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Inverse logarithmic coefficient bounds for starlike functions subordinated to the exponential functions

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Abstract

In recent years, many subclasses of univalent functions, directly or not directly related to the exponential functions, have been introduced and studied. In this paper, we consider the class of $\mathcal{S}_e^*$ for which $zf'(z)/f(z)$ is subordinate to e^z in the open unit disk. The classic concept of Hankel determinant is generalized by replacing the inverse logarithmic coefficient of functions belonging to certain subclasses of univalent functions. In particular, we obtain the best possible bounds for the second Hankel determinant of logarithmic coefficients of inverse starlike functions subordinated to exponential functions. This work may inspire to pay more attention to the coefficient properties with respect to the inverse functions of various classes of univalent functions.

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1 Introduction and definitions

Let $\mathcal{H}(\mathbb{D})$ represent the family of analytic functions defined in the region of unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$, the normalized functions taking the form of

$$f(z) = z + \sum_{s=2}^{\infty} b_s z^s, \quad z \in \mathbb{D}, \tag{1}$$

are belonging to the class $\mathcal{A}$. Assuming also that $\mathcal{S} \subset \mathcal{A}$ be the set of all univalent functions in $\mathbb{D}$. In the theory of univalent functions, the Carathéodory function [1] is well studied. It is holomorphic in $\mathbb{D}$ with positive real part, i.e., $\Re(p(z)) > 0$ and taking the series representation of

$$p(z) = 1 + \sum_{s=1}^{\infty} \mu_s z^s, \quad z \in \mathbb{D}. \tag{2}$$

We denote by $\mathcal{P}$ the set of these functions.

A basic relationship in geometry function theory is subordination. We write $g \prec \widetilde{g}$ to illustrate that g is subordinate to $\widetilde{g}$. It is explained that for given two functions $g, \widetilde{g} \in \mathcal{H}(\mathbb{D})$, a Schwarz function ω is existing such that $g(z) = \widetilde{g}(\omega(z))$ for $z \in \mathbb{D}$. Once $\widetilde{g}$ is univalent in $\mathbb{D}$, then this relation is equivalent to saying that

$$g(z) \prec \widetilde{g}(z), \quad (z \in \mathbb{D}) \quad \Longleftrightarrow \quad g(0) = \widetilde{g}(0) \quad \text{and} \quad g(\mathbb{D}) \subset \widetilde{g}(\mathbb{D}).$$

The three classic classes of univalent functions are $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$, which are known as starlike functions, convex functions and the bounded turning functions. These classes are characterized as

$$\mathcal{C}(q^*) = \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec q^*(z) \right\},$$
$$\mathcal{S}^*(q*) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q^*(z) \right\},$$
$$\mathcal{R}(q^*) = \left\{ f \in \mathcal{A} : f'(z) \prec q^*(z) \right\},$$

with $q^*(z) = \frac{1+z}{1-z}$, which maps $\mathbb{D}$ to the right half plane. By choosing q^* as some other special functions, various interesting subfamilies of $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{R}$ were studied, the interested readers can refer to [2].

Define

$$\mathcal{C}_e := \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec e^z \right\},$$
$$\mathcal{S}_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z \right\},$$
$$\mathcal{R}_e := \left\{ f \in \mathcal{A} : f'(z) \prec e^z \right\}.$$

The class $\mathcal{S}_e^*$ was introduced and studied by Mendiratta et al. [3]. Later, many interesting classes of univalent functions associated the exponential functions were intensively investigated. Cho et al. [4] introduced a class of starlike functions connected with sin function by letting that $q^* = 1 + \sin z$. In [5], the authors considered a subclass of starlike function given by choosing $q^* = \cos z$. Kumar et al. [6] defined a new class of starlike function by taking $q^* = 1 + ze^z$. For univalent functions defined by modified sigmoid functions [7], it uses the functions $q^* = \frac{2}{1+e^{-z}}$. As it can be seen, all these specially chosen functions are closely related with the exponential function.

Let $\iota, n \in \mathbb{N} = \{1, 2, \ldots\}$. Then, the Hankel determinant $H_{\iota,n}(f)$, introduced by Pommerenke [8, 9], for $f \in \mathcal{S}$ in the form of

$$H_{\iota,n}(f) := \begin{vmatrix} b_n & b_{n+1} & \ldots & b_{n+\iota-1} \\ b_{n+1} & b_{n+2} & \ldots & b_{n+\iota} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n+\iota-1} & b_{n+q} & \ldots & b_{n+2\iota-2} \end{vmatrix}. \tag{3}$$

When investigating power series with integral coefficients and singularities, this method has proven to be effective by considering Hankel determinant, see [10]. In recent years, the bounds of $H_{\iota,n}(f)$ for various types of univalent functions has been studied. For example, the absolute bounds of the second Hankel determinant $H_{2,2}(f) = b_2b_4 - b_3^2$ were calculated in [11, 12] for various subsets of univalent functions. However, there are still many unsolved problems in the exact estimation of this determinant, like the family of close-to-convex functions [13]. For the third Hankel determinant, the sharp bound of $|H_{3,1}(f)|$ for convex functions $\mathcal{C}$ was obtained in [14]. For $f \in \mathcal{S}^*$, it is proved that $|H_{3,1}(f)| \leq \frac{4}{9}$ by Kowalczyk et al. [15]. For the bounded turning functions $\mathcal{R}$, the upper bound was calculated to be $\frac{1}{4}$ in [16]. For some subclasses of univalent or non-univalent functions, some interesting results on bounds on the Hankel determinant were also found in the studies like [17–22].

The Bieberbach conjecture is based on logarithmic coefficients γ_s of f, where γ_s is defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{s=1}^{\infty}\gamma_s z^s, \qquad \log 1 = 0. \tag{4}$$

These coefficients are well studied in the theory of univalent functions and it has been proven that

$$\sum_{s=1}^{\infty}|\gamma_s|^2 \leq \frac{\pi^2}{6}, \tag{5}$$

the bound is sharp for the Koebe function, see [23]. Recently, many authors have investigated the logarithmic coefficients related problems for various classes of univalent functions, e.g., [24–28]. But the best upper bounds for $|\gamma_s|$ $(s \geq 3)$ of univalent functions and some of their subfamilies are still open. In 2021, Kowalczyk et al. [29, 30] introduced the Hankel determinant using logarithmic coefficients for the first time, i.e., it replaces b_n as γ_n as entry.

