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Ground state normalized solutions to the Kirchhoff equation with general nonlinearities: mass supercritical case

Qun Wang[1] and Aixia Qian[1*]

*Correspondence:
qaixia@qfnu.edu.cn
[1]School of Mathematical Sciences, Qufu Normal University, Shandong 273165, P.R. China

Abstract

We study the following nonlinear mass supercritical Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u+\mu u=f(u)\quad \text{in } \mathbb{R}^N,$$

where $a,b,m>0$ are prescribed, and the normalized constrain $\int_{\mathbb{R}^N}|u|^2\,dx=m$ is satisfied in the case $1\le N\le 3$. The nonlinearity f is more general and satisfies weak mass supercritical conditions. Under some mild assumptions, we establish the existence of ground state when $1\le N\le 3$ and obtain infinitely many radial solutions when $2\le N\le 3$ by constructing a particular bounded Palais–Smale sequence.

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1 Introduction and main results

In this paper, we are concerned with the existence of ground normalized solutions and infinitely many radial normalized solutions to the following nonlinear Kirchhoff-type problem

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u+\mu u=f(u) \tag{1.1}$$

having the prescribed mass

$$\int_{\mathbb{R}^N}|u|^2\,dx=m \tag{1.2}$$

for a priori given $a,b,m>0$ and f satisfying appropriate assumptions.

To find the solutions of equation (1.1), we have two different choices. We can fix $\mu \in \mathbb{R}$ and look for solutions as critical points of the associated energy functional

$$I_\mu(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\mu\|u\|_2^2 - \int_{\mathbb{R}^N} F(u)\,dx,$$

where $F(t) = \int_0^t f(s)\,ds$. Alternatively, we can look for solutions to (1.1) having prescribed mass (1.2). In this case the frequency $\mu \in \mathbb{R}$ is part of the unknown and will appear as a Lagrange multiplier. Similarly to the first case, the solutions of (1.1) are critical points of the functional

$$I(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^N} F(u)\,dx \tag{1.3}$$

under the constraint

$$u \in S_m = \left\{u \in H^1\left(\mathbb{R}^N\right) : \|u\|_2^2 = m\right\}. \tag{1.4}$$

This case seems particularly interesting from the physical point of view, and hence we focus on this issue.

Recall that in the case where $f(u) = |u|^{p-2}u$, Ye [1] first proposed that the mass critical exponent for the Kirchhoff constraint minimization problem should be

$$\overline{p} = 2 + \frac{8}{N},$$

which is the threshold exponent for many dynamical properties. The mass critical exponent divides the fixed mass problem into the following three cases: the mass subcritical, critical, and supercritical cases. At first, Ye [1] made a detailed analysis of the existence behavior of the normalized solution in these three cases. Luo and Wang [2] considered the multiplicity of solutions in the mass supercritical case where $N = 3$. Later, Ye [3] made a further study in the mass critical case, obtaining a mountain pass critical point. Afterward, the problem involving potentials such as trapping and periodic potentials has been exploited and further developed in other contexts; we refer to [4–8].

Recently, normalized solutions for Kirchhoff equation with general nonlinearity attracted much more attention. Chen and Xie [9] first generalized the special case to general nonlinearities f satisfying $\lim_{t\to 0}\frac{f(t)}{|t|} = 0$ and $\lim_{t\to\infty}\frac{F(t)}{|t|^{\frac{14}{3}}} = +\infty$ and proved the existence and multiplicity of solutions under some mild conditions on f in the case $N = 3$. Under a sequence of technical assumptions, Tang and Chen [10] studied the nonautonomous case with indefinite potential $K(x)f(u)$, establishing the existence of normalized solutions for both mass supercritical and subcritical cases. More recently, He, Lv, Zhang, and Zhong [11] considered the general nonlinearities for mass supercritical case in the dimension $1 \le N \le 3$. To make it more precise, it is convenient to recall the following assumptions.

(H_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd.

(H_1) There exists $(\alpha, \beta) \in \mathbb{R}_+^2$ satisfying $2 + \frac{8}{N} < \alpha \le \beta < 2^*$ such that

$$0 < \alpha F(t) \le f(t)t \le \beta F(t) \quad \text{for all } t \in \mathbb{R}\backslash\{0\},$$

where $2^* := \frac{2N}{N-2}$ for $N = 3$ and $2^* := +\infty$ for $N = 1, 2$.

