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A study on reversed dynamic inequalities of Hilbert-type on time scales nabla calculus

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Abstract

In this paper, we establish some reversed dynamic inequalities of Hilbert type on time scales nabla calculus by applying reversed Hölder's inequality, chain rule on time scales, and the mean inequality. As particular cases of our results (when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), we get the reversed form of discrete and continuous inequalities proved by Chang-Jian, Lian-Ying and Cheung (*Math. Slovaca* 61(1):15–28, 2011).

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1 Introduction

David Hilbert proved Hilbert's double-series inequality without exact determination of the constant in his lectures (see [14]) and proved that if $\{a_m\}_{m=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two real sequences such that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \tag{1.1}$$

In 1911, Schur [27] discovered the integral analogue of (1.1), which became known as the Hilbert integral inequality

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy \le \pi\left(\int_0^{\infty} f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^{\infty} g^2(y)\,dy\right)^{\frac{1}{2}} \tag{1.2}$$

for real functions f and g such that $0 < \int_0^{\infty} f^2(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^2(y)\,dy < \infty$. The constant π in (1.1) and (1.2) is the best possible constant factor.

In 1925, by introducing a pair of conjugate exponents (p,q) ($p,q > 1$ with $1/p + 1/q = 1$) Hardy [13] gave an extension of (1.1) as follows. If p, $q > 1$ and a_m, $b_n \ge 0$ are such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1.3}$$

Hardy and Reisz [14] proved the equivalent integral analogue of (1.3)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}} \tag{1.4}$$

for nonnegative functions f and g such that $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(y)\,dy < \infty$. The constant factor $\pi/\sin(\pi/p)$ in (1.3) and (1.4) is the best possible. In 1998, Pachpatte [17] gave a new inequality close to that of Hilbert: Let $a(s)$: $\{0,1,2,\dots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $b(\vartheta)$: $\{0,1,2,\dots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $a(0) = b(0) = 0$. Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{|a_s||b_\vartheta|}{s+\vartheta} \le C(p,q)\left(\sum_{s=1}^{p}(p-s+1)|\nabla a_s|^2\right)^{\frac{1}{2}} \times \left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)|\nabla b_\vartheta|^2\right)^{\frac{1}{2}}, \tag{1.5}$$

where

$$\nabla a_s = a_s - a_{s-1} \quad \text{and} \quad C(p,q) = \frac{1}{2}\sqrt{pq}.$$

In 2002, Kim et al. [15] proved that if λ, $\mu > 1$ and $a(s)$: $\{0,1,2,\dots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $b(\vartheta)$: $\{0,1,2,\dots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $a(0) = b(0) = 0$, then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{q}\frac{|a_s||b_\vartheta|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \le D^*(\lambda,\mu,p,q)\left(\sum_{s=1}^{p}(p-s+1)|\nabla a_s|^\lambda\right)^{\frac{1}{\lambda}} \times \left(\sum_{\vartheta=1}^{q}(q-\vartheta+1)|\nabla b_\vartheta|^\mu\right)^{\frac{1}{\mu}}, \tag{1.6}$$

where

$$\nabla a_s = a_s - a_{s-1} \quad \text{and} \quad D^*(\lambda,\mu,p,q) = \frac{1}{\lambda+\mu}p^{\frac{\lambda-1}{\lambda}}q^{\frac{\mu-1}{\mu}}.$$

Also, Kim and Kim [15] proved the continuous analogue of (1.6): Let λ, $\mu > 1$, and let f and g be real continuous functions on the intervals $(0,x)$ and $(0,y)$, respectively, with $f(0) = g(0) = 0$. Then

$$\int_0^x\int_0^y \frac{|f(s)||g(t)|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\,ds\,dt \le M(\lambda,\mu,x,y)\left(\int_0^x (x-s)\left|f'(s)\right|^\lambda ds\right)^{\frac{1}{\lambda}}\left(\int_0^y (y-t)\left|g'(t)\right|^\mu dt\right)^{\frac{1}{\mu}} \tag{1.7}$$

for $x, y \in (0,\infty)$, where

$$M(\lambda,\mu,x,y) = \frac{1}{\lambda+\mu}x^{\frac{\lambda-1}{\lambda}}y^{\frac{\mu-1}{\mu}}.$$

In 2011, Chang-Jian et al. [8] generalized (1.5) as follows. Let $p_i > 1$ and $1/p_i + 1/p_i^* = 1$, and let $a_i(s_i)$ be real sequences defined for $s_i = 0, 1, 2, \dots, m_i$ such that $a_i(0) = 0$, $i = 1, 2, \dots, n$. Define the operator ∇ by $\nabla a_i(s_i) = a_i(s_i) - a_i(s_i - 1)$ for any function $a_i(s_i)$, $i = 1, 2, \dots, n$. Then

$$\sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} \frac{(n - \sum_{i=1}^{n} \frac{1}{p_i})^{n-\sum_{i=1}^{n} 1/p_i}}{\prod_{i=1}^{n} m_i^{1/p_i^*}} \frac{\prod_{i=1}^{n} |a_i(s_i)|}{(\sum_{i=1}^{n} s_i/p_i^*)^{\sum_{i=1}^{n} 1/p_i^*}} \leq \prod_{i=1}^{n} \left(\sum_{s_i=1}^{m_i} (m_i - s_i + 1) \left| \nabla a_i(s_i) \right|^{p_i} \right)^{\frac{1}{p_i}}. \tag{1.8}$$

