrak{T}w^*(\varsigma)| \to 0$ , which implies  $||\mathfrak{T}w_n^* - \mathfrak{T}w^*||_{\mathfrak{D}} \to 0$ . Hence,  $\mathfrak{T}$  is continuous.

Assume  $w_1^*, w_2^* \in \mathfrak{D}$ . Then,

$$\begin{split} \left| \mathfrak{T} w_{n}^{*}(\varsigma) - \mathfrak{T} w^{*}(\varsigma) \right| \\ &\leq \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big( \mu(\varsigma) - \mu(q) \big)^{\pi^{*}-1} \big| \mathfrak{h} \big( q, w_{n}^{*}(q) \big) - \mathfrak{h} \big( q, w^{*}(q) \big) \big| \, \mathrm{d}q \\ &+ \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) \big( \mu(T) - \mu(q) \big)^{\pi^{*}-1} \big| \mathfrak{h} \big( q, w_{n}^{*}(q) \big) - \mathfrak{h} \big( q, w^{*}(q) \big) \big| \, \mathrm{d}q \\ &+ \sum_{i=1}^{n} \beta_{i} \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) \big( \mu(\zeta_{i}) - \mu(q) \big)^{p_{i} + \pi^{*}-1} \\ &\times \big| \mathfrak{h} \big( q, w_{n}^{*}(q) \big) - \mathfrak{h} \big( q, w^{*}(q) \big) \big| \, \mathrm{d}q. \end{split}$$

Assume  $w_1^*$ ,  $w_2^* \in \mathfrak{D}$ . Then,

$$\begin{split} \mathfrak{T}w_{1}^{*}(\varsigma) &- \mathfrak{T}w_{2}^{*}(\varsigma) \Big| \\ &\leq \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big( \mu(\varsigma) - \mu(q) \big)^{\pi^{*-1}} \frac{\lambda^{*}}{\sqrt{2\theta}} \Big| w_{1}^{*} - w_{2}^{*} \Big| \, \mathrm{d}q \\ &+ \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) \big( \mu(T) - \mu(q) \big)^{\pi^{*-1}} \frac{\lambda^{*}}{\sqrt{2\theta}} \Big| w_{1}^{*} - w_{2}^{*} \Big| \, \mathrm{d}q \\ &+ \sum_{i=1}^{n} \beta_{i} \frac{|(\mu(\varsigma) - \mu(s_{0}))|}{|\Lambda|\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) \big( \mu(\zeta_{i}) - \mu(q) \big)^{p_{i} + \pi^{*-1}} \frac{\lambda^{*}}{\sqrt{2\theta}} \Big| w_{1}^{*} - w_{2}^{*} \Big| \, \mathrm{d}q \\ &\leq \frac{\frac{\lambda^{*}}{\sqrt{2\theta}} \| w_{1}^{*} - w_{2}^{*} \| (|\mu(\varsigma) - \mu(s_{0})|)^{\pi^{*}}}{\Gamma(\pi^{*} + 1)} + \frac{\frac{\lambda^{*}}{\sqrt{2\theta}} \| w_{1}^{*} - w_{2}^{*} \| (|(\mu(\varsigma) - \mu(s_{0})|)|)^{\pi^{*+1}}}{|\Lambda|\Gamma(\pi^{*} + 1)} \\ &+ \sum_{i=1}^{n} \beta_{i} \frac{\frac{\lambda^{*}}{\sqrt{2\theta}} \| w_{1}^{*} - w_{2}^{*} \| (|(\mu(\varsigma) - \mu(q))|)|^{p_{i} + \pi^{*+1}}}{|\Lambda|\Gamma(p_{i} + \pi^{*} + 1)} \end{split}$$

$$= \frac{\lambda^{*}}{\sqrt{2\theta}} \left\{ \frac{(|\mu(T) - \mu(s_{0})|)^{\pi^{*}}}{\Gamma(\pi^{*} + 1)} + \frac{(|(\mu(T) - \mu(s_{0}))|)^{\pi^{*} + 1}}{|\Lambda|\Gamma(\pi^{*} + 1)} \right. \\ \left. + \sum_{i=1}^{n} \beta_{i} \frac{n(|(\mu(T) - \mu(s_{0}))|)^{p_{i} + \pi^{*} + 1}}{|\Lambda|\Gamma(p_{i} + \pi^{*} + 1)} \right\} \left\| w_{1}^{*} - w_{2}^{*} \right\|.$$

