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Approximating fixed points of weak enriched contractions using Kirk's iteration scheme of higher order

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Abstract

In this paper, we introduce two types of weak enriched contractions, namely weak enriched $\mathcal{F}$-contraction, weak enriched $\mathcal{F}'$-contraction, and k-fold averaged mapping based on Kirk's iterative algorithm of order k. The types of contractions introduced herein unify, extend, and generalize several existing classes of enriched and weak enriched contraction mappings. Moreover, K-fold averaged mappings can be viewed as a generalization of the averaged mappings and double averaged mappings. We then prove the existence of a unique fixed point of the k-fold averaged mapping associated with weak enriched contractions introduced herein. We study necessary conditions that guarantee the equality of the sets of fixed points of the k-fold averaged mapping and weak enriched contractions. We show that an appropriate Kirk's iterative algorithm can be used to approximate a fixed point of a k-fold averaged mapping and of the two weak enriched contractions. We also study the well-posedness, limit shadowing property, and Ulam–Hyers stability of the k-fold averaged mapping. We provide necessary conditions that ensure the periodic point property of each illustrated weak enriched contraction. Some examples are presented to show that our results are a potential generalization of the comparable results in the existing literature.

Mathematics Subject Classification: Primary 47H10; secondary 47H09; 54H25

Keywords: Fixed point; Weak enriched $\mathcal{F}$-contraction; Weak enriched $\mathcal{F}'$-contraction; Kirk's iteration; Well-posedness; Limit shadowing property; Ulam–Hyers stability; Periodic point property

1. Introduction and preliminaries

Let X be a Banach space, and let K be a nonempty subset of X. A self-mapping T on K is called nonexpansive if for all $x, y \in K$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

An element $x^* \in K$ is called a fixed point of T if it is a solution of the operator equation $Tx^* = x^*$. The set of all fixed points of T is denoted by $Fix(T)$. The nth iterate of the mapping T is defined as $T^n = T^{n-1} \circ T$ for $n \geq 1$, where $T^0 = I$ (the identity map on X). Let x_0

be an arbitrary point of X. The set $\{x_0, T^n(x_0) : n \geq 1\}$ is called the forward orbit of x_0 and is denoted by $O(T, x_0, \infty)$. For $n \geq 1$, we denote the set $\{x, Tx, \dots, T^n x\}$ by $O(T, x, n)$. The mapping T is said to be a Picard operator if (i) $Fix(T) = \{x^*\}$; (ii) the forward orbit of each $x_0 \in X$ converges to x^* as $n \to \infty$. A self-mapping T on X is called a Banach contraction mapping if there exists a constant $c \in [0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$. If we take $c = 1$, then T is called a nonexpansive mapping. The nonexpansive mappings can be viewed as the limiting case of Banach contraction mappings. The nth iterates of a Banach contraction mapping are called Picard's iterates.

According to the Banach contraction principle [1], any Banach contraction mapping defined on a complete metric space (X, d) is a Picard operator. Moreover, for each $x_0 \in X$, the fixed point of the mapping T can be approximated by Picard's iterates. However, the forward orbit generated by a nonexpansive mapping T does not converge to the fixed point of T. Indeed, if K is a closed nonempty subset of a Banach space X and $T : K \to K$ is nonexpansive, then the mapping T may not have a fixed point, may have more than one fixed point, or even have a unique fixed point, whereas the forward orbit induced by a nonexpansive mapping fails to converge to its fixed point. Hence, to approximate the fixed points of non-expansive mappings, other approximation techniques are needed. Moreover, the existence of fixed points of nonexpansive mappings requires a rich geometric structure of underlying spaces. These aspects have made the study of nonexpansive mappings as one of the major and most active research areas of nonlinear analysis.

Many authors have employed an explicit averaged iteration of the form $x_{n+1} = f(x_n, Tx_n)$, $n \geq 1$. One of the famous techniques is to form an averaged mapping: For a given operator T on X and $\lambda \in (0, 1]$, an operator T_λ associated with T and identity mapping I is called an averaged mapping if $T_\lambda := (1 - \lambda)I + \lambda T$. This term was introduced in [2], where it was shown that under certain conditions the forward orbit induced by T_λ converges to a fixed point of T. The first interesting result in this direction was obtained by Krasnoselskii [3]: if K is a closed convex subset of a uniformly convex Banach space and T is a nonexpansive mapping on K into a compact subset of K, then the forward orbit of any x in K for $\lambda = \frac{1}{2}$ converges to a fixed point of T.

It was proved by Schaefer [4] that the above results for arbitrary $\lambda \in (0, 1)$. Subsequently, Edelstein [5] proved the corresponding result in the framework of a strictly convex Banach space, which is more general than a uniformly convex Banach space. Obviously, Krasnoselskii's iteration is a generalization of Picard's iteration process.

Another important iteration scheme is Kirk's iteration scheme introduced by Kirk [6], which is a sequence $\{x_n\}$ defined by

$$x_n = \alpha_0 x_{n-1} + \alpha_1 Tx_{n-1} + \alpha_2 T^2 x_{n-1} + \cdots + \alpha_k T^k x_{n-1},$$

where $x_0 \in K$, $\alpha_1 > 0$, and $\alpha_i \geq 0$ for $i = 1, 2, \dots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$.

Indeed, Kirk's iteration scheme is a forward orbit of the mapping $S : K \to K$ [6] given by

$$S := \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k,$$

where $\alpha_1 > 0$ and $\alpha_i \geq 0$ for $i = 1, 2, \dots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$. Obviously, the mapping S is a generalization of an averaged mapping T_λ.

Kirk proved that the set of fixed points of the mapping S coincides with $Fix(T)$ under certain suitable conditions, Kirk's iteration scheme converges to the fixed point of T:

Theorem 1.1 [6] *Let K be a convex subset of a Banach space X, and let T be a nonexpansive self-mapping on K. Define the mapping $S : K \to K$ by*

$$S := \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k,$$

where $\alpha_1 > 0$ and $\alpha_i \geq 0$ for $i = 0, 1, \ldots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$. Then $S(x) = x$ if and only if $T(x) = x$.

Corollary 1.1 [6] *Let X be a uniformly convex Banach space, and let T be a nonexpansive compact mapping of X into X, that is, T maps bounded subsets of X into relatively compact subsets of X. If $Fix(T)$ is nonempty, then each forward orbit induced by the mapping S as given in Theorem 1.1 converges to a fixed point of T.*

Recently, Berinde and Păcurar [7] introduced the notion of enriched contractive mappings. A self-mapping T on a Banach space X is called an enriched contraction mapping if there exist $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that for all $x, y \in X$,

$$\left\| b(x - y) + Tx - Ty \right\| \leq \theta \|x - y\|.$$

They proved the existence of fixed point of an enriched contraction, which can be approximated by means of an appropriate Krasnoselskii iterative scheme. Specifically, the fixed point of the averaged mapping T_λ with $\lambda \in (0, 1]$ is also a fixed point of T and can be approximated by the sequence $\{T_\lambda^n x_0\}$ for any $x_0 \in X$.

Theorem 1.2 [7] *Let T be an enriched contraction mapping defined on a Banach space X into itself. Then $|Fix(T)| = 1$, and there exists $\lambda \in (0, 1]$ such that for each $x_0 \in X$, the Krasnoselskii iteration scheme $\{x_n\}$ given by*

$$x_n := (1 - \lambda) x_{n-1} + \lambda T x_{n-1}, \quad n \geq 0,$$

converges to a unique fixed point of T.

It is worth mentioning that the enriched contraction mapping introduced by Berinde and Păcurar [7] involves the displacements $\|Tx - Ty\|$ and $\|x - y\|$ only. However, for any two distinct points $x, y \in X$, there exist other four displacements associated with a self-mapping T given by $\|x - Tx\|$, $\|y - Ty\|$, $\|x - Ty\|$, and $\|y - Tx\|$. There have been several well-known contraction mappings that involve two or more displacements (see, for example, Bianchini [8], Chatterjea [9], Ćirić [10, 11], Kannan [12], Khan [13], and Reich [14–18]). Motivated by the work of Berinde and Păcurar, many authors have applied enrichment techniques to different classical contraction mappings. Lately, Berinde [19, 20], Górnicki and Bisht [21], Berinde and Păcurar [22–24], Anjum and Abbas [25], Abbas et al. [26] introduced the so-called enriched contractions related to nonexpansive, Kannan, Chatterjea, Ćirić–Reich–Rus, and interpolative Kannan contractions and gave the existence of fixed points of such enriched contractions with the help of Krasnoselskii iterative scheme.

Recently, Nithiarayaphaks and Sintunavarat [27] introduced the notion of weak enriched contraction mappings and a generalization of the averaged mappings called double averaged mappings. Let $\alpha_1 > 0$, $\alpha_2 \geq 0$, and $\alpha_1 + \alpha_2 = 1$, and let T be a self-mapping on a Banach space X. A double averaged mapping denoted by T_{α_1,α_2} is a mapping associated with I, T, and T^2 and is defined by

$$T_{\alpha_1,\alpha_2} := (1-\alpha_1-\alpha_2)I + \alpha_1 T + \alpha_2 T^2.$$

It is easy to see that T_{α_1,α_2} is a generalization of T_λ. Indeed, $T_\lambda = T_{\alpha_1,0}$. Moreover, the double averaged mapping T_{α_1,α_2} is a particular case of the mapping S in [6] of order $k = 2$. A self-mapping T on a Banach space X is called a weak enriched contraction mapping if there exist nonnegative real numbers a, b, and $w \in [0, a+b+1)$ such that for all $x, y \in X$,

$$\left\| a(x-y) + Tx - Ty + b\left(T^2x - T^2y\right)\right\| \le w\|x-y\|.$$

Nithiarayaphaks and Sintunavarat [27] proved that for each self-mapping T on a closed convex subset of a Banach space satisfying the weak enriched contraction condition, there are $\alpha_1 > 0$ and $\alpha_2 \ge 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that T_{α_1,α_2} has a unique fixed point, and an appropriate Kirk's iteration scheme can approximate it. We refer to the following statement.