Also, the authors of [8] proved that if $h_i \geq 1$ and $p_i > 1$ are constants, $1/p_i + 1/p_i^* = 1$, and $f_i(s_i)$ are real-valued differentiable functions on $[0, x_i)$, where $x_i \in (0, \infty)$, such that $f_i(0) = 0$ for $i = 1, 2, \dots, n$, then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{(n - \sum_{i=1}^{n} \frac{1}{p_i})^{n-\sum_{i=1}^{n} 1/p_i}}{\prod_{i=1}^{n} h_i x_i^{1/p_i^*}} \frac{\prod_{i=1}^{n} |f_i^{h_i}(s_i)|}{(\sum_{i=1}^{n} s_i/p_i^*)^{\sum_{i=1}^{n} 1/p_i^*}} \, ds_n \dots ds_1 \leq \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left| f_i^{h_i - 1}(s_i) . f_i'(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}. \tag{1.9}$$

In the last decades, a new theory has been discovered to unify the continuous and discrete calculi. It is called a time scale theory. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The results in this paper contain the classical continuous and discrete inequalities as particular cases where $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, respectively. In addition, these inequalities can be extended to the corresponding inequalities on various time scales such as $\mathbb{T} = h\mathbb{N}$, $h > 0$, and $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$. Many authors studied the dynamic inequalities on time scales. For more details about the dynamic inequalities on time scales, see [4–6, 16, 18–26, 28–30], and for applications of Hilbert-type inequalities, see [2, 9–11].

The aim of this paper is to determine assumptions for establishing some new reversed forms of inequalities (1.8) and (1.9) on time scales by establishing some new Hilbert-type inequalities on time scales nabla calculus. Our results will be proved by applying the integration by parts, reverse Hölder's inequality on time scales, and the reverse of mean inequality.

The organization of the paper as follows. In Sect. 2, we present some definitions, properties, and some lemmas on time scales needed in Sect. 3, where we prove our results. These results as particular cases where $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$ give the reversed of inequalities (1.8) and (1.9), respectively.

2 Preliminaries and basic lemmas

In 2001, Bohner and Peterson [7] defined the time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Also, they defined the backward jump operator as $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$. For any function $f : \mathbb{T} \to \mathbb{R}$, $f^\rho(\tau)$ denotes $f(\rho(\tau))$. We define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$.

Definition 2.1 ([3]) A function $\lambda : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous if it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$. The space of ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([3]) A function $\lambda : \mathbb{T} \to \mathbb{R}$ is said to be ∇-differentiable at $t \in \mathbb{T}$ if ψ is defined in a neighborhood U of t and there exists a unique real number $\psi^{\nabla}(t)$, called the nabla derivative of ψ at t, such that for each $\epsilon > 0$, there exists a neighborhood N of t with $N \subseteq U$, and

$$\left|\psi\left(\rho(t)\right) - \psi(s) - \psi^{\nabla}(t)\left[\rho(t) - s\right]\right| \leq \epsilon\left|\rho(t) - s\right| \quad \text{for all } s \in N.$$

Lemma 2.1 (*Chain rule for nabla derivative* [12]) *Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $\chi : \mathbb{T} \to \mathbb{R}$ is continuous and nabla differentiable. Then $\psi \circ \chi : \mathbb{T} \to \mathbb{R}$ is nabla differentiable, and there exists d in the real interval $[\rho(t), t]$ such that*

$$(\psi \circ \chi)^{\nabla}(t) = \psi'\left(\chi(d)\right)\chi^{\nabla}(t). \tag{2.1}$$

Definition 2.3 ([3]) A function $\Lambda : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $\psi : \mathbb{T} \to \mathbb{R}$ if $\Lambda^{\nabla}(t) = \psi(t)$ for all $t \in \mathbb{T}$. We then define the nabla integral of ψ by

$$\int_a^t \psi(s)\nabla s = \Lambda(t) - \Lambda(a) \quad \text{for all } t \in \mathbb{T}.$$

Theorem 2.1 ([3]) *Let a, b, $c \in \mathbb{T}$ and α, $\beta \in \mathbb{R}$, and let ψ, $\chi : \mathbb{T} \to \mathbb{R}$ be ld-continuous. Then we have the following properties*:

(1) $\int_a^b [\alpha\psi(t) + \beta\chi(t)]\nabla t = \alpha \int_a^b \psi(t)\nabla t + \beta \int_a^b \chi(t)\nabla t$,
(2) $\int_a^b \psi(t)\nabla t = -\int_b^a \psi(t)\nabla t$,
(3) $\int_a^c \psi(t)\nabla t = \int_a^b \psi(t)\nabla t + \int_b^c \psi(t)\nabla t$,
(4) $|\int_a^b \psi(t)\nabla t| \leq \int_a^b |\psi(t)|\nabla t$,
(5) *If* $\psi(t) \geq 0$ *for all* $t \in [a,b]_{\mathbb{T}}$, *then* $\int_a^b \psi(t)\nabla t \geq 0$,
(6) *If* $\psi(t) \geq \chi(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$, *then* $\int_a^b \psi(t)\nabla t \geq \int_a^b \chi(t)\nabla t$.