According to 1/4-theorem proposed by Koebe, we know that the inverse function F of f defined in a neighborhood of origin exists with certainty. We may write as

$$F(w) := f^{-1}(w) = w + B_2w^2 + B_3w^3 + \cdots, \quad |w| < \frac{1}{4}. \tag{6}$$

Then the logarithmic coefficient Γ_n of F is given by

$$\log\left(\frac{F(z)}{z}\right) = 2\sum_{s=1}^{\infty}\Gamma_s w^s, \quad |w| < \frac{1}{4}. \tag{7}$$

The logarithmic coefficients of the inverses of univalent functions was studied by Ponnusamy et al. [31].

Motivated by the above works, it seems natural to consider the Hankel determinant with Γ_n replacing b_n, see [32]. Using this idea, we have

$$\mathcal{H}_{2,1}(F_{f^{-1}}/2) = \Gamma_1\Gamma_3 - \Gamma_2^2 \tag{8}$$

and

$$\mathcal{H}_{2,2}(F_{f^{-1}}/2) = \Gamma_2\Gamma_4 - \Gamma_3^2. \tag{9}$$

In [33], it was calculated that

$$\Gamma_1 = -\frac{1}{2}b_2, \tag{10}$$

$$\Gamma_2 = -\frac{1}{2}b_3 + \frac{3}{4}b_2^2, \tag{11}$$

$$\Gamma_3 = -\frac{1}{2}b_4 + 2b_2b_3 - \frac{5}{3}b_2^3, \tag{12}$$

$$\Gamma_4 = -\frac{1}{2}b_5 + \frac{5}{2}b_2b_4 - \frac{15}{2}b_2^2b_3 + \frac{5}{4}b_3^2 + \frac{35}{8}b_2^4. \tag{13}$$

Thus, we have

$$\mathcal{H}_{2,1}(F_{f^{-1}}/2) = \frac{1}{4}\left(b_2b_4 - b_2^2b_3 - b_3^2 + \frac{13}{12}b_2^4\right) \tag{14}$$

and

$$\begin{aligned}\mathcal{H}_{2,2}(F_{f^{-1}}/2) = {} & \frac{145}{288}b_2^6 - \frac{55}{48}b_2^4b_3 + \frac{5}{24}b_2^3b_4 + \frac{11}{16}b_2^2b_3^2 - \frac{5}{8}b_2^2b_5 \\ & + \frac{3}{4}b_2b_3b_4 - \frac{1}{4}b_2^4 + \frac{1}{4}b_3b_5 - \frac{5}{8}b_3^3.\end{aligned} \tag{15}$$

Let $f_\alpha = e^{-i\alpha}f(e^{i\alpha}z)$. It is noted that

$$\mathcal{H}_{2,1}(F_{f_\alpha^{-1}}/2) = e^{4i\alpha}\mathcal{H}_{2,1}(F_{f^{-1}}/2) \tag{16}$$

and

$$\mathcal{H}_{2,2}(F_{f_\alpha^{-1}}/2) = e^{6i\alpha}\mathcal{H}_{2,2}(F_{f^{-1}}/2). \tag{17}$$

Hence, the functional $\phi_f = |\mathcal{H}_{2,1}(F_{f^{-1}}/2)|$ and $\varphi_f = |\mathcal{H}_{2,2}(F_{f^{-1}}/2)|$ are all rotation invariant. Recently, the upper bound of Hankel determinant for the class $\mathcal{S}_e^*$ and $\mathcal{C}_e$ were studied, including [34, 35]. On the inverse coefficient problem for the class $\mathcal{R}_e$, it was investigated in [36]. In this article, we aim to calculate the sharp bounds on $|\mathcal{H}_{2,1}(F_{f^{-1}}/2)|$ and $|\mathcal{H}_{2,2}(F_{f^{-1}}/2)|$ for the class $\mathcal{S}_e^*$.

2 A set of lemmas

We use the following lemmas to obtain our main results.

Lemma 2.1 ([37]) *Assume $p \in \mathcal{P}$ be the form of* (2). *Then*

$$2\mu_2 = \mu_1^2 + \kappa\left(4 - \mu_1^2\right), \tag{18}$$

$$4\mu_3 = \mu_1^3 + 2\left(4 - \mu_1^2\right)\mu_1\kappa - \mu_1\left(4 - \mu_1^2\right)\kappa^2 + 2\left(4 - \mu_1^2\right)\left(1 - |\kappa|^2\right)\delta, \tag{19}$$

$$\begin{aligned}8\mu_4 = {} & \mu_1^4 + \left(4 - \mu_1^2\right)\kappa\left[c_1^2\left(\kappa^2 - 3\kappa + 3\right) + 4\kappa\right] - 4\left(4 - \mu_1^2\right)\left(1 - |\kappa|^2\right) \\ & \times \left[\mu_1(\kappa - 1)\delta + \overline{\kappa}\delta^2 - \left(1 - |\delta|^2\right)\varrho\right],\end{aligned} \tag{20}$$

for some $\kappa, \delta, \varrho \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 2.2 (See [38]) *For given real numbers A, B, C, let*

$$Y(A,B,C) = \max_{z\in\overline{\mathbb{D}}}\left\{\left|A + Bz + Cz^2\right| + 1 - |z|^2\right\}.$$

If $A > 0$ and $C < 0$, then

$$Y(A,B,C) = \begin{cases} 1 - A + \frac{B^2}{4(1+C)}, & \text{if } B^2 \geq -\frac{4AC^3}{1-C^2},\ |B| < 2(1+C), \\ 1 + A + \frac{B^2}{4(1-C)}, & \text{if } B^2 < \min\{4(1-C)^2, -\frac{4AC^3}{1-C^2}\}, \\ R(A,B,C), & \text{otherwise,} \end{cases}$$

where

$$R(A,B,C) = \begin{cases} A + |B| + C, & \text{if } -C(4A + |B|) \leq A|B|, \\ -A + |B| - C, & \text{if } -C(-4A + |B|) \geq A|B|, \\ (A - C)\sqrt{1 - \frac{B^2}{4AC}}, & \text{otherwise.} \end{cases}$$

Lemma 2.3 *Define $\rho : [0,4] \to \mathbb{R}$ by*

$$\rho(t) := h_1(t)\sqrt{h_2(t)},$$

where

$$h_1(t) = 23t^2 - 96t + 576, \qquad h_2(t) = \frac{18 - t}{12 + t}.$$

Then ρ is convex on $[0,4]$.

