Srivastava et al. *Journal of Inequalities and Applications* (2024) 2024:16
https://doi.org/10.1186/s13660-024-03090-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Faber polynomial coefficient inequalities for bi-Bazilevič functions associated with the Fibonacci-number series and the square-root functions

H.M. Srivastava[1,2,3,4,5], Shahid Khan[6], Sarfraz Nawaz Malik[7], Fairouz Tchier[8], Afis Saliu[9*] and Qin Xin[10]

*Correspondence:
asaliu@utg.edu.gm
[9]Department of Mathematics, University of the Gambia, Birkama Campus, MDI Road, Kanifing, Serrekunda, P.O. Box 3530, The Gambia
Full list of author information is available at the end of the article

Abstract

Two new subclasses of the class of bi-Bazilevič functions, which are related to the Fibonacci-number series and the square-root functions, are introduced and studied in this article. Under a special choice of the parameter involved, these two classes of Bazilevič functions reduce to two new subclasses of star-like biunivalent functions related with the Fibonacci-number series and the square-root functions. Using the Faber polynomial expansion (FPE) technique, we find the general coefficient bounds for the functions belonging to each of these classes. We also find bounds for the initial coefficients for bi-Bazilevič functions and demonstrate how unexpectedly these initial coefficients behave in relation to the square-root functions and the Fibonacci-number series.

Mathematics Subject Classification: 30C45; 30C50

Keywords: Analytic functions; Univalent functions; Biunivalent functions; Bazilevič functions; Fibonacci numbers; Faber polynomials expansions; Fekete; Szegö problem

1 Introduction and preliminaries

The study of finding bounds on the coefficients is and has remained a major problem in Geometric Function Theory of Complex Analysis. The size of the coefficients of a given analytic function can have an impact on a variety of characteristics, including univalence, rate of growth, and distortion. Formulation of coefficient problems contains the estimation of the general or nth coefficient bounds, the Fekete–Szegö problem, Hankel determinants, and many other entities. A number of researchers have tackled the aforementioned coefficient problems by using different techniques. For instance, Bieberbach [1] provided the estimation on the second coefficient of univalent functions and conjectured a corresponding estimate on the nth coefficient of a univalent function, which was finally settled by de Branges [2]. Another interesting problem, which has a close relationship with the Bieberbach conjecture, was tackled by Littlewood and Paley (see [3]) by deriving the coefficient bounds for odd univalent functions. Further, Fekete and Szegö [4] obtained sharp bounds for the difference of the first two coefficients of univalent functions. Then, the problem of finding bounds on coefficients received much attention for many other subclasses of univalent functions. Additional information on this topic can be found in [5–7] and the references cited therein.

Just like univalent functions, finding coefficient estimates for biunivalent functions has been a subject of substantial interest in recent years. Evidently, researchers working on this topic drew a lot of motivation from the groundbreaking research of Srivastava et al. [8]. Numerous fascinating examples of functions falling under the class of biunivalent functions can be found in the work of Srivastava et al. [8]. Since bounds on these functions can be utilized for predicting their geometry, coefficient problems are also crucial in the study of biunivalent functions. For most of the subclasses of the biunivalent functions, finding the bounds for the general coefficients is still an open problem. However, under some assumptions, the general coefficient estimates for some particular subclasses of biunivalent functions were obtained in recent years (see, for details [9–16]).

In this study, by using the Faber polynomial (see [17, 18]), we obtain coefficient bounds for some classes of biunivalent functions. This polynomial has been extensively studied in the past few years. This polynomial plays an important role in the mathematical sciences, particularly in Geometric Function Theory of Complex Analysis. By employing the Faber polynomial expansion technique, Hamidi and Jahangiri [19, 20] as well as Srivastava et al. [21] developed new subclasses of biunivalent functions and discovered some novel and useful characteristics. Several different subclasses of the analytic and biunivalent function class were introduced and studied analogously by many other authors (see, for example, [10, 22–27]).

Let $\mathcal{A}$ stand for the set of all holomorphic functions ξ in the open unit disk:

$$\mathbb{E} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

which are normalized by

$$\xi(0) = 0 \quad \text{and} \quad \xi'(0) = 1.$$

Each function $\xi \in \mathcal{A}$ can, therefore, be expressed in the series form given by

$$\xi(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Additionally, the members of the class $\mathcal{S}$, which is a subclass of $\mathcal{A}$, are also univalent in $\mathbb{E}$.

Each function $\xi \in \mathcal{S}$ has an inverse function $\xi^{-1} = g$, which is defined as follows:

$$g(\xi(z)) = z \quad (z \in \mathbb{E})$$

and

$$\xi(g(w)) = w \quad \left(|w| < r_0(\xi); r_0(\xi) \geqq \frac{1}{4}\right).$$

The series expansion of the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

An analytic function ξ is called biunivalent in $\mathbb{E}$ if the function ξ and its inverse ξ^{-1} are both univalent in $\mathbb{E}$. The class of all biunivalent functions in $\mathbb{E}$ is denoted by Σ. By the same token, a function ξ is said to be bi-Bazilevič in $\mathbb{E}$ if both the function and its inverse are Bazilevič in $\mathbb{E}$ (see [28]). The behavior of these types of functions is unpredictable and, in fact, not much is known about their coefficients.

For $\xi_1, \xi_2 \in \mathcal{A}$, if the function ξ_1 is subordinate to the function ξ_2 in $\mathbb{E}$, denoted by

$$\xi_1(z) \prec \xi_2(z) \quad (z \in \mathbb{E}),$$

then we have a function $u_0 \in \mathcal{A}$ such that $|u_0(z)| < 1$, $u_0(0) = 0$ and

$$\xi_1(z) = \xi_2\big(u_0(z)\big) \quad (z \in \mathbb{E}).$$

The set of star-like functions of order α in $\mathbb{E}$ is denoted by the symbol $\mathcal{S}^*(\alpha)$ and we have

$$\mathcal{S}^*(\alpha) = \left\{\xi : \xi \in \mathcal{S} \text{ and } \Re\left(\frac{z\xi'(z)}{\xi(z)}\right) > \alpha \ (0 \leqq \alpha < 1)\right\}.$$

