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Hankel determinant for a general subclass of m-fold symmetric biunivalent functions defined by Ruscheweyh operators

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Abstract

Making use of the Hankel determinant and the Ruscheweyh derivative, in this work, we consider a general subclass of m-fold symmetric normalized biunivalent functions defined in the open unit disk. Moreover, we investigate the bounds for the second Hankel determinant of this class and some consequences of the results are presented. In addition, to demonstrate the accuracy on some functions and conditions, most general programs are written in Python V.3.8.8 (2021).

Mathematics Subject Classification: 30C45; 30C50; 26A51; 26B05; 15A15

Keywords: Analytic and univalent functions; m-fold symmetric univalent functions; m-fold symmetric biunivalent functions; Ruscheweyh derivative; Hankel determinant

1 Introduction

Let $\mathcal{A}$ denote the class of the analytic functions f in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$ of the Taylor–Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_k z^k. \tag{1.1}$$

Further, assume that $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ that contains all univalent functions in $\mathbb{U}$ satisfying (1.1) and $\mathcal{P}$ represents the subclass of all functions $h(z)$ of the form

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U}$ and $\operatorname{Re}(h(z)) > 0$, $z \in \mathbb{U}$.

For a function $f \in \mathcal{A}$ defined by (1.1), the Ruscheweyh derivative operator (see [23]) is defined by

$$\mathcal{R}^{\gamma} f(z) = z + \sum_{k=2}^{\infty} \Omega(\gamma, k) a_k z^k,$$

where $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $z \in \mathbb{U}$, and

$$\Omega(\gamma, k) = \frac{\Gamma(\gamma + k)}{\Gamma(k)\Gamma(\gamma + 1)}.$$

The Koebe 1/4-theorem (see [12]) asserts that every univalent function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U}) \quad \text{and} \quad f\big(f^{-1}(w)\big) = w \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \big(2a_2^2 - a_3\big)w^3 - \big(5a_2^3 - 5a_2 a_3 + a_4\big)w^4 + \cdots. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be biunivalent if both f and f^{-1} satisfy the univalent property. The class of biunivalent functions in $\mathbb{U}$ is denoted by Σ. Some examples of functions in the class Σ are given as follows:

$$\frac{z}{1-z}, \qquad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

with the corresponding inverse functions

$$\frac{e^w - 1}{e^w}, \qquad \frac{w}{1+w} \quad \text{and} \quad \frac{e^{2w}-1}{e^{2w}+1},$$

respectively.

Determination of the estimates for the Taylor–Maclaurin coefficients a_n is a crucial problem in geometric function theory and provides knowledge about the geometric characteristics of these functions. Lewin [17] investigated the class Σ of biunivalent functions and showed that $|a_2| < 1.51$ for the functions belonging to the class Σ. Brannan and Clunie [8] conjectured that $|a_2| \leq \sqrt{2}$. Subsequently, Netanyahu [20] showed that max $|a_2| = \frac{4}{3}$ for $f \in \Sigma$. Srivastava et al. [26] improved the investigation for various subclasses of the biunivalent function class Σ and established bounds on $|a_2|$ and $|a_3|$ in recent years. Many recent studies are devoted to studying the biunivalent functions class Σ and obtaining nonsharp bounds on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (see, for example, [1, 7, 18, 29, 30]). However, the coefficient estimates bound of $|a_n| (n \in \{4, 5, 6, \ldots\})$ for a function $f \in \Sigma$ defined by (1.1) remains an open problem. In fact, there is no natural way to obtain the upper bound for coefficients greater than three. In exceptional cases, there are some articles in which Faber polynomial techniques were used for finding upper bounds for higher-order coefficients (see, for example, [4, 6, 31]).

The Hankel determinant is a valuable tool in studying univalent functions whose components are coefficients of functions in the subclasses of $\mathcal{S}$. The Hankel determinants $H_q(n)$ $(n, q \in \mathbb{N})$ of the function f are defined by (see [21])

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n + 2q - 2 \end{vmatrix} \quad (a_1 = 1).$$

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Estimates for the upper bounds of $|H_2(1)| = |a_3 - a_2^2|$ and $|H_2(2)| = |a_2a_4 - a_3^2|$ are called Fekete–Szegö and second Hankel determinant problems, respectively. Additionally, Fekete and Szegö [13] proposed the summarized functional $a_3 - \mu a_2^2$, in which μ is some real number. Lee et al. [16] presented a concise overview of Hankel determinants for analytic univalent functions and obtained bounds for $\mathrm{H}_2(2)$ for functions belonging to some classes defined by subordination. The estimation of $|\mathrm{H}_2(2)|$ has been the focus of recent Hankel determinant papers (see, for example, [5, 11, 22, 25, 32]).

For each function $f \in \mathcal{S}$, the function

$$h(z) = \left(f\left(z^m\right)\right)^{\frac{1}{m}} \quad (z \in \mathbb{U}, m \in \mathbb{N}) \tag{1.4}$$

is univalent and maps the unit disk into a region with m-fold symmetry. A function f is said to be m-fold symmetric (see [15]) and denoted by $\mathcal{A}_m$, if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.5}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (1.5). In fact, the functions in the class $\mathcal{S}$ are 1-fold symmetric. In view of the work of Koepf [15] the m-fold symmetric function $h \in \mathcal{P}$ is of the form

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots. \tag{1.6}$$

Analogous to the concept of m-fold symmetric univalent functions, Srivastava et al. [27] defined the concept of m-fold symmetric biunivalent functions in a direct way. Each function $f \in \Sigma$ generates an m-fold symmetric biunivalent function for each $m \in \mathbb{N}$. The normalized form of f is given as (1.5) and the extension $g = f^{-1}$ is as follows:

$$\begin{aligned} g(w) = {} & w - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots. \end{aligned} \tag{1.7}$$

We denote by Σ_m the class of m-fold symmetric biunivalent functions in $\mathbb{U}$. For $m = 1$, the series (1.7) coincides with the series (1.3) of the class Σ. Some examples of m-fold symmetric biunivalent functions are given as follows:

$$\left[\frac{z^m}{1-z^m}\right]^{\frac{1}{m}}, \qquad \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}} \quad \text{and} \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}},$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \qquad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}},$$

respectively.