Hence,

$$\left\|\mathfrak{T}w_1^*(\varsigma) - \mathfrak{T}w_2^*(\varsigma)\right\| \leq \frac{\lambda^*}{\sqrt{2\theta}}\Omega_1 \left\|w_1^* - w_2^*\right\|.$$

Therefore,

$$\|\mathfrak{T}w_{1}^{*}(\varsigma) - \mathfrak{T}w_{2}^{*}(\varsigma)\|^{2} \leq \frac{\lambda^{*2}}{2\theta}\Omega_{1}^{2} \|w_{1}^{*} - w_{2}^{*}\|^{2}.$$

Apply the assumptions to obtain

$$\|\mathfrak{T}w_{1}^{*}(\varsigma) - \mathfrak{T}w_{2}^{*}(\varsigma)\|^{2} \leq \frac{1}{2\theta} \|w_{1}^{*} - w_{2}^{*}\|^{2}.$$

Hence,

$$\ln \left\| \mathfrak{T} w_{1}^{*}(\varsigma) - \mathfrak{T} w_{2}^{*}(\varsigma) \right\|^{2} \leq -\ln \theta + \ln \frac{\left\| w_{1}^{*} - w_{2}^{*} \right\|^{2}}{2}.$$

Therefore,

$$\ln\theta + \ln \left\|\mathfrak{T}w_1^*(\varsigma) - \mathfrak{T}w_2^*(\varsigma)\right\|^2 \le \ln \frac{\|w_1^* - w_2^*\|^2}{2}.$$

Now, define  $h : \mathbb{R}^+ \to \mathfrak{R}$  by  $h(u) = \ln u$  and put  $\tau = \ln \theta$ ,  $\lambda = \frac{1}{2}$  and  $k = \frac{1}{3}$ . Then, it is easy to show that  $\mathfrak{T}$  is a convex F-contraction. Thus, applying Theorem 2.4,  $\mathfrak{T}$  possesses  $w^* \in \mathfrak{D}$  as a FP that turns out to be a solution of BVP (1). The proof is complete.

In the following, the existence of solutions of problem (2) will be discussed. Call function  $w^* \in C_{\mathfrak{D}}(\mathbb{I}, \mathfrak{D})$  the solution of (2) if it satisfies all the boundary conditions and  $\exists w^* \in L^1(\mathbb{I})$  such that  $w^*(\varsigma) \in \mathfrak{J}(\varsigma, w^*(\varsigma))$  for almost all  $\varsigma \in \mathbb{I}$  and

$$w^{*}(\varsigma) = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{\pi^{*}-1} w^{*}(q) dq$$
  
$$- \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \left[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) (\mu(T) - \mu(q))^{\pi^{*}-1} w^{*}(q) dq$$
  
$$+ \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) (\mu(\zeta_{i}) - \mu(q))^{p_{i} + \pi^{*}-1} w^{*}(q) dq \right],$$
(17)

for all  $\varsigma \in \mathbb{I}$ . For each  $w^* \in \mathfrak{D}$ , we demonstrate the selections' set of  $\mathfrak{J}$  by

$$\mathfrak{S}_{\mathfrak{J},w^*} = \left\{ w^* \in L^1(\mathbb{I}) : w^*(\varsigma) \in \mathfrak{J}(\varsigma, w^*(\varsigma)) \text{ for all most all } \varsigma \in \mathbb{I} \right\}.$$

Now, consider  $\mathfrak{L}:\mathfrak{D}\to\mathbb{P}(\mathfrak{D})$  as

$$\mathfrak{L}(w^*) = \big\{ \mathfrak{F} \in \mathfrak{D} : \text{there exists } w^* \in \mathfrak{S}_{\mathfrak{J}, w^*} \text{ such that } \mathfrak{F}(\varsigma) = \pi(\varsigma) \text{ for all } \varsigma \in \mathbb{I} \big\}, \qquad (18)$$

where

$$\pi(\varsigma) = \frac{1}{\Gamma(\pi^*)} \int_{s_0}^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{\pi^* - 1} w^*(q) \, \mathrm{d}q$$

$$- \frac{(\mu(\varsigma) - \mu(s_0))}{\Lambda} \left[ \frac{1}{\Gamma(\pi^*)} \int_{s_0}^{T} \mu'(q) (\mu(T) - \mu(q))^{\pi^* - 1} w^*(q) \, \mathrm{d}q$$

$$+ \sum_{i=1}^n \beta_i \frac{1}{\Gamma(p_i + \pi^*)} \int_{s_0}^{\zeta_i} \mu'(q) (\mu(\zeta_i) - \mu(q))^{p_i + \pi^* - 1} w^*(q) \, \mathrm{d}q \right].$$
(19)