Proof By observing that

$$\begin{aligned} h_2^{\frac{3}{2}}(t)\rho''(t) &= h_1''(t)\big(h_2(t)\big)^2 + h_1'(t)h_2(t)h_2'(t) - \frac{1}{4}h_1(t)h_2''(t) + \frac{1}{2}h_1h_2(t)h_2''(t)(t) \\ &= \frac{46t^4 + 138t^3 - 19{,}251t^2 - 209{,}088t + 2{,}949{,}696}{(t + 12)^4} \geq 0, \end{aligned}$$

we have $\rho''(t) \geq 0$ on $[0,4]$. The assertion in Lemma 2.3 thus follows. □

3 Main results

We first determine the bounds of $|\mathcal{H}_{2,1}(f^{-1}/2)|$ for $f \in \mathcal{S}_e^*$.

Theorem 3.1 *Let $f \in \mathcal{S}_e^*$. Then*

$$\left|\mathcal{H}_{2,1}\big(f^{-1}/2\big)\right| = \left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \leq \frac{29}{428}.$$

The result is sharp.

Proof Let $f \in \mathcal{S}_e^*$. From [35], we know

$$b_2 = \frac{1}{4}\mu_1, \tag{21}$$

$$b_3 = \frac{1}{3}\left(\frac{1}{2}\mu_2 - \frac{1}{8}\mu_1^2\right), \tag{22}$$

$$b_4 = \frac{1}{4}\left(\frac{1}{2}\mu_3 + \frac{1}{48}\mu_1^3 - \frac{1}{4}\mu_1\mu_2\right), \tag{23}$$

$$b_5 = \frac{1}{5}\left(\frac{1}{2}\mu_4 - \frac{1}{8}\mu_2^2 + \frac{1}{384}\mu_1^4 + \frac{1}{16}\mu_1^2\mu_2 - \frac{1}{4}\mu_1\mu_3\right) \tag{24}$$

for some $p \in \mathcal{P}$ in the form of

$$p(z) = 1 + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \cdots, \quad z \in \mathbb{D}. \tag{25}$$

Using (14), we get

$$\mathcal{H}_{2,1}(f^{-1}/2) = \frac{107}{9216}\mu_1^4 - \frac{7}{384}\mu_1^2\mu_2 + \frac{1}{48}\mu_1\mu_3 - \frac{1}{64}\mu_2^2. \tag{26}$$

By the rotation invariant property for the class $\mathcal{S}_e^*$ and the functional $|\mathcal{H}_{2,1}(f^{-1}/2)|$, we can assume that $\mu_1 = \mu \in [0,2]$. Using (18) and (19) to express μ_2 and μ_3, we obtain

$$\begin{aligned}\left|\mathcal{H}_{2,1}(f^{-1}/2)\right| = \bigg| &\frac{35}{9216}\mu^4 - \frac{5}{768}\mu^2(4-\mu^2)\kappa - \frac{1}{768}(4-\mu^2)(12+\mu^2)\kappa^2 \\ &+ \frac{1}{96}\mu(4-\mu^2)(1-|\kappa|^2)\delta\bigg|\end{aligned}$$

for some $\kappa, \delta \in \overline{\mathbb{D}}$.

When $\mu = 0$, it is clear that $|\mathcal{H}_{2,1}(f^{-1}/2)| \leq \frac{1}{16} \approx 0.0625$. If $\mu = 2$, we have $|\mathcal{H}_{2,1}(f^{-1}/2)| = \frac{35}{576} \approx 0.0608$. For the case of $\mu \in (0,2)$, using $|\delta| \leq 1$, we get that

$$\begin{aligned}\left|\mathcal{H}_{2,1}(f^{-1}/2)\right| &\leq \frac{\mu(4-\mu^2)}{96}\left(\left|\frac{35\mu^3}{96(4-\mu^2)} - \frac{5}{8}\mu\kappa - \frac{12+\mu^2}{8\mu}\kappa^2\right| + 1 - |\kappa|^2\right) \\ &=: \frac{\mu(4-\mu^2)}{96}\Psi(A,B,C),\end{aligned}$$

where

$$\Psi(A,B,C) = \left|A + B\kappa + C\kappa^2\right| + 1 - |\kappa|^2, \tag{27}$$

with

$$A = \frac{35\mu^3}{96(4-\mu^2)}, \qquad B = -\frac{5}{8}\mu, \qquad C = -\frac{12+\mu^2}{8\mu}. \tag{28}$$

Obviously, $A > 0$, $C < 0$ and we can apply Lemma 2.2 to find the maximum of Ψ.

It is easy to be verified that $|B| \geq 2(1+C)$ and $B^2 \geq 4(1-C)^2$ and thus we only need to consider the cases of $R(A,B,C)$.

Noting that $-C(4A + |B|) \leq A|B|$ is equivalent to

$$\frac{5(-19\mu^4 + 240\mu^2 + 576)}{768(4-\mu^2)} \leq 0, \tag{29}$$

which is impossible to hold for all $\mu \in (0,2)$. Hence, it is left to check the condition $-C(-4A+|B|) \geq A|B|$.