Recently, some authors have studied the m-fold symmetric biunivalent function class Σ_{m} (see, for example, [9, 19, 28, 33]) and obtained nonsharp bound estimates on the first two Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. In this respect, Altinkaya and Yalçin [3] obtained nonsharp estimates on the second Hankel determinant for the subclass $H_{\Sigma_m}(\beta)$ of the m-fold symmetric biunivalent function class Σ_{m}.

For a function $f \in \mathcal{A}_m$ defined by (1.5), analogous to the Ruscheweyh derivative $\mathcal{R}^\gamma$: $\mathcal{A} \to \mathcal{A}$, the m-fold Ruscheweyh derivative $\mathcal{R}^\gamma : \mathcal{A}_m \to \mathcal{A}_m$ is defined as follows (see [24]):

$$\mathcal{R}^\gamma f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\gamma+k+1)}{\Gamma(k+1)\Gamma(\gamma+1)} a_{mk+1} z^{mk+1} \quad (\gamma \in \mathbb{N}_0, m \in \mathbb{N}, z \in \mathbb{U}).$$

Considering the significant role of the Hankel determinant in recent years, the object of this paper is to study estimates for $|H_2(2)|$ of a general subclass of m-fold symmetric biunivalent functions in $\mathbb{U}$ by applying the m-fold Ruscheweyh derivative operator and to obtain upper bounds on $|a_{m+1}a_{3m+1} - a_{2m+1}^2|$ for functions in the subclass $\Xi_{\Sigma_m}(\lambda, \gamma; \beta)$.

In order to derive our main results, we need to define the following lemmas that will be useful in proving the basic theorem of Sect. 2.

Lemma 1.1 [12] *If the function $h \in \mathcal{P}$ is given by the series* (1.2), *then*

$$|h_k| \le 2 \quad (k \in \mathbb{N}) \tag{1.8}$$

and

$$\left|h_2 - \frac{h_1^2}{2}\right| \le 2 - \frac{|h_2|^2}{2}. \tag{1.9}$$

Lemma 1.2 [14] *If the function $h \in \mathcal{P}$ is given by the series* (1.2), *then*

$$2h_2 = h_1^2 + x\left(4 - h_1^2\right) \tag{1.10}$$

and

$$4h_3 = h_1^3 + 2\left(4 - h_1^2\right)h_1 x - h_1\left(4 - h_1^2\right)x^2 + 2\left(4 - h_1^2\right)\left(1 - |x|^2\right)z, \tag{1.11}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

2 The main result and consequences

Our main aim in this section is to study estimates for the second Hankel determinant of the subclass $\Xi_{\Sigma_m}(\lambda,\gamma;\beta)$ of m-fold symmetric biunivalent functions in $\mathbb{U}$, and we show that our results are an improvement on the existing coefficient estimates.

Definition 2.1 A function $f \in \Sigma_m$ given by (1.5) is said to be in the class $\Xi_{\Sigma_m}(\lambda,\gamma;\beta)$ ($\lambda \geq 1$, $\gamma \in \mathbb{N}_0$, $0 \leq \beta < 1$ and $m \in \mathbb{N}$) if it satisfies the conditions

$$\operatorname{Re}\left\{(1-\lambda)\frac{\mathcal{R}^\gamma f(z)}{z} + \lambda\big(\mathcal{R}^\gamma f(z)\big)'\right\} > \beta \tag{2.1}$$

and

$$\operatorname{Re}\left\{(1-\lambda)\frac{\mathcal{R}^\gamma f(w)}{w} + \lambda\big(\mathcal{R}^\gamma f(w)\big)'\right\} > \beta, \tag{2.2}$$

where $z, w \in \mathbb{U}$ and the function $g = f^{-1}$ is given by (1.7).

Theorem 2.1 *Let $f \in \Xi_{\Sigma_m}(\lambda,\gamma;\beta)$ be given by (1.5). Then,*

$$\begin{aligned}
&\left|a_{m+1}a_{3m+1} - a_{2m+1}^2\right| \\
&\quad\leq \begin{cases}
\frac{4(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)}\left[\frac{(m+1)^2(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)^3} + \frac{6}{(\gamma+2)(\gamma+3)(3m\lambda+1)}\right], & \beta \in [0,\tau], \\
\frac{4(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}\left[4 - \frac{[\omega_2(1-\beta)+9\omega_3-4\omega_4]^2}{\omega_4[\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4]}\right], & \beta \in [\tau,1),
\end{cases}
\end{aligned}$$

where

$$\omega_1 := (m+1)^2(\gamma+2)^2(\gamma+3)(2m\lambda+1)^2(3m\lambda+1), \tag{2.3}$$

$$\omega_2 := m(\gamma+1)(\gamma+2)(\gamma+3)(m\lambda+1)^2(2m\lambda+1)(3m\lambda+1), \tag{2.4}$$

$$\omega_3 := (\gamma+1)^2(\gamma+2)(m\lambda+1)^3(2m\lambda+1)^2, \tag{2.5}$$

$$\omega_4 := (\gamma+1)^2(\gamma+3)(m\lambda+1)^4(3m\lambda+1) \tag{2.6}$$

and

$$\tau := 1 - \frac{\omega_2 + \sqrt{\omega_2^2 + 12\omega_1\omega_3}}{2\omega_1}.$$

Proof It follows from (2.1) and (2.2) that there exist p and q in the class $\mathcal{P}$ such that

$$(1-\lambda)\frac{\mathcal{R}^\gamma f(z)}{z} + \lambda\big(\mathcal{R}^\gamma f(z)\big)' = \beta + (1-\beta)p(z) \tag{2.7}$$

and

$$(1-\lambda)\frac{\mathcal{R}^\gamma f(w)}{w} + \lambda\big(\mathcal{R}^\gamma f(w)\big)' = \beta + (1-\beta)q(z), \tag{2.8}$$

where p and q are given by the series (1.6).