(H_2) The function defined by $\tilde{F}(t) := f(t)t - 2F(t)$ is of class C^1 and satisfies

$$\tilde{F}'(t)t > \alpha \tilde{F}(t) \quad \text{for all } t \in \mathbb{R}.$$

In [11], under (H_0)–(H_2), they obtained a radial ground state normalized solution by constructing a min-max structure and compactness analysis. Afterward, [12] discussed the same results by a global branch approach. Here the normalized ground state solution is a solution of (1.1) having the minimal energy among all the solutions belonging to S_m.

Our aim in this paper is relaxing the growth assumptions (H_0)–(H_2) in the mass supercritical case and extending the previous results on the existence of ground state in $H^1(\mathbb{R}^N)$ and the multiplicity in $H^1_r(\mathbb{R}^N)$. Compared with the mass subcritical case, the constrained functional $I|_{S_m}$ is no longer bounded from below and coercive, and the weaker and more natural mass supercritical assumptions make the problem more complicated. Indeed, since the functional has no global minimum on S_m, we need to identify a suspected critical level; the Palais–Smale sequences may not be bounded and have no convergent subsequence in $H^1(\mathbb{R}^N)$, which brings great difficulties to our proof. Before stating our main results, we present the following weaker assumptions on f.

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous.
(f_1) $\lim_{t\to 0} f(t)/|t|^{1+\frac{8}{N}} = 0$.
(f_2) $\lim_{t\to\infty} f(t)/|t|^5 = 0$ if $N = 3$; $\lim_{t\to\infty} f(t)/e^{\gamma t^2} = 0$ for all $\gamma > 0$ if $N = 2$.
(f_3) $\lim_{t\to\infty} F(t)/|t|^{2+\frac{8}{N}} = +\infty$.
(f_4) $t \mapsto \tilde{F}(t)/|t|^{2+\frac{8}{N}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0,\infty)$.
(f_5) $f(t)t < 6F(t)$ for all $t \in \mathbb{R}\backslash\{0\}$ if $N = 3$.

It is not difficult to see that (H_0)–(H_2) are somehow similar to those in [13]. Notice that the first part of (H_1) is the known Ambrosetti–Rabinowitz condition, which not only plays an important role in the mass supercritical case, but also helps to obtain the boundedness of constrained Palais–Smale sequence. Inspired by [14–16], where a Nehari-type condition is used instead of the Ambrosetti–Rabinowitz condition, we propose the weaker versions of (f_3) and (f_4) instead of the first parts of (H_1) and (H_2). In addition, (f_5) is a weaker version of the second part, which guarantees the positivity of the Lagrange multiplier μ. The following example shows that (f_0)–(f_5) are weaker than (H_0)–(H_2). Let

$$\alpha_N := \begin{cases} 1 & \text{for } N = 1,2, \\ \frac{2}{3} & \text{for } N = 3, \end{cases}$$

$$f(t) := \left[\left(2 + \frac{8}{N}\right)\ln(1 + \alpha_N) + \frac{\alpha_N |t|^{\alpha_N}}{1 + |t|^{\alpha_N}}\right]|t|^{\frac{8}{N}} t$$

with the primitive function $F(t) := |t|^{2+\frac{8}{N}} \ln(1 + \alpha_N)$. By a straightforward calculation we can easily see that f satisfies (f_0)–(f_5) instead of (H_1).

Theorem 1.1 *Let* $1 \le N \le 3$ *and assume that* (f_0)–(f_5) *hold. Then* (1.1)–(1.2) *admits a ground state for any* $m > 0$. *In particular, if* f *is odd, then* (1.1)–(1.2) *admits a positive ground state for any* $m > 0$. *Moveover, the associated Lagrange multiplier* μ *is positive.*

Remark 1.2 The Lagrange multiplier $\mu > 0$ is crucial to prove the compactness of embedding. When $N = 1, 2$, we only need assumptions (f_0)–(f_4) for this purpose.