Lemma 2.2 (*The integration by parts on time scales* [3]) *If a, $b \in \mathbb{T}$ and $f, g : \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

$$\int_a^b f(t)g^{\nabla}(t)\nabla t = f(t)g(t)|_a^b - \int_a^b f^{\nabla}(t)g^{\rho}(t)\nabla t. \tag{2.2}$$

Lemma 2.3 (*Reverse Hölder's inequality* [1]) *If $a, b \in \mathbb{T}$, $f, g \in C_{ld}(\mathbb{T}, \mathbb{R})$, $\gamma < 0$, and $1/\gamma + 1/\nu = 1$, then*

$$\int_a^b \left|f(\tau)g(\tau)\right|\nabla\tau \geq \left[\int_a^b \left|f(\tau)\right|^{\gamma}\nabla\tau\right]^{\frac{1}{\gamma}}\left[\int_a^b \left|g(\tau)\right|^{\nu}\nabla\tau\right]^{\frac{1}{\nu}}. \tag{2.3}$$

Lemma 2.4 *Let $a_i, b_i \in \mathbb{T}$, and let either $\psi_i \in C_{ld}([a_i, b_i]_{\mathbb{T}}, (-\infty, 0])$ be nonincreasing functions or $\psi_i \in C_{ld}([a_i, b_i]_{\mathbb{T}}, [0, \infty))$ be nondecreasing functions with $\psi_i(a_i) = 0$, $i = 1, 2, \ldots, n$. Then*

$$\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right|\nabla t_i = \left|\psi_i(\xi_i)\right|, \quad \xi_i \in [a_i, b_i]_{\mathbb{T}}. \tag{2.4}$$

Proof Firstly, if $\psi_i \in C_{ld}([a_i, b_i]_{\mathbb{T}}, (-\infty, 0])$ is a nonincreasing function with $\psi_i(a_i) = 0$, then we see that $\psi_i^{\nabla}(t_i) \leq 0$, and then

$$\begin{aligned} \int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i &= -\int_{a_i}^{\xi_i} \psi_i^{\nabla}(t_i) \nabla t_i \\ &= -\left[\psi_i(\xi_i) - \psi_i(a_i)\right] = -\psi_i(\xi_i) = \left|\psi_i(\xi_i)\right|. \end{aligned} \tag{2.5}$$

Secondly, if $\psi_i \in C_{ld}([a_i, b_i]_{\mathbb{T}}, [0, \infty))$ is a nondecreasing function with $\psi_i(a_i) = 0$, then we observe that $\psi_i^{\nabla}(t_i) \geq 0$, and then

$$\begin{aligned} \int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i &= \int_{a_i}^{\xi_i} \psi_i^{\nabla}(t_i) \nabla t_i \\ &= \psi_i(\xi_i) - \psi_i(a_i) = \psi_i(\xi_i) = \left|\psi_i(\xi_i)\right|. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we get

$$\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i = \left|\psi_i(\xi_i)\right|,$$

which is (2.4). The proof is complete. □

Lemma 2.5 (*Mean inequality* [14]) *If α_i, $\beta_i > 0$ for $i = 1, 2, \ldots, n$, then*

$$\prod_{i=1}^{n} \alpha_i^{\beta_i} \leq \frac{(\sum_{i=1}^{n} \alpha_i \beta_i)^{\sum_{i=1}^{n} \beta_i}}{(\sum_{i=1}^{n} \beta_i)^{\sum_{i=1}^{n} \beta_i}}. \tag{2.7}$$

Lemma 2.6 *Let $q_i < 0$ with $1/p_i + 1/q_i = 1$, and let $s_i > 0$, $i = 1, 2, \ldots, n$. Then*

$$\prod_{i=1}^{n} s_i^{1/q_i} \geq \frac{(\sum_{i=1}^{n} s_i/q_i)^{\sum_{i=1}^{n} 1/q_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{(n - \sum_{i=1}^{n} 1/p_i)}}. \tag{2.8}$$

Proof Applying Lemma 2.5 with $\alpha_i = s_i$ and $\beta_i = -1/q_i$, we observe that

$$\prod_{i=1}^{n} s_i^{1/q_i} \geq \left(\frac{\sum_{i=1}^{n} s_i/q_i}{\sum_{i=1}^{n} 1/q_i}\right)^{\sum_{i=1}^{n} 1/q_i}. \tag{2.9}$$

Since $1/q_i = 1 - 1/p_i$, we have that

$$\sum_{i=1}^{n} 1/q_i = \sum_{i=1}^{n} (1 - 1/p_i) = n - \sum_{i=1}^{n} 1/p_i,$$

and then inequality (2.9) becomes

$$\prod_{i=1}^{n} s_i^{1/q_i} \geq \frac{(\sum_{i=1}^{n} s_i/q_i)^{\sum_{i=1}^{n} 1/q_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{n - \sum_{i=1}^{n} 1/p_i}},$$

which is (2.8). The proof is complete. □

3 Main results

Throughout the paper, we will assume that the integrals considered exist.

Now we can state and prove our results.

Theorem 3.1 *Let $a_i, b_i \in \mathbb{T}$, let $0 < p_i < 1$ and $q_i < 0$ be such that $1/p_i + 1/q_i = 1$, and let either $\psi_i \in \mathrm{C}_{ld}([a_i, b_i]_{\mathbb{T}}, (-\infty, 0])$ be nonincreasing functions or $\psi_i \in \mathrm{C}_{ld}([a_i, b_i]_{\mathbb{T}}, [0, \infty))$ be nondecreasing functions with $\psi_i(a_i) = 0$, $i = 1, 2, \ldots, n$. Then*

$$\begin{aligned}\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} &\frac{(n - \sum_{i=1}^{n} 1/p_i)^{n-\sum_{i=1}^{n} 1/p_i}}{\prod_{i=1}^{n} (b_i - a_i)^{\frac{1}{q_i}}} \frac{\prod_{i=1}^{n} |\psi_i(\xi_i)|}{(\sum_{i=1}^{n} (\xi_i - a_i)/q_i)^{\sum_{i=1}^{n} 1/q_i}} \nabla\xi_1 \ldots \nabla\xi_n \\ &\geq \prod_{i=1}^{n} \left(\int_{a_i}^{b_i} \left|\psi_i^{\nabla}(\xi_i)\right|^{p_i} \left[b_i - \rho(\xi_i)\right] \nabla\xi_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{3.1}$$