Let $\mu_0 = \sqrt{\frac{16\sqrt{39}-72}{25}} \approx 1.0568$ be the only positive root of the equation $-25\mu^4 - 144\mu^2 + 192 = 0$. For $\mu \in (0,\mu_0]$, we have

$$-C\bigl(-4A+|B|\bigr) - A|B| = \frac{5(-25\mu^4 - 144\mu^2 + 192)}{256(4-\mu^2)} \geq 0. \tag{30}$$

Then $\Psi(A,B,C) \leq (-A+|B|-C)$ and thus

$$\begin{aligned}\bigl|\mathcal{H}_{2,1}(f^{-1}/2)\bigr| &\leq \frac{\mu(4-\mu^2)}{96}\bigl(-A+|B|-C\bigr)\\ &= \frac{-107\mu^4 + 144\mu^2 + 576}{9216}\\ &=: \varrho_1(\mu).\end{aligned}$$

It is an elementary work to get that ϱ_1 attains its maximum value $\frac{29}{428}$ at $\mu_1 = \frac{6\sqrt{214}}{107} \approx 0.8203$. Therefore, we obtain $|\mathcal{H}_{2,1}(f^{-1}/2)| \leq \frac{29}{428} \approx 0.0678$ when $\mu \in (0,\mu_0]$.

If $\mu \in (\mu_0,2)$, then $\Psi(A,B,C) \leq (A-C)\sqrt{1-\frac{B^2}{4AC}}$ and we have

$$\begin{aligned}\bigl|\mathcal{H}_{2,1}(f^{-1}/2)\bigr| &\leq \frac{\mu(4-\mu^2)}{96}(A-C)\sqrt{1-\frac{B^2}{4AC}}\\ &= \frac{23\mu^4 - 96\mu^2 + 576}{32{,}256}\sqrt{\frac{14(18-\mu^2)}{12+\mu^2}}\\ &=: \frac{1}{32{,}256}\varrho_2\bigl(\mu^2\bigr),\end{aligned}$$

where

$$\varrho_2(t) = \bigl(23t^2 - 96t + 576\bigr)\sqrt{\frac{14(18-t)}{12+t}}, \quad t \in \bigl(\mu_0^2, 4\bigr). \tag{31}$$

From Lemma 2.3, we know ϱ_2 is convex on $[\mu_0^2, 4]$. Hence, we obtain

$$\varrho_2(t) \leq \max\bigl\{\varrho_2\bigl(\mu_0^2\bigr), 4\bigr\} = \varrho_2\bigl(\mu_0^2\bigr). \tag{32}$$

It follows that

$$\bigl|\mathcal{H}_{2,1}(f^{-1}/2)\bigr| \leq \frac{1}{32{,}256}\varrho_2\bigl(\mu_0^2\bigr) \approx 0.0655. \tag{33}$$

Combining all the above, we conclude that

$$\bigl|\mathcal{H}_{2,1}(f^{-1}/2)\bigr| \leq \frac{29}{428} \approx 0.0678.$$

The equality is achieved by the extremal function given by

$$f(z) = z\exp\biggl(\int_0^z \frac{e^{\omega(t)}-1}{t}\,dt\biggr), \tag{34}$$

with

$$\omega(z)=\frac{q(z)-1}{q(z)+1}, \tag{35}$$

and

$$q(z)=\frac{1+\frac{6\sqrt{214}}{107}z+z^2}{1-z^2}. \tag{36}$$

This completes the proof of Theorem 3.1. □

Now we will calculate the upper bounds of $|\mathcal{H}_{2,2}(f^{-1}/2)|$ for the class $\mathcal{S}_e^*$.

Theorem 3.2 *Let $f\in\mathcal{S}_e^*$. Then*

$$\left|\mathcal{H}_{2,2}\left(f^{-1}/2\right)\right|=\left|\Gamma_2\Gamma_4-\Gamma_3^2\right|\leq\frac{9}{192}.$$

This result is the best possible.

Proof By putting (22), (23), and (24) with $\mu_1=\mu$ into (15), we obtain

$$\begin{aligned}\mathcal{H}_{2,2}\left(f^{-1}/2\right)=\frac{1}{2{,}654{,}208}&\left(9733\mu^6-31{,}020\mu^4\mu_2+20{,}736\mu_2\mu_4\right.\\&-25{,}920\mu_2^3+16{,}560\mu^2\mu_2^2+35{,}712\mu\mu_2\mu_3\\&\left.-25{,}920\mu^2\mu_4+18{,}336\mu^3\mu_3-18{,}432\mu_3^2\right).\end{aligned} \tag{37}$$

Using $\lambda=4-\mu^2$ in (18), (19), and (20) of Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{H}_{2,2}\left(f^{-1}/2\right)=\frac{1}{2{,}654{,}208}&\left\{1075\mu^6-1944\mu^4\kappa^3\lambda-912\mu^4\kappa^2\lambda\right.\\&+144\mu^2\kappa^4\lambda^2-3744\mu^2\kappa^3\lambda^2+2628\mu^2\kappa^2\lambda^2-7776\mu^2\kappa^2\lambda\\&-3534\mu^4\kappa\lambda+5184\kappa^3\lambda^2-3240\kappa^3\lambda^3+4896\mu\kappa\lambda^2\left(1-|\kappa|^2\right)\delta\\&-576\mu\kappa^2\lambda^2\left(1-|\kappa|^2\right)\delta+7776\mu^3\kappa\lambda\left(1-|\kappa|^2\right)\delta\\&+5712\mu^3\lambda\left(1-|\kappa|^2\right)\delta-4608\lambda^2\left(1-|\kappa|^2\right)^2\delta^2\\&-5184\lambda^2|\kappa|^2\left(1-|\kappa|^2\right)\delta^2+7776\mu^2\lambda\overline{\kappa}\left(1-|\kappa|^2\right)\delta^2\\&+5184\kappa\lambda^2\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho\\&\left.-7776\mu^2\lambda\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho\right\}.\end{aligned}$$

We note that it can be written in the form of

$$H_{2,3}(f)=\frac{1}{2{,}654{,}208}\left[\zeta_1(\mu,\kappa)+\zeta_2(\mu,\kappa)\delta+\zeta_3(\mu,\kappa)\delta^2+\Phi(\mu,\kappa,\delta)\varrho\right].$$