Now we briefly describe the difficulties encountered in the present paper and sketch our strategy to find normalized ground state solutions to (1.1). Define

$$E_m := \inf_{u \in \mathcal{P}_m} I(u), \tag{1.5}$$

where

$$\mathcal{P}_m := \left\{ u \in S_m \;\middle|\; P(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u)\,dx \right\}. \tag{1.6}$$

Since $\mathcal{P}_m$ is a natural constraint of $I|_{S_m}$, by proving that $E_m > 0$ we identify the possible critical level E_m. Notice that I is coercive on $\mathcal{P}_m$ if we can construct a Palais–Smale sequence on $\mathcal{P}_m$, then it is bounded. To find a Palais–Smale sequence satisfying $P_m(u) = 0$, we adopt the arguments of [17, 18]. Notice that here $\tilde{F}$ is not C^1, so we apply the techniques introduced in [19, 20]. Thus we manage to construct a bounded Palais–Smale sequence $\{u_n\}$ by Lemma 4.1. Although the work space in general does not embed compactly into any space $L^p(\mathbb{R}^N)$, inspired by [21], we finally overcome the lack of compactness by a series of arguments.

Remark 1.3 By the definition of E_m we can see that any minimizer $u \in \mathcal{P}_m$ of E_m is a normalized ground state solution of (1.1)–(1.2). However, despite this fact, it seems not a good choice to prove Theorem 1.1 by solving directly the minimization problem. The main reason why we choose a Palais–Smale sequence $\{u_n\} \in \mathcal{P}_m$ instead of a minimizing sequence of E_m is that the nontrivial weak limit $u \in H^1(\mathbb{R}^N)$ with $s := \|u\|_2^2 \in (0, m]$ of minimizing sequence may not be in the Pohozaev manifold $\mathcal{P}_s$.

Theorem 1.4 *Let* $1 \le N \le 3$ *and assume that* (f_0)–(f_5) *hold. Then the function* $m \mapsto E_m$ *is positive and continuous, and* $\lim_{m\to 0^+} E_m = +\infty$.

Theorem 1.5 *Let* $2 \le N \le 3$ *and assume that* f *is odd satisfying* (f_0)–(f_5). *Then* (1.1)–(1.2) *has infinitely many radial solutions* $\{u_k\}_{k=1}^{\infty}$ *for any* $m > 0$.

To prove Theorem 1.5, we work in $H_r^1(\mathbb{R}^N)$. Since f is odd, it follows that I is even on S_m. Combining this with the genus theory, we can construct an infinite sequence of critical level $E_{m,k}$. Meanwhile, it is not difficult to show that $E_{m,k}$ is nondecreasing and positive. Then by an argument similar to the proof of Theorem 1.1 we obtain the existence of infinitely many radial solutions.

This paper is organized as follows. In Sect. 2, we collect some preliminary results. Section 3 is devoted to the proof of Theorem 1.4. The proof of Theorem 1.1 is given in Sect. 4. In Sect. 5, we prove Theorem 1.5. In this paper, ":=" denotes definition; $\|u\|_p$ denotes the L^p-norm; $\|u\|$ is used only for the norm in $H^1(\mathbb{R}^N)$; $H_r^1(\mathbb{R}^N)$ stands for the space of radially symmetric functions in $H^1(\mathbb{R}^N)$; "$\rightharpoonup$" denotes weak convergence in the related function space; and $u^- = -\min\{0, u\}$ stands for the negative part of u.

2 Preliminaries and functional setting

In this section, we give several preliminary lemmas. For the notational convenience, we define

$$B_m := \{u \in H^1(\mathbb{R}^N) \mid \|u\|_2^2 \le m\}$$

for $m > 0$.

Lemma 2.1 *Let $1 \le N \le 3$. Assume that f satisfies (f_0)–(f_2).*

(i) *For any $m > 0$, there exists $\delta = \delta(N,m) > 0$ sufficiently small such that*

$$\frac{a}{4}\|\nabla u\|_2^2 \le I(u) \le a\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4$$

for $u \in B_m$ with $\|\nabla u\|_2 \le \delta$.

(ii) *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. If $\lim_{n\to\infty}\|u_n\|_{2+\frac{8}{N}} = 0$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n)\,dx = 0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}\tilde{F}(u_n)\,dx. \tag{2.1}$$

(iii) *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in $H^1(\mathbb{R}^N)$. Assume that $\lim_{n\to\infty}\|u_n\|_{2+\frac{8}{N}} = 0$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)v_n\,dx = 0.$$

Proof (i) It suffices to show that there exists a sufficiently small $\delta = \delta(N,m) > 0$ such that

$$\int_{\mathbb{R}^N}|F(u)|\,dx \le \frac{a}{4}\|\nabla u\|_2^2 \quad \text{for all } u \in B_m \text{ with } \|\nabla u\|_2 \le \delta. \tag{2.2}$$