We also find that

$$\begin{aligned}
&(1-\lambda)\frac{\mathcal{R}^{\gamma}f(z)}{z}+\lambda\big(\mathcal{R}^{\gamma}f(z)\big)' \\
&\quad=1+(\gamma+1)(m\lambda+1)a_{m+1}z^{m}+\frac{1}{2}(\gamma+1)(\gamma+2)(2\lambda m+1)a_{2m+1}z^{2m} \\
&\qquad+\frac{1}{6}(\gamma+1)(\gamma+2)(\gamma+3)(3\lambda m+1)a_{3m+1}z^{3m}+\cdots
\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}
&(1-\lambda)\frac{\mathcal{R}^{\gamma}g(w)}{w}+\lambda\big(\mathcal{R}^{\gamma}g(w)\big)' \\
&\quad=1-(\gamma+1)(m\lambda+1)a_{m+1}w^{m}+\frac{1}{2}(\gamma+1)(\gamma+2)(2m\lambda+1) \\
&\qquad\times\big[(m+1)a_{m+1}^{2}-a_{2m+1}\big]w^{2m}-\frac{1}{6}(\gamma+1)(\gamma+2)(\gamma+3)(3m\lambda+1) \\
&\qquad\times\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^{3}-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]w^{3m}+\cdots.
\end{aligned}\tag{2.10}$$

Equating coefficients in (2.7) and (2.8) we have

$$(\gamma+1)(m\lambda+1)a_{m+1}=(1-\beta)p_{m},\tag{2.11}$$

$$\frac{1}{2}(\gamma+1)(\gamma+2)(2\lambda m+1)a_{2m+1}=(1-\beta)p_{2m},\tag{2.12}$$

$$\frac{1}{6}(\gamma+1)(\gamma+2)(\gamma+3)(3m\lambda+1)a_{3m+1}=(1-\beta)p_{3m}\tag{2.13}$$

and

$$-(\gamma+1)(m\lambda+1)a_{m+1}=(1-\beta)q_{m},\tag{2.14}$$

$$\frac{1}{2}(\gamma+1)(\gamma+2)(2m\lambda+1)\big[(m+1)a_{m+1}^{2}-a_{2m+1}\big]=(1-\beta)q_{2m},\tag{2.15}$$

$$\begin{aligned}
&-\frac{1}{6}(\gamma+1)(\gamma+2)(\gamma+3)(3m\lambda+1) \\
&\quad\times\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^{3}-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]=(1-\beta)q_{3m}.
\end{aligned}\tag{2.16}$$

From (2.11) and (2.14), we obtain

$$p_{m}=-q_{m}\tag{2.17}$$

and

$$a_{m+1}=\frac{1-\beta}{(\gamma+1)(m\lambda+1)}p_{m}.\tag{2.18}$$

Now, from (2.12), (2.15), and (2.18), we obtain

$$a_{2m+1}=\frac{(m+1)(1-\beta)^{2}}{2(\gamma+1)^{2}(m\lambda+1)^{2}}p_{m}^{2}+\frac{(1-\beta)}{(\gamma+1)(\gamma+2)(2m\lambda+1)}(p_{2m}-q_{2m}).\tag{2.19}$$

Also, from (2.13), (2.16), (2.18), and (2.19), we find that

$$\begin{aligned} a_{3m+1} &= \frac{(3m+2)(1-\beta)^2}{2(\gamma+1)^2(\gamma+2)(m\lambda+1)(2m\lambda+1)} p_m(p_{2m}-q_{2m}) \\ &\quad + \frac{3(1-\beta)}{(\gamma+1)(\gamma+2)(\gamma+3)(3m\lambda+1)}(p_{3m}-q_{3m}). \end{aligned} \tag{2.20}$$

Then, from (2.18), (2.19), and (2.20) we have that

$$\begin{aligned} & a_{m+1}a_{3m+1}-a_{2m+1}^2 \\ &\quad = -\frac{(m+1)^2(1-\beta)^4}{4(\gamma+1)^4(m\lambda+1)^4} p_m^4 \\ &\qquad + \frac{m(1-\beta)^3}{2(\gamma+1)^3(\gamma+2)(m\lambda+1)^2(2m\lambda+1)} p_m^2(p_{2m}-q_{2m}) \\ &\qquad + \frac{3(1-\beta)^2}{(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} p_m(p_{3m}-q_{3m}) \\ &\qquad - \frac{(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}(p_{2m}-q_{2m})^2. \end{aligned} \tag{2.21}$$

According to Lemma 1.2 and (2.17), we can write

$$p_{2m}-q_{2m} = \frac{4-p_m^2}{2}(x-y) \tag{2.22}$$

and

$$\begin{aligned} p_{3m}-q_{3m} &= \frac{p_m^3}{2} + \frac{p_m(4-p_m^2)}{2}(x+y) - \frac{p_m(4-p_m^2)}{4}\left(x^2+y^2\right) \\ &\quad + \frac{4-p_m^2}{2}\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right], \end{aligned} \tag{2.23}$$

$$p_{2m}+q_{2m} = p_m^2 + \frac{4-p_m^2}{2}(x+y), \tag{2.24}$$

for some x, y, z, and w with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$. Using (2.22) and (2.23) in (2.21) we obtain