A class $\mathcal{B}(\gamma,\rho)$ of analytic functions ξ was first studied by Bazilevič [28] in 1955. Its definition in an open unit disk $\mathbb{E}$ is as follows:

$$\xi(z) = \left(\frac{\rho}{1+\gamma^2}\int_0^z \big(p(t) - i\gamma\big)t^{(-\frac{\gamma\rho i}{1+\gamma^2}-1)}\big[h(t)\big]^{\frac{\rho}{1+\gamma^2}}\,\mathrm{d}t\right)^{\frac{1+i\gamma}{\rho}}, \tag{1.3}$$

where $h \in \mathcal{S}^*$, $p \in \mathcal{P}$, $\rho \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}$. Bazilevič [28] also showed that $\mathcal{B}(\gamma,\rho)$ is a subclass of $\mathcal{S}$. The following inequality results upon taking $\gamma = 0$ in (1.3) and then differentiating each side:

$$\Re\left(\frac{z\xi'(z)}{\xi^{1-\rho}(\xi(z))^{\rho}}\right) > 0, \tag{1.4}$$

wherein all the powers are considered to be principal values. Thomas [29] gave the name "Bazilevič functions of type ρ" for such type of functions that satisfy (1.4). Despite the fact that the class $\mathcal{B}(\rho)$ is the largest subclass of univalent functions and contains a large number of known subclasses of $\mathcal{S}$, little is known about the class $\mathbb{B}(\gamma,\rho)$ of functions defined by (1.3) or for the class $\mathbb{B}(\rho)$ in general. The class of Bazilevič functions and the associated counterparts have undergone substantial research in a variety of areas. For further information, see [30–33].

The idea of subordination was used in order to define many subclasses of analytic functions. For instance, the Carathéodory class $\mathcal{P}$ of functions with positive real part can be defined as follows:

$$\mathcal{P} = \left\{p : p(0) = 1 \text{ and } p(z) \prec \frac{1+z}{1-z}(z \in \mathbb{E})\right\}.$$

Geometric Function Theory of Complex Analysis is especially intriguing in the geometric structure of the image domain. On an understanding of the ranges of these functions, other classes of analytic functions have been developed and explored. Several well-known subclasses of the class $\mathcal{P}$ can be created by substituting the function $\frac{1+z}{1-z}$ with appropriate functions. For instance, we cite the following special cases:

1. By taking

$$\varphi_1(z) = \frac{1+(1-2\alpha)z}{1-z} \quad (0 \leqq \alpha < 1),$$

we obtain the plane to the right of the vertical line $u = \alpha$ (see [34]).

2. By taking

$$\varphi_1(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1),$$

we have the circular domain centered at $\frac{1-AB}{1-B^2}$ and with radius $\frac{A-B}{1-B^2}$ (see [35]).

3. If we take

$$\varphi_2(z) = z + \sqrt{1+z^2},$$

we obtain the crescent-shaped region that was studied in [36].

4. For

$$\varphi_3(z) = 1 + z - \frac{z^3}{3},$$

we have the nephroid domain that was investigated in [37].

5. If we set

$$\varphi_4(z) = \left(\frac{1+z}{1+\frac{1-\beta}{\beta}z}\right)^{\alpha} \quad \left(\alpha \geqq 1; \beta \geqq \frac{1}{2}\right),$$

we obtain the leaf-like domain (see [38]).

6. By taking $\phi_5(z) = 1 + \sin z$, we obtain the eight-shaped region (see [39]).

7. For

$$\phi_6(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \quad \left(\tau := \frac{1-\sqrt{5}}{2}\right),$$

we have the shell-like domains studied in [40, 41]. The function

$$\phi_6(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

generates the shell-like curve. In a more detailed way, commonly known as a shell-like curve, is produced by mapping the unit circle through the function $\phi_6(z)$ given by:

$$\phi_6\left(e^{i\theta}\right) = \frac{\sqrt{5}}{2(3-2\cos\theta_1)} + i\frac{\sin\theta_1(4\cos\theta_1-1)}{2(3-2\cos\theta_1)2(1+\cos\theta_1)} \quad (0 \leqq \theta_1 < 2\pi).$$

The series representation for this significant function is as follows:

$$\varphi_6(z) = 1 + \sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n$$

$$= 1 + \sum_{n=1}^{\infty} T_n z^n,$$

where

$$T_n = (u_{n-1} + u_{n+1})\tau^n \tag{1.5}$$

and

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}} \quad \left(\tau = \frac{1-\sqrt{5}}{2}\right),$$

which brings it closer to the Fibonacci numbers by producing a series of constant coefficients. For indepth research on the aforementioned functions and a large number of other similar functions, see ([42–48]) and the references cited therein.

8. The function $\varphi_7(z) = \sqrt{1+z}$ yields the right-half of the lemniscate of Bernoulli, which was introduced and studied in [49, 50]. Moreover, $\phi_7(z) = \sqrt{1+z}$ is an analytic function with positive real part in the unit disk $\mathbb{E}$, which satisfies the following conditions:

$$\varphi_7(0) = 1 \quad \text{and} \quad \varphi_7'(0) > 0$$

such that the series expansion of the form $\varphi_7(z) = \sqrt{1+z}$ is given as follows:

$$\begin{aligned}\varphi_7(z) &= 1 + \sum_{n=1}^{\infty} \frac{(2n-2)!(-1)^{n-1}}{(n-1)!n!2^{2n-1}} z^n \\ &= 1 + \sum_{n=1}^{\infty} Q_n z^n \\ &= 1 + Q_1 z + Q_2 z^2 + \cdots,\end{aligned}$$

where

$$Q_n = \frac{(2n-2)!(-1)^{n-1}}{(n-1)!n!2^{2n-1}}. \tag{1.6}$$

In our work, we shall use the function:

$$\phi_6(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \quad \left(\tau = \frac{1-\sqrt{5}}{2}\right)$$

and the function

$$\phi_7(z) = \sqrt{1+z}.$$

Definition 1.1 Let ξ be an analytic function and be of the form (1.1). Then, $\xi \in \mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$ if and only if

$$\left(\frac{z}{\xi(z)}\right)^{1-\beta} \xi'(z) \prec \varphi_6(z)$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) \prec \varphi_6(w),$$

where $0 \leqq \beta < 1$ and $z, w \in \mathbb{E}$.