Proof From (2.4) we have that for $\xi_i \in [a_i, b_i]_{\mathbb{T}}$,

$$\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i = \left|\psi_i(\xi_i)\right|,$$

and then

$$\prod_{i=1}^{n} \left|\psi_i(\xi_i)\right| = \prod_{i=1}^{n} \int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i. \tag{3.2}$$

Applying (2.3) to $\int_{a_i}^{\xi_i} |\psi_i^{\nabla}(t_i)| \nabla t_i$, we observe that

$$\begin{aligned} \int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i &\geq \left(\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right|^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \left(\int_{a_i}^{\xi_i} \nabla t_i \right)^{\frac{1}{q_i}} \\ &= (\xi_i - a_i)^{\frac{1}{q_i}} \left(\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right|^{p_i} \nabla t_i \right)^{\frac{1}{p_i}}, \end{aligned}$$

and then

$$\prod_{i=1}^{n} \int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right| \nabla t_i \geq \prod_{i=1}^{n} (\xi_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right|^{p_i} \nabla t_i \right)^{\frac{1}{p_i}}. \tag{3.3}$$

Substituting (3.3) into (3.2), we see that

$$\prod_{i=1}^{n} \left|\psi_i(\xi_i)\right| \geq \prod_{i=1}^{n} (\xi_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{\xi_i} \left|\psi_i^{\nabla}(t_i)\right|^{p_i} \nabla t_i \right)^{\frac{1}{p_i}}. \tag{3.4}$$

Applying Lemma 2.6 with $s_i = \xi_i - a_i$, we have that

$$\prod_{i=1}^{n} (\xi_i - a_i)^{1/q_i} \geq \frac{(\sum_{i=1}^{n} (\xi_i - a_i)/q_i)^{\sum_{i=1}^{n} 1/q_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{n-\sum_{i=1}^{n} 1/p_i}},$$

and so inequality (3.4) becomes

$$\prod_{i=1}^{n}\left|\psi_i(\xi_i)\right| \geq \frac{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}. \tag{3.5}$$

Multiplying (3.5) by $(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}/(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}$ and integrating over ξ_i from a_i to b_i, $i=1,2,\dots,n$, we observe that

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}\prod_{i=1}^{n}|\psi_i(\xi_i)|}{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}\nabla\xi_1\dots\nabla\xi_n\\
&\quad\geq \int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_1\dots\nabla\xi_n\\
&\quad=\prod_{i=1}^{n}\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i.
\end{aligned}\tag{3.6}$$

Applying (2.3) to $\int_{a_i}^{b_i}(\int_{a_i}^{\xi_i}|\psi_i^{\nabla}(t_i)|^{p_i}\nabla t_i)^{\frac{1}{p_i}}\nabla\xi_i$, we have that

$$\begin{aligned}
\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i &\geq \left(\int_{a_i}^{b_i}\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\nabla\xi_i\right)^{\frac{1}{p_i}}\left(\int_{a_i}^{b_i}\nabla\xi_i\right)^{\frac{1}{q_i}}\\
&= (b_i-a_i)^{\frac{1}{q_i}}\left(\int_{a_i}^{b_i}\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\nabla\xi_i\right)^{\frac{1}{p_i}},
\end{aligned}$$

and then

$$\begin{aligned}
&\prod_{i=1}^{n}\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i\\
&\quad\geq\prod_{i=1}^{n}(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\nabla\xi_i\right)^{\frac{1}{p_i}}.
\end{aligned}\tag{3.7}$$

Substituting (3.7) into (3.6), we see that

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}\prod_{i=1}^{n}|\psi_i(\xi_i)|}{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}\nabla\xi_1\dots\nabla\xi_n\\
&\quad\geq\prod_{i=1}^{n}(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\nabla\xi_i\right)^{\frac{1}{p_i}}.
\end{aligned}\tag{3.8}$$

Applying (2.2) on $\int_{a_i}^{b_i}(\int_{a_i}^{\xi_i}|\psi_i^{\nabla}(t_i)|^{p_i}\nabla t_i)\nabla\xi_i$, we get

$$\begin{aligned}
&\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)\nabla\xi_i\\
&\quad=\left(\int_{a_i}^{\xi_i}\left|\psi_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)g_1(\xi_i)\Big|_{a_i}^{b_i}-\int_{a_i}^{b_i}\left|\psi_i^{\nabla}(\xi_i)\right|^{p_i}g_1^{\rho}(\xi_i)\nabla\xi_i
\end{aligned}$$

$$= \int_{a_i}^{b_i} \left|\psi_i^{\nabla}(\xi_i)\right|^{p_i} \left[b_i - \rho(\xi_i)\right] \nabla \xi_i, \tag{3.9}$$

where $g_1(\xi_i) = \xi_i - b_i$. Substituting (3.9) into (3.8), we obtain

$$\begin{aligned} &\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \frac{(n - \sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i} \prod_{i=1}^n |\psi_i(\xi_i)|}{(\sum_{i=1}^n (\xi_i - a_i)/q_i)^{\sum_{i=1}^n 1/q_i}} \nabla \xi_1 \ldots \nabla \xi_n \\ &\quad \geq \prod_{i=1}^n (b_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_{a_i}^{b_i} \left|\psi_i^{\nabla}(\xi_i)\right|^{p_i} \left[b_i - \rho(\xi_i)\right] \nabla \xi_i \right)^{\frac{1}{p_i}}, \end{aligned} \tag{3.10}$$

and then

$$\begin{aligned} &\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \frac{(n - \sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}}{\prod_{i=1}^n (b_i - a_i)^{\frac{1}{q_i}}} \frac{\prod_{i=1}^n |\psi_i(\xi_i)|}{(\sum_{i=1}^n (\xi_i - a_i)/q_i)^{\sum_{i=1}^n 1/q_i}} \nabla \xi_1 \ldots \nabla \xi_n \\ &\quad \geq \prod_{i=1}^n \left(\int_{a_i}^{b_i} \left|\psi_i^{\nabla}(\xi_i)\right|^{p_i} \left[b_i - \rho(\xi_i)\right] \nabla \xi_i \right)^{\frac{1}{p_i}}, \end{aligned}$$

which is (3.1). The proof is complete. □

Remark 3.1 As a particular case of Theorem 3.1, if $\mathbb{T} = \mathbb{N}_0$, $\rho(\xi_i) = \xi_i - 1$, and $a_i = 0$ for $i = 1, 2, \ldots, n$, then we get the reversed form of (1.8).