If $N = 3$, then by (f_0)–(f_2)

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0: \quad |F(t)| \le \varepsilon|t|^{\frac{10}{3}} + C_\varepsilon|t|^6.$$

For any $u \in B_m$, by the Gagliardo–Nirenberg inequality we deduce

$$\begin{aligned}\int_{\mathbb{R}^3}|F(u)|\,dx &\le \varepsilon\|u\|_{\frac{10}{3}}^{\frac{10}{3}} + C_\varepsilon\|u\|_6^6\\ &\le \varepsilon C m^{\frac{2}{3}}\|\nabla u\|_2^2 + C_\varepsilon C'\|\nabla u\|_2^6\\ &= \left[\varepsilon C m^{\frac{2}{3}} + C_\varepsilon C'\|\nabla u\|_2^4\right]\|\nabla u\|_2^2,\end{aligned}$$

where $C, C' > 0$. Then we derive (2.2) by taking

$$\varepsilon := \frac{a}{8Cm^{\frac{2}{3}}} \quad \text{and} \quad \delta := \left(\frac{a}{8C_\varepsilon C'}\right)^{\frac{1}{4}}.$$

If $N = 2$, then let $\gamma := \frac{1}{m+1}$. For any $\varepsilon > 0$, by (f_0)–(f_2) there exists $C'_\varepsilon > 0$ such that $|f(t)| \le \varepsilon|t|^3 + C'_\varepsilon|t|^5 e^{\gamma t^2/2}$ for all $t \in \mathbb{R}$. Since

$$\begin{aligned}\int_0^t \tau^5 e^{\gamma\tau^2/2}\,d\tau &= \frac{1}{\gamma}t^4\big(e^{\gamma t^2/2}-1\big) - \frac{4}{\gamma}\int_0^t \tau^3\big(e^{\gamma\tau^2/2}-1\big)\,d\tau \\ &\le \frac{1}{\gamma}t^4\big(e^{\gamma t^2/2}-1\big) \quad \text{for all } t \ge 0,\end{aligned}$$

it follows that $|F(t)| \le \varepsilon|t|^4 + \frac{1}{\gamma}C'_\varepsilon|t|^4(e^{\gamma t^2/2}-1)$ for all $t \in \mathbb{R}$. This, together with the Moser–Trudinger inequality, implies that there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^2}\big(e^{\gamma u^2}-1\big)\,dx \le C_1^2 \quad \text{for all } u \in B_m \text{ with } \|\nabla u\|_2 \le 1.$$

For all $\delta \in (0,1)$ and $u \in B_m$ with $\|\nabla u\|_2 \le \delta$, by the Hölder and Gagliardo–Nirenberg inequalities we have

$$\begin{aligned}\int_{\mathbb{R}^2}\big|F(u)\big|\,dx &\le \varepsilon\|u\|_4^4 + \frac{1}{\gamma}C'_\varepsilon\int_{\mathbb{R}^2}u^4\big(e^{\gamma u^2/2}-1\big)\,dx \\ &\le \varepsilon\|u\|_4^4 + \frac{1}{\gamma}C'_\varepsilon\|u\|_8^4\bigg[\int_{\mathbb{R}^2}\big(e^{\gamma u^2/2}-1\big)^2\,dx\bigg]^{\frac{1}{2}} \\ &\le \varepsilon\|u\|_4^4 + \frac{1}{\gamma}C'_\varepsilon\|u\|_8^4\bigg[\int_{\mathbb{R}^2}\big(e^{\gamma u^2}-1\big)\,dx\bigg]^{\frac{1}{2}} \\ &\le \varepsilon C_2 m\|\nabla u\|_2^2 + C_1C_3C'_\varepsilon m^{\frac{1}{2}}(m+1)\|\nabla u\|_2^3 \\ &\le \big[\varepsilon C_2 m + C_1C_3C'_\varepsilon m^{\frac{1}{2}}(m+1)\delta\big]\|\nabla u\|_2^2,\end{aligned}$$

where $C_2, C_3 > 0$ are independent of m, ε, δ, and u. Taking $\varepsilon > 0$ and $\delta \in (0,1)$ small enough, we derive (2.2) with $N = 2$.