Definition 1.2 Let ξ be an analytic function and be of the form (1.1). Then, $\xi \in \mathcal{B}_{\Sigma}^{\beta}(\varphi_7(z))$ if and only if

$$\left(\frac{z}{\xi(z)}\right)^{1-\beta} \xi'(z) \prec \varphi_7(z)$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) \prec \varphi_7(w),$$

where $0 \leqq \beta < 1$ and $z, w \in \mathbb{E}$.

For $\beta = 0$, the above two classes of Bazilevič functions reduce to two new subclasses of star-like biunivalent functions related with the Fibonacci-number series and the square-root function.

Definition 1.3 Assume that ξ is an analytic function and has the form given by (1.1). Then, $\xi \in \mathcal{S}_{\Sigma}(\varphi_6(z))$ if and only if

$$\frac{z\xi'(z)}{\xi(z)} \prec \varphi_6(z)$$

and

$$\frac{wg'(w)}{g(w)} \prec \varphi_6(w),$$

where $z, w \in \mathbb{E}$.

Definition 1.4 Suppose that ξ is an analytic function and has the form (1.1). Then, $\xi \in \mathcal{S}_{\Sigma}(\varphi_7(z))$ if and only if

$$\frac{z\xi'(z)}{\xi(z)} \prec \varphi_7(z)$$

and

$$\frac{wg'(w)}{g(w)} \prec \varphi_7(w),$$

where $z, w \in \mathbb{E}$.

Throughout this article, we will be presuming that $0 \leqq \beta < 1$.

Next, with a view to introducing the Faber polynomial expansion (FPE) method and its applications, we assume that the coefficients of the inverse map g of the analytic function ξ can be expressed as follows by using the FPE method (see [51, 52]):

$$g(w) = \xi^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} \mathfrak{S}_{n-1}^{n}(a_2, a_3, \dots, a_n) w^n,$$

where

$$\begin{aligned}
\mathfrak{S}_{n-1}^{-n} &= \mathfrak{S}_{n-1}^{-n}(a_2, a_3, \dots, a_n) \\
&= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\
&\quad + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\
&\quad + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \bigl[a_5 + (-n+2) a_3^2\bigr] \\
&\quad + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \bigl[a_6 + (-2n+5) a_3 a_4\bigr] \\
&\quad + \sum_{\mathrm{j} \geqq 7} a_2^{j-\mathrm{n}} \mathcal{Q}_{\mathrm{n}},
\end{aligned}$$

in which $\mathcal{Q}_{\mathrm{n}}$ is a homogeneous polynomial in the variables $a_2, a_3, \dots, a_n$ for $7 \leqq \mathrm{j} \leqq n$ such as (for example) $(-n)!$ are symbolically interpreted as follows:

$$(-n)! \equiv \Gamma(1-n) = (-n)(-n-1)(-n-2)\cdots \quad \bigl(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}\bigr).$$

In particular, the first three terms of $\mathfrak{S}_{n-1}^{-n}$ are given by:

$$\begin{aligned}
&\frac{1}{2}\mathfrak{S}_1^{-2} = -a_2, \\
&\frac{1}{3}\mathfrak{S}_2^{-3} = 2a_2^2 - a_3
\end{aligned}$$

and

$$\frac{1}{4}\mathfrak{S}_3^{-4} = -\bigl(5a_2^3 - 5a_2 a_3 + a_4\bigr).$$

In general, for $\mathfrak{r} \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ and $n \geqq 2$, an expansion of $\mathfrak{S}_n^{\mathfrak{r}}$ is of the following form:

$$\mathfrak{S}_n^{\mathfrak{p}} = \mathfrak{r} a_n + \frac{\mathfrak{r}(\mathfrak{r}-1)}{2} \mathcal{D}_n^2 + \frac{\mathfrak{r}!}{(\mathfrak{r}-3)!3!} \mathcal{D}_n^3 + \cdots + \frac{\mathfrak{r}!}{(\mathfrak{r}-n)!n!} \mathcal{D}_n^n,$$

where

$$\mathcal{D}_n^{\mathfrak{r}} = \mathcal{D}_n^{\mathfrak{r}}(a_2, a_3, \dots).$$

Also, by [51, 52], we have

$$\mathcal{D}_n^{\nu}(a_2,\dots,a_{n+1}) = \sum_{n=1}^{\infty} \frac{\nu!(a_2)^{\mu_1}\cdots(a_n)^{\mu_n}}{\mu_{1!},\dots,\mu_n!} \quad (a_1 = 1; \nu \leqq n).$$

The sum is taken over all nonnegative integers $\mu_1,\dots,\mu_n$ that satisfy the following conditions:

$$\mu_1 + \mu_2 + \cdots + \mu_n = \nu$$

and

$$\mu_1 + 2\mu_2 + \cdots + n\mu_n = n.$$

Clearly, we have

$$\mathcal{D}_n^n(a_1,\dots,a_n) = a_2^n.$$

In this paper, we introduce new subclasses of analytic biunivalent functions connected with the Fibonacci-number series and the square-root functions. We also obtain the estimates for the initial Taylor–Maclaurin coefficients for functions in each of these subclasses. Moreover, we investigate estimates for the general coefficients a_n for functions in these subclasses of biunivalent functions by using the Faber polynomial expansion (FBE) method.

2 Main results

To prove our main results, we shall need the following known results (see [1]).

Lemma 2.1 *Let*

$$\psi(z) = \sum_{n=1}^{\infty} c_n z^n$$

be a Schwarz function so that

$$\left|\psi(z)\right| < 1 \quad \bigl(|z| < 1\bigr).$$

If $\gamma \geqq 0$, *then*

$$\left|c_2 + \gamma c_1^2\right| \leqq 1 + (\gamma - 1)|c_1|^2.$$

In the following theorem, we use the Faber polynomial expansion method and determine the unpredictable behavior of the coefficients' estimates of the functions class $\mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$ defined by the Fibonacci-number series.

Theorem 2.2 *Let* $\xi \in \mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$ *be given by* (1.1). *Also, let* $a_k = 0$ $(2 \leqq k \leqq n-1)$. *Then,*

$$|a_n| \leqq \frac{|T_1|}{n+\beta} \quad \bigl(n \in \mathbb{N} \setminus \{1,2\}\bigr),$$

where T_1 *is given in* (1.5) *for all values of* n.