Here $\rho, \kappa, \delta \in \overline{\mathbb{D}}$ and

$$\begin{aligned}
\zeta_1(\mu,\kappa) &= 1075\mu^6 + (4-\mu^2)\big[144\mu^2(4-\mu^2)\kappa^4 - 288(5\mu^4 - 20\mu^2 + 108)\kappa^3 \\
&\quad + 12\mu^2(228 - 295\mu^2)\kappa^2 - 3534\mu^4\kappa\big], \\
\zeta_2(\mu,\kappa) &= 48\mu(4-\mu^2)(1-|\kappa|^2)\big[-12(4-\mu^2)\kappa^2 + (60\mu^2 + 408)\kappa + 119\mu^2\big], \\
\zeta_3(\mu,\kappa) &= 576(4-\mu^2)(1-|\kappa|^2)\left[(4-\mu^2)(-|\kappa|^2 - 8) + \frac{27}{2}\mu^2\overline{\kappa}\right], \\
\Phi(\mu,\kappa,\delta) &= 2592(4-\mu^2)(1-|\kappa|^2)(1-|\delta|^2)\big[3\mu^2 + 2\kappa(4-\mu^2)\big].
\end{aligned}$$

By taking $|\kappa| = x$, $|\delta| = y$ along with $|\varrho| \leq 1$, we obtain

$$\begin{aligned}
\left|\mathcal{H}_{2,2}(f^{-1}/2)\right| &\leq \frac{1}{2{,}654{,}208}\big[\left|\zeta_1(\mu,x)\right| + \left|\zeta_2(\mu,x)\right|y + \left|\zeta_3(\mu,x)\right|y^2 + \left|\Phi(\mu,x,\delta)\right|\big]. \\
&\leq \frac{1}{2{,}654{,}208}\big[\Theta(\mu,x,y)\big],
\end{aligned} \tag{38}$$

where we set

$$\Theta(\mu,x,y) = r_1(\mu,x) + r_2(\mu,x)y + r_3(\mu,x)y^2 + r_4(\mu,x)(1-y^2),$$

with

$$\begin{aligned}
r_1(\mu,x) &= 1075\mu^6 + (4-\mu^2)\big[144\mu^2(4-\mu^2)x^4 + 288(5\mu^4 - 20\mu^2 + 108)\mu^2x^3 \\
&\quad + 12\mu^2\left|288 - 295\mu^2\right|x^2 + 3534\mu^4x\big], \\
r_2(\mu,x) &= 48\mu(4-\mu^2)(1-x^2)\big[12(4-\mu^2)x^2 + (60\mu^2 + 408)x + 119\mu^2\big], \\
r_3(\mu,x) &= 576(4-\mu^2)(1-x^2)\left[(4-\mu^2)(x^2+8) + \frac{27}{2}\mu^2x\right], \\
r_4(\mu,x) &= 2592(4-\mu^2)(1-x^2)\big[3\mu^2 + 2x(4-\mu^2)\big].
\end{aligned}$$

Then all that remains for us is to find the maximum value of Θ in the closed domain defined by $\Omega := [0,2] \times [0,1] \times [0,1]$. In light of $\Theta(0,1,1) = 124{,}416$, it is seen that

$$\max_{(\mu,x,y)\in\Omega}\big\{\Theta(\mu,x,y)\big\} \geq 124{,}416. \tag{39}$$

Now we aim to illustrate that the maximum value of Θ with $(\mu,x,y) \in \Omega$ is equal to 1,224,416.

When $x = 1$, it reduces to

$$\begin{aligned}
\Theta(\mu,1,y) &= 1075\mu^6 + (4-\mu^2)\left(4830\mu^4 - 5184\mu^2 + 31{,}104 + 12\mu^2\left|288 - 295\mu^2\right|\right) \\
&=: \varpi(\mu).
\end{aligned}$$

Let $\epsilon = \sqrt{\frac{295}{288}} \approx 0.9881$. If $\mu \geq \epsilon$, then

$$\varpi(\mu) = -7295\mu^6 + 42{,}120\mu^4 - 65{,}664\mu^2 + 124{,}416, \tag{40}$$

which has a maximum about 110,662.6 at $\mu \approx 1.6624$ on $[\epsilon, 2]$. When $\mu \in [0,\epsilon]$, we obtain

$$\varpi(\mu) = -215\mu^6 + 6888\mu^4 - 38{,}016\mu^2 + 124{,}416, \tag{41}$$

which achieves its maximum 124,416 at $\mu = 0$ on $[0,\epsilon]$. Taking $\mu = 2$, we have $\Theta(2,x,y) \equiv$ 68,800. In these cases, Θ gains a maximum 124,416 and thus we may assume $\mu < 2$ and $x < 1$ in the next discussions.

Let $(\mu,x,y) \in [0,2) \times [0,1) \times (0,1)$. Then

$$\frac{\partial\Theta}{\partial y} = r_2(\mu,x) + 2\big[r_3(\mu,x) - r_4(\mu,x)\big]y. \tag{42}$$

Plugging $\frac{\partial\Theta}{\partial y} = 0$ yields

$$\widehat{y} = \frac{\mu[12(4-\mu^2)x^2 + (60\mu^2 + 408)x + 119\mu^2]}{12(1-x)[2(4-\mu^2)x + 43\mu^2 - 64]}.$$

If $\widehat{y} \in (0,1)$, then we must have the following inequalities:

$$\begin{aligned} &12(2-\mu)(\mu+2)^2x^2 + \big(60\mu^3 + 540\mu^2 + 408\mu - 864\big)x \\ &\quad + 119\mu^3 - 516\mu^2 + 768 < 0, \end{aligned} \tag{43}$$

$$\mu^2 > \frac{8(8-x)}{43-2x} =: h(x). \tag{44}$$

It is not difficult to prove that the inequality in Equation (43) is false for $x \in [\frac{1}{3}, 1)$. Therefore, for the existence of a critical point $(\widehat{\mu},\widehat{x},\widehat{y})$ and $\widehat{y} \in (0,1)$, we must have $\widehat{x} < \frac{1}{3}$. By observing that h is decreasing on $(0,1)$, then $\widehat{\mu}^2 > \frac{46}{31}$. As $\widehat{x} < \frac{1}{3}$, we know

$$r_1(\widehat{\mu},\widehat{x}) \le r_1\left(\widehat{\mu}, \frac{1}{3}\right) =: \phi_1(\widehat{\mu}). \tag{45}$$