**Theorem 3.3** Consider a set-valued map  $\mathfrak{J} : \mathbb{I} \times \mathfrak{D} \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{D})$ . Suppose that:

- (i)  $\mathfrak{J}$  is bounded and integrable, and  $\mathfrak{J}(\cdot, w_1^*) : \mathbb{I} \to \mathbb{P}_{\mathbb{CP}}(\mathfrak{D})$  is measurable for  $w_1^* \in \mathfrak{D}$ .
- (ii) There is a member  $\omega \in C(\mathbb{I}, [0, \infty))$  such that

$$\mathbb{H}_{d}\big(\mathfrak{J}\big(\varsigma, w_{1}^{*}\big), \mathfrak{J}\big(a, \dot{w_{1}^{*}}\big)\big) \leq \frac{1}{\theta\sqrt{2}} \frac{\omega(\varsigma)\lambda^{*}}{\|\omega\|} |w_{1}^{*} - \dot{w_{1}^{*}}|, \qquad (20)$$

for all  $\varsigma \in \mathbb{I}$  and  $w_1^*, \hat{w_1^*} \in \mathfrak{D}$ , where  $\lambda^* = \frac{1}{\Omega_1}$ . Then, BVP (2) has a solution.

*Proof* Eventually, the fixed point of  $\mathfrak{L}$  will be characterized as a solution of BVP (2). Since the set-valued map  $\varsigma \mapsto \mathfrak{J}(\varsigma, w^*(\varsigma))$  is measurable closed-valued, there is a measurable selection of  $\mathfrak{J}$  and  $\mathfrak{S}_{\mathfrak{J},w^*}$  is nonempty. We aim to prove that  $\mathfrak{L}(w^*)$  is a closed subset of  $\mathfrak{D}$ . Consider a convergent sequence  $\{w_n^*\}$  of  $\mathfrak{L}(w^*)$  tending to  $w^*$ . Corresponding to every n,  $\Upsilon_n \in \mathfrak{S}_{\mathfrak{J},w^*}$  exists such that

$$w_{n}^{*}(\varsigma) = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{\pi^{*}-1} \Upsilon_{n}(q) \, \mathrm{d}q$$
  
$$- \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \Biggl[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) (\mu(T) - \mu(q))^{\pi^{*}-1} \Upsilon_{n}(q) \, \mathrm{d}q$$
  
$$+ \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) (\mu(\zeta_{i}) - \mu(q))^{p_{i} + \pi^{*}-1} \Upsilon_{n}(q) \, \mathrm{d}q \Biggr],$$
(21)

for all  $\varsigma \in \mathbb{I}$ . Note that since  $w_n^* \to w^*$  in  $L^1(\mathbb{I})$ , the values of  $\mathfrak{J}$  are compact. Hence,  $\Upsilon \in \mathfrak{S}_{\mathfrak{J},w^*}$  and

$$w_{n}^{*}(\varsigma) \to w^{*}(\varsigma) = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{\pi^{*}-1} \Upsilon(q) \, \mathrm{d}q$$
  
$$- \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \left[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) (\mu(T) - \mu(q))^{\pi^{*}-1} \Upsilon(q) \, \mathrm{d}q \right]$$
  
$$+ \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) (\mu(\zeta_{i}) - \mu(q))^{p_{i} + \pi^{*}-1} \Upsilon(q) \, \mathrm{d}q \right], \quad (22)$$

which implies  $w^* \in \mathfrak{L}(w^*)$  and therefore the values of  $\mathfrak{L}$  are closed. Since  $\mathfrak{J}$  is compactvalued, it is indeed easy to show that  $\mathfrak{L}(w^*)$  is bounded. For all  $w^*, w^* \in \mathfrak{D}$  and  $\mathfrak{F}_1^* \in \mathfrak{L}(w^*)$ choose  $\Upsilon_1 \in \mathfrak{S}_{\mathfrak{J}, w^*}$  such that