Proof For $\xi \in \mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$, then the FPE for $(\frac{z}{\xi(z)})^{1-\beta}\xi'(z)$ is given by

$$\left(\frac{z}{\xi(z)}\right)^{1-\beta}\xi'(z) = 1 + \sum_{n=1}^{\infty}\left(1+\frac{n}{\beta}\right)K_k^{\beta}(a_2,a_3,\ldots,a_{n+1})z^n. \tag{2.1}$$

For the inverse mapping $g = \xi^{-1}$, the FPE for $(\frac{w}{g(w)})^{1-\beta}g'(w)$ is

$$\left(\frac{w}{g(w)}\right)^{1-\beta}g'(w) = 1 + \sum_{n=1}^{\infty}\left(1+\frac{n}{\beta}\right)K_k^{\beta}(b_2,b_3,\ldots,b_{n+1})w^n, \tag{2.2}$$

where

$$b_{n+1} = \frac{1}{n+1}K_k^{-(n+1)}(a_2,a_3,\ldots,a_{n+1}) \quad (k \geqq 1).$$

Furthermore, since $\xi \in \mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$, by the definition of subordination, there exists a Schwarz function $\psi(z)$ given by

$$\psi(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{E})$$

such that

$$\begin{aligned}\left(\frac{z}{\xi(z)}\right)^{1-\beta}\xi'(z) &= \varphi_6\big(\psi(z)\big)\\ &= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} T_k \mathfrak{S}_n^k(c_1,c_2,\ldots,c_n)z^n.\end{aligned} \tag{2.3}$$

Similarly, we have

$$\left(\frac{w}{g(w)}\right)^{1-\beta}g'(w) \quad (w \in \mathbb{E}).$$

There exists a Schwarz function $q(z)$ given by

$$q(w) = \sum_{n=1}^{\infty} d_n w^n$$

such that

$$\begin{aligned}\left(\frac{w}{g(w)}\right)^{1-\beta}g'(w) &= \varphi_1\big(q(w)\big)\\ &= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} T_k \mathfrak{S}_n^k(d_1,d_2,\ldots,d_n)z^n.\end{aligned} \tag{2.4}$$

Evaluating the coefficients of equations (2.1) and (2.3), for any $n \geqq 2$, we obtain

$$\left(1+\frac{n}{\beta}\right)\mathfrak{S}_n^{\beta}(a_2,a_3,\ldots,a_{n+1})a_n = \sum_{k=1}^{n-1} T_k \mathfrak{S}_n^k(c_1,c_2,\ldots,c_{n-1}). \tag{2.5}$$

Evaluating the coefficients of equations (2.2) and (2.4), for any $n \geqq 2$, we have

$$\left(1+\frac{n}{\beta}\right)\mathfrak{S}_k^{\beta}(b_2,b_3,\dots,b_n)=\sum_{k=1}^{n-1}T_k\mathfrak{S}_n^k(d_1,d_2,\dots,d_n). \tag{2.6}$$

Solving for a_2 and taking the moduli of the coefficients of the Schwarz functions ψ and q, we obtain (see [1])

$$|c_n| \leqq 1 \quad \text{and} \quad |d_n| \leqq 1.$$

Also, from the assumptions that $2 \leqq k \leqq n-1$ and $a_k = 0$, respectively, we obtain

$$b_n = -a_n$$

as well as

$$\left(1+\frac{n}{\beta}\right)\beta a_n = T_1 c_{n-1} \tag{2.7}$$

and

$$-\left(1+\frac{n}{\beta}\right)\beta a_n = T_1 d_{n-1}. \tag{2.8}$$

By solving equations (2.7) and (2.8) for a_n and determining the moduli, and by the coefficients of the Schwarz functions $\psi(z)$ and $q(w)$, we have $|c_n| \leq 1$and $|d_n| \leq 1$, (see [1]):

$$|a_n| \leqq \frac{|T_1|}{n+\beta}.$$

This completes the proof of Theorem 2.2. □

Theorem 2.3 *If an analytic function ξ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{\beta}(\varphi_6(z))$, and if*

$$T_2 = \alpha T_1 \quad (\alpha > 0),$$

then

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2T_1}{(1+\beta)(\beta+2)}} & \left(0 < T_1 \leqq \frac{2(\beta+1)}{(\beta+2)}\right), \\ \frac{T_1}{\beta+1} & \left(T_1 \geqq \frac{2(\beta+1)}{(\beta+2)}\right), \end{cases}$$

$$|a_3| \leqq \frac{T_1}{\beta+2}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{T_1}{(\beta+1)(\beta+2)},$$

where T_1 and T_2 are given by (1.5).

Proof Taking $n = 2$ in (2.5) and (2.6), we have

$$(\beta + 1)a_2 = T_1 c_1 \tag{2.9}$$

and

$$-(\beta + 1)a_2 = T_1 d_1. \tag{2.10}$$

From (2.9) and (2.10), we find that

$$c_1 = -d_1. \tag{2.11}$$

If we take the moduli of both sides of any of these two equations, for the coefficients of the Schwarz functions ψ and q given by (see [1])

$$|c_n| \leqq 1 \quad \text{and} \quad |d_n| \leqq 1,$$

we have

$$|a_2| \leqq \frac{T_1}{\beta + 1}. \tag{2.12}$$

For $n = 3$, equations (2.5) and (2.6) yield

$$\frac{(\beta - 1)(\beta + 2)}{2} a_2^2 + (\beta + 2)a_3 = T_1 c_2 + \alpha T_1 c_1^2 \tag{2.13}$$

and

$$\frac{(3 + \beta)(\beta + 2)}{2} a_2^2 - (\beta + 2)a_3 = T_1 d_2 + \alpha T_1 d_1^2, \tag{2.14}$$

respectively. Thus, upon adding (2.13) and (2.14), we obtain

$$(1 + \beta)(\beta + 2)a_2^2 = T_1\big[\big(c_2 + \alpha c_1^2\big) + \big(d_2 + \alpha d_1^2\big)\big].$$

Again, for the coefficients of the Schwarz functions ψ and q, we have

$$(1 + \beta)(\beta + 2)|a_2|^2 \leqq T_1\big[\big|c_2 + \alpha c_1^2\big| + \big|d_2 + \alpha d_1^2\big|\big].$$