Theorem 1.3 [27] *Let C be a closed convex subset of a Banach space $(X, \|\cdot\|)$, and let T be a weak enriched contraction self-mapping on C. Then there are $\alpha_1 > 0$ and $\alpha_2 \ge 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that the following statements hold*:

(*i*) $|Fix(T_{\alpha_1,\alpha_2})| = 1$;

(*ii*) *for any $x_0 \in C$, the iteration scheme $\{x_n\} \subset C$ defined as*

$$x_n = (1-\alpha_1-\alpha_2)x_{n-1} + \alpha_1 Tx_{n-1} + \alpha_2 T^2 x_{n-1}$$

for $n \in \mathbb{N}$ converges to a unique fixed point of T_{α_1,α_2}.

It seems useful to unify the fixed point results mentioned by using Kirk's iteration scheme of order k generated by a generalized enriched contraction mapping. This is a twofold unification: (1) generalization of enriched contraction mappings so that the several existing enriched contraction mappings are deduced as particular cases and (2) a consideration of Kirk's iteration scheme of order greater than 2. The first objective can be achieved using the notion of implicit relations, a useful technique for the unification of contraction conditions (see, for example, [28–30] and references therein).

The important contributions from this work are highlighted as follows:

1. The notions of two weak enriched contractions, weak enriched $\mathcal{F}$-contraction and weak enriched $\mathcal{F}'$-contraction, are defined.
2. Generalization of averaged mappings, k-fold averaged mappings, is introduced.
3. The existence of a unique fixed point of the k-fold averaged mapping associated with two weak enriched contractions is proved in the framework of Banach spaces.
4. It is shown that the fixed point theorems for weak enriched or enriched versions due to Kannan, Chatterjea, and Ćirić, Reich, and Rus can be derived from the results presented in this paper.
5. The well-posedness, limit shadowing property, and Ulam–Hyers stability of k-fold averaged mappings are studied.

6. Necessary conditions are investigated for the equality of the fixed point sets of the k-fold averaged mapping and the weak enriched contraction mapping.
7. Necessary conditions that guarantee the periodic point property of weak enriched contractions are provided.
8. Examples are presented to support the validity of our results.

2. Weak enriched $\mathcal{F}$-contraction and weak enriched $\mathcal{F}'$-contraction

First, we introduce two families of mappings.

Let $\mathcal{F}$ be the family of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying the following conditions:

$(\mathcal{F}_1)$ f is continuous in each argument;

$(\mathcal{F}_2)$ there exists $k \in [0, 1)$ such that if $r < f(s, r, s)$ or $r < f(r, s, s)$, then $r \leq ks$ for all $r, s \in \mathbb{R}_+$;

$(\mathcal{F}_3)$ for $\lambda > 0$ and for all $r, s, t \in \mathbb{R}_+$, $\lambda f(r, s, t) \leq f(\lambda r, \lambda s, \lambda t)$.

We now give some examples to show that the family $\mathcal{F}$ is nonempty.

Example 2.1 It is straightforward to verify that the mappings defined below belong to the class $\mathcal{F}$:

(1) $f(r, s, t) = \alpha(s + t)$, where $\alpha \in [0, 1/2)$.

(2) $f(r, s, t) = \alpha(r + s)$, where $\alpha \in [0, 1/2)$.

(3) $f(r, s, t) = \alpha r$, where $\alpha \in [0, 1)$.

(4) $f(r, s, t) = \alpha \max\{r + s, s + t, r + t\}$, where $\alpha \in [0, 1/2)$.

(5) $f(r, s, t) = \max \alpha\{r, s, t\}$, where $\alpha \in [0, 1)$.

(6) $f(r, s, t) = \max \alpha\{s, t\}$, where $\alpha \in [0, 1)$.

(7) $f(r, s, t) = ar + bs + ct$, where $0 \leq a, b, c < 1$ and $a + b + c = 1$.

(8) $f(r, s, t) = r^\alpha s^\beta t^{1-\alpha-\beta}$, where $\alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$.

(9) $f(r, s, t) = s^\alpha t^{1-\alpha}$, where $\alpha \in (0, 1)$.

Let $\mathcal{F}'$ be the family of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying the following conditions:

$(\mathcal{F}'_1)$ f is continuous in each argument;

$(\mathcal{F}'_2)$ there exists $k \in [0, 1)$ such that if $r < f(s, s, r)$ or $r < f(r, s, s)$ or $r < f(s, 0, r + s)$, then $r \leq ks$ for all $r, s \in \mathbb{R}_+$;

$(\mathcal{F}'_3)$ for $\lambda > 0$ and for all $r, s, t \in \mathbb{R}_+$, $\lambda f(r, s, t) \leq f(\lambda r, \lambda s, \lambda t)$;

$(\mathcal{F}'_4)$ if $t \leq u$, then $f(r, s, t) \leq f(r, s, u)$ for all $r, s, t, u \in \mathbb{R}_+$;

$(\mathcal{F}'_5)$ if $r \leq f(r, r, r)$, then $r = 0$.

Example 2.2 The following mappings belong to the class $\mathcal{F}'$:

(1) $f(r, s, t) = \alpha(s + t)$, where $\alpha \in [0, 1/2)$.

(2) $f(r, s, t) = \alpha(r + s)$, where $\alpha \in [0, 1/2)$.

(3) $f(r, s, t) = \alpha r$, where $\alpha \in [0, 1)$.

(4) $f(r, s, t) = \alpha \max\{r + s, s + t, r + t\}$, where $\alpha \in [0, 1/2)$.

(5) $f(r, s, t) = \alpha\sqrt{rs}$, where $\alpha \in [0, 1)$.

(6) $f(r, s, t) = \alpha(rst)^{\frac{1}{3}}$, where $\alpha \in [0, 1)$.

(7) $f(r, s, t) = \alpha(r + s + t)$, where $\alpha \in [0, 1/3)$.

Nithiarayaphaks and Sintunavarat [27] introduced the notion of double averaged mapping and proved the existence of a unique fixed point of such mappings using Kirk's iterative scheme. Clearly, double averaged mappings coincide with the averaged mappings S [6] for the order $k = 2$.

Using the mapping S [6], we introduce the k-fold averaged mapping as follows.

Definition 2.1 Let X be a Banach space, let K be a nonempty subset of X, and let T be a self-mapping on X. Define the self-mapping $\bar{T}$ on K associated with T by

$$\bar{T} := (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k,$$

where $\alpha_i > 0$, and $\sum_{i=1}^{k} \alpha_i \in (0, 1]$. Such a mapping $\bar{T}$ is called a k-fold averaged mapping ($k \geq 3, k \in \mathbb{N}$).

We now provide two notions of weak enriched contractions.

Definition 2.2 Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \to X$ is called a weak enriched $\mathcal{F}$-contraction if there exists $f \in \mathcal{F}$ such that for all $x, y \in X$, $a_i \in (0, \infty)$, $i = 1, 2, \ldots, k$, $k \geq 3$, $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\| a_1(x - y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad \leq f\left(\left(\sum_{i=1}^{k} a_i + 1 \right) \|x - y\|, \left\| (x - Tx) + a_2\left(x - T^2x\right) + \cdots + a_k\left(x - T^kx\right) \right\|, \right. \\
&\qquad \left. \left\| (y - Ty) + a_2\left(y - T^2y\right) + \cdots + a_k\left(y - T^ky\right) \right\| \right).
\end{aligned} \tag{1}$$

Definition 2.3 Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \to X$ is called a weak enriched $\mathcal{F}'$-contraction if there exists $f \in \mathcal{F}'$ such that for all $x, y \in X$, $a_i \in (0, \infty)$, $i = 1, 2, \ldots, k$, $k \geq 3$, $k \in \mathbb{N}$, we have:

$$\begin{aligned}
&\left\| a_1(x - y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad \leq f\left(\left(\sum_{i=1}^{k} a_i + 1 \right) \|x - y\|, \right. \\
&\qquad \left\| \left(\sum_{i=1}^{k} a_i + 1 \right)(y - x) + (x - Tx) + a_2\left(x - T^2x\right) + \cdots + a_k\left(x - T^kx\right) \right\|, \\
&\qquad \left. \left\| \left(\sum_{i=1}^{k} a_i + 1 \right)(x - y) + (y - Ty) + a_2\left(y - T^2y\right) + \cdots + a_k\left(y - T^ky\right) \right\| \right).
\end{aligned} \tag{2}$$

Now we give some examples of such mappings.

Example 2.3 Let $X = \mathbb{R}$ be a usual normed space, and let T be a self-mapping on $[0, \infty)$ defined by $Tx = \frac{x}{2}$ for $x \in [0, \infty)$. Note that the mapping T is a weak enriched $\mathcal{F}$-contraction for $a_i = \frac{1}{2}$, $i = 1, 2, \ldots, k$, $k \geq 3$, $k \in \mathbb{N}$, and $f(r, s, t) = \alpha r$, $\alpha = \frac{4}{5} \in [0, 1)$.

Indeed, it follows from Definition 2.2 that

$$\left\| a_1(x - y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\|$$

$$
\begin{aligned}
&= \left\| \frac{1}{2}(x-y) + \left(\frac{x}{2}-\frac{y}{2}\right) + \frac{1}{2}\left(\frac{x}{4}-\frac{y}{4}\right) + \cdots + \frac{1}{2}\left(\frac{x}{2^k}-\frac{y}{2^k}\right) \right\| \\
&= \left\| \frac{1}{2}(x-y) + \frac{1}{2}(x-y) + \frac{1}{8}(x-y) + \cdots + \frac{1}{2^{k+1}}(x-y) \right\| \\
&\le 2\|x-y\|
\end{aligned}
$$

and

$$
\begin{aligned}
&f\Bigg(\left(\sum_{i=1}^{k} a_i + 1\right)\|x-y\|, \\
&\quad \left\| \left(\sum_{i=1}^{k} a_i + 1\right)(y-x) + (x-Tx) + a_2\left(x-T^2x\right) + \cdots + a_k\left(x-T^kx\right) \right\|, \\
&\quad \left\| \left(\sum_{i=1}^{k} a_i + 1\right)(x-y) + (y-Ty) + a_2\left(y-T^2y\right) + \cdots + a_k\left(y-T^ky\right) \right\| \Bigg) \\
&= \alpha\left(\sum_{i=1}^{k} a_i + 1\right)\|x-y\| \\
&= \frac{4}{5}\left(1+\frac{k}{2}\right)\|x-y\| \\
&\ge \frac{4}{5}\left(1+\frac{3}{2}\right)\|x-y\| \\
&= 2\|x-y\|.
\end{aligned}
$$

This implies that (1) holds. Hence T is a weak enriched $\mathcal{F}$-contraction.