Corollary 3.1 *As a particular case of Theorem 3.1, if* $\mathbb{T} = \mathbb{R}$, $\rho(\xi_i) = \xi_i$, $a_i = 0$, $0 < p_i < 1$ *and* $q_i < 0$ *are such that* $1/p_i + 1/q_i = 1$, *and either* ψ_i *is a nonpositive nonincreasing function, or* ψ_i *is a nonnegative nondecreasing function with* $\psi_i(0) = 0$, $i = 1, 2, \ldots, n$, *then*

$$\begin{aligned} &\int_0^{b_n} \cdots \int_0^{b_1} \frac{K \prod_{i=1}^n |\psi_i(\xi_i)|}{(\sum_{i=1}^n \frac{\xi_i}{q_i})^{\sum_{i=1}^n 1/q_i}} d\xi_1 \ldots d\xi_n \\ &\quad \geq \prod_{i=1}^n \left(\int_0^{b_i} \left|\psi_i'(\xi_i)\right|^{p_i} [b_i - \xi_i] \, d\xi_i \right)^{\frac{1}{p_i}}, \end{aligned}$$

where $K = (n - \sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i} / \prod_{i=1}^n b_i^{\frac{1}{q_i}}$.

In the following theorem, we generalize Theorem 3.1 by replacing the function $\psi_i(\xi_i)$ with $\psi_i^{h_i}(\xi_i)$, $h_i \geq 1$.

Theorem 3.2 *Let* $a_i, b_i \in \mathbb{T}$, $0 < h_i \leq 1$, *let* $0 < p_i < 1$ *and* $q_i < 0$ *be such that* $1/p_i + 1/q_i = 1$, *and let* $\psi_i \in \mathrm{C}_{ld}([a_i, b_i]_{\mathbb{T}}, \mathbb{R}^+ \cup \{0\})$ *be increasing functions with* $\psi_i(a_i) = 0$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\begin{aligned} &\int_{a_n}^{b_n} .. \int_{a_1}^{b_1} \frac{(n - \sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}}{\prod_{i=1}^n h_i (b_i - a_i)^{\frac{1}{q_i}}} \frac{\prod_{i=1}^n \psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n (\xi_i - a_i)/q_i)^{\sum_{i=1}^n 1/q_i}} \nabla \xi_1 .. \nabla \xi_n \\ &\quad \geq \prod_{i=1}^n \left(\int_{a_i}^{b_i} (b_i - \rho(\xi_i)) \left(\psi_i^{h_i - 1}(\xi_i) \psi_i^{\nabla}(\xi_i)\right)^{p_i} \nabla \xi_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{3.11}$$

Proof Applying (2.1) to $\psi_i^{h_i}(t_i)$, we get

$$\left[\psi_i^{h_i}(t_i)\right]^\nabla = h_i \psi_i^{h_i-1}(\zeta_i)\psi_i^\nabla(t_i), \tag{3.12}$$

where $\zeta_i \in [\rho(t_i), t_i]$. Since ψ_i is an increasing function, $0 < h_i \le 1$, and $\zeta_i \le t_i$, we have from (3.12) that

$$\left[\psi_i^{h_i}(t_i)\right]^\nabla \ge h_i \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)$$

(note that this statement holds with equality for $h_i = 1$), and then integrating the last inequality over t_i from a_i to ξ_i, where $\psi_i(a_i) = 0$, we observe that

$$\begin{aligned} & h_i \int_{a_i}^{\xi_i} \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\nabla t_i \\ & \quad \le \int_{a_i}^{\xi_i} \left[\psi_i^{h_i}(t_i)\right]^\nabla \nabla t_i = \psi_i^{h_i}(\xi_i) - \psi_i^{h_i}(a_i) = \psi_i^{h_i}(\xi_i). \end{aligned}$$

Thus

$$\prod_{i=1}^n h_i \int_{a_i}^{\xi_i} \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\nabla t_i \le \prod_{i=1}^n \psi_i^{h_i}(\xi_i). \tag{3.13}$$

Applying (2.3) to $\int_{a_i}^{\xi_i} \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\nabla t_i$, we have that

$$\begin{aligned} & \int_{a_i}^{\xi_i} \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\nabla t_i \\ & \quad \ge \left(\int_{a_i}^{\xi_i} \nabla t_i\right)^{\frac{1}{q_i}} \left(\int_{a_i}^{\xi_i} \left(\psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\right)^{p_i} \nabla t_i\right)^{\frac{1}{p_i}} \\ & \quad = (\xi_i - a_i)^{\frac{1}{q_i}} \left(\int_{a_i}^{\xi_i} \left(\psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\right)^{p_i} \nabla t_i\right)^{\frac{1}{p_i}}, \end{aligned}$$

and then

$$\begin{aligned} & \prod_{i=1}^n \int_{a_i}^{\xi_i} \psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\nabla t_i \\ & \quad \ge \prod_{i=1}^n (\xi_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_{a_i}^{\xi_i} \left(\psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\right)^{p_i} \nabla t_i\right)^{\frac{1}{p_i}}. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we get

$$\prod_{i=1}^n \psi_i^{h_i}(\xi_i) \ge \prod_{i=1}^n h_i (\xi_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_{a_i}^{\xi_i} \left(\psi_i^{h_i-1}(t_i)\psi_i^\nabla(t_i)\right)^{p_i} \nabla t_i\right)^{\frac{1}{p_i}}. \tag{3.15}$$