Thus, by Lemma 2.1, we have

$$(1 + \beta)(\beta + 2)|a_2|^2 \leqq T_1\big[1 + (\alpha - 1)|c_1|^2 + 1 + (\alpha - 1)|d_1|^2\big].$$

Now, by using the moduli of the coefficients of the Schwarz functions, we obtain

$$|a_2| \leqq \sqrt{\frac{2T_1}{(1 + \beta)(\beta + 2)}}. \tag{2.15}$$

Consequently, we note that

$$\sqrt{\frac{2T_1}{(1+\beta)(\beta+2)}} < \frac{T_1}{\beta+1}.$$

Multiplying equation (2.13) by $(3+\beta)$ and equation (2.14) by $(\beta-1)$, we obtain

$$2(\beta+1)(\beta+2)a_3 = T_1\big[(3+\beta)\big(c_2+(\alpha)c_1^2\big)-(\beta-1)\big(d_2+(-B)d_1^2\big)\big].$$

From Lemma 2.1, we have

$$|a_3| \leqq \frac{T_1}{2(\beta+1)(\beta+2)}\big[(3+\beta)\big(1+(\alpha-1)\big)|c_1|^2+(\beta-1)\big(1+(\alpha-1)\big)|d_1|^2\big].$$

Applying the moduli of the coefficients of the Schwarz functions to the above equation, we have

$$|a_3| \leqq \frac{T_1}{(\beta+2)}.$$

Lastly, upon subtracting equation (2.13) from equation (2.14), if we use equation (2.11), we find that

$$(\beta+1)(\beta+2)\big(a_3-a_2^2\big) = T_1|c_2-d_2|,$$

which can be rewritten as follows:

$$\big|a_3-a_2^2\big| \leqq \frac{T_1}{(\beta+1)(\beta+2)}.$$

This completes the proof. □

Taking $\beta=0$ in Theorem 2.2, we obtain Theorem 2.4 below for the class $\mathcal{S}_\Sigma(\varphi_6(z))$.

Theorem 2.4 *Let $\xi \in \mathcal{S}_\Sigma(\varphi_6(z))$ be an analytic function. Also, let $a_k = 0\ 2(\leqq k \leqq n-1)$. Then,*

$$|a_n| \leqq \frac{|T_1|}{n} \quad \big(n \in \mathbb{N}\setminus\{1,2\}\big). \tag{2.16}$$

Taking $\beta = 0$ in Theorem 2.3, we obtain the following Theorem 2.5 for the class $\mathcal{S}_\Sigma(\varphi_6(z))$.

Theorem 2.5 *If an analytic function ξ given by (1.1) belongs to the class $\mathcal{S}_\Sigma(\varphi_6(z))$, and if*

$$T_2 = \alpha T_1 \quad (\alpha > 0),$$

then

$$|a_2| \leqq \begin{cases} \sqrt{T_1} & (0 < T_1 \leqq 1), \\ \frac{T_1}{2} & (T_1 \geqq 1), \end{cases} \tag{2.17}$$

$$|a_3| \leqq \frac{T_1}{2} \tag{2.18}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{T_1}{2}. \tag{2.19}$$

In the following theorem, we use the Faber polynomial expansion method and determine the unpredictable behavior of coefficient estimates of the functions class $\mathcal{B}_\Sigma^\beta(\varphi_7(z))$ defined by the series of the square-root function.

Theorem 2.6 *Let $\xi \in \mathcal{B}_\Sigma^\beta(\varphi_7(z))$ be an analytic function. Also, let $a_k = 0$ $(2 \leqq k \leqq n-1)$. Then,*

$$|a_n| \leqq \frac{|Q_1|}{n+\beta} \quad \bigl(n \in \mathbb{N} \setminus \{1,2\}\bigr), \tag{2.20}$$

where the value of Q_1 is given by (1.6) for all values of n.

Proof For $\xi \in \mathcal{B}_\Sigma^\beta(\varphi_7(z))$, the FPE for $(\frac{z}{\xi(z)})^{1-\beta}\xi'(z)$ is given by

$$\left(\frac{z}{\xi(z)}\right)^{1-\beta} \xi'(z) = 1 + \sum_{n=1}^{\infty}\left(1+\frac{n}{\beta}\right)K_k^\beta(a_2, a_3, \dots, a_{n+1})z^n. \tag{2.21}$$

For the inverse mappings $g = \xi^{-1}$, the FPE for $(\frac{w}{g(w)})^{1-\beta}g'(w)$ is:

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = 1 + \sum_{n=1}^{\infty}\left(1+\frac{n}{\beta}\right)K_k^\beta(b_2, b_3, \dots, b_{n+1})w^n, \tag{2.22}$$

where

$$b_{n+1} = \frac{1}{n+1}K_k^{-(n+1)}(a_2, a_3, \dots, a_{n+1}) \quad (k \geqq 1).$$

Furthermore, since $\xi \in \mathcal{B}_\Sigma^\beta(\varphi_7(z))$, by the definition of subordination, there exists a Schwarz function $\psi(z)$ given by

$$\psi(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in E$$

such that

$$\begin{aligned}\left(\frac{z}{\xi(z)}\right)^{1-\beta} \xi'(z) &= \varphi_7\bigl(\psi(z)\bigr) \\ &= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} Q_k \mathfrak{S}_n^k(c_1, c_2, \dots, c_n)z^n. \end{aligned} \tag{2.23}$$

Similarly, for $(\frac{w}{g(w)})^{1-\beta}g'(w)$, there exists another Schwarz function $q(w)$ given by

$$q(w) = \sum_{n=1}^{\infty} d_n w^n$$

such that

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = \varphi_2\big(q(w)\big)$$
$$= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} Q_k \mathfrak{S}_n^k(d_1, d_2, \ldots, d_n) z^n. \tag{2.24}$$

Evaluating the coefficients of equations (2.21) and (2.23), for any $n \geqq 2$, we obtain

$$\left(1 + \frac{n}{\beta}\right)\mathfrak{S}_n^{\beta}(a_2, a_3, \ldots, a_{n+1})a_n = \sum_{k=1}^{n-1} Q_k \mathfrak{S}_n^k(c_1, c_2, \ldots, c_{n-1}). \tag{2.25}$$