Example 2.4 Let $X = \mathbb{R}$ be a usual normed space, and let T be a self-mapping on $[0,\infty)$ defined by $Tx = 1-x$ for $x \in [0,\infty)$. Then the mapping T is a weak enriched $\mathcal{F}$-contraction for $a_i = \frac{1}{2^i}$, $i = 1,2,\dots,k$, $k \ge 3$, $k \in \mathbb{N}$, and $f(r,s,t) = r$.

Indeed, from Definition 2.2 we have

$$
\begin{aligned}
&\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad = \left\| \frac{1}{2}(x-y) + (y-x) + \frac{1}{4}(x-y) + \frac{1}{8}(y-x) + \cdots + \frac{1}{2^k}(-1)^k(x-y) \right\| \\
&\quad \le \sum_{i=0}^{k}\left(\frac{1}{2^i}\right)\|x-y\|
\end{aligned}
$$

and

$$
\begin{aligned}
&f\Bigg(\left(\sum_{i=1}^{k} a_i + 1\right)\|x-y\|, \left\|(x-Tx) + a_2\left(x-T^2x\right) + \cdots + a_k\left(x-T^kx\right)\right\|, \\
&\quad \left\|(y-Ty) + a_2\left(y-T^2y\right) + \cdots + a_k\left(y-T^ky\right)\right\| \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{k} a_i + 1 \right) \|x - y\| \\
&= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k} \right) \|x - y\|.
\end{aligned}
$$

This implies that (1) holds. Hence T is a weak enriched $\mathcal{F}$-contraction.

Example 2.5 Let $X = \mathbb{R}$ be a usual normed space, and let T be a self-mapping on $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \subseteq X$ defined by

$$
Tx = \begin{cases} -x, & x \in [-1, -\frac{1}{2}], \\ 1 - x, & x \in [\frac{1}{2}, 1]. \end{cases}
$$

Then the mapping T is a weak enriched $\mathcal{F}$-contraction for $a_i = 1$, $i = 1, 2, \ldots, k$, $k \geq 3$, and $k \in \mathbb{N}$, and $f(r, s, t) = \frac{1}{4}(s + t)$. Without loss of generality, we may assume that $x, y \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ with $x \leq y$. We consider three cases for x, y: First, for each $x, y \in [-1, -\frac{1}{2}]$, it follows from Definition 2.3 that

$$
\begin{aligned}
&\left\| a_1(x - y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad = \left\| (x - y) + (y - x) + (x - y) + \cdots + (-1)^k(x - y) \right\| \\
&\quad = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \|x - y\| & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

Now, for each $x, y \in [\frac{1}{2}, 1]$, we have

$$
\begin{aligned}
&\left\| a_1(x - y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad = \left\| (x - y) + (y - x) + (x - y) + \cdots + (-1)^k(x - y) \right\| \\
&\quad = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \|x - y\| & \text{if } k \text{ is even,} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&f\Bigg(\left(\sum_{i=1}^{k} a_i + 1 \right) \|x - y\|, \\
&\qquad \left\| \left(\sum_{i=1}^{k} a_i + 1 \right)(y - x) + (x - Tx) + a_2\left(x - T^2x\right) + \cdots + a_k\left(x - T^kx\right) \right\|, \\
&\qquad \left\| \left(\sum_{i=1}^{k} a_i + 1 \right)(x - y) + (y - Ty) + a_2\left(y - T^2y\right) + \cdots + a_k\left(y - T^ky\right) \right\| \Bigg) \\
&\quad = \begin{cases} \frac{1}{4}\big[\|(k+1)(y - x) + (k+1)x - \frac{k+1}{2}\| + \|(k+1)(x - y) \\ \qquad + (k+1)y - \frac{k+1}{2}\|\big] & \text{if } k \text{ is odd,} \\ \frac{1}{4}\big[\|(k+1)(y - x) + kx - \frac{k}{2}\| + \|(k+1)(x - y) + ky - \frac{k}{2}\|\big] & \text{if } k \text{ is even,} \end{cases}
\end{aligned}
$$

$$
\geq \begin{cases} \frac{k+1}{4}\|y-x\| & \text{if } k \text{ is odd,} \\ \frac{k+2}{4}\|y-x\| & \text{if } k \text{ is even,} \end{cases}
$$

$$
\geq \begin{cases} \|y-x\| & \text{if } k \text{ is odd,} \\ \frac{5}{4}\|y-x\| & \text{if } k \text{ is even.} \end{cases}
$$

Finally, for all $x \in [-1, -\frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$, we have

$$
\begin{aligned}
&\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) + \cdots + a_k\left(T^kx - T^ky\right) \right\| \\
&\quad = \left\| (x-y) + (y-x-1) + (1+x-y) + \cdots + (-1)^k(1+x-y) \right\| \\
&\quad = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ \|x-y\| & \text{if } k \text{ is even,} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\left(\sum_{i=1}^{k} a_i + 1\right)\|x-y\|,\right. \\
&\qquad \left\| \left(\sum_{i=1}^{k} a_i + 1\right)(y-x) + (x-Tx) + a_2\left(x-T^2x\right) + \cdots + a_k\left(x-T^kx\right) \right\|, \\
&\qquad \left.\left\| \left(\sum_{i=1}^{k} a_i + 1\right)(x-y) + (y-Ty) + a_2\left(y-T^2y\right) + \cdots + a_k\left(y-T^ky\right) \right\|\right) \\
&\quad = \begin{cases} \frac{1}{4}[\|(k+1)(y-x) + (k+1)x - \frac{k-1}{2}\| + \|(k+1)(x-y) & \\ \quad + (k+1)y - \frac{k-1}{2}\| & \text{if } k \text{ is odd,} \\ \frac{1}{4}[\|(k+1)(y-x) + kx - \frac{k}{2}\| + \|(k+1)(x-y) + ky - \frac{k}{2}\| & \text{if } k \text{ is even,} \end{cases} \\
&\quad \geq \begin{cases} \frac{1}{4}\|(k+1)(y-x)\| & \text{if } k \text{ is odd,} \\ \frac{1}{4}\|(k+2)(y-x)\| & \text{if } k \text{ is even,} \end{cases} \\
&\quad \geq \begin{cases} \|y-x\| & \text{if } k \text{ is odd,} \\ \frac{5}{4}\|y-x\| & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

It is straightforward to check that in the above cases that (2) holds. Hence T is a weak enriched $\mathcal{F}'$-contraction.

Remark 2.1 If we choose $f(r,s,t) = \alpha r$, $\alpha \in [0,1)$, and set $a_i = 0$ for $i = 3,4,\ldots,k$ in Definition 2.2 or 2.3, then we get the weak enriched contraction mapping in [27].

Consequently, by choosing appropriate functions f and values $a_i, i = 1,2,\ldots,k$, we can obtain some weak enriched versions of the classical contractions mentioned that, to the best of our knowledge, have not been considered so far.

Definition 2.4 Defining $f \in \mathcal{F}$ by $f(r,s,t) = \alpha(s+t)$, $\alpha \in [0,1/2)$, and setting $a_i = 0$ for $i = 3,4,\ldots,k$ in Definition 2.2, we say that the mapping T is a weak enriched Kannan-contraction mapping, that is, there exist $a_i > 0$, $i = 1, 2$, and $\alpha \in [0, 1/2)$ such that

$$\begin{aligned} &\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) \right\| \\ &\quad \le \alpha\left[\left\| (x - Tx) + a_2\left(x - T^2x\right) \right\| + \left\| (y - Ty) + a_2\left(y - T^2y\right) \right\| \right] \end{aligned} \tag{3}$$

for all $x, y \in X$.

Definition 2.5 Defining $f \in \mathcal{F}'$ by $f(r,s,t) = \alpha(s+t)$, $\alpha \in [0,1/2)$, and setting $a_i = 0$ for $i = 3, 4, \dots, k$ in Definition 2.3, we say that the mapping T is a weak enriched Chatterjea-contraction mapping, that is, there exist $a_i > 0$, $i = 1, 2$, and $\alpha \in [0, 1/2)$ such that

$$\begin{aligned} &\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) \right\| \\ &\quad \le \alpha\left[\left\| (a_1 + a_2 + 1)(y - x) + (x - Tx) + a_2\left(x - T^2x\right) \right\| \right. \\ &\qquad \left. + \left\| (a_1 + a_2 + 1)(x - y) + (y - Ty) + a_2\left(y - T^2y\right) \right\| \right] \end{aligned} \tag{4}$$

for all $x, y \in X$.

Definition 2.6 Defining $f \in \mathcal{F}$ by $f(r,s,t) = kr + l(s+t), k, l \ge 0$, $k + 2l < 1$ and setting $a_i = 0$ for $i = 3, 4, \dots, k$ in Definition 2.2, we say that the mapping T is a weak enriched Ćirić–Reich–Rus-contraction mapping, that is, there exist $a_i > 0$, $i = 1, 2$, and $k, l \ge 0$, $k + 2l < 1$ such that

$$\begin{aligned} &\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) \right\| \\ &\quad \le k\|x - y\| + l\left[\left\| (x - Tx) + a_2\left(x - T^2x\right) \right\| + \left\| (y - Ty) + a_2\left(y - T^2y\right) \right\| \right] \end{aligned} \tag{5}$$

for all $x, y \in X$.