$$\widetilde{\mathfrak{F}}_{1}^{*} = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big( \mu(\varsigma) - \mu(q) \big)^{\pi^{*}-1} \Upsilon_{1}(q) \, \mathrm{d}q \\ - \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \Bigg[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) \big( \mu(T) - \mu(q) \big)^{\pi^{*}-1} \Upsilon_{1}(q) \, \mathrm{d}q \\ + \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) \big( \mu(\zeta_{i}) - \mu(q) \big)^{p_{i} + \pi^{*}-1} \Upsilon_{1}(q) \, \mathrm{d}q \Bigg],$$
(23)

for all  $\varsigma \in \mathbb{I}$ . Thus,

$$\mathbb{H}_{d}(\mathfrak{J}(\varsigma, w^{*}(\varsigma)), \mathfrak{J}(\varsigma, w^{*}(\varsigma)) \leq \frac{1}{\theta\sqrt{2}} \frac{\omega(\varsigma)\lambda^{*}}{\|\omega\|} |w_{1}^{*} - w_{1}^{*}|,$$
(24)

for all  $w^*, w^* \in \mathfrak{D}$ . Therefore, an element  $\pi \in \mathfrak{J}(\varsigma, w^*(\varsigma))$  exists such that

$$\left|w_{1}^{*}(\varsigma) - \pi\right| \leq \frac{1}{\theta\sqrt{2}} \frac{\omega(\varsigma)\lambda^{*}}{\|\omega\|} \left|w_{1}^{*} - \hat{w_{1}^{*}}\right|.$$

$$(25)$$

Now, consider map  $\mathfrak{B}^* : \mathbb{I} \to \mathbb{P}(\mathfrak{D})$ , which is defined by

$$\mathfrak{B}^*(\varsigma) = \left\{ \pi \in \mathfrak{D} : \left| w_1^* - \pi(\varsigma) \right| \le \frac{1}{\theta \sqrt{2}} \frac{\omega(\varsigma) \lambda^*}{\|\omega\|} \left| w_1^* - \hat{w_1^*} \right| \right\}.$$

Note that  $\mathfrak{B}^*(\cdot) \cap \mathfrak{J}(\varsigma, w^*(\varsigma))$  is measurable since  $w_1^*$  and  $\tau = \frac{1}{\theta\sqrt{2}} \frac{\omega(\varsigma)\lambda^*}{\|\omega\|} |w_1^* - w_1^*|$  are measurable. Now, let  $w_2^* \in \mathfrak{J}(\varsigma, w^*(\varsigma))$ . Therefore,

$$\left|w_{1}^{*}(\varsigma) - w_{2}^{*}(\varsigma)\right| \leq \frac{1}{\theta\sqrt{2}} \frac{\omega(\varsigma)\lambda^{*}}{\|\omega\|} \left|w_{1}^{*} - w_{1}^{*}\right|,\tag{26}$$

for all  $\varsigma \in \mathbb{I}$ . Let us define  $\mathfrak{F}_2^* \in \mathfrak{L}(\varsigma)$  by

$$\mathfrak{F}_{2}^{*}(\varsigma) = \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{\pi^{*}-1} \Upsilon_{2}(q) \, \mathrm{d}q \\ - \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \Biggl[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) (\mu(T) - \mu(q))^{\pi^{*}-1} \Upsilon_{2}(q) \, \mathrm{d}q \\ + \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\zeta_{i}} \mu'(q) (\mu(\zeta_{i}) - \mu(q))^{p_{i}+\pi^{*}-1} \Upsilon_{2}(q) \, \mathrm{d}q \Biggr],$$
(27)

for all  $\varsigma \in \mathbb{I}$  and put  $\|\omega\| = \sup_{a \in \mathbb{I}} |\omega(\varsigma)|$ . Then,

$$\begin{split} \left| \mathfrak{F}_{1}^{*} - \mathfrak{F}_{2}^{*} \right| &\leq \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{\varsigma} \mu'(q) \big( \mu(\varsigma) - \mu(q) \big)^{\pi^{*}-1} \big| \Upsilon_{1}(q) - \Upsilon_{2}(q) \big| \, \mathrm{d}q \\ &- \frac{(\mu(\varsigma) - \mu(s_{0}))}{\Lambda} \Biggl[ \frac{1}{\Gamma(\pi^{*})} \int_{s_{0}}^{T} \mu'(q) \big( \mu(T) - \mu(q) \big)^{\pi^{*}-1} \big| \Upsilon_{1}(q) - \Upsilon_{2}(q) \big| \, \mathrm{d}q \end{split}$$