$$\begin{aligned} & \left|a_{m+1}a_{3m+1}-a_{2m+1}^2\right| \\ &\quad = \left| -\frac{(m+1)^2(1-\beta)^4}{4(\gamma+1)^4(m\lambda+1)^4} p_m^4 + \frac{m(1-\beta)^3}{4(\gamma+1)^3(\gamma+2)(m\lambda+1)^2(2m\lambda+1)} \right. \\ &\qquad \times p_m^2\left(4-p_m^2\right)(x-y) \\ &\qquad + \frac{3(1-\beta)^2}{(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} p_m\left[\frac{p_m^3}{2} + \frac{p_m(4-p_m^2)}{2}(x+y) \right. \\ &\qquad - \frac{p_m(4-p_m^2)}{4}\left(x^2+y^2\right) + \frac{4-p_m^2}{2}\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right] \end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\beta)^2}{4(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}\left(4-p_m^2\right)^2(x-y)^2 \Bigg| \\
&\leq \frac{(m+1)^2(1-\beta)^4}{4(\gamma+1)^4(m\lambda+1)^4}p_m^4 + \frac{3(1-\beta)^2}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)}p_m^4 \\
&\quad + \frac{3(1-\beta)^2}{(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)}p_m\left(4-p_m^2\right) \\
&\quad + \left[\frac{m(1-\beta)^3}{4(\gamma+1)^3(\gamma+2)(m\lambda+1)^2(2m\lambda+1)}p_m^2\left(4-p_m^2\right)\right. \\
&\quad \left. + \frac{3(1-\beta)^2}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)}p_m^2\left(4-p_m^2\right)\right]\left(|x|+|y|\right) \\
&\quad + \left[\frac{3(1-\beta)^2}{4(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)}p_m^2\left(4-p_m^2\right)\right. \\
&\quad \left. - \frac{3(1-\beta)^2}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)}p_m\left(4-p_m^2\right)\right]\left(|x|^2+|y|^2\right) \\
&\quad + \frac{(1-\beta)^2}{4(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}\left(4-p_m^2\right)^2\left(|x|+|y|\right)^2.
\end{aligned}
$$

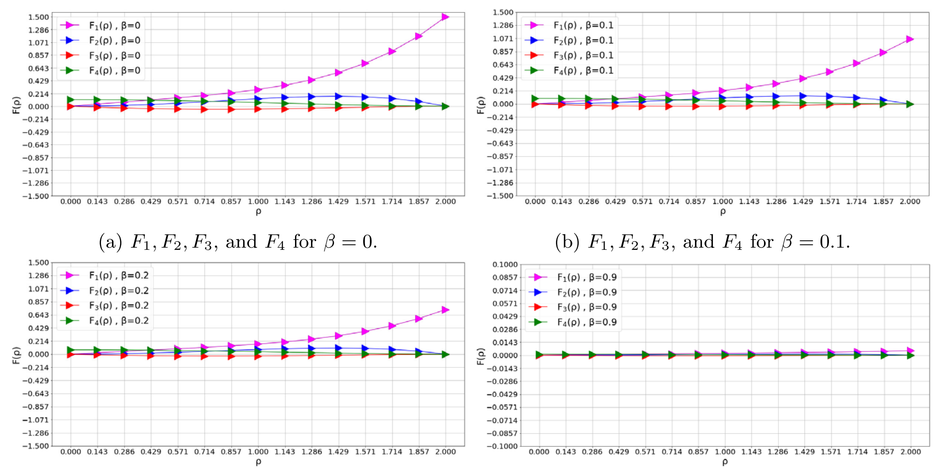
Since p in the class $\mathcal{P}$, we have (Lemma 1.1) $|p_m| \leq 2$. Letting $p_m = \rho$, we may assume, without loss of generality, that $\rho \in [0,2]$. Thus, for $\mu_1 = |x| \leq 1$ and $\mu_2 = |y| \leq 1$, we obtain

$$
\left|a_{m+1}a_{3m+1} - a_{2m+1}^2\right| \leq F_1 + F_2(\mu_1+\mu_2) + F_3\left(\mu_1^2+\mu_2^2\right) + F_4(\mu_1+\mu_2)^2,
$$

where

$$
\begin{aligned}
F_1 &= F_1(\rho) \\
&= \frac{(m+1)^2(1-\beta)^4\rho^4}{4(\gamma+1)^4(m\lambda+1)^4} + \frac{3(1-\beta)^2\rho^4}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} \\
&\quad + \frac{3(1-\beta)^2\rho(4-\rho^2)}{(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} \geq 0, \\
F_2 &= F_2(\rho) \\
&= \frac{m(1-\beta)^3\rho^2(4-\rho^2)}{4(\gamma+1)^3(\gamma+2)(m\lambda+1)^2(2m\lambda+1)} \\
&\quad + \frac{3(1-\beta)^2\rho^2(4-\rho^2)}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} \geq 0, \\
F_3 &= F_3(\rho) \\
&= \frac{3(1-\beta)^2\rho^2(4-\rho^2)}{4(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} \\
&\quad - \frac{3(1-\beta)^2\rho(4-\rho^2)}{2(\gamma+1)^2(\gamma+2)(\gamma+3)(m\lambda+1)(3m\lambda+1)} \leq 0, \\
F_4 &= F_4(\rho) = \frac{(1-\beta)^2(4-\rho^2)^2}{4(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2} \geq 0.
\end{aligned}
$$

Figure 1 demonstrates that F_1, F_2, F_4 are non-negatives, and F_3 is non-positive.



(a) F_1, F_2, F_3, and F_4 for $\beta = 0$.

(b) F_1, F_2, F_3, and F_4 for $\beta = 0.1$.

(c) F_1, F_2, F_3, and F_4 for $\beta = 0.2$.

(d) F_1, F_2, F_3, and F_4 for $\beta = 0.9$.