Applying Lemma 2.6 with $s_i = \xi_i - a_i$, we have that

$$\prod_{i=1}^n (\xi_i - a_i)^{1/q_i} \ge \frac{\left(\sum_{i=1}^n (\xi_i - a_i)/q_i\right)^{\sum_{i=1}^n 1/q_i}}{\left(n - \sum_{i=1}^n 1/p_i\right)^{n - \sum_{i=1}^n 1/p_i}}. \tag{3.16}$$

Substituting (3.16) into (3.15), we see that

$$\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i) \geq \frac{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}} \times \prod_{i=1}^{n} h_i\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}. \tag{3.17}$$

Multiplying (3.17) by $(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}/(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}$ and integrating over ξ_i from a_i to b_i, $i=1,2,\dots,n$, we observe that

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}\nabla\xi_1\dots\nabla\xi_n \\
&\quad\geq \int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\prod_{i=1}^{n}h_i\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_1\dots\nabla\xi_n \\
&\quad= \prod_{i=1}^{n}h_i\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i.
\end{aligned} \tag{3.18}$$

Applying (2.3) to $\int_{a_i}^{b_i}(\int_{a_i}^{\xi_i}(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i))^{p_i}\nabla t_i)^{\frac{1}{p_i}}\nabla\xi_i$, we have that

$$\begin{aligned}
&\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i \\
&\quad\geq (b_i-a_i)^{\frac{1}{q_i}}\left(\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)\nabla\xi_i\right)^{\frac{1}{p_i}},
\end{aligned}$$

and then

$$\begin{aligned}
&\prod_{i=1}^{n}\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla\xi_i \\
&\quad\geq \prod_{i=1}^{n}(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)\nabla\xi_i\right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we see that

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^{n}(\xi_i-a_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}\nabla\xi_1\dots\nabla\xi_n \\
&\quad\geq \prod_{i=1}^{n}h_i(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)\nabla\xi_i\right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.20}$$

Applying (2.2) to $\int_{a_i}^{b_i}(\int_{a_i}^{\xi_i}(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i))^{p_i}\nabla t_i)\nabla\xi_i$, we get

$$\begin{aligned}
&\int_{a_i}^{b_i}\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)\nabla\xi_i \\
&\quad = g_2(\xi_i)\left(\int_{a_i}^{\xi_i}\left(\psi_i^{h_i-1}(t_i)\psi_i^{\nabla}(t_i)\right)^{p_i}\nabla t_i\right)\Big|_{a_i}^{b_i} - \int_{a_i}^{b_i} g_2^{\rho}(\xi_i)\left(\psi_i^{h_i-1}(\xi_i)\psi_i^{\nabla}(\xi_i)\right)^{p_i}\nabla\xi_i \\
&\quad = \int_{a_i}^{b_i}\left(b_i-\rho(\xi_i)\right)\left(\psi_i^{h_i-1}(\xi_i)\psi_i^{\nabla}(\xi_i)\right)^{p_i}\nabla\xi_i,
\end{aligned} \tag{3.21}$$

where $g_2(\xi_i) = \xi_i - b_i$. Substituting (3.21) into (3.20), we observe that

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}}{\prod_{i=1}^n h_i(b_i-a_i)^{\frac{1}{q_i}}}\frac{\prod_{i=1}^n \psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n(\xi_i-a_i)/q_i)^{\sum_{i=1}^n 1/q_i}}\nabla\xi_1\ldots\nabla\xi_n \\
&\quad \geq \prod_{i=1}^n\left(\int_{a_i}^{b_i}\left(b_i-\rho(\xi_i)\right)\left(\psi_i^{h_i-1}(\xi_i)\psi_i^{\nabla}(\xi_i)\right)^{p_i}\nabla\xi_i\right)^{\frac{1}{p_i}},
\end{aligned}$$

which satisfies (3.11). The proof is complete. □

Remark 3.2 As a particular case of Theorem 3.2, if $h_i = 1$ for $i = 1, 2, \ldots, n$, then we observe that Theorem 3.1 holds.

Remark 3.3 If $\mathbb{T} = \mathbb{R}$ and $a_i = 0$, then we get the reverse of inequality (1.9) under the following conditions: $0 < h_i \leq 1$, $q_i < 0$, and $\psi_i \in C([a_i, b_i], \mathbb{R}^+ \cup \{0\})$ are increasing functions with $\psi_i(0) = 0$ for $i = 1, 2, \ldots, n$.