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$$+ \sum_{i=1}^{n} \beta_{i} \frac{1}{\Gamma(p_{i} + \pi^{*})} \int_{s_{0}}^{\varsigma_{i}} \mu'(q) (\mu(\varsigma_{i}) - \mu(q))^{p_{i} + \pi^{*} - 1} |\Upsilon_{1}(q) - \Upsilon_{2}(q)| dq$$

$$\leq \frac{(|\mu(T) - \mu(s_{0})|)^{\pi^{*}}}{\Gamma(\pi^{*} + 1)} \|\omega\| \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| \frac{\lambda^{*}}{\|\omega\|}$$

$$+ \frac{(|(\mu(T) - \mu(s_{0}))|)^{\pi^{*} + 1}}{|\Lambda|\Gamma(\pi^{*} + 1)} \|\omega\| \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| \frac{\lambda^{*}}{\|\omega\|}$$

$$+ \sum_{i=1}^{n} \beta_{i} \frac{n(|(\mu(T) - \mu(s_{0}))|)^{p_{i} + \pi^{*} + 1}}{|\Lambda|\Gamma(p_{i} + \pi^{*} + 1)} \|\omega\| \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| \frac{\lambda^{*}}{\|\omega\|}$$

$$= \left[ \frac{(|\mu(T) - \mu(s_{0})|)^{\pi^{*}}}{\Gamma(\pi^{*} + 1)} + \frac{(|(\mu(T) - \mu(s_{0}))|)^{\pi^{*} + 1}}{|\Lambda|\Gamma(\pi^{*} + 1)} \right] \|\omega\| \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| \frac{\lambda^{*}}{\|\omega\|}$$

$$= \Omega_{1} \|\omega\| \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| \frac{\lambda^{*}}{\|\omega\|}$$

$$= \Omega_{1} \lambda^{*} \frac{1}{\theta\sqrt{2}} \|w^{*} - w^{*}\| .$$

$$(28)$$

If  $\tau = \ln \theta$  and define  $\mathbb{F}(u) = \ln u$ , then the inequality  $2\tau + \mathbb{F}(2\mathbb{H}_{d}(\mathfrak{L}(w^{*}), \mathfrak{L}(\hat{w^{*}}))) \leq \mathbb{F}(||w^{*} - \hat{w^{*}}||)$  holds for all  $w^{*}, \hat{w^{*}} \in \mathfrak{D}$ . As a conclusion, applying Theorem 2.5,  $\mathfrak{L}$  admits a FP that is the solution for BVP (2).

#### 4 Examples

In this section two examples are provided to illustrate the theoretical results.

Example 4.1 Consider BVP

$$\begin{cases} {}^{\mathcal{C}}D_{0}^{1.79,\exp(\varsigma+1)}w^{*}(\varsigma) = \frac{a\cos(\varsigma)|w^{*}(\varsigma)|}{340(|w^{*}(\varsigma)|+1)}, \quad \varsigma \in [0,1]; \\ w^{*}(0) = 0, \\ w^{*}(1) = 0.71^{RL}I_{0}^{1.69,\exp(\varsigma+1)}w^{*}(0.37) + 0.85^{RL}I_{0}^{1.93,\exp(\varsigma+1)}w^{*}(0.39), \end{cases}$$