Figure 1 Graph of F_1, F_2, F_3, and F_4 for $\gamma = 0$ and $\lambda = m = 1$

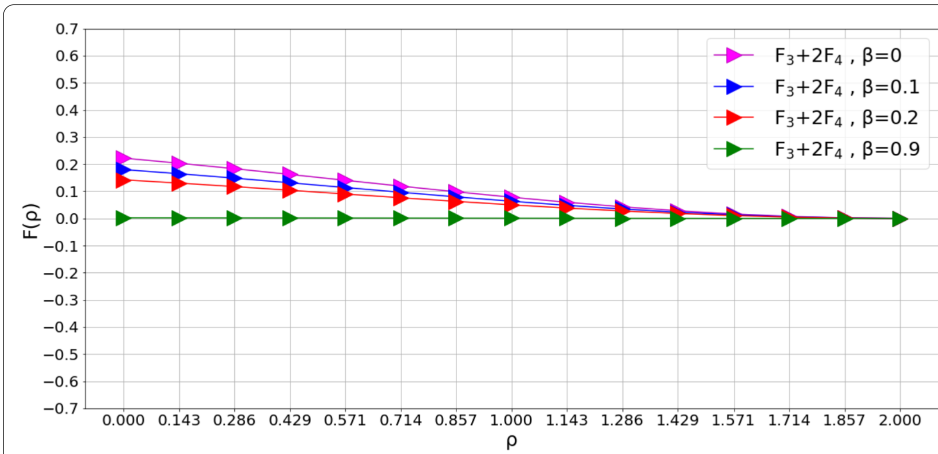


Figure 2 Graph of $F_3 + 2F_4$ for $\gamma = 0$ and $\lambda = m = 1$

Now, we need to maximize

$$F(\mu_1, \mu_2) = F_1 + F_2(\mu_1 + \mu_2) + F_3\left(\mu_1^2 + \mu_2^2\right) + F_4(\mu_1 + \mu_2)^2$$

in the closed square $\mathbb{S} = [0,1] \times [0,1]$ for $\rho \in [0,2]$. We investigate the maximum of $F(\mu_1, \mu_2)$ when $\rho \in (0,2)$, $\rho = 0$ and $\rho = 2$, keeping in mind the sign of

$$F_{\mu_1\mu_1}F_{\mu_2\mu_2} - (F_{\mu_1\mu_2})^2$$

(according to the Second Derivative Test for functions of the two dependent variables μ_1 and μ_2).

First, let $\rho \in (0,2)$. Since $F_3 < 0$ and $F_3 + 2F_4 > 0$ for $\rho \in (0,2)$ (see Fig. 2), we see that

$$F_{\mu_1\mu_1}F_{\mu_2\mu_2} - (F_{\mu_1\mu_2})^2 = 4F_3\left(F_3 + 2F_4\right) < 0.$$

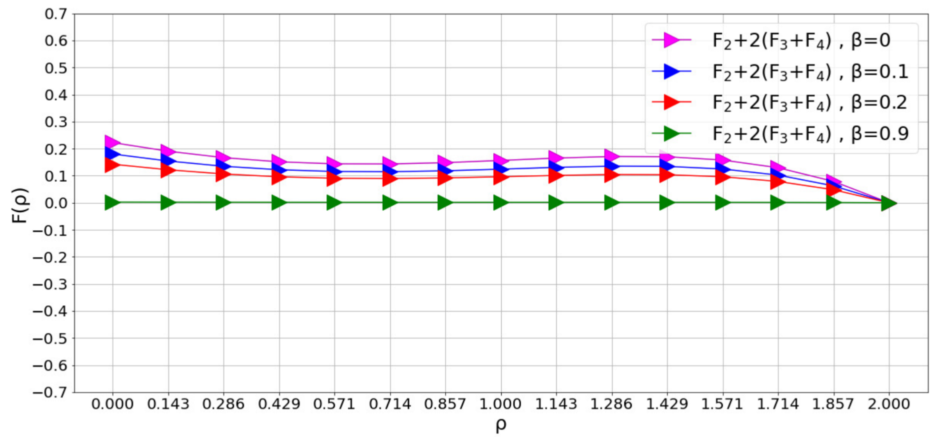


Figure 3 Graph of $F_2 + 2(F_3 + F_4)$ for $\gamma = 0$ and $\lambda = m = 1$

Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

Case 1. For $\mu_1 = 0$ and $\mu_2 \in [0, 1]$ (a similar argument can be applied for $\mu_2 = 0$ and $\mu_1 \in [0, 1]$, so we omit the details in that case), we obtain

$$F(0, \mu_2) = G(\mu_2) \equiv F_1 + F_2\mu_2 + (F_3 + F_4)\mu_2^2.$$

Subcase 1. Let $F_3 + F_4 \geq 0$. In this case, for $0 < \mu_2 < 1$ we have that

$$G'(\mu_2) = F_2 + 2(F_3 + F_4)\mu_2 > 0,$$

that is, $G(\mu_2)$ is an increasing function. Hence, the maximum of $G(\mu_2)$ occurs at $\mu_2 = 1$ and

$$\max\{F(0, \mu_2) : \mu_2 \in [0, 1]\} = \max\{G(\mu_2) : \mu_2 \in [0, 1]\} = G(1) = F_1 + F_2 + F_3 + F_4.$$

Subcase 2. Let $F_3 + F_4 < 0$. Note that (see Fig. 3):

$$F_2 + 2(F_3 + F_4) \geq 0.$$

For $\mu_2 \in (0, 1)$ since $F_3 + F_4 < 0$ we have that

$$F_2 + 2(F_3 + F_4)\mu_2 > F_2 + 2(F_3 + F_4) \geq 0,$$

so $G'(\mu_2) > 0$. Thus, $\max\{G(\mu_2) : \mu_2 \in [0, 1]\} = G(1)$.