Furthermore, by evaluating the coefficients of equations (2.22) and (2.24), for any $n \geqq 2$, we obtain

$$\left(1 + \frac{n}{\beta}\right)\mathfrak{S}_k^{\beta}(b_2, b_3, \ldots, b_n) = \sum_{k=1}^{n-1} Q_k \mathfrak{S}_n^k(d_1, d_2, \ldots, d_n). \tag{2.26}$$

Solving for a_2 and taking the moduli of the coefficients of the Schwarz functions ψ and q, we have (see [1])

$$|c_n| \leqq 1 \quad \text{and} \quad |d_n| \leqq 1.$$

Also, from the assumptions that $2 \leqq k \leqq n-1$ and $a_k = 0$; respectively, we obtain

$$b_n = -a_n$$

as well as

$$\left(1 + \frac{n}{\beta}\right)\beta a_n = Q_1 c_{n-1} \tag{2.27}$$

and

$$-\left(1 + \frac{n}{\beta}\right)\beta a_n = Q_1 d_{n-1}. \tag{2.28}$$

Thus, by solving equations (2.27) and (2.28) for a_n and determining the moduli with the coefficients of the Schwarz functions $\psi(z)$ and $q(w)$, which are $|c_n| \leq 1$ and $|d_n| \leq 1$, respectively (see [1]), we obtain

$$|a_n| \leqq \frac{|Q_1|}{n+\beta}.$$

This completes the proof of Theorem 2.6. □

Theorem 2.7 *If an analytic function ξ given by* (1.1) *belongs to the class $\mathcal{B}_{\Sigma}^{\beta}(\varphi_7(z))$, and if*

$$Q_2 = \alpha Q_1 \quad (\alpha > 0),$$

then

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2Q_1}{(1+\beta)(\beta+2)}} & (0 < Q_1 \leqq \frac{2(\beta+1)}{(\beta+2)}), \\ \frac{Q_1}{\beta+1} & (Q_1 \geqq \frac{2(\beta+1)}{(\beta+2)}), \end{cases} \tag{2.29}$$

$$|a_3| \leqq \frac{Q_1}{\beta+2}$$

and

$$|a_3 - a_2^2| \leqq \frac{Q_1}{(\beta+1)(\beta+2)},$$

where Q_1 *and* Q_2 *are given by* (1.6).

Proof Taking $n = 2$ in (2.25) and (2.26), we have

$$(\beta+1)a_2 = Q_1 c_1 \tag{2.30}$$

and

$$-(\beta+1)a_2 = Q_1 d_1. \tag{2.31}$$

From (2.30) and (2.31), we obtain

$$c_1 = -d_1. \tag{2.32}$$

If we take the moduli of both sides of any of these two equations, for the coefficients of the Schwarz functions ψ and q given by

$$|c_n| \leqq 1 \quad \text{and} \quad |d_n| \leqq 1,$$

we obtain

$$|a_2| \leqq \frac{Q_1}{\beta+1}. \tag{2.33}$$

For $n = 3$, equations (2.5) and (2.6) yield

$$\frac{(\beta-1)(\beta+2)}{2}a_2^2 + (\beta+2)a_3 = Q_1 c_2 + \alpha Q_1 c_1^2 \tag{2.34}$$

and

$$\frac{(3+\beta)(\beta+2)}{2}a_2^2 - (\beta+2)a_3 = Q_1 d_2 + \alpha Q_1 d_1^2, \tag{2.35}$$

respectively. Thus, upon adding (2.34) and (2.35), we obtain

$$(1+\beta)(\beta+2)a_2^2 = Q_1\big[\big(c_2 + \alpha c_1^2\big) + \big(d_2 + \alpha d_1^2\big)\big].$$

Using the moduli of the coefficients of the Schwarz functions, we have

$$(1+\beta)(\beta+2)|a_2|^2 \leqq Q_1\big[\big|c_2+\alpha c_1^2\big|+\big|d_2+\alpha d_1^2\big|\big].$$

Thus, by Lemma 2.1, we have

$$(1+\beta)(\beta+2)|a_2|^2 \leqq Q_1\big[1+(\alpha-1)|c_1|^2+1+(\alpha-1)|d_1|^2\big].$$

Taking the moduli on both sides and using the fact that

$$|c_n| \leqq 1 \quad \text{and} \quad |d_n| \leqq 1,$$

we obtain

$$|a_2| \leqq \sqrt{\frac{2Q_1}{(1+\beta)(\beta+2)}}.$$

Consequently, we note that

$$\sqrt{\frac{2Q_1}{(1+\beta)(\beta+2)}} < \frac{Q_1}{\beta+1}.$$

Multiplying equation (2.34) by $(3+\beta)$ and equation (2.35) by $(\beta-1)$, we obtain

$$2(\beta+1)(\beta+2)a_3 = Q_1\big[(3+\beta)\big(c_2+(\alpha)c_1^2\big)-(\beta-1)\big(d_2+(-B)d_1^2\big)\big].$$

Thus, by Lemma 2.1, we have

$$|a_3| \leqq \frac{Q_1}{2(\beta+1)(\beta+2)}\big[(3+\beta)\big(1+(\alpha-1)\big)|c_1|^2+(\beta-1)\big(1+(\alpha-1)\big)|d_1|^2\big].$$

Also, for the coefficients of the Schwarz functions, we obtain

$$|a_3| \leqq \frac{Q_1}{\beta+2}.$$

Lastly, upon subtracting equation (2.34) from equation (2.35), if we use relation (2.32), we obtain

$$(\beta+1)(\beta+2)\big(a_3-a_2^2\big) = Q_1|c_2-d_2|.$$

For the coefficients of the Schwarz functions $\psi(z)$ and $q(w)$, we have $|c_n| \leq 1$ and $|d_n| \leq 1$, respectively (see [1]), we obtain

$$\big|a_3-a_2^2\big| \leqq \frac{Q_1}{(\beta+1)(\beta+2)}. \qquad \square$$

Taking $\beta = 0$ in Theorem 2.6, we obtain the following result for the class $\mathcal{S}_\Sigma(\varphi_7(z))$.