Using $1 - \widehat{x}^2 < 1$ and $\widehat{x} < \frac{1}{3}$, we obtain

$$r_j(\widehat{\mu},\widehat{x}) \le \frac{9}{8} r_j\left(\widehat{\mu}, \frac{1}{3}\right) =: \phi_j(\widehat{\mu}) \quad j = 2,3,4. \tag{46}$$

Therefore, we deduce that

$$\Theta(\widehat{\mu},\widehat{x},\widehat{y}) \le \phi_1(\widehat{\mu}) + \phi_4(\widehat{\mu}) + \phi_2(\widehat{\mu})\widehat{y} + \big[\phi_3(\widehat{\mu}) - \phi_4(\widehat{\mu})\big]\widehat{y}^2 =: \Xi(\widehat{\mu},\widehat{y}).$$

As $\phi_3(\widehat{\mu}) - \phi_4(\widehat{\mu}) = 64(4-\widehat{\mu}^2)(184 - 127\widehat{\mu}^2) \le 0$, we see $\frac{\partial^2\Xi}{\partial\widehat{y}^2} \le 0$. Then it is found that

$$\frac{\partial\Theta}{\partial\widehat{y}} \ge \frac{\partial\Theta}{\partial\widehat{y}}\Big|_{\widehat{y}=1} = \phi_2(\widehat{\mu}) + 2\big[\phi_3(\widehat{\mu}) - \phi_4(\widehat{\mu})\big] \ge 0, \quad \widehat{\mu} \in \left(\sqrt{\frac{46}{31}}, 2\right).$$

This means that

$$\Xi(\widehat{\mu},\widehat{y}) \le \Xi(\widehat{\mu},1) = \phi_1(\widehat{\mu}) + \phi_2(\widehat{\mu}) + \phi_3(\widehat{\mu}) =: \chi_0(\widehat{\mu}).$$

Because χ_0 takes a maximum value 107,665.6, we have $\Theta(\widehat{\mu},\widehat{x},\widehat{y}) < 124{,}416$. Therefore, we have to check the cases of $y = 0$ and $y = 1$ to get the maximum point of Θ.

Suppose that $y = 0$. As

$$\Theta(\mu,x,0) = r_1(\mu,x) + r_4(\mu,x) \tag{47}$$

and

$$\Theta(\mu,x,1) = r_1(\mu,x) + r_2(\mu,x) + r_3(\mu,x), \tag{48}$$

we have

$$\begin{aligned}\Theta(\mu,x,1) - \Theta(\mu,x,0) &= r_2(\mu,x) + r_3(\mu,x) - r_4(\mu,x)\\ &= 48\bigl(4 - c^2\bigr)\bigl(1 - x^2\bigr)\Pi(\mu,x),\end{aligned}$$

where

$$\begin{aligned}\Pi(\mu,x) = {}& 12(1+\mu)\bigl(4-\mu^2\bigr)x^2 + 6\bigl(10\mu^3 + 45\mu^2 + 68\mu - 72\bigr)x\\ &+ 119\mu^3 - 258\mu^2 + 384.\end{aligned} \tag{49}$$

It is a tedious basic work to show that $\Pi(\mu,x) \geq 0$ on $[0,2] \times [0,1]$. Thus, we know $\Theta(\mu,x,1) \geq \Theta(\mu,x,0)$ for all $(\mu,x) \in [0,2] \times [0,1]$. Hence, we can only discuss the maximum of Θ in the case of $y = 1$.

Now it is time to find to find the maximum value of Θ on $y = 1$. Actually, when $\mu \in [\epsilon,2)$, we have

$$\begin{aligned}\Theta(\mu,x,1) &= 1075\mu^6 + \bigl(4-\mu^2\bigr)\bigl(m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0\bigr)\\ &= 1075\mu^6 + \bigl(4-\mu^2\bigr)Q(\mu,x)\\ &=: W(\mu,x),\end{aligned}$$

where

$$Q(\mu,x) = m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0 \tag{50}$$

with

$$\begin{aligned}m_4 &= 144\bigl(4-\mu^2\bigr)\bigl(\mu^2 - 4\mu - 4\bigr),\\ m_3 &= 288\bigl(5\mu^4 - 10\mu^3 - 47\mu^2 - 68\mu + 108\bigr),\\ m_2 &= 12\bigl(295\mu^4 - 524\mu^3 + 48\mu^2 + 192\mu - 1344\bigr),\\ m_1 &= 6\mu\bigl((589\mu^3 + 480\mu^2 + 1296\mu + 3264\bigr),\\ m_0 &= 48\bigl(119\mu^3 - 96\mu^2 + 384\bigr).\end{aligned}$$

In view of $[\epsilon,2)\subset[\frac{9}{10},2]$, we next prove that $W<124{,}416$ on the rectangle $R_1:=[\frac{9}{10},2]\times[0,1]$. When $\mu=\frac{9}{10}$, we obtain

$$W\left(\frac{9}{10},x\right)=:\varsigma_1(x). \tag{51}$$

It is calculated that ς_1 attains its maximum about 98,883.6 at $x\approx 0.7837$. When $\mu=2$, we have $W(2,x)\equiv 68{,}800$ for all $x\in[0,1]$. If $x=0$, then

$$\begin{aligned}W(\mu,0)&=1075\mu^6-5712\mu^5+4608\mu^4+22{,}848\mu^3-36{,}864\mu^2+73{,}728\\&=:\varsigma_2(\mu).\end{aligned} \tag{52}$$

It is observed that ς_2 achieves its maximum 68,800 at $\mu=2$ on $[\frac{9}{10},2]$. If $x=1$, then

$$W(\mu,1)=-7295\mu^6+42{,}120\mu^4-65{,}664\mu^2+124{,}416=:\varsigma_3(\mu). \tag{53}$$

We see ς_3 obtains its maximum about 110,662.6 at $\mu\approx 1.6624$ on $[\frac{9}{10},2]$. By numerical calculation, it is noted that the system of the equations

$$\frac{\partial W}{\partial\mu}=0\quad\text{and}\quad\frac{\partial W}{\partial x}=0 \tag{54}$$

has no positive roots in $(\frac{9}{10},2)\times(0,1)$. Thus, there are no critical points of W in the interior of R_1. Based on these facts, we conclude that $W<124{,}416$ when $\mu\in[\epsilon,2)$.