Case 2. For $\mu_1 = 1$ and $\mu_2 \in [0, 1]$ (a similar argument can be applied for $\mu_2 = 1$ and $\mu_1 \in [0, 1]$, so we omit the details in that case), we obtain

$$F(1, \mu_2) = H(\mu_2) = F_1 + F_2 + F_3 + F_4 + (F_2 + 2F_4)\mu_2 + (F_3 + F_4)\mu_2^2.$$

Thus, an argument like in Subcases 1 and 2 yields

$$\max\{F(1, \mu_2) : \mu_2 \in [0, 1]\} = \max\{H(\mu_2) : \mu_2 \in [0, 1]\} = H(1) = F_1 + 2(F_2 + F_3) + 4F_4.$$

Next, let $\rho = 2$. Now, let $(\mu_1, \mu_2) \in \mathbb{S}$ and note that

$$F(\mu_1,\mu_2) = \frac{4(\gamma+2)(\gamma+3)(3m\lambda+1)(m+1)^2(1-\beta)^4 + 24(\gamma+1)^2(m\lambda+1)^3(1-\beta)^2}{(\gamma+1)^4(\gamma+2)(\gamma+3)(m\lambda+1)^4(3m\lambda+1)}. \tag{2.25}$$

Keeping in mind the constant value in (2.25) we have

$$\max\{F(\mu_1,\mu_2) : \mu_1 \in [0,1], \mu_2 \in [0,1]\} = F(1,1) = F_1 + 2(F_2 + F_3) + 4F_4.$$

Finally, let $\rho = 0$. Now, let $(\mu_1, \mu_2) \in \mathbb{S}$ and note that

$$F(\mu_1,\mu_2) = \frac{4(1-\beta)^2(\mu_1+\mu_2)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}.$$

We see that the maximum of $F(\mu_1, \mu_2)$ occurs at $\mu_1 = \mu_2 = 1$ and

$$\max\{F(\mu_1,\mu_2) : \mu_1 \in [0,1], \mu_2 \in [0,1]\} = F(1,1) = F_1 + 2(F_2 + F_3) + 4F_4.$$

Combining all cases, note that since $F_1 + 2(F_2 + F_3) + 4F_4 \geq 0$ when $\rho \in [0, 2]$ (see Fig. 4), we have

$$\max\{F(\mu_1,\mu_2) : \mu_1 \in [0,1], \mu_2 \in [0,1]\} = F(1,1).$$

Let $K : [0, 2] \to \mathbb{R}$ be given by

$$K(\rho) = F(1,1) = F_1 + 2(F_2 + F_3) + 4F_4. \tag{2.26}$$

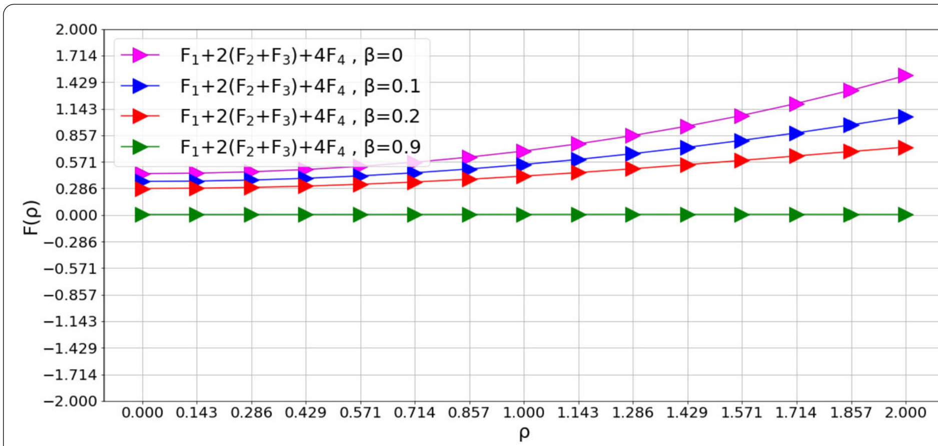


Figure 4 Graph of $F_1 + 2(F_2 + F_3) + 4F_4$ for $\gamma = 0$ and $\lambda = m = 1$

Substituting the values of F_1, F_2, F_3, and F_4 in the function K defined by (2.26), yields

$$\begin{aligned}
&K(\rho)\\
&\quad= \frac{(1-\beta)^2}{4(\gamma+1)^4(\gamma+2)^2(\gamma+3)(1+m\lambda)^4(1+2m\lambda)^2(1+3m\lambda)}\\
&\qquad\times \Big[\big[(m+1)^2(\gamma+2)^2(\gamma+3)(2m\lambda+1)^2(3m\lambda+1)(1-\beta)^2\\
&\qquad- 2m(\gamma+1)(\gamma+2)(\gamma+3)(m\lambda+1)^2(2m\lambda+1)(3m\lambda+1)(1-\beta)\\
&\qquad- 12(\gamma+1)^2(\gamma+2)(m\lambda+1)^3(2m\lambda+1)^2 + 4(\gamma+1)^2(\gamma+3)(m\lambda+1)^4(3m\lambda+1)\big]\rho^4\\
&\qquad+ \big[8m(\gamma+1)(\gamma+2)(\gamma+3)(1+m\lambda)^2(1+2m\lambda)(1+3m\lambda)(1-\beta)\\
&\qquad+ 72(\gamma+1)^2(\gamma+2)(m\lambda+1)^3(2m\lambda+1)^2 - 32(\gamma+1)^2(\gamma+3)(m\lambda+1)^4(3m\lambda+1)\big]\rho^2\\
&\qquad+ 64(\gamma+1)^2(\gamma+3)(1+m\lambda)^4(1+3m\lambda)\Big].
\end{aligned}$$

Now, the maximum of $K(\rho)$ occurs either at $\rho = 0$, $\rho \in (0,2)$ or $\rho = 2$. Suppose first the maximum of $K(\rho)$ occurs at some $\rho \in (0,2)$. Note that for any $\rho \in (0,2)$ we have

$$\begin{aligned}
&K'(\rho)\\
&\quad= \frac{(1-\beta)^2}{(\gamma+1)^4(\gamma+2)^2(\gamma+3)(1+m\lambda)^4(1+2m\lambda)^2(1+3m\lambda)}\\
&\qquad\times \Big[\big[(m+1)^2(\gamma+2)^2(\gamma+3)(2m\lambda+1)^2(3m\lambda+1)(1-\beta)^2\\
&\qquad- 2m(\gamma+1)(\gamma+2)(\gamma+3)(m\lambda+1)^2(2m\lambda+1)(3m\lambda+1)(1-\beta)\\
&\qquad- 12(\gamma+1)^2(\gamma+2)(m\lambda+1)^3(2m\lambda+1)^2 + 4(\gamma+1)^2(\gamma+3)(m\lambda+1)^4(3m\lambda+1)\big]\rho^3\\
&\qquad+ \big[4m(\gamma+1)(\gamma+2)(\gamma+3)(\lambda m+1)^2(1+2m\lambda)(1+3m\lambda)(1-\beta)\\
&\qquad+ 36(\gamma+1)^2(\gamma+2)(m\lambda+1)^3(2m\lambda+1)^2 - 16(\gamma+1)^2(\gamma+3)(m\lambda+1)^4(3m\lambda+1)\big]\rho\Big].
\end{aligned}$$