Theorem 2.8 *Let* $\xi \in \mathcal{S}_\Sigma(\varphi_7(z))$ *be an analytic function given by* (1.1). *Also, let* $a_k = 0$ $(2 \leqq k \leqq n-1)$. *Then,*

$$|a_n| \leqq \frac{|Q_1|}{n} \quad \bigl(n \in \mathbb{N} \setminus \{1,2\}\bigr).$$

Taking $\beta = 0$ in Theorem 2.7, we deduce Theorem 2.9 below for the class $\mathcal{S}_\Sigma(\varphi_7(z))$.

Theorem 2.9 *If an analytic function* ξ *given by* (1.1) *belongs to the class* $\mathcal{S}_\Sigma(\varphi_7(z))$, *and if*

$$Q_2 = \alpha Q_1 \quad (\alpha > 0),$$

then

$$|a_2| \leqq \begin{cases} \sqrt{Q_1} & (0 < Q_1 \leqq 1), \\ \frac{Q_1}{2} & (Q_1 \geqq 1), \end{cases}$$

$$|a_3| \leqq \frac{Q_1}{2}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{Q_1}{2}.$$

3 Conclusion

In our present investigation, we have introduced and studied the properties and characteristics of functions belonging to two new subclasses of the class of bi-Bazilevič functions, which are related to the Fibonacci-number series and the square-root functions. Under a special choice of the parameters involved, these two classes of Bazilevič functions are shown to reduce to two new subclasses of star-like biunivalent functions related with the Fibonacci-number series and the square-root functions. We have applied the Faber polynomial expansion (FPE) technique in order to find the general coefficient bounds for the functions belonging to each of these functions' classes. We have also established bounds for the initial coefficients for bi-Bazilevič functions and have demonstrated the unexpected behavior of these initial coefficients in relation to the square-root functions and the Fibonacci-number series. The technique described in this article allows us to define several new subclasses of analytic functions and meromorphic functions connected with Fibonacci numbers and square-root functions. For these classes, we can determine the nth coefficient bounds by using the technique of the Faber polynomial. Furthermore, we can expand this work by incorporating Chebyshev polynomials, quantum calculus, symmetric quantum calculus, sigmoid activation functions, linear operators, and differential operators.

Acknowledgements
This research was supported by the researchers Supporting Project Number (RSP2024R401), King Saud University, Riyadh, Saudi Arabia.

Funding
No external funding was received.

Data availability
No data or material was used in this study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
H.M. Srivastava, S. Khan and Q. Xin contributed in conceptualization, methodology and investigation. S. N. Malik, F. Tchier and A. Saliu did formal analysis, wrote the main manuscript and check for validation of results.

Author details
[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada. [2]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan, Republic of China. [3]Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, 02447, Republic of Korea. [4]Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007, Baku, Azerbaijan. [5]Section of Mathematics, International Telematic University Uninettuno, I-00186, Rome, Italy. [6]Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad, 22500, Pakistan. [7]Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt, 47040, Pakistan. [8]Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh, 11495, Saudi Arabia. [9]Department of Mathematics, University of the Gambia, Birkama Campus, MDI Road, Kanifing, Serrekunda, P.O. Box 3530, The Gambia. [10]Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100, Torshavn, Faroe Islands, Denmark.

Received: 20 May 2023 Accepted: 11 January 2024 Published online: 31 January 2024

References
1. Duren, P.L.: Univalent Functions. Grundlehren der Mathematischen Wissenschaften, vol. 259. Springer, New York (1983)
2. de Branges, L.: A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152 (1985)
3. Dragomir, S.S., Sofo, A.: Advances in Inequalities for Series. Nova Publishers, New York (2008)
4. Fekete, M., Szegö, G.: Eine Bemerkung Über Ungerade Schlichte Funktionen. J. Lond. Math. Soc. **8**, 85–89 (1933)
5. Khan, M.F., Khan, N., Araci, S., Khan, S., Khan, B.: Coefficient inequalities for a subclass of symmetric q-starlike functions involving certain conic domains. J. Math. **2022**, 9446672 (2022)
6. Jia, Z., Kan, N., Khan, S., Khan, B.: Faber polynomial coefficients estimates for certain subclasses of q-Mittag-Leffler-type analytic and bi-univalent functions. AIMS Math. **7**, 2512–2528 (2021)
7. Srivastava, H.M., Khan, N., Darus, M., Khan, S., Ahmad, Q.Z., Hussain, S.: Fekete–Szegö type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points. Mathematics **8**, 842 (2020)
8. Srivastava, H.M., Mishra, A.K., Gochhayat, P.: Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. **23**, 1188–1192 (2010)
9. Altınkaya, Ş., Yalçın, S.: Faber polynomial coefficient bounds for a subclass of bi-univalent functions. C. R. Math. Acad. Sci. Paris, Sér. I **353**, 1075–1080 (2015)
10. Altınkaya, Ş., Yalçın, S.: Faber polynomial coefficient bounds for a subclass of bi-univalent functions. Stud. Univ. Babeş–Bolyai, Math. **61**, 37–44 (2016)
11. Srivastava, H.M., Eker, S.S., Ali, R.M.: Coeffcient bounds for a certain class of analytic and bi-univalent functions. Filomat **29**, 1839–1845 (2015)
12. Xu, Q.-H., Xiao, H.-G., Srivastava, H.M.: A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Appl. Math. Comput. **18**, 11461–11465 (2012)
13. Xu, Q.-H., Gui, Y.-C., Srivastava, H.M.: Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Appl. Math. Lett. **25**(6), 990–994 (2012)
14. Sivasubramanian, S., Sivakumar, R., Kanas, S., Kim, S.-A.: Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions. Ann. Pol. Math. **113**(3), 295–304 (2015)
15. Srivastava, H.M., Murugusundaramoorthy, G., El-Deeb, S.M.: Faber polynomial coefficient estimates of bi-close-to-convex functions connected with the Borel distribution of the Mittag-Leffler type. J. Nonlinear Var. Anal. **5**, 103–118 (2021)
16. Oros, G.I., Cotîrla, L.I.: Coefficient estimates and the Fekete–Szegö problem for new classes of m-fold symmetric bi-univalent functions. Mathematics **10**, 129 (2022)
17. Faber, G.: Über polynomische Entwickelungen. Math. Ann. **57**, 389–408 (1903)
18. Faber, G.: Über Tschebyscheffsche polynome. J. Reine Angew. Math. **150**, 79–106 (1919)
19. Hamdi, S.G., Jahangiri, J.M.: Faber polynomials coefficient estimates for analytic bi-close-to-convex functions. C. R. Math. Acad. Sci. Paris, Sér. I **352**, 17–20 (2014)
20. Hamdi, S.G., Jahangiri, J.M.: Faber polynomial coefficient estimates for bi-univalent functions defined by subordinations. Bull. Iran. Math. Soc. **41**, 1103–1119 (2015)
21. Srivastava, H.M., Eker, S.S., Hamidi, S.G., Jahangiri, J.M.: Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator. Bull. Iran. Math. Soc. **8**, 149–157 (2018)
22. Bulut, S.: Faber polynomial coefficients estimates for a comprehensive subclass of analytic bi-univalent functions. C. R. Math. Acad. Sci. Paris, Sér. I **352**, 479–484 (2014)
23. Gong, S.: The Bieberbach Conjecture. AMS/IP Studies in Advanced Mathematics, vol. 12. Am. Math. Soc., Providence (1999). Translated from the 1989 Chinese original and revised by the author