In the case $N = 1$, since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, there exists $K > 0$ such that

$$\|u\|_\infty \le K \quad \text{for all } u \in B_m \text{ with } \|\nabla u\|_2 \le 1.$$

Let $\varepsilon > 0$ and $\delta \in (0,1)$. By (f_0) and (f_1) there exists $C''_\varepsilon > 0$ such that $|F(t)| \le \varepsilon t^6 + C''_\varepsilon t^{10}$ for $|t| \le K$. Therefore, for all $u \in B_m$ with $\|\nabla u\|_2 \le \delta$, the Gagliardo–Nirenberg inequality implies that

$$\begin{aligned}\int_{\mathbb{R}}\big|F(u)\big|\,dx &\le \varepsilon\|u\|_6^6 + C''_\varepsilon\|u\|_{10}^{10} \\ &\le \varepsilon C_4 m^2\|\nabla u\|_2^2 + C_5 C''_\varepsilon m^3\|\nabla u\|_2^4 \\ &\le \big(\varepsilon C_4 m^2 + C_5 C''_\varepsilon m^3\delta^2\big)\|\nabla u\|_2^2,\end{aligned}$$

where $C_4, C_5 > 0$ are independent of m, ε, δ, and u. Choosing $\varepsilon > 0$ and $\delta \in (0,1)$ small enough, (2.2) holds with $N = 1$.

We only prove (ii) and (iii) in the case $N = 2$; the other cases can be presented similarly.

(ii) Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we can take a sufficiently large $L > 0$ such that $\sup_{n\geq 1} \|u_n\| \leq L$. Let $\gamma := \frac{1}{L^2}$. It follows by the Moser–Trudinger inequality that there exists $D > 0$ such that

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} \left(e^{\gamma u_n^2} - 1\right) dx \leq D. \tag{2.3}$$

For given $\varepsilon > 0$, from (f_0)–(f_2) we obtain the existence of $D_\varepsilon > 0$ such that

$$\left|\tilde{F}(t)\right| \leq \varepsilon\left(e^{\gamma t^2} - 1\right) + D_\varepsilon t^6$$

for $t \in \mathbb{R}$. Then

$$\int_{\mathbb{R}^2} \left|\tilde{F}(u_n)\right| dx \leq \varepsilon \int_{\mathbb{R}^2} \left(e^{\gamma u_n^2} - 1\right) dx + D_\varepsilon \|u_n\|_6^6 \leq \varepsilon D + D_\varepsilon \|u_n\|_6^6.$$

By the arbitrariness of ε and $\lim_{n\to\infty} \|u_n\|_6 = 0$, we derive $\lim_{n\to\infty} \int_{\mathbb{R}^2} \tilde{F}(u_n)\, dx = 0$. By a similar argument we can show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F(u_n)\, dx = 0 \quad \text{if } \lim_{n\to\infty} \|u_n\|_6 = 0.$$

(iii) In view of (2.3), it follows that

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} \left(e^{\gamma u_n^2/2} - 1\right)^2 dx \leq \sup_{n\geq 1} \int_{\mathbb{R}^2} \left(e^{\gamma u_n^2} - 1\right) dx \leq D.$$

Let $\varepsilon > 0$ be arbitrary. By (f_0)–(f_2) there exists $D'_\varepsilon > 0$ such that

$$\left|f(t)\right| \leq \varepsilon\left(e^{\gamma t^2/2} - 1\right) + D'_\varepsilon t^5$$

for $t \in \mathbb{R}$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|f(u_n)v_n\right| dx &\leq \varepsilon \int_{\mathbb{R}^2} \left(e^{\gamma u_n^2/2} - 1\right)|v_n|\, dx + D'_\varepsilon \int_{\mathbb{R}^2} |u_n|^5|v_n|\, dx \\
&\leq \varepsilon \left[\int_{\mathbb{R}^2} \left(e^{\gamma u_n^2/2} - 1\right)^2 dx\right]^{\frac{1}{2}} \|v_n\|_2 + D'_\varepsilon \|u_n\|_6^5 \|v_n\|_6 \\
&\leq \varepsilon\sqrt{D}\|v_n\|_2 + D'_\varepsilon \|u_n\|_6^5 \|v_n\|_6.
\end{aligned}$$

As a result, $\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)v_n\, dx = 0$. □

Remark 2.2 Under the assumptions of Lemma 2.1, for any $m > 0$, as in the proof of (2.2) with minor changes, there exists $\delta = \delta(N, m) > 0$ sufficiently small such that

$$\int_{\mathbb{R}^N} \left|\tilde{F}(u)\right| dx \leq \frac{a}{N}\|\nabla u\|_2^2$$

for $u \in B_m$ with $\|\nabla u\|_2 \leq \delta$. Thus it follows that

$$P(u) := a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u)\, dx \geq \frac{a}{2}\|\nabla u\|_2^2$$

for $u \in B_m$ with $\|\nabla u\|_2 \leq \delta$.