where  $\varsigma \in [0, 1]$ ,  $\pi^* = 1.79$ ,  $s_0 = 0$ , T = 1, n = 2,  $p_1 = 1.69$ ,  $p_2 = 1.93$ ,  $\xi_1 = 0.37$ ,  $\xi_2 = 0.39$ ,  $\beta_1 = 0.71$ ,  $\beta_2 = 0.85$ , and  $\mu = \exp(\varsigma + 1)$ . Here,  ${}^{C}D_0^{1.79,\exp(a+1)}$  and  ${}^{RL}I_0^{p_i}$  are the FD of Caputo type of order 1.79 and the FI of R–L type of order  $p_i$ , respectively. Then, we have  $\Lambda = 1.6094$  and  $\Omega_1 = 5.0139$ . Consider the continuous mapping  $\mathfrak{h}(\varsigma, w^*(\varsigma)) = \frac{\varsigma \cos(\varsigma)|w^*(\varsigma)|}{340(|w^*(\varsigma)|+1)}$ . Then,

$$\left|\mathfrak{h}(\varsigma, w_1^*(\varsigma)) - \mathfrak{h}(\varsigma, w_2^*(\varsigma))\right| \leq \frac{\varsigma}{340} \left|w_1^* - w_2^*\right| \leq \frac{\sqrt{2}}{4\Omega_1} \left|w_1^* - w_2^*\right|.$$

Now, by Theorem 3.2, the BVP has a solution.

Example 4.2 Consider BVP

$$\begin{cases} {}^{\mathcal{C}}D_0^{1.3,\exp(\varsigma/2)}w^*(\varsigma) \in [0, \frac{\exp(\frac{3}{\varsigma}+1)}{30} + \frac{\sqrt{\pi}\cos(\coth(\varsigma))}{2+\exp(\varsigma)} + \frac{\varsigma\sin(\varsigma)|w^*(\varsigma)|}{126(\varsigma+7)}], \quad \varsigma \in [0,3]; \\ w^*(0) = 0, \\ w^*(3) = 0.69^{RL}I_0^{1.37,\exp(2\varsigma)}w^*(0.37) + 0.72^{RL}I_0^{1.82,\exp(2\varsigma)}w^*(0.38), \end{cases}$$

where  $\pi^* = 1.3$ ,  $s_0 = 0$ , T = 3, n = 2,  $p_1 = 1.37$ ,  $p_2 = 1.82$ ,  $\xi_1 = 0.37$ ,  $\xi_2 = 0.38$ ,  $\beta_1 = 0.69$ , and  $\beta_2 = 0.72$ . Here,  ${}^{C}D_0^{1.3,\exp(\varsigma/2)}$ ,  ${}^{RL}I_0^{1.37,\exp(\varsigma/2)}$ , and  ${}^{RL}I_0^{1.82,\exp(\varsigma/2)}$  are the FD of Caputo type of order 0.3, the FIs of R–L type of order 1.37 and 1.82, respectively. Now, define the continuous set-valued mapping  $\mathfrak{J} : [0,3] \times \mathfrak{R} \to \mathfrak{R}$  by  $\mathfrak{J}(\varsigma, w^*(\varsigma)) = [\varsigma, \frac{\exp(\sqrt[3]{\varsigma}+1)}{30} + \frac{\sqrt{\pi} \cos(\coth(\varsigma))}{2+\exp(\varsigma)} + \frac{\varsigma \sin(\varsigma)|w^*(\varsigma)|}{126(\varsigma+7)}]$ . For  $w_1^*, w_2^* \in \mathfrak{R}$  we have

$$\mathbb{H}\left(\mathfrak{J}\left(\varsigma, w_{1}^{*}(\varsigma)\right) - \mathfrak{J}\left(\varsigma, w_{2}^{*}(\varsigma)\right)\right) \leq \frac{a}{63} \frac{1}{2} \left[\left|\sin\left(w_{1}^{*}(\varsigma)\right) - \sin\left(w_{2}^{*}(\varsigma)\right)\right|\right]$$
$$\leq \frac{1}{\theta\sqrt{2}\Omega_{1}} \left|w_{1}^{*}(\varsigma) - w_{2}^{*}(\varsigma)\right|. \tag{29}$$