24. Srivastava, H.M., Khan, S., Ahmad, Q.Z., Khan, N., Hussain, S.: The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator. Stud. Univ. Babeş–Bolyai, Math. **63**, 419–436 (2018)
25. Srivastava, H.M., Motamednezhad, A., Adegani, E.A.: Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator. Mathematics **8**, 172 (2020)
26. Tang, H., Srivastava, H.M., Sivasubramanian, S., Gurusamy, P.: Fekete–Szegö functional problems of m-fold symmetric bi-univalent functions. J. Math. Inequal. **10**, 1063–1092 (2016)
27. Srivastava, H.M., Motamednezhad, A., Salehian, S.: Coefficients of a comprehensive subclass of meromorphic bi-univalent functions associated with the Faber polynomial expansion. Axioms **10**, 27 (2021)
28. Bazilevič, I.E.: On a class of integrability in quadratures of the Löwner Kufarev equation. Math. Sb. **37**, 471–476 (1955)
29. Thomas, D.K.: On Bazilevič functions. Transl. Am. Math. Soc. **132**, 353–361 (1968)
30. Keough, F.R., Miller, S.S.: On the coefficients of Bazilevič functions. Proc. Am. Math. Soc. **30**, 492–496 (1971)
31. Noor, K.I.: Bazilevič functions of type ρ. Int. J. Math. Math. Sci. **5**, 411–415 (1982)
32. Noor, K.I., Al-Bany, S.A.: On Bazilevič functions. Int. J. Math. Math. Sci. **10**, 79–88 (1987)
33. Owa, S., Obradović, M.: Certain subclasses of Bazilevič functions of type ρ. Int. J. Math. Math. Sci. **9**, 347–359 (1986)
34. Goodman, A.W.: Univalent Functions. Volumes I and II. Polygonal Publishing House, Washington (1983)
35. Janowski, W.: Some extremal problems for certain families of analytic functions. Ann. Pol. Math. **28**, 297–326 (1973)
36. Raina, R.K., Sokół, J.: Some properties related to a certain class of starlike functions. C. R. Acad. Sci. Paris, Ser. I **353**, 973–978 (2015)
37. Wani, L.A., Swaminathan, A.: Starlike and convex functions associated with a nephroid domain. Bull. Malays. Math. Sci. Soc. **44**, 79–104 (2021)
38. Paprocki, E., Sokół, J.: The extremal problems in some subclass of strongly starlike functions. Folia Sci. Univ. Technol. Resoviensis **157**, 89–94 (1996)
39. Cho, N.E., Kumar, V., Kumar, S.S., Ravichandran, V.: Radius problems for starlike functions associated with the sine function. Bull. Iran. Math. Soc. **45**, 213–232 (2019)
40. Dziok, J., Raina, R.K., Sokół, J.: Certain results for a class of convex functions related to shell-like curve connected with Fibonacci numbers. Comput. Math. Appl. **61**, 2606–2613 (2011)
41. Sokół, J., Stankiewicz, J.: Radius of convexity of some subclasses of strongly starlike functions. Zeszyty Nauk. Politech. Rzeszowskiej Mat. **19**, 101–105 (1996)
42. Kanas, S., Masih, V.S., Ebadian, A.: Coefficients problems for families of holomorphic functions related to hyperbola. Math. Slovaca **70**, 605–616 (2020)
43. Kargar, R., Ebadian, A., Sokół, J.: On booth lemniscate and starlike functions. Anal. Math. Phys. **9**, 143–154 (2019)
44. Malik, S.N., Mahmood, S., Raza, M., Farman, S., Zainab, S., Muhammad, N.: Coefficient inequalities of functions associated with hyperbolic domains. Mathematics **7**, 88 (2019)
45. Raza, M., Mushtaq, S., Malik, S.N., Sokół, J.: Coefficient inequalities for analytic functions associated with cardioid domains. Hacet. J. Math. Stat. **49**, 2017–2027 (2020)
46. Yunus, Y., Halim, S.A., Akbarally, A.B.: Subclass of starlike functions associated with a limaçon. In: Proceedings of the AIP, Conference 2018, Maharashtra, India, 5–6 July 2018. AIP Publishing, New York (2018)
47. Liu, L., Liu, J.-L.: Properties of certain multivalent analytic functions associated with the lemniscate of Bernoulli. Axioms **10**, 160 (2021)
48. Shafiq, M., Srivastava, H.M., Khan, N., Ahmad, Q.Z., Darus, M., Kiran, S.: An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers. Symmetry **12**, 1043 (2020). https://doi.org/10.3390/sym12061043
49. Sokół, J., Stankiewicz, J.: Radious of convexity of some subclasses of strongly starlike functions. Folia Sci. Univ. Tech. Resoviensis **147**, 101–105 (1996)
50. Sokół, J.: On starlike functions connected with Fibonacci numbers. Folia Sci. Univ. Technol. Resoviensis **175**, 111–116 (1999)
51. Airault, H.: Remarks on Faber polynomials. Int. Math. Forum **3**, 449–456 (2008)
52. Airault, H., Bouali, H.: Differential calculus on the Faber polynomials. Bull. Sci. Math. **130**, 179–222 (2006)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.