For $u \in H^1(\mathbb{R}^N)$ and $s \in \mathbb{R}$, define the radial dilation

$$(s \star u)(x) := e^{\frac{Ns}{2}} u(e^s x) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

It is straightforward to check that $s \star u \in H^1(\mathbb{R}^N)$ and $\|s \star u\|_2^2 = \|u\|_2^2$ for every $s \in \mathbb{R}$.

Lemma 2.3 *Let $1 \le N \le 3$. Assume that f satisfies (f_0)–(f_3). Then for all $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, we have*

(i) *$I(s \star u) \to 0^+$ as $s \to -\infty$ and*

(ii) *$I(s \star u) \to -\infty$ as $s \to +\infty$.*

Proof (i) Let $m := \|u\|_2^2 > 0$. The fact that $s \star u \in S_m \subset B_m$ and $\|\nabla(s \star u)\|_2 = e^s \|\nabla u\|_2$, combined with Lemma 2.1(i), implies

$$\frac{a}{4} e^{2s} \|\nabla u\|_2^2 \le I(s \star u) \le a e^{2s} \|\nabla u\|_2^2 + \frac{b}{4} e^{4s} \|\nabla u\|_2^4.$$

Then $\lim_{s \to -\infty} I(s \star u) = 0^+$.

(ii) For $\lambda \ge 0$, we define the function

$$h_\lambda(t) := \begin{cases} \frac{F(t)}{|t|^{2+\frac{8}{N}}} + \lambda & \text{for } t \neq 0, \\ \lambda & \text{for } t = 0. \end{cases} \tag{2.4}$$

Obviously, $F(t) = h_\lambda(t)|t|^{2+\frac{8}{N}} - \lambda |t|^{2+\frac{8}{N}}$ for $t \in \mathbb{R}$. By (f_0), (f_1), and (f_3), we can easily see that h_λ is continuous and

$$h_\lambda(t) \to +\infty \quad \text{as } t \to \infty.$$

Taking $\lambda > 0$ sufficiently large such that $h_\lambda(t) \ge 0$ for all $t \in \mathbb{R}$, by Fatou's lemma we deduce

$$\lim_{s \to +\infty} \int_{\mathbb{R}^N} h_\lambda\left(e^{\frac{Ns}{2}} u\right) |u|^{2+\frac{8}{N}} \, dx = +\infty.$$

From

$$\begin{aligned} I(s \star u) &= \frac{a}{2} \left\| \nabla(s \star u) \right\|_2^2 + \frac{b}{4} \left\| \nabla(s \star u) \right\|_2^4 + \lambda \|s \star u\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} - \int_{\mathbb{R}^N} h_\lambda(s \star u) |s \star u|^{2+\frac{8}{N}} \, dx \\ &= e^{4s} \left[e^{-2s} \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \lambda \|u\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} - \int_{\mathbb{R}^N} h_\lambda\left(e^{\frac{Ns}{2}} u\right) |u|^{2+\frac{8}{N}} \, dx \right] \end{aligned} \tag{2.5}$$

we conclude that $I(s \star u) \to -\infty$ as $s \to +\infty$. □

Remark 2.4 Let $1 \le N \le 3$. Assume that f satisfies (f_0), (f_1), and (f_4). Define g by

$$g(t) := \begin{cases} \frac{f(t)t - 2F(t)}{|t|^{2+\frac{8}{N}}} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases} \tag{2.6}$$

Clearly, g is continuous, strictly decreasing on $(-\infty, 0]$, and strictly increasing on $[0, \infty)$.

Lemma 2.5 *Let* $1 \leq N \leq 3$. *If* f *satisfies* (f_0), (f_1), (f_3), *and* (f_4),, *then*

$$f(t)t > \left(2 + \frac{8}{N}\right)F(t) > 0 \quad \textit{for all } t \neq 0.$$

Proof The proof of the lemma