Next, we conclude the following results:
Result 1. Let

$$\omega_1(1-\beta)^2 - 2\omega_2(1-\beta) - 12\omega_3 + 4\omega_4 \geq 0,$$

that is,

$$\beta \in \left[0, 1 - \frac{\omega_2 + \sqrt{\omega_2^2 + \omega_1[12\omega_3 - 4\omega_4]}}{\omega_1}\right],$$

where ω_1, ω_2, ω_3, and ω_4 are given by (2.3), (2.4), (2.5), and (2.6), respectively.
Note that $K'(\rho) > 0$ for every $\rho \in (0,2)$. Thus,

$$\begin{aligned}
&\max\{K(\rho) : 0 < \rho < 2\}\\
&\quad= K(2^-) = \frac{4(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)}\left[\frac{(m+1)^2(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)^3} + \frac{6}{(\gamma+2)(\gamma+3)(3m\lambda+1)}\right].
\end{aligned}$$

Result 2. Let

$$\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4<0,$$

that is,

$$\beta\in\left(1-\frac{\omega_2+\sqrt{\omega_2^2+\omega_1[12\omega_3-4\omega_4]}}{\omega_1},1\right).$$

Then, $K'(\rho)=0$ gives the critical point $\rho_1=0$ or

$$\rho_2=\sqrt{\frac{16\omega_4-4\omega_2(1-\beta)-36\omega_3}{\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4}}.$$

When

$$\beta\in\left(1-\frac{\omega_2+\sqrt{\omega_2^2+\omega_1[12\omega_3-4\omega_4]}}{\omega_1},1-\frac{\omega_2+\sqrt{\omega_2^2+12\omega_1\omega_3}}{2\omega_1}\right],$$

we observe that $\rho_2\geq 2$. Then, the maximum value of $K(\rho)$ occurs at 0^+ or 2^-. This is a contradiction since we assumed the maximum of $K(\rho)$ occurs at some $\rho\in(0,2)$.

When

$$\beta\in\left(1-\frac{\omega_2+\sqrt{\omega_2^2+12\omega_1\omega_3}}{2\omega_1},1\right),$$

we observe that $\rho_2\in(0,2)$. Since $K''(\rho_2)<0$, the maximum value of $K(\rho)$ occurs at $\rho=\rho_2$. Thus, we have

$$\begin{aligned}\max&\{K(\rho):\rho\in(0,2)\}\\&=K(\rho_2)\\&=\frac{4(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}\\&\quad\times\left[4-\frac{[\omega_2(1-\beta)+9\omega_3-4\omega_4]^2}{\omega_4[\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4]}\right].\end{aligned}\tag{2.27}$$

Next, suppose if $\beta\in[0,\tau]$ and the maximum of $K(\rho)$ occurs at $\rho=2$. Then,

$$\begin{aligned}\max&\{K(\rho):\rho\in[0,2]\}\\&=K(2)\\&=\frac{4(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)}\left[\frac{(m+1)^2(1-\beta)^2}{(\gamma+1)^2(m\lambda+1)^3}+\frac{6}{(\gamma+2)(\gamma+3)(3m\lambda+1)}\right].\end{aligned}$$

We only now need to note that (see the idea in the second part of Result 2) if $\beta \in (\tau, 1)$ then the maximum of $K(\rho)$ cannot occur at $\rho = 2$ since

$$K(2) \le \frac{4(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2} \times \left[4 - \frac{[\omega_2(1-\beta)+9\omega_3-4\omega_4]^2}{\omega_4[\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4]}\right] \bigl(=K(\rho_2)\bigr).$$

Finally, let us consider $\beta \in [0,1)$ and the maximum of $K(\rho)$ occurring at $\rho = 0$. Then,

$$\max\bigl\{K(\rho):\rho\in[0,2]\bigr\} = K(0) = \frac{16(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2}.$$

We note that (see the ideas in the second part of Result 2) if $\beta \in (\tau, 1)$ then the maximum of $K(\rho)$ cannot occur at $\rho = 0$ since

$$K(0) \le \frac{4(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2m\lambda+1)^2} \times \left[4 - \frac{[\omega_2(1-\beta)+9\omega_3-4\omega_4]^2}{\omega_4[\omega_1(1-\beta)^2-2\omega_2(1-\beta)-12\omega_3+4\omega_4]}\right] \bigl(=K(\rho_2)\bigr).$$

Finally, note that (see the ideas in Result 1 and the details in Result 2) if

$$\beta \in \left[0, 1 - \frac{\omega_2 + \sqrt{\omega_2^2 + \omega_1[12\omega_3 - 4\omega_4]}}{\omega_1}\right],$$

or

$$\beta \in \left(1 - \frac{\omega_2 + \sqrt{\omega_2^2 + \omega_1[12\omega_3 - 4\omega_4]}}{\omega_1}, 1\right),$$

then the maximum of $K(\rho)$ cannot occur at $\rho = 0$ since $K(0) \le K(2)$.

This completes the proof. □

By setting $\lambda = 1$ and $\gamma = 0$ in Theorem 2.1, we obtain the following consequence.