Then, we have  $\mathbb{H}_d(\mathfrak{J}(\varsigma, w_1^*(\varsigma)) - \mathfrak{J}(a, w_2^*(\varsigma))) \le \omega(\varsigma) |w_1^*(\varsigma) - w_2^*(\varsigma)| \frac{1}{\Omega_1}$ . Thus,  $\Omega_1 = 27.2258$ ,  $\Lambda = 3.4493$ , and  $\lambda^* = 0.0367$ . Consider  $\mathfrak{L} : \mathfrak{D} \to \mathcal{P}(\mathfrak{D})$  as

$$\mathfrak{L}(w^*) = \{\mathfrak{F} \in \mathfrak{D} : \text{there exists } w^* \in \mathfrak{S}_{\mathfrak{J}, w^*} \text{ such that } \mathfrak{F}(z) = \pi(z) \text{ for all } \varsigma \in [0, 3] \}, (30)$$

where

$$\pi(\varsigma) = \frac{1}{\Gamma(1.3)} \int_0^{\varsigma} \mu'(q) (\mu(\varsigma) - \mu(q))^{0.3} \Upsilon(q) \, \mathrm{d}q$$
  
$$- \frac{(\mu(\varsigma) - 1)}{3.4493} \left[ \frac{1}{\Gamma(1.3)} \int_0^3 \mu'(q) (\mu(T) - \mu(q))^{0.3} \Upsilon(q) \, \mathrm{d}q \right]$$
  
$$+ \sum_0^2 \beta_i \frac{1}{\Gamma(p_i + 1.3)} \int_0^{\zeta_i} \mu'(q) (\mu(\zeta_i) - \mu(q))^{p_i + 0.3} w^*(q) \, \mathrm{d}q \left].$$
(31)

Now, by Theorem 3.3, the BVP has a solution.

The importance of the  $\mu$ -Caputo derivative can be seen in the following example.

*Example* 4.3 Here, we study a model explained by a fractional differential equation, and we show how fractional derivatives with respect to another function may be helpful. The significant and historical model to describe population growth is the Malthusian law, which was proposed in 1798 by the English economist Thomas Malthus. It is given by  $N'(t) = \lambda N(t)$ , where  $\lambda$  is the population growth rate (equal to the difference between the birth and mortality rates), sometimes called the Malthusian parameter, and N(t) is the number of individuals in a population at time *t*. It is assumed that  $\lambda$  is constant, and so if  $N_0$  denotes the initial population size, then the solution to this Cauchy problem is the exponential function  $N(t) = N_0 \exp(\lambda t)$ .

Consider the FDE obtained from the Malthusian law of population growth, by replacing the first-order derivative by the  $\mu$ -Caputo fractional derivative with respect to  $\mu$  as  $CD_{0^+}^{\alpha,\mu}N(t) = \lambda N(t)$ . From [7], the solution of this FDE, together with the initial condition  $N(0) = N_0$ , is the function

$$N(t) = N_0 E_\alpha \left( \lambda \left( \mu(t) - \mu(0) \right)^\alpha \right). \tag{32}$$

For  $\mu(x) = x$ , with the usual Caputo fractional derivative, we obtain  $N(t) = N_0 E \alpha(\lambda t^{\alpha})$  (see [8]), and it was proven that the FDE was more efficient in modeling the population growth

than the ODE. This work further exemplifies that, when considering function N as given by (32), a better accuracy on the model is obtained. The purpose is to determine the best-fitting curve N (that is, find the parameters  $\lambda$  and  $\alpha$ ) by minimizing the sum of the squares of the offsets of the points from the curve. For more details and aspects of the numerical process see [7] and [27].

#### 5 Conclusion

In this manuscript, applying some *F*-contraction, convex *F*-contraction, and some fixedpoint theorems, the existence of solutions of a  $\mu$ -Caputo FDE and an inclusion problem equipped with nonlocal  $\mu$ -integral boundary conditions have been investigated. Some examples have also been provided to illustrate the results. Among the important problems that can be investigated in the continuation of this research, we can refer to topics such as: checking the existence and uniqueness of boundary value problems with different boundary and initial conditions and checking their application in important subjects such as insurance, economics, and modeling many important diseases.

One other possible extension is to consider the fractional order as a function of time  $\pi^*$ , and determine the order that fits closer to the mathematical model. These and other questions will be treated in the future.

#### Acknowledgements

The authors would like to thank the anonymous referees for their careful reading of the manuscript and their valuable comments that have improved the manuscript significantly.

Funding

Not applicable

#### Declarations

Ethics approval and consent to participate Not applicable

#### **Consent for publication**

Not applicable

#### **Competing interests**

The authors declare no competing interests.

#### Author contributions

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#### Received: 1 November 2023 Accepted: 31 January 2024 Published online: 27 February 2024

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