Definition 2.7 Defining $f \in \mathcal{F}$ by $f(r,s,t) = s^\alpha t^{1-\alpha}$, $0 < \alpha < 1$, and setting $a_i = 0$ for $i = 3, 4, \dots, k$ in Definition 2.2, we say that the mapping T is a weak enriched interpolative Kanan-contraction mapping, that is, there exist $a_i > 0$, $i = 1, 2$, and $0 < \alpha < 1$ such that

$$\begin{aligned} &\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) \right\| \\ &\quad \le \left\| (x - Tx) + a_2\left(x - T^2x\right) \right\|^\alpha \left\| (y - Ty) + a_2\left(y - T^2y\right) \right\|^{1-\alpha} \end{aligned} \tag{6}$$

for all $x, y \in X$.

Definition 2.8 Defining $f \in \mathcal{F}$ by $f(r,s,t) = r^\alpha s^\beta t^{1-\alpha-\beta}$, $0 < \alpha, \beta < 1$, and setting $a_i = 0$ for $i = 3, 4, \dots, k$ in Definition 2.2, we say that the mapping T is a weak enriched interpolative Ćirić–Reich–Rus-contraction mapping, that is, there exist $a_i > 0$, $i = 1, 2$, and $0 < \alpha + \beta < 1$ such that

$$\begin{aligned} &\left\| a_1(x-y) + Tx - Ty + a_2\left(T^2x - T^2y\right) \right\| \\ &\quad \le \|x - y\|^\alpha \left\| (x - Tx) + a_2\left(x - T^2x\right) \right\|^\beta \left\| (y - Ty) + a_2\left(y - T^2y\right) \right\|^{1-\alpha-\beta} \end{aligned} \tag{7}$$

for all $x, y \in X$.

Remark 2.2 By setting $a_2 = 0$ in Definitions 2.2 and 2.4–2.8 we obtain the enriched versions of the contraction mappings of Banach [7], Kanan [22], Chatterjea [23], Ćirić, Reich, and Rus [24], enriched interpolative Kanan-contractions [26], and enriched interpolative Ćirić–Reich–Rus–contractions, respectively.

Next, we recall the definitions of well-posedness, limit shadowing property of a mapping, and Ulam–Hyers stability of the fixed point equation.

Let T be a self-mapping on a metric space (X,d).

Definition 2.9 The fixed point problem $Fix(T)$ is said to be well posed if T has a unique fixed point x^* and for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have $\lim_{n\to\infty} d(x_n, x^*) = 0$.

Definition 2.10 The fixed point problem $Fix(T)$ is said to possess the limit shadowing property in X if for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists $z \in X$ such that $\lim_{n\to\infty} d(T^n z, x_n) = 0$.

Definition 2.11 The fixed point equation $x = Tx$ is Ulam–Hyers stable if there exists a constant $K > 0$ such that for each $\epsilon > 0$ and each $v^* \in X$ with $d(v^*, Tv^*) \leq \epsilon$, there exists $x^* \in X$ with $Tx^* = x^*$ such that $d(x^*, v^*) \leq K\epsilon$.

Let us start with the result dealing with the existence and uniqueness of a fixed point of an n-fold averaged mapping related to these two types of weak enriched contractions in the setting of a Banach space,

Theorem 2.1 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction. Then there exist $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$ with $\sum_{i=1}^{k} \alpha_i \in (0,1]$ such that the following statements hold*:

(*i*) *the n-fold averaged mapping $\bar{T}$ associated with T has a unique fixed point*;

(*ii*) *for any $x_0 \in X$, Kirk's iteration $\{x_n\}$ given by $x_n = \bar{T}x_{n-1}$, that is,*

$$x_n = (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)x_{n-1} + \alpha_1 Tx_{n-1} + \alpha_2 T^2 x_{n-1} + \cdots + \alpha_k T^k x_{n-1} \tag{8}$$

for $n \in \mathbb{N}$, converges to the unique fixed point of $\bar{T}$.

Proof As T is a weak enriched $\mathcal{F}$-contraction, there are $a_i \geq 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$, satisfying inequality (1). Define $\alpha_1 = \frac{1}{\sum_{i=1}^{k} a_i + 1} > 0$ and $\alpha_r = \frac{a_r}{\sum_{i=1}^{k} a_i + 1} \geq 0$, $r = 2,3,\dots,k$. Then inequality (1) becomes

$$\begin{aligned}
&\left\| \left(\frac{1 - \alpha_2 - \alpha_3 - \cdots - \alpha_k}{\alpha_1} - 1 \right)(x - y) + Tx - Ty + \frac{\alpha_2}{\alpha_1}\left(T^2 x - T^2 y\right) + \cdots \right. \\
&\quad \left. + \frac{\alpha_k}{\alpha_1}\left(T^k x - T^k y\right) \right\| \\
&\leq f\left(\frac{1}{\alpha_1}\|x - y\|, \left\| (x - Tx) + \frac{\alpha_2}{\alpha_1}\left(x - T^2 x\right) + \frac{\alpha_3}{\alpha_1}\left(x - T^3 x\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(x - T^k x\right) \right\|, \right. \\
&\quad \left. \left\| (y - Ty) + \frac{\alpha_2}{\alpha_1}\left(y - T^2 y\right) + \frac{\alpha_3}{\alpha_1}\left(y - T^3 y\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(y - T^k y\right) \right\| \right)
\end{aligned}$$

for all $x,y\in X$. As $\alpha_1>0$ and $\mathcal{F}_3$ holds, the above inequality can be rewritten as follows:

$$\begin{aligned}
&\big\|(1-\alpha_1-\alpha_2-\alpha_3-\cdots-\alpha_k)(x-y)+\alpha_1(Tx-Ty)+\alpha_2\big(T^2x-T^2y\big)+\cdots\\
&\quad+\alpha_k\big(T^kx-T^ky\big)\big\|\\
&\le\alpha_1 f\left(\frac{1}{\alpha_1}\|x-y\|,\left\|(x-Tx)+\frac{\alpha_2}{\alpha_1}\big(x-T^2x\big)+\frac{\alpha_3}{\alpha_1}\big(x-T^3x\big)+\cdots+\frac{\alpha_k}{\alpha_1}\big(x-T^kx\big)\right\|,\right.\\
&\quad\left.\left\|(y-Ty)+\frac{\alpha_2}{\alpha_1}\big(y-T^2y\big)+\frac{\alpha_3}{\alpha_1}\big(y-T^3y\big)+\cdots+\frac{\alpha_k}{\alpha_1}\big(y-T^ky\big)\right\|\right)\\
&\le f\big(\|x-y\|,\big\|\alpha_1(x-Tx)+\alpha_2\big(x-T^2x\big)+\alpha_3\big(x-T^3x\big)+\cdots+\alpha_k\big(x-T^kx\big)\big\|,\\
&\quad\big\|\alpha_1(y-Ty)+\alpha_2\big(y-T^2y\big)+\alpha_3\big(y-T^3y\big)+\cdots+\alpha_k\big(y-T^ky\big)\big\|\big),
\end{aligned}$$

which, together with Definition 2.1, implies that

$$\|\bar{T}x-\bar{T}y\|\le f\big(\|x-y\|,\|x-\bar{T}x\|,\|y-\bar{T}y\|\big) \tag{9}$$

for all $x,y\in X$.

Let x_0 be an arbitrary element in X. Define $x_n=\bar{T}^nx_0$ for $n\ge1$, $n\in\mathbb{N}$. Putting $x=x_n$, $y=x_{n-1}$ in (9), we have

$$\|x_{n+1}-x_n\|\le f\big(\|x_n-x_{n-1}\|,\|x_n-x_{n+1}\|,\|x_{n-1}-x_n\|\big).$$

By $\mathcal{F}_2$ there exits $\beta\in[0,1)$ such that

$$\|x_{n+1}-x_n\|\le\beta\|x_n-x_{n-1}\|.$$

By repeating this process we obtain that

$$\|x_{n+1}-x_n\|\le\beta^n\|x_1-x_0\|.$$

Now for all $m,n\ge1$, we have

$$\begin{aligned}
\|x_{n+m}-x_n\|&\le\|x_{n+m}-x_{n+m-1}\|+\|x_{n+m-1}-x_{n+m-2}\|+\cdots+\|x_{n+1}-x_n\|\\
&\le\big(\beta^{n+m-1}+\beta^{n+m-2}+\cdots+\beta^n\big)\|x_1-x_0\|\\
&=\frac{\beta^n(1-\beta^m)}{1-\beta}\|x_1-x_0\|,
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$.

Taking $x=x^*$, $y=x_n$ in inequality (9), we have

$$\big\|\bar{T}x^*-\bar{T}x_n\big\|\le f\big(\big\|x^*-x_n\big\|,\big\|x^*-\bar{T}x^*\big\|,\|x_n-\bar{T}x_n\|\big).$$

Taking the limit as $n\to\infty$ in the above inequality, by $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain that

$$\big\|\bar{T}x^*-x^*\big\|\le f\big(\big\|x^*-x^*\big\|,\big\|x^*-\bar{T}x^*\big\|,\big\|x^*-x^*\big\|\big)$$

$$\leq \beta \left\| x^* - \bar{T}x^* \right\| = 0.$$

Thus $\bar{T}x^* = x^*$.