Theorem 3.3 *Let $a_i, b_i \in \mathbb{T}$, $h_i \geq 1$, $q_i < 0$, $p_i = q_i/(q_i-1)$, and let $\psi_i \in C_{ld}([a_i, b_i]_{\mathbb{T}}, \mathbb{R}^+ \cup \{0\})$ be decreasing functions with $\psi_i(b_i) = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\begin{aligned}
&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}\prod_{i=1}^n \psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n(b_i-\xi_i)/q_i)^{\sum_{i=1}^n 1/q_i}}\nabla\xi_1\ldots\nabla\xi_n \\
&\quad \geq \prod_{i=1}^n h_i(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^n\left(\int_{a_i}^{b_i}\left(\rho(\xi_i)-a_i\right)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\psi_i^{\nabla}(\xi_i)\right]\right)^{p_i}\nabla\xi_i\right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.22}$$

Proof Applying the chain rule formula (2.1) to the term $\psi_i^{h_i}(t_i)$, $h_i \geq 1$, we get

$$\left[-\psi_i^{h_i}(t_i)\right]^{\nabla} = h_i\psi_i^{h_i-1}(\zeta_i)\left[-\psi_i^{\nabla}(t_i)\right], \tag{3.23}$$

where $\zeta_i \in [\rho(t_i), t_i]$. Since ψ_i is a decreasing function, $h_i \geq 1$, and $\zeta_i \leq t_i$, we have from (3.23) that

$$\left[-\psi_i^{h_i}(t_i)\right]^{\nabla} \geq h_i\psi_i^{h_i-1}(t_i)\left[-\psi_i^{\nabla}(t_i)\right],$$

and then by integrating the last inequality over t_i from ξ_i to b_i, where $\psi_i(b_i) = 0$, we see that

$$h_i\int_{\xi_i}^{b_i}\psi_i^{h_i-1}(t_i)\left[-\psi_i^{\nabla}(t_i)\right]\nabla t_i \leq \int_{\xi_i}^{b_i}\left[-\psi_i^{h_i}(t_i)\right]^{\nabla}\nabla t_i = -\psi_i^{h_i}(b_i) + \psi_i^{h_i}(\xi_i) = \psi_i^{h_i}(\xi_i).$$

Therefore

$$\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i)\geq\prod_{i=1}^{n}h_i\int_{\xi_i}^{b_i}\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\nabla t_i. \tag{3.24}$$

Applying reverse Hölder's inequality (2.3) to the term

$$\int_{\xi_i}^{b_i}\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\nabla t_i$$

with $q_i<0$, $p_i=q_i/(q_i-1)$, $f(t_i)=\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)]$, and $g(t_i)=1$, we have that

$$\begin{aligned}&\int_{\xi_i}^{b_i}\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\nabla t_i\\&\quad\geq\left(\int_{\xi_i}^{b_i}\nabla t_i\right)^{\frac{1}{q_i}}\left(\int_{\xi_i}^{b_i}\big(\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\big)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\\&\quad=(b_i-\xi_i)^{\frac{1}{q_i}}\left(\int_{\xi_i}^{b_i}\big(\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\big)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}},\end{aligned}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n}\int_{\xi_i}^{b_i}\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\nabla t_i\\&\quad\geq\prod_{i=1}^{n}(b_i-\xi_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{\xi_i}^{b_i}\big(\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\big)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}.\end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.24), we get

$$\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i)\geq\prod_{i=1}^{n}h_i(b_i-\xi_i)^{\frac{1}{q_i}}\prod_{i=1}^{n}\left(\int_{\xi_i}^{b_i}\big(\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\big)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}. \tag{3.26}$$

Applying Lemma 2.6 with $s_i=b_i-\xi_i$, we have that

$$\prod_{i=1}^{n}(b_i-\xi_i)^{1/q_i}\geq\frac{(\sum_{i=1}^{n}(b_i-\xi_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}}. \tag{3.27}$$

Substituting (3.27) into (3.26), we see that

$$\begin{aligned}\prod_{i=1}^{n}\psi_i^{h_i}(\xi_i)\geq{}&\frac{(\sum_{i=1}^{n}(b_i-\xi_i)/q_i)^{\sum_{i=1}^{n}1/q_i}}{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}}\\&\times\prod_{i=1}^{n}h_i\left(\int_{\xi_i}^{b_i}\big(\psi_i^{h_i-1}(t_i)\big[-\psi_i^{\nabla}(t_i)\big]\big)^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}.\end{aligned} \tag{3.28}$$

Multiplying (3.28) by the term

$$\left(n-\sum_{i=1}^{n}1/p_i\right)^{n-\sum_{i=1}^{n}1/p_i}\Bigg/\left(\sum_{i=1}^{n}(b_i-\xi_i)/q_i\right)^{\sum_{i=1}^{n}1/q_i}$$

and integrating over ξ_i from a_i to b_i, $i = 1, 2, \dots, n$, we observe that

$$\begin{aligned}
&\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \frac{(n - \sum_{i=1}^{n} 1/p_i)^{n-\sum_{i=1}^{n} 1/p_i} \prod_{i=1}^{n} \psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^{n} (b_i - \xi_i)/q_i)^{\sum_{i=1}^{n} 1/q_i}} \nabla\xi_1 \dots \nabla\xi_n \\
&\geq \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \prod_{i=1}^{n} h_i \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \nabla\xi_1 \dots \nabla\xi_n \\
&= \prod_{i=1}^{n} h_i \int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \nabla\xi_i.
\end{aligned} \tag{3.29}$$

Applying reverse Hölder's inequality (2.3) to the term

$$\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \nabla\xi_i$$

with $q_i < 0$, $p_i = q_i/(q_i - 1)$,

$$g(\xi_i) = 1, \quad \text{and} \quad f(\xi_i) = \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}},$$

we have that

$$\begin{aligned}
&\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \nabla\xi_i \\
&\geq (b_i - a_i)^{\frac{1}{q_i}} \left(\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right) \nabla\xi_i \right)^{\frac{1}{p_i}},
\end{aligned}$$

and then

$$\begin{aligned}
&\prod_{i=1}^{n} \int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right)^{\frac{1}{p_i}} \nabla\xi_i \\
&\geq \prod_{i=1}^{n} (b_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right) \nabla\xi_i \right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29), we see that

$$\begin{aligned}
&\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \frac{(n - \sum_{i=1}^{n} 1/p_i)^{n-\sum_{i=1}^{n} 1/p_i} \prod_{i=1}^{n} \psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^{n} (b_i - \xi_i)/q_i)^{\sum_{i=1}^{n} 1/q_i}} \nabla\xi_1 \dots \nabla\xi_n \\
&\geq \prod_{i=1}^{n} h_i (b_i - a_i)^{\frac{1}{q_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right) \nabla\xi_i \right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.31}$$