It remains to consider $\mu\in[0,\epsilon)$. In this case, we obtain

$$\begin{aligned}\Theta(\mu,x,1)&=1075\mu^6+\left(4-\mu^2\right)\left(l_4x^4+l_3x^3+l_2x^2+l_1x+l_0\right)\\&=1075\mu^6+\left(4-\mu^2\right)L(\mu,x)\\&=:K(\mu,x),\end{aligned}$$

where

$$L(\mu,x)=l_4x^4+l_3x^3+l_2x^2+l_1x+l_0 \tag{55}$$

with

$$\begin{aligned}l_4&=144\left(4-\mu^2\right)\left(\mu^2-4\mu-4\right),\\l_3&=288\left(5\mu^4-10\mu^3-47\mu^2-68\mu+108\right),\\l_2&=12\left(-295\mu^4-524\mu^3+624\mu^2+192\mu-1344\right),\\l_1&=6\mu(\left(589\mu^3+480\mu^2+1296\mu+3264\right),\\l_0&=48\left(119\mu^3-96\mu^2+384\right).\end{aligned}$$

As $[0,\epsilon)\subset[0,1]$, we next prove that $W\leq 124{,}416$ on the rectangle $R_2:=[0,1]\times[0,1]$. When $\mu=0$, we obtain

$$K(0,x)=-9216x^4+124{,}416x^3-64{,}512x^2+73{,}728:=\sigma_1(x). \tag{56}$$

It is calculated that σ_1 attains its maximum 124,416 at $x = 1$. When $\mu = 2$, we have

$$K(1,x) = -9072x^4 - 10{,}368x^3 - 48{,}492x^2 + 101{,}322x + 59{,}683 =: \sigma_2(x). \tag{57}$$

It is noted that σ_2 has a maximum about 11,197.1 attained at $x \approx 0.7292$. If $x = 0$, then

$$K(\mu,0) = 1075\mu^6 - 5712\mu^5 + 4608\mu^4 + 22{,}848\mu^3 - 36{,}864\mu^2 + 73{,}728 =: \sigma_3(\mu). \tag{58}$$

It is observed that σ_3 achieves its maximum 73,728 at $\mu = 0$ on $[0,1]$. If $x = 1$, then

$$K(\mu,1) = -215\mu^6 + 6888\mu^4 - 38{,}016\mu^2 + 124{,}416 =: \sigma_4(\mu). \tag{59}$$

We see σ_4 obtains its maximum about 124,416 at $\mu = 0$ on $[0,1]$. By numerical calculation, we get that the system of the equations

$$\frac{\partial K}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial K}{\partial x} = 0 \tag{60}$$

has two positive roots in $(0,1) \times (0,1)$, i.e., $(0.0903, 0.0721)$ and $(0.9180, 0.7954)$. The two critical points of K have the values about 73,686.7 and 100,800.0. From these discussions, we get that $K(\mu,x) \leq 124{,}416$ if $\mu \in [0,\epsilon)$. Therefore, we obtain the inequality that $\Theta(\mu,x,y) \leq 124{,}416$ on the whole domain Ω, which leads to

$$\left|\mathcal{H}_{2,2}(f^{-1}/2)\right| \leq \frac{124{,}416}{2{,}654{,}208} = \frac{9}{192} \approx 0.0469.$$

The equality is obtained by the extremal function f given by

$$f(z) = z\exp\left(\int_0^z \frac{e^{t^2}-1}{t}\,dt\right) = z + \frac{1}{2}z^3 + \frac{1}{4}z^5 + \cdots, \quad z \in \mathbb{D}. \tag{61}$$

□

4 Conclusion

In this paper, we consider the Hankel determinant by taking coefficients of logarithmic coefficients of inverse functions for certain subclasses of univalent functions. This is a natural generalization of the classic concept and may help to understand more properties of the inverse functions. We have obtained the sharp bounds for the second Hankel determinant of logarithmic coefficients of inverse functions with respect to the subclass of starlike functions defined by subordination to the exponential functions. Given the importance of the logarithmic coefficients and inverse coefficients of univalent functions, our work provides a new basis for the study of the Hankel determinant. It could also inspire further similar results investigating other subfamilies of univalent functions or taking the bounds of higher-order Hankel determinants.