Finally, we assume that $\bar{T}$ has two fixed points w and z with $w \neq z$. Then

$$\begin{aligned}
\|w-z\| &= \|\bar{T}w - \bar{T}z\| \\
&\leq f\left(\|w-z\|, \|w-\bar{T}w\|, \|z-\bar{T}z\|\right) \\
&= f\left(\|w-z\|, 0, 0\right) \\
&\leq \beta \cdot 0 = 0,
\end{aligned}$$

which shows that $z = w$. Hence the result follows. □

Theorem 2.2 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}'$-contraction. Then there exist $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0,1]$ such that the following statements hold*:

(*i*) *the n-fold averaged mapping $\bar{T}$ associated with the mapping T has a unique fixed point*;

(*ii*) *for any $x_0 \in X$, Kirk's iteration $\{x_n\}$ given by $x_n = \bar{T}x_{n-1}$ converges to the unique fixed point of $\bar{T}$.*

Proof As the mapping T is a weak enriched $\mathcal{F}'$-contraction, there are $a_i \geq 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$, satisfying inequality (2). Define $\alpha_1 = \frac{1}{\sum_{i=1}^{k} a_i + 1} > 0$ and $\alpha_r = \frac{a_r}{\sum_{i=1}^{k} a_i + 1} \geq 0$, $r = 2,3,\dots,k$. Then inequality (2) becomes

$$\begin{aligned}
&\left\| \left(\frac{1-\alpha_2-\alpha_3-\cdots-\alpha_k}{\alpha_1} - 1 \right)(x-y) + Tx - Ty + \frac{\alpha_2}{\alpha_1}\left(T^2x - T^2y\right) + \cdots \right. \\
&\quad \left. + \frac{\alpha_k}{\alpha_1}\left(T^kx - T^ky\right) \right\| \\
&\leq f\left(\frac{1}{\alpha_1}\|x-y\|, \left\| \frac{1}{\alpha_1}(y-x) + (x-Tx) + \frac{\alpha_2}{\alpha_1}\left(x-T^2x\right) + \frac{\alpha_3}{\alpha_1}\left(x-T^3x\right) + \cdots \right.\right. \\
&\quad \left. + \frac{\alpha_k}{\alpha_1}\left(x-T^kx\right) \right\|, \\
&\quad \left.\left\| \frac{1}{\alpha_1}(x-y) + (y-Ty) + \frac{\alpha_2}{\alpha_1}\left(y-T^2y\right) + \frac{\alpha_3}{\alpha_1}\left(y-T^3y\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(y-T^ky\right) \right\| \right)
\end{aligned}$$

for all $x, y \in X$. By $\alpha_1 > 0$ and $\mathcal{F}'_3$ the above inequality can be rewritten as follows:

$$\begin{aligned}
&\big\| (1-\alpha_1-\alpha_2-\alpha_3-\cdots-\alpha_k)(x-y) \\
&\quad + \alpha_1(Tx-Ty) + \alpha_2\left(T^2x-T^2y\right) + \cdots + \alpha_k\left(T^kx-T^ky\right) \big\| \\
&\leq \alpha_1 f\left(\frac{1}{\alpha_1}\|x-y\|, \right. \\
&\quad \left\| (y-x) + \alpha_1(x-Tx) + \frac{\alpha_2}{\alpha_1}\left(x-T^2x\right) + \frac{\alpha_3}{\alpha_1}\left(x-T^3x\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(x-T^kx\right) \right\|, \\
&\quad \left. \left\| (x-y) + \alpha_1(y-Ty) + \frac{\alpha_2}{\alpha_1}\left(y-T^2y\right) + \frac{\alpha_3}{\alpha_1}\left(y-T^3y\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(y-T^ky\right) \right\| \right)
\end{aligned}$$

$$\begin{aligned}&\le f\big(\|x-y\|,\\ &\quad \big\|(y-x)+\alpha_1(x-Tx)+\alpha_2\big(x-T^2x\big)+\alpha_3\big(x-T^3x\big)+\cdots+\alpha_k\big(x-T^kx\big)\big\|,\\ &\quad \big\|(x-y)+\alpha_1(y-Ty)+\alpha_2\big(y-T^2y\big)+\alpha_3\big(y-T^3y\big)+\cdots+\alpha_k\big(y-T^ky\big)\big\|\big),\end{aligned}$$

which, together with Definition 2.1, implies that

$$\|\bar{T}x-\bar{T}y\|\le f\big(\|x-y\|,\|y-\bar{T}x\|,\|x-\bar{T}y\|\big) \tag{10}$$

for all $x, y \in X$.

Let x_0 be an arbitrary element in X. Define $x_n = \bar{T}^n x_0$ for $n \ge 1$, $n \in \mathbb{N}$. Taking $x = x_n$, $y = x_{n-1}$ in (10), by $\mathcal{F}'_4$ we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le f\big(\|x_n-x_{n-1}\|,\|x_n-x_n\|,\|x_{n-1}-x_{n+1}\|\big)\\ &\le f\big(\|x_n-x_{n-1}\|,\|x_n-x_n\|,\|x_{n-1}-x_n\|+\|x_n-x_{n+1}\|\big).\end{aligned}$$

By $\mathcal{F}'_2$ there exits $\beta \in [0,1)$ such that

$$\|x_{n+1}-x_n\|\le\beta\|x_n-x_{n-1}\|.$$

By repeating this process we obtain that

$$\|x_{n+1}-x_n\|\le\beta^n\|x_1-x_0\|.$$

Now for all $m, n \ge 1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \|x_{n+m}-x_{n+m-1}\|+\|x_{n+m-1}-x_{n+m-2}\|+\cdots+\|x_{n+1}-x_n\|\\ &\le \big(\beta^{n+m-1}+\beta^{n+m-2}+\cdots+\beta^n\big)\|x_1-x_0\|\\ &= \frac{\beta^n(1-\beta^m)}{1-\beta}\|x_1-x_0\|,\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now taking $x = x^*$, $y = x_n$ in inequality (10), we have

$$\big\|\bar{T}x^*-\bar{T}x_n\big\|\le f\big(\big\|x^*-x_n\big\|,\big\|x^*-\bar{T}x_n\big\|,\big\|x_n-\bar{T}x^*\big\|\big).$$

Taking the limit as $n \to \infty$ in this inequality, by $\mathcal{F}'_1$ and $\mathcal{F}'_2$ we obtain that

$$\begin{aligned}\big\|\bar{T}x^*-x^*\big\| &\le f\big(\big\|x^*-x^*\big\|,\big\|x^*-x^*\big\|,\big\|x^*-\bar{T}x^*\big\|\big)\\ &\le\beta\big\|x^*-x^*\big\|=0\end{aligned}$$

and $\bar{T}x^* = x^*$.

Finally, we assume that $\bar{T}$ has fixed points w and z with $w \ne z$. Then

$$\|w-z\|=\|\bar{T}w-\bar{T}z\|$$

$$\leq f\big(\|w-z\|, \|w-\bar{T}z\|, \|z-\bar{T}w\|\big)$$
$$= f\big(\|w-z\|, \|w-z\|, \|z-w\|\big),$$

which by $\mathcal{F}'_5$ gives that $\|z-w\| = 0$. Therefore $\bar{T}$ has a unique fixed point. □

Remark 2.3 If we choose $f(r,s,t) = \alpha r$, $\alpha \in [0,1)$, and set $a_i = 0$ for $i = 3,4,\ldots,k$ in Theorem 2.1 or 2.2, then we get Theorem 2.3 in [27].

Choosing $f(r,s,t) = \alpha(s+t)$, where $\alpha \in [0,1/2)$; $f(r,s,t) = kr + l(s+t)$, where $0 \leq k, l < 1$, $k+2l < 1$; $f(r,s,t) = r^{\alpha}t^{1-\alpha}$, where $\alpha \in (0,1)$; $f(r,s,t) = r^{\alpha}s^{\beta}t^{1-\alpha-\beta}$, where $\alpha, \beta \in (0,1)$ and $0 < \alpha+\beta < 1$ in Theorem 2.1; and $f(r,s,t) = \alpha(s+t)$, where $\alpha \in [0,1/2)$ in Theorem 2.2 and $a_i = 0$ for $i = 3,4,\ldots,k$, we obtain the fixed point theorems for weak enriched Kannan/Ćirić–Reich–Rus/interpolative Kannan/interpolative Ćirić–Reich–Rus/Chatterjea-contractions given in the following corollary.

Corollary 2.1 *Let X be a Banach space, and let T be a weak enriched Kannan-contraction (or Ćirić–Reich–Rus-contraction, Chatterjea-contraction, interpolative Kannan-contraction, interpolative Ćirić–Reich–Rus-contraction). Then there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ with $\alpha_1 + \alpha_2 \in (0,1]$ such that the following statements hold*:

(*i*) *the 2-fold averaged mapping T_{α_1,α_2} has a unique fixed point*;

(*ii*) *for any $x_0 \in X$, Kirk's iteration $\{x_n\}$ given by $x_n = T_{\alpha_1,\alpha_2}x_{n-1}$, that is,*

$$x_n = (1-\alpha_1-\alpha_2)x_{n-1} + \alpha_1 Tx_{n-1} + \alpha_2 T^2 x_{n-1} \tag{11}$$

for $n \in \mathbb{N}$, converges to the unique fixed point of T_{α_1,α_2}.

Remark 2.4 Setting $\alpha_2 = 0$ in Corollary 2.1, we obtain the fixed point theorems corresponding to enriched Kannan/Chatterjea/Ćirić–Reich–Rus/interpolative Kannan-contraction introduced in [22–24] and [26].

Now we need the following definitions and notations.

Definition 2.12 [31] Let T be a self-mapping on a normed space $(X, \|\cdot\|)$, and let A be a bounded subset of X. The diameter of a set A is denoted by $\Lambda[A]$ and defined as $\sup\{\|x-y\| : x,y \in A\}$.

A normed space $(X, \|\cdot\|)$ is said to be a T-orbital Banach space if every Cauchy sequence in $O(T,x,\infty)$ for some $x \in X$ converges in X.

We now prove the following lemmas for the class of weak enriched $\mathcal{F}$-contractions (resp., weak enriched $\mathcal{F}'$-contractions).