Corollary 2.1 [3] *Let $f \in \Xi_{\Sigma_m}(\beta)$ $(0 \le \beta < 1)$ be given by* (1.5). *Then,*

$$\left|a_{m+1}a_{3m+1} - a_{2m+1}^2\right| \le \begin{cases} \frac{4(1-\beta)^2}{m+1}\left[\frac{(1-\beta)^2}{m+1} + \frac{1}{3m+1}\right], & \beta \in [0,\nu], \\ \frac{(1-\beta)^2}{(2m+1)^2}\left[4 - \frac{[m(1-\beta)\psi_1+3\psi_2-2\psi_3]^2}{\psi_3[(2m+1)(1-\beta)^2\psi_1-m(1-\beta)\psi_1+\psi_3-2\psi_2]}\right], & \beta \in [\nu,1), \end{cases}$$

where

$$\psi_1 := (2m+1)(3m+1),$$

$$\psi_2 := (m+1)(2m+1)^2,$$

$$\psi_3 := (m+1)^2(3m+1)$$

and

$$\nu := \frac{(3m+1)(7m+4) - \sqrt{m^2(3m+1)^2 + 8\psi_2(3m+1)}}{4\psi_1}.$$

By taking $m = 1$ in Theorem 2.1, we conclude the following result.

Corollary 2.2 *Let $f \in \Xi_\Sigma(\lambda, \gamma; \beta)$ $(\lambda \geq 1, \gamma \in \mathbb{N}_0, 0 \leq \beta < 1)$ be given by (1.1). Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{8(1-\beta)^2}{(\gamma+1)^2(\lambda+1)}\left[\frac{2(1-\beta)^2}{(\gamma+1)^2(\lambda+1)^3} + \frac{3}{(\gamma+2)(\gamma+3)(3\lambda+1)}\right], & \beta \in [0, \xi], \\ \frac{4(1-\beta)^2}{(\gamma+1)^2(\gamma+2)^2(2\lambda+1)^2}\left[4 - \frac{[\vartheta_2(1-\beta)+9\vartheta_3-4\vartheta_4]^2}{\vartheta_4[4\vartheta_1(1-\beta)^2-2\vartheta_2(1-\beta)-12\vartheta_3+4\vartheta_4]}\right], & \beta \in [\xi, 1), \end{cases}$$

where

$$\vartheta_1 := (\gamma + 2)^2(\gamma + 3)(2\lambda + 1)^2(3\lambda + 1),$$

$$\vartheta_2 := (\gamma + 1)(\gamma + 2)(\gamma + 3)(\lambda + 1)^2(2\lambda + 1)(3\lambda + 1),$$

$$\vartheta_3 := (\gamma + 1)^2(\gamma + 2)(\lambda + 1)^3(2\lambda + 1)^2,$$

$$\vartheta_4 := (\gamma + 1)^2(\gamma + 3)(\lambda + 1)^4(3\lambda + 1)$$

and

$$\xi := 1 - \frac{\vartheta_2 + \sqrt{\vartheta_2^2 + 48\vartheta_1\vartheta_3}}{8\vartheta_1}.$$

Remark 2.1 Corollary 2.2 improves a result in Altinkaya and Yalçin [2, Theorem 3].

By putting $\gamma = 0$ in Corollary 2.2, we obtain the following result.

Corollary 2.3 *Let $f \in \Xi_\Sigma(\lambda; \beta)$ $(\lambda \geq 1, 0 \leq \beta < 1)$ be given by (1.1). Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{8(1-\beta)^2}{\lambda+1}\left[\frac{2(1-\beta)^2}{(\lambda+1)^3} + \frac{1}{2(3\lambda+1)}\right], & \beta \in [0, \epsilon], \\ \frac{(1-\beta)^2}{(2\lambda+1)^2}\left[4 - \frac{[\eta_2(1-\beta)+3\eta_3-2\eta_4]^2}{\eta_4[8\eta_1(1-\beta)^2-2\eta_2(1-\beta)-4\eta_3+2\eta_4]}\right], & \beta \in [\epsilon, 1), \end{cases}$$

where

$$\eta_1 := (2\lambda + 1)^2(3\lambda + 1),$$

$$\eta_2 := (\lambda + 1)^2(2\lambda + 1)(3\lambda + 1),$$

$$\eta_3 := (\lambda + 1)^3(2\lambda + 1)^2,$$

$$\eta_4 := (\lambda + 1)^4(3\lambda + 1)$$

and

$$\epsilon := 1 - \frac{(\lambda + 1)^2(3\lambda + 1) + \sqrt{(\lambda + 1)^4(3\lambda + 1)^2 + 32(\lambda + 1)^3(2\lambda + 1)^2(3\lambda + 1)}}{16(2\lambda + 1)(3\lambda + 1)}.$$

Remark 2.2 Corollary 2.3 improves a result in Altinkaya and Yalçin [2, Corollary 5].

By setting $\lambda = 1$ in Corollary 2.3, we obtain the following consequence.

Corollary 2.4 [10] *Let $f \in \Xi_\Sigma(\beta)$ $(0 \le \beta < 1)$ be given by* (1.1). *Then,*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} (1-\beta)^2[(1-\beta)^2 + \frac{1}{2}], & \beta \in [0, \frac{11-\sqrt{37}}{12}], \\ \frac{(1-\beta)^2}{32}[\frac{60\beta^2-84\beta-25}{9\beta^2-15\beta+1}], & \beta \in [\frac{11-\sqrt{37}}{12}, 1). \end{cases}$$

Remark 2.3 Corollary 2.4 recovers a result in Altinkaya and Yalçin [2, Corollary 4].

3 Concluding remarks

In this investigation, we consider a constructed subclass $\Xi_{\Sigma_m}(\lambda, \gamma; \beta)$ of the class Σ_m of m-fold symmetric biunivalent functions and several properties of the results are discussed. Moreover, with a specialization of the parameters, some consequences of the class are mentioned and they improve some existing upper bounds for $H_2(2)$ on certain subclasses of 1-fold symmetric biunivalent functions.

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