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Author contributions
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References
1. Carathéodory, C.: BÜber den Variabilitätsbereich der Fourier'schen Konstanten von position harmonischen Funktionen. C. R. Math. **32**(1), 193–217 (1911)
2. Ma, W.C., Minda, D.: A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis, Tianjin. Conf. Proc. Lecture Notes Anal., vol. I, pp. 157–169. International Press, Cambridge (1992)
3. Mendiratta, R., Nagpal, S., Ravichandran, V.: On a subclass of strongly starlike functions associated with exponential function. Bull. Malays. Math. Sci. Soc. **38**(1), 365–386 (2015)
4. Cho, N.E., Kumar, V., Kumar, S.S., Ravichandran, V.: Radius problems for starlike functions associated with the sine function. Bull. Iran. Math. Soc. **45**, 213–232 (2019)
5. Khadija, B., Raza, M.: Starlike functions associated with cosine functions. Bull. Iran. Math. Soc. **47**, 1513–1532 (2021)
6. Kumar, S.S., Kamaljeet, G.: A cardioid domain and starlike functions. Anal. Math. Phys. **11**, 1–34 (2021)
7. Riaz, A., Raza, M., Thomas, D.K.: The third Hankel determinant for starlike functions associated with sigmoid functions. Forum Math. **34**, 137–156 (2022)
8. Pommerenke, C.: On the coefficients and Hankel determinants of univalent functions. Bull. Aust. Math. Soc. **41**(1), 111–122 (1966)
9. Pommerenke, C.: On the Hankel determinants of univalent functions. Mathematika **14**(1), 108–112 (1967)
10. Dienes, P.: The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable, Dover, New York (1957)
11. Bansal, D.: Upper bound of second Hankel determinant for a new class of analytic functions. Appl. Math. Lett. **26**(1), 103–107 (2013)
12. Deniz, E., Cağlar, M., Orhan, H.: Second Hankel determinant for bi-starlike and bi-convex functions of order β. Appl. Math. Comput. **271**, 301–307 (2015)
13. Răducanu, D., Zaprawa, P.: Second Hankel determinant for close-to-convex functions. C. R. Math. Acad. Sci. Paris **355**(10), 1063–1071 (2017)
14. Kowalczyk, B., Lecko, A., Sim, Y.J.: The sharp bound of the Hankel determinant of the third kind for convex functions. Bull. Aust. Math. Soc. **97**, 435–445 (2018)
15. Kowalczyk, B., Lecko, A., Thomas, D.K.: The sharp bound of the third Hankel determinant for starlike functions. Forum Math. **34**(5), 1249–1254 (2022)
16. Kowalczyk, B., Lecko, A.: The sharp bound of the third Hankel determinant for functions of bounded turning. Bol. Soc. Mat. Mexicana **27**(3), 1–13 (2021)
17. Kowalczyk, B., Lecko, A., Sim, Y.J.: The sharp bound of the third Hankel determinant for some classes of analytic functions. Bull. Aust. Math. Soc. **55**, 1859–1868 (2018)
18. Lecko, A., Sim, Y.J., Śmiarowska, B.: The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. Complex Anal. Oper. Theory **13**, 2231–2238 (2019)
19. Arif, M., Rani, L., Raza, M., Zaprawa, P.: Fourth Hankel determinant for the family of functions with bounded turning. Bull. Korean Math. Soc. **55**(6), 1703–1711 (2018)
20. Ullah, K., Srivastava, H.M., Rafiq, A., Arif, M., Arjika, S.: A study of sharp coefficient bounds for a new subfamily of starlike functions. J. Inequal. Appl. **2021**, 194 (2021)
21. Shi, L., Arif, M., Abbas, M., Ihsan, M.: Sharp bounds of Hankel determinant for the inverse functions on a subclass of bounded turning functions. Mediterr. J. Math. **20**, 156 (2023)
22. Wang, Z.G., Hussain, M., Wang, X.Y.: On sharp solutions to majorization and Fekete-Szegö problems for starlike functions. Miskolc Math. Notes **24**, 1003–1019 (2023)
23. Duren, P.L., Leung, Y.J.: Logarithmic coefficients of univalent functions. J. Anal. Math. **36**, 36–43 (1979)
24. Thomas, D.K.: On logarithmic coefficients of close to convex functions. Proc. Am. Math. Soc. **144**, 1681–1687 (2016)

25. Obradović, M., Ponnusamy, S., Wirths, K.J.: Logarithmic coeffcients and a coefficient conjecture for univalent functions. Monatshefte Math. **185**, 489–501 (2018)
26. Vasudevarao, A., Arora, V., Shaji, A.: On the second Hankel determinant of logarithmic coefficients for certain univalent functions. Mediterr. J. Math. **20**, 81 (2023)
27. Adegani, E.A., Motamednezhad, A., Jafari, M., Bulboacă, T.: Logarithmic coefficients inequality for the family of functions convex in one direction. Mathematics **11**(9), 2140 (2023)
28. Adegani, E.A., Alimohammadi, D., Bulboacă, T., Cho, N.E., Bidkham, M.: The logarithmic coefficients for some classes defined by subordination. AIMS Math. **8**(9), 21732–21745 (2023)
29. Kowalczyk, B., Lecko, A.: Second Hankel determinant of logarithmic coefficients of convex and starlike functions. Bull. Aust. Math. Soc. **105**, 458–467 (2021)
30. Kowalczyk, B., Lecko, A.: Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha. Bull. Malays. Math. Sci. Soc. **45**, 727–740 (2022)
31. Ponnusamy, S., Sharma, N.L., Wirths, K.J.: Logarithmic coefficients of the inverse of univalent functions. Results Math. **73**, 160 (2018)
32. Lecko, A., Śmiarowska, B.: Zalcman functional of logarithmic coefficients of inverse functions in certain classes of analytic functions. Anal. Math. Phys. (2022)
33. Guo, D., Tang, H., Li, Z., Xu, Q., Ao, E.: Coefficient problems for a class of univalent functions. Mathematics **11**, 1835 (2023)
34. Zaprawa, P.: Hankel determinants for univalent functions related to the exponential function. Symmetry **11**, 1211 (2019)
35. Shi, L., Arif, M., Iqbal, J., Ullah, K., Ghufran, S.M.: Sharp bounds of Hankel determinant on logarithmic coefficients for functions starlike with exponential function. Fractal Fract. **6**, 645 (2022)
36. Shi, L., Srivastava, H.M., Rafiq, A., Arif, M., Ihsan, M.: Results on Hankel determinants for the inverse of certain analytic functions subordinated to the exponential function. Mathematics **10**, 3429 (2022)
37. Kwon, O.S., Lecko, A., Sim, Y.J.: On the fourth coefficient of functions in the Carathéodory class. Comput. Methods Funct. Theory **18**, 307–314 (2018)
38. Choi, J.H., Kim, Y.C., Sugawa, T.: A general approach to the Fekete-Szegö problem. J. Math. Soc. Jpn. **59**, 707–727 (2007)

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