Lemma 2.1 *Let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contractions) on a normed space $(X, \|\cdot\|)$, and let n be a positive integer. Suppose that the following assumption holds*:

(*Q*) *for each weak enriched $\mathcal{F}$-contraction, there exists $c \in [0,1)$ such that*

$$f\left(\left(\sum_{i=1}^{k} a_i + 1\right)\|x-y\|, \left\|(x-Tx) + a_2\left(x-T^2x\right) + \cdots + a_k\left(x-T^kx\right)\right\|,\right. \tag{12}$$

$$
\begin{aligned}
&\left\|(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|\Bigg) \\
&\le c\max\Bigg\{\left(\sum_{i=1}^k a_i+1\right)\|x-y\|, \\
&\left\|(x-Tx)+a_2\left(x-T^2x\right)+\cdots+a_k\left(x-T^kx\right)\right\|, \\
&\left\|(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|, \\
&\left\|\left(\sum_{i=1}^k a_i+1\right)(y-x)+(x-Tx)+a_2\left(x-T^2x\right)+\cdots+a_k\left(x-T^kx\right)\right\|, \\
&\left\|\left(\sum_{i=1}^k a_i+1\right)(x-y)+(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|\Bigg\}
\end{aligned}
$$

or

(Q') *for each weak enriched $\mathcal{F}'$-contraction, there exists $c \in [0,1)$ such that*

$$
\begin{aligned}
&f\Bigg(\left(\sum_{i=1}^k a_i+1\right)\|x-y\|, \\
&\quad\left\|\left(\sum_{i=1}^k a_i+1\right)(y-x)+(x-Tx)+a_2\left(x-T^2x\right)+\cdots+a_k\left(x-T^kx\right)\right\|, \\
&\quad\left\|\left(\sum_{i=1}^k a_i+1\right)(x-y)+(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|\Bigg) \\
&\le c\max\Bigg\{\left(\sum_{i=1}^k a_i+1\right)\|x-y\|, \left\|(x-Tx)+a_2\left(x-T^2x\right)+\cdots+a_k\left(x-T^kx\right)\right\|, \\
&\quad\left\|(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|, \\
&\quad\left\|\left(\sum_{i=1}^k a_i+1\right)(y-x)+(x-Tx)+a_2\left(x-T^2x\right)+\cdots+a_k\left(x-T^kx\right)\right\|, \\
&\quad\left\|\left(\sum_{i=1}^k a_i+1\right)(x-y)+(y-Ty)+a_2\left(y-T^2y\right)+\cdots+a_k\left(y-T^ky\right)\right\|\Bigg\}
\end{aligned}
\tag{13}
$$

for all $x, y \in X$, $a_i \in (0,\infty)$, $i = 1,2,\dots,k$, $k \ge 3$, $k \in \mathbb{N}$.

Then there exist $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \ge 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^k \alpha_i \in (0,1]$ such that for each $x \in X$ and for all s, t in $\{1,2,\dots,n\}$, we have

$$
\left\|\bar{T}^s x-\bar{T}^t x\right\| \le c\Lambda\left[O(\bar{T},x,n)\right],
$$

where $\bar{T}$ is the k-fold averaged mapping associated with the weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contractions).

Proof Since T is a weak enriched $\mathcal{F}$-contraction, there exist $a_i \in (0,\infty)$, $i = 1,2,\dots,k$, $k \ge 3$, $k \in \mathbb{N}$, satisfying inequality (1). Define $\alpha_1 = \frac{1}{\sum_{i=1}^k a_i+1} > 0$ and $\alpha_r = \frac{a_r}{\sum_{i=1}^k a_i+1} \ge 0$,

$r = 2, 3, \ldots, k$. Note that for all $x, y \in X$, inequality (1) reduces to

$$\begin{aligned}
&\left\| \left(\frac{1 - \alpha_2 - \alpha_3 - \cdots - \alpha_k}{\alpha_1} - 1 \right)(x - y) + Tx - Ty + \frac{\alpha_2}{\alpha_1}\left(T^2x - T^2y\right) + \cdots \right. \\
&\qquad \left. + \frac{\alpha_k}{\alpha_1}\left(T^kx - T^ky\right) \right\| \\
&\quad \le f\left(\frac{1}{\alpha_1}\|x - y\|, \left\| (x - Tx) + \frac{\alpha_2}{\alpha_1}\left(x - T^2x\right) + \frac{\alpha_3}{\alpha_1}\left(x - T^3x\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(x - T^kx\right) \right\|, \right. \\
&\qquad \left. \left\| (y - Ty) + \frac{\alpha_2}{\alpha_1}\left(y - T^2y\right) + \frac{\alpha_3}{\alpha_1}\left(y - T^3y\right) + \cdots + \frac{\alpha_k}{\alpha_1}\left(y - T^ky\right) \right\| \right).
\end{aligned}$$

Together with assumption (Q), this inequality can be written as

$$\begin{aligned}
\|\bar{T}x - \bar{T}y\| &\le f\left(\|x - y\|, \|x - \bar{T}x\|, \|y - \bar{T}\|\right) \\
&\le c \max\left\{\|x - y\|, \|x - \bar{T}x\|, \|y - \bar{T}\|, \|x - \bar{T}y\|, \|y - \bar{T}x\|\right\}.
\end{aligned} \tag{14}$$

Let $x \in X$ be arbitrary, and let n be a fixed positive integer. From (14) we have

$$\begin{aligned}
\left\|\bar{T}^sx - \bar{T}^tx\right\| &= \left\|\bar{T}\bar{T}^{s-1}x - \bar{T}\bar{T}^{t-1}x\right\| \\
&\le c \max\left\{ \left\|\bar{T}^{s-1}x - \bar{T}^{t-1}x\right\|, \left\|\bar{T}^{s-1}x - \bar{T}^sx\right\|, \left\|\bar{T}^{t-1}x - \bar{T}^tx\right\|, \right. \\
&\qquad \left. \left\|\bar{T}^{s-1}x - \bar{T}^tx\right\|, \left\|\bar{T}^{t-1}x - \bar{T}^sx\right\| \right\}.
\end{aligned}$$

This implies that $\|\bar{T}^sx - \bar{T}^tx\| \le c\Lambda[O(\bar{T}, x, n)]$.

The same conclusion for weak enriched $\mathcal{F}'$-contraction with assumption (Q') can be drawn by following arguments similar to those above. □

Remark 2.5 It follows from Lemma 2.1 that if T is a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction) and $x \in X$, then for any positive integer n, there exists $r \le n$ such that

$$\left\|x - \bar{T}^rx\right\| = \Lambda\left[O(\bar{T}, x, n)\right].$$

Lemma 2.2 *Let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction) on a normed space $(X, \|\cdot\|)$, and let n be a positive integer. Suppose that there exists $c \in [0, 1)$ such that assumption (Q) (resp., Q') is satisfied. Then there exist $\alpha_i > 0$, $i = 1, 2, \ldots, k$, $k \ge 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^k \alpha_i \in (0, 1]$ such that*

$$\Lambda\left[O(\bar{T}, x, \infty)\right] \le \frac{1}{1 - c}\|x - \bar{T}x\|$$

for all $x \in X$, where $\bar{T}$ is the k-fold averaged mapping associated with the weak enriched $\mathcal{F}$-contraction (weak enriched $\mathcal{F}'$-contraction).

Proof Since T is a weak enriched $\mathcal{F}$-contraction, there exist $a_i \in (0, \infty)$, $i = 1, 2, \ldots, k$, $k \ge 3$, $k \in \mathbb{N}$, satisfying inequality (1). Define $\alpha_1 = \frac{1}{\sum_{i=1}^k a_i + 1} > 0$ and $\alpha_r = \frac{a_r}{\sum_{i=1}^k a_i + 1} \ge 0$, $r = 2, 3, \ldots, k$.

Let x be an arbitrary element in X. As the sequence $\{\Lambda[O(\bar{T},x,n)]\}$ is increasing, we have

$$\Lambda\big[O(\bar{T},x,\infty)\big] = \sup\big\{\Lambda\big[O(\bar{T},x,n)\big] : n \in \mathbb{N}\big\}.$$

Then (14) follows if we show that

$$\Lambda\big[O(\bar{T},x,n)\big] \le \frac{1}{1-c}\|x-\bar{T}x\|, \quad n \in \mathbb{N}.$$

Let n be a positive integer. By Remark 2.5 there exists $\bar{T}^r x \in O(\bar{T},x,n)(1 \le r \le n)$ such that

$$\big\|x-\bar{T}^r x\big\| = \Lambda\big[O(\bar{T},x,n)\big].$$

Using Lemma 2.1 and the triangle inequality, we have

$$\begin{aligned}\big\|x-\bar{T}^r x\big\| &\le \|x-\bar{T}x\| + \big\|\bar{T}x-\bar{T}^r x\big\| \\ &\le \|x-\bar{T}x\| + c\Lambda\big[O(\bar{T},x,n)\big] \\ &= \|x-\bar{T}x\| + c\big\|x-\bar{T}^r x\big\|.\end{aligned}$$

Therefore

$$\Lambda\big[O(\bar{T},x,n)\big] = \big\|x-\bar{T}^r x\big\| \le \frac{1}{1-c}\|x-\bar{T}x\|, \quad \forall n \in \mathbb{N}.$$

The same conclusion for weak enriched $\mathcal{F}'$-contraction with assumption (Q') can be drawn by following arguments similar to those above. □

Theorem 2.3 *Let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction) on a normed space $(X,\|\cdot\|)$. Suppose that there exists $c \in [0,1)$ such that assumption (Q) (resp., Q') is satisfied. Then there exist $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \ge 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^k \alpha_i \in (0,1]$ such that the following statements hold, provided that X is a $\bar{T}$-orbital Banach space:*

(i) the n-fold averaged mapping $\bar{T}$ associated with mapping T has a unique fixed point;

(ii) for any $x_0 \in X$, Kirk's iteration $\{x_n\}$ given by $x_n = \bar{T}x_{n-1}$ converges to the unique fixed point of $\bar{T}$.

Proof Applying arguments similar to those in the proof of Lemma 2.1, for $\alpha_1 = \frac{1}{\sum_{i=1}^k a_i+1} > 0$ and $\alpha_r = \frac{a_r}{\sum_{i=1}^k a_i+1} \ge 0$, $r = 2,3,\dots,k$, we have

$$\|\bar{T}x-\bar{T}x\| \le c\max\big\{\|x-y\|,\|x-\bar{T}x\|,\|y-\bar{T}y\|,\|x-\bar{T}y\|,\|y-\bar{T}x\|\big\}. \tag{15}$$

Let $x_0 \in X$. Define the Kirk iteration $\{x_n\}$ by $x_n = \bar{T}x_{n-1} = \bar{T}^n x_0$, $n \in \mathbb{N}$.