Applying the integration-by-parts formula (2.2) to the term

$$\int_{a_i}^{b_i} \left(\int_{\xi_i}^{b_i} (\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i} \nabla t_i \right) \nabla\xi_i$$

with $f_3(\xi_i) = \int_{\xi_i}^{b_i}(\psi_i^{h_i-1}(t_i)[-\psi_i^{\nabla}(t_i)])^{p_i}\nabla t_i$ and $g_3^{\nabla}(\xi_i) = 1$, we get

$$\begin{aligned}\int_{a_i}^{b_i}\left(\int_{\xi_i}^{b_i}\left(\psi_i^{h_i-1}(t_i)\left[-\psi_i^{\nabla}(t_i)\right]\right)^{p_i}\nabla t_i\right)\nabla\xi_i \\ = g_3(\xi_i)\left(\int_{\xi_i}^{b_i}\left(\psi_i^{h_i-1}(t_i)\left[-\psi_i^{\nabla}(t_i)\right]\right)^{p_i}\nabla t_i\right)\Big|_{a_i}^{b_i} \\ + \int_{a_i}^{b_i} g_3^{\rho}(\xi_i)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\psi_i^{\nabla}(\xi_i)\right]\right)^{p_i}\nabla\xi_i,\end{aligned} \tag{3.32}$$

where $g_3(\xi_i) = \xi_i - a_i$. Since $g_3(a_i) = 0$, we have from (3.32) that

$$\begin{aligned}\int_{a_i}^{b_i}\left(\int_{\xi_i}^{b_i}\left(\psi_i^{h_i-1}(t_i)\left[-\psi_i^{\nabla}(t_i)\right]\right)^{p_i}\nabla t_i\right)\nabla\xi_i \\ = \int_{a_i}^{b_i}\left(\rho(\xi_i) - a_i\right)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\psi_i^{\nabla}(\xi_i)\right]\right)^{p_i}\nabla\xi_i.\end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.31), we observe that

$$\begin{aligned}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}\prod_{i=1}^n\psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n(b_i-\xi_i)/q_i)^{\sum_{i=1}^n 1/q_i}}\nabla\xi_1\ldots\nabla\xi_n \\ \geq \prod_{i=1}^n h_i(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^n\left(\int_{a_i}^{b_i}\left(\rho(\xi_i)-a_i\right)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\psi_i^{\nabla}(\xi_i)\right]\right)^{p_i}\nabla\xi_i\right)^{\frac{1}{p_i}},\end{aligned}$$

which is (3.22). The proof is complete. □

Corollary 3.2 *As a particular case of Theorem 3.3, if $\mathbb{T} = \mathbb{R}$, a_i, $b_i \in \mathbb{T}$, $h_i \geq 1$, $q_i < 0$, $p_i = q_i/(q_i - 1)$, and $\psi_i \in C([a_i, b_i], \mathbb{R}^+ \cup \{0\})$ is a decreasing function with $\psi_i(b_i) = 0$ for $i = 1, 2, \ldots, n$, then $\rho(\xi_i) = \xi_i$, and*

$$\begin{aligned}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\frac{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}\prod_{i=1}^n\psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n(b_i-\xi_i)/q_i)^{\sum_{i=1}^n 1/q_i}}\,d\xi_1\ldots d\xi_n \\ \geq \prod_{i=1}^n h_i(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^n\left(\int_{a_i}^{b_i}(\xi_i-a_i)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\psi_i'(\xi_i)\right]\right)^{p_i}d\xi_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

Corollary 3.3 *As a particular case of Theorem 3.3, if $\mathbb{T} = \mathbb{N}_0$, a_i, $b_i \in \mathbb{T}$, $h_i \geq 1$, $q_i < 0$, $p_i = q_i/(q_i - 1)$, and ψ_i is a nonnegative decreasing sequence with $\psi_i(b_i) = 0$, $i = 1, 2, \ldots, n$, then $\rho(\xi_i) = \xi_i - 1$, and*

$$\begin{aligned}\sum_{\xi_n=1+a_n}^{b_n}\cdots\sum_{\xi_1=1+a_1}^{b_1}\frac{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}\prod_{i=1}^n\psi_i^{h_i}(\xi_i)}{(\sum_{i=1}^n(b_i-\xi_i)/q_i)^{\sum_{i=1}^n 1/q_i}} \\ \geq \prod_{i=1}^n h_i(b_i-a_i)^{\frac{1}{q_i}}\prod_{i=1}^n\left(\sum_{\xi_i=1+a_i}^{b_i}(\xi_i-a_i-1)\left(\left[\psi_i(\xi_i)\right]^{h_i-1}\left[-\nabla\psi_i(\xi_i)\right]\right)^{p_i}\right)^{\frac{1}{p_i}},\end{aligned}$$

where $\nabla\psi_i(\xi_i) = \psi_i(\xi_i) - \psi_i(\xi_i - 1)$.

4 Conclusion

In this paper, we establish some reversed dynamic inequalities of Hilbert type on time scales nabla calculus by applying reversed Hölder's inequality, chain rule on time scales, and the mean inequality. Also, we can get the reversed form of discrete and continuous inequalities proved by Chang-Jian, Lian-Ying, and Cheung.

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Data availability
The data used to support this study are included within the paper.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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