We now show that the sequence of iterates $\{x_n\}$ is a Cauchy sequence. Let n and m $(m < n)$ be positive integers. By Lemma 2.1 we have

$$\begin{aligned}\|x_m-x_n\| &= \big\|\bar{T}^m x_0-\bar{T}^n x_0\big\| \\ &= \big\|\bar{T}\bar{T}^{m-1}x_0-\bar{T}\bar{T}^{n-1}x_0\big\|\end{aligned}$$

$$
\begin{aligned}
&= \left\| \bar{T}x_{m-1} - \bar{T}^{n-m+1}x_{m-1} \right\| \\
&\le c\Lambda\left[O(\bar{T}, x_{m-1}, n-m+1)\right].
\end{aligned}
$$

By Remark 2.5 there exists an integer $p, 1 \le p \le n-m+1$, such that

$$
\|x_{m-1} - x_{m+p-1}\| = \Lambda\left[O(\bar{T}, x_{m-1}, n-m+1)\right].
$$

It follows from Lemma 2.1 that

$$
\begin{aligned}
\|x_{m-1} - x_{m+p-1}\| &= \left\| \bar{T}x_{m-2} - \bar{T}^{p+1}x_{m-2} \right\| \\
&\le c\Lambda\left[O(\bar{T}, x_{m-2}, p+1)\right],
\end{aligned}
$$

which implies that

$$
\|x_{m-1} - x_{m+p-1}\| \le c\Lambda\left[O(\bar{T}, x_{m-2}, n-m+2)\right].
$$

Therefore we conclude that

$$
\|x_m - x_n\| \le c\Lambda\left[O(\bar{T}, x_{m-1}, n-m+1)\right] \le c^2\Lambda\left[O(\bar{T}, x_{m-2}, n-m+2)\right].
$$

Continuing this procedure, we have

$$
\|x_m - x_n\| \le c\Lambda\left[O(\bar{T}, x_{m-1}, n-m+1)\right] \le \cdots \le c^m\Lambda\left[O(\bar{T}, x_0, m)\right].
$$

From Lemma 2.2 we obtain that

$$
\|x_m - x_n\| \le \frac{c^m}{1-c}\|x_0 - \bar{T}x_0\|.
$$

Taking the limit as $n \to \infty$ in this inequality, we see that the sequence $\{x_n\}$ is a Cauchy sequence.

Since X is a $\bar{T}$-orbital Banach space, there exists $x^* \in X$ such that $\lim x_{n n\to\infty} = x^*$.

Note that

$$
\begin{aligned}
\left\|x^* - \bar{T}x^*\right\| &\le \left\|x^* - x_{n+1}\right\| + \left\|x_{n+1} - \bar{T}x^*\right\| \\
&= \left\|x^* - x_{n+1}\right\| + \left\|\bar{T}x_n - \bar{T}x^*\right\| \\
&\le \left\|x^* - x_{n+1}\right\| + c\max\left\{\left\|x_n - x^*\right\|, \|x_n - x_{n+1}\|, \left\|x^* - \bar{T}x^*\right\|, \left\|x^* - x_{n+1}\right\|,\right. \\
&\qquad \left.\left\|x_n - \bar{T}x^*\right\|\right\} \\
&\le \left\|x^* - x_{n+1}\right\| + c\left\{\left\|x_n - x^*\right\| + \|x_n - x_{n+1}\| + \left\|x^* - \bar{T}x^*\right\| + \left\|x^* - x_{n+1}\right\|\right. \\
&\qquad \left.+ \left\|x_n - \bar{T}x^*\right\|\right\}.
\end{aligned}
$$

Hence

$$
\left\|x^* - \bar{T}x^*\right\| \le \frac{1}{1-c}\left\{(1+c)\left\|x^* - x_{n+1}\right\| + c\left\|x_n - x^*\right\| + c\|x_n - x_{n+1}\|\right\}.
$$

Using $\lim_{n\to\infty} x_n = x^*$, we have $\|x^* - \bar{T}x^*\| = 0$, which shows that x^* is a fixed point of $\bar{T}$. The uniqueness of a fixed point follows from (15).

The same conclusion follows for the weak enriched $\mathcal{F}'$-contraction under assumption (Q') by applying the arguments similar to those above. □

Next, we will study the well-posedness and limit shadowing property for each type of weak enriched contractions defined herein.

Theorem 2.4 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction. Then $Fix(\bar{T})$ is well posed.*

Proof It follows from Theorem 2.1 that $\bar{T}$ has a unique fixed point x^* in X.

Suppose that $\lim_{n\to\infty} \|\bar{T}x_n - x_n\| = 0$. Using (9), we have

$$\begin{aligned}\|x_n - x^*\| &\le \|x_n - \bar{T}x_n\| + \|\bar{T}x_n - x^*\| \\ &= \|x_n - \bar{T}x_n\| + \|\bar{T}x_n - \bar{T}x^*\| \\ &\le \|x_n - \bar{T}x_n\| + f\big(\|x_n - x^*\|, \|x_n - \bar{T}x_n\|, \|x^* - \bar{T}x^*\|\big).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \|x_n - x^*\| \le f\Big(\lim_{n\to\infty} \|x_n - x^*\|, 0, 0\Big).$$

By $\mathcal{F}_2$ there exists $\beta \in [0,1)$ such that $\lim_{n\to\infty} \|x_n - x^*\| \le \beta \cdot 0$, which implies that $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and hence the result. □

Theorem 2.5 *Let X be a Banach space. Suppose that T is a weak enriched $\mathcal{F}'$-contraction. Then $Fix(\bar{T})$ is well posed.*

Proof The result follows using arguments similar to those in the proof of Theorem 2.4. □

Theorem 2.6 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction). Then $Fix(\bar{T})$ possesses the limit shadowing property in X.*

Proof It follows from Theorem 2.1 (resp., Theorem 2.2) that $\bar{T}$ has a unique fixed point x^* in X. Then for any $n \in \mathbb{N}$, $\bar{T}^n x^* = x^*$. Suppose that $\lim_{n\to\infty} \|\bar{T}x_n - x_n\| = 0$. Note that

$$\begin{aligned}\|x_n - \bar{T}^n x^*\| &= \|x_n - x^*\| \\ &\le \|x_n - \bar{T}x_n\| + \|\bar{T}x_n - \bar{T}x^*\| \\ &\le \|x_n - \bar{T}x_n\| + f\big(\|x_n - x^*\|, \|x_n - \bar{T}x_n\|, \|x^* - \bar{T}x^*\|\big) \\ &\quad \big(\text{resp.}, f\big(\|x_n - x^*\|, \|x_n - \bar{T}x^*\|, \|x^* - \bar{T}x_n\|\big)\big).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \bar{T}^n x^*\| &\le f\Big(\lim_{n\to\infty} \|x_n - x^*\|, 0, 0\Big) \\ &\quad \Big(\text{resp.}\, f\Big(\lim_{n\to\infty} \|x_n - x^*\|, \lim_{n\to\infty} \|x_n - x^*\|, \lim_{n\to\infty} \|x_n - x^*\|\Big)\Big).\end{aligned}$$

It follows from $\mathcal{F}_2$ (resp., $\mathcal{F}'_5$) that $\lim_{n\to\infty}\|x_n - \bar{T}^n x^*\| = 0$. Hence $Fix(\bar{T})$ possesses the limit shadowing property in X. □

We conclude this section with the following theorem.

Theorem 2.7 *Let X be a Banach space. Suppose that T is a weak enriched $\mathcal{F}$-contraction and satisfies the following assumption:*

$\mathcal{F}_4$: *there exists $\beta \in (0,1)$ such that $f(r,s,t) \le \beta r + t$ for all $r,s,t \in \mathbb{R}_+$.*

Then the fixed point equation $\bar{T}x = x$ is Ulam–Hyers stable.

Proof It follows from Theorem 2.1 that $\bar{T}$ has a unique fixed point x^* in X.

Let $\epsilon > 0$, and let $v^* \in X$ be an ϵ-solution, that is,

$$\left\|v^* - \bar{T}v^*\right\| \le \epsilon.$$

Since $x^* \in X$ and $\|x^* - \bar{T}x^*\| = 0 \le \epsilon$, it follows that x^* is also an ϵ-solution. By $\mathcal{F}_4$ there exists $\beta \in (0,1)$ such that

$$\begin{aligned}
\left\|x^* - v^*\right\| &= \left\|\bar{T}x^* - v^*\right\| \\
&\le \left\|\bar{T}x^* - \bar{T}v^*\right\| + \left\|\bar{T}v^* - v^*\right\| \\
&\le f\left(\left\|x^* - v^*\right\|, \left\|x^* - \bar{T}x^*\right\|, \left\|v^* - \bar{T}v^*\right\|\right) + \left\|\bar{T}v^* - v^*\right\| \\
&= f\left(\left\|x^* - v^*\right\|, 0, \left\|v^* - \bar{T}v^*\right\|\right) + \left\|\bar{T}v^* - v^*\right\| \\
&\le \beta\left\|x^* - v^*\right\| + 2\left\|v^* - \bar{T}v^*\right\| \\
&\le \beta\left\|x^* - v^*\right\| + 2\epsilon,
\end{aligned}$$

which implies that

$$\left\|x^* - v^*\right\| \le K\epsilon,$$

where $K = \frac{2}{1-\beta}$. Hence the result follows. □

Theorem 2.8 *Suppose that T is a weak enriched $\mathcal{F}'$-contraction on a Banach space X and satisfies the following assumption:*

$\mathcal{F}'_6$: *there exists $\beta \in (0,\frac{1}{3})$ such that $f(r,s,r) \le \beta(2r+s)$ for all $r,s,t \in \mathbb{R}_+$.*

Then the fixed point equation $\bar{T}x = x$ is Ulam–Hyers stable.

Proof It follows from Theorem 2.2 that $\bar{T}$ has a unique fixed point x^* in X.

Let $\epsilon > 0$, and let $v^* \in X$ be an ϵ-solution, that is,

$$\left\|v^* - \bar{T}v^*\right\| \le \epsilon.$$

Note that x^* is also an ϵ-solution. By $\mathcal{F}'_6$ there exists $\beta \in (0,\frac{1}{3})$ such that

$$\left\|x^* - v^*\right\| = \left\|\bar{T}x^* - v^*\right\|$$

$$
\begin{aligned}
&\leq \|\bar{T}x^* - \bar{T}v^*\| + \|\bar{T}v^* - v^*\| \\
&\leq f\big(\|x^* - v^*\|, \|x^* - \bar{T}v^*\|, \|v^* - \bar{T}x^*\|\big) + \|\bar{T}v^* - v^*\| \\
&= f\big(\|x^* - v^*\|, \|x^* - \bar{T}v^*\|, \|v^* - x^*\|\big) + \|\bar{T}v^* - v^*\| \\
&\leq \beta\big(2\|x^* - v^*\| + \|x^* - \bar{T}v^*\|\big) + \|v^* - \bar{T}v^*\| \\
&\leq \beta\big(2\|x^* - v^*\| + \big(\|x^* - v^*\| + \|v^* - \bar{T}v^*\|\big)\big) + \|v^* - \bar{T}v^*\| \\
&\leq 3\beta\|x^* - v^*\| + (1+\beta)\epsilon,
\end{aligned}
$$

which implies that

$$
\|x^* - v^*\| \leq K\epsilon,
$$

where $K = \frac{1+\beta}{1-3\beta}$. Hence the result follows. □

3. The equality of *Fix*(T) and *Fix*($\bar{T}$)

Suppose the existence of a fixed point of a k-fold averaged mapping associated with a weak enriched $\mathcal{F}$-contraction mapping T (resp., weak enriched $\mathcal{F}'$-contraction) is known. We will study necessary conditions for the equality between the set of fixed points of the k-fold averaged mapping and of the associated weak enriched contraction.

Let us begin with the following remark, which is known for averaged mappings T_λ and double averaged mappings T_{α_1,α_2}.

Remark 3.1 For a self-mapping T on a normed space X and $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^k \alpha_i \in (0,1]$, the k-fold averaged mapping $\bar{T} : X \to X$ associated with T given by

$$
\bar{T} := (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k
$$

has the property $Fix(T) \subseteq Fix(\bar{T})$.

We now study the conditions guaranteeing the equality of $Fix(T)$ and $Fix(\bar{T})$.

Theorem 3.1 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction* (*resp., $\mathcal{F}'$-contraction*). *Suppose that there exist $\alpha_i > 0$, $i = 1,2,\dots,k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^k \alpha_i \in$* $(0,1]$ *satisfying the following assumption*:

(A_1) *for all $b_i \in (0,1)$, $i = 1,2,\dots,k$, with $\sum_{i=1}^k b_i \in [0,1)$ and $z \in Fix(\bar{T})$,*

$$
\|z - Tz\| \leq \left\| z - \left(1 - \sum_{i=2}^k b_i\right) Tz - b_2 T^2 z - \cdots - b_k T^k z \right\|. \tag{16}
$$

Then $Fix(T) = Fix(\bar{T})$.

Proof It follows from Remark 3.1 that $Fix(T) \subseteq Fix(\bar{T})$. To prove the converse, assume that $Fix(\bar{T}) \neq \emptyset$. Otherwise, the result is obvious. From Theorem 2.1 (resp., Theorem 2.2) we have $Fix(\bar{T}) \neq \emptyset$. If $z \in Fix(\bar{T})$, then there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0, 1]$ such that

$$z = (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)z + \alpha_1 Tz + \alpha_2 T^2 z + \cdots + \alpha_k T^k z.$$

Set $b_i = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$, $i = 1, 2, \dots, k$, in (16) to obtain

$$\begin{aligned}
&\|z - Tz\| \\
&\quad \leq \left\| z - \frac{\alpha_1}{\sum_{i=1}^{k} \alpha_i} Tz - \frac{\alpha_2}{\sum_{i=1}^{k} \alpha_i} T^2 z - \cdots - \frac{\alpha_k}{\sum_{i=1}^{k} \alpha_i} T^k z \right\| \\
&\quad = \frac{1}{\sum_{i=1}^{k} \alpha_i} \| z - (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)z - \alpha_1 Tz - \alpha_2 T^2 z - \cdots - \alpha_k T^k z \| \\
&\quad = \|z - \bar{T}z\| \\
&\quad = 0,
\end{aligned}$$

which implies that $z \in Fix(T)$. Hence $Fix(\bar{T}) = Fix(T)$. □

The following result also guarantees the equality of $Fix(T)$ and $Fix(\bar{T})$.

Theorem 3.2 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction). Suppose that there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0, 1]$ and $\lambda \in [0, 1)$ such that*

(A_2) for all $x \in X$, we have

$$\|\bar{T}x - Tx\| \leq \lambda \|x - Tx\|.$$

Then $Fix(T) = Fix(\bar{T})$.

Proof It follows from Remark 3.1 that $Fix(T) \subseteq Fix(\bar{T})$. From Theorem 2.1 (resp., Theorem 2.2) we have $Fix(\bar{T}) \neq \emptyset$. Now for each $z \in Fix(\bar{T})$,

$$\|z - Tz\| = \|\bar{T}z - Tz\| \leq \lambda \|z - Tz\|$$

implies that $\|z - Tz\| = 0$, that is, $z \in Fix(T)$, that is, $Fix(\bar{T}) \subseteq Fix(T)$. Therefore $Fix(\bar{T}) = Fix(T)$. □

Next, we obtain an approximation of a fixed point of weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction) using Kirk's iteration scheme for $\bar{T}$.

Theorem 3.3 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction). Suppose that (A_1) or (A_2) holds. Then*

(i) T has a unique fixed point in X;

(ii) for any $x_0 \in X$, Kirk's iteration $\{x_n\}$ given by $x_n = \bar{T}x_{n-1}$ converges to a unique fixed point of T.

Proof By Theorem 2.1 (resp., Theorem 2.2) there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0, 1]$ such that $\bar{T}$ defined by

$$\bar{T} := (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)I + \alpha_1 T + \alpha_2 T^2 z + \cdots + \alpha_k T^k$$

has a unique fixed point $x^* \in X$, which can be obtained by Kirk's iteration (6) for any element $x_0 \in X$. As α_i $(i = 1, 2, \dots, k)$ satisfy assumption (A_1) or (A_2), the result follows from Theorem 3.1 or Theorem 3.2. □

Let us now recall the notion of the periodic point property of a self-mapping T defined on a set X.

Definition 3.1 A self-mapping T on X is said to have the periodic point property P if $Fix(T) = Fix(T^n)$ for every $n \in \mathbb{N}$.

Note that $Fix(T) \subset Fix(T^n)$ for every $n \in \mathbb{N}$. However, the converse is not true in general.

Note that the mapping T has the periodic point property P if and only if T_λ has the periodic point property P. Indeed, $Fix(T) = Fix(T_\lambda)$.

Now we study the conditions that guarantee that the self-mapping T satisfying the weak enriched contraction condition has the periodic point property P.

Lemma 3.1 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction). Suppose that there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0, 1]$ and*

(C) for any $\epsilon > 0$, there exist $x, y \in X$ such that $\|x - \bar{T}y\| < \epsilon$ implies that $\|x - T^i y\| < \frac{\epsilon}{i}$, $i = 1, 2, \dots, k$.

Then the fixed point of $\bar{T}$ coincides with the one of T^i $(i = 1, 2, \dots, k, k \geq 3, k \in \mathbb{N})$.

Proof It follows from Theorem 2.1 (resp., Theorem 2.2) that there exist $\alpha_i > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, with $\sum_{i=1}^{k} \alpha_i \in (0, 1]$ such that $\bar{T}$ admits a unique fixed point x^* in X and the iteration $\{x_n\}$ defined by $x_n = \bar{T} x_{n-1}$, $n \in \mathbb{N}$, converges to x^*. Then for any $\frac{\epsilon}{i} > 0$, $i = 1, 2, \dots, k$, $k \geq 3$, $k \in \mathbb{N}$, there exists $N(i) \in \mathbb{N}$ such that for $n(i) \geq N(i)$, we have

$$0 < \left\|x^* - \bar{T} x_{n(i)}\right\| \leq \frac{\epsilon}{i}, \quad i = 1, 2, \dots, k, k \geq 3, k \in \mathbb{N}.$$

From assumption (C), for $n(i) \geq N(i)$, we obtain that

$$\left\|x^* - T^i x_{n(i)}\right\| \leq \frac{\epsilon}{i}, \quad i = 1, 2, \dots, k, k \geq 3, k \in \mathbb{N}.$$

Set $K = \max\{N(1), N(2), \dots, N(k)\}$. For $n > K$, we have

$$\begin{aligned}\left\|x^* - \bar{T} x_n\right\| &= \left\|\sum_{i=1}^{k} \alpha_i \left(x^* - T^i x_n\right)\right\| \\ &\leq \sum_{i=1}^{k} \left\|\alpha_i \left(x^* - T^i x_n\right)\right\|\end{aligned}$$

$$\leq \sum_{i=1}^{k} \alpha_i \frac{\epsilon}{i} \leq \sum_{i=1}^{k} \alpha_i \epsilon = \epsilon.$$

This shows that $\|x^* - T^i x_n\| \to 0$, $i = 1, 2, \ldots, k$, $k \geq 3$, $k \in \mathbb{N}$, by taking the limit in this inequality as $n \to \infty$ and the arbitrariness of ϵ. Hence x^* is also a fixed point of $T^i, i = 1, 2, \ldots, k$, $k \geq 3$, $k \in \mathbb{N}$, which coincides with the fixed point of $\bar{T}$. □

Theorem 3.4 *Let X be a Banach space, and let T be a weak enriched $\mathcal{F}$-contraction (resp., weak enriched $\mathcal{F}'$-contraction). If conditions (A_1) or (A_2) and (C) hold, then T has the periodic property P.*

Proof The conclusion follows from Theorem 3.3 and Lemma 3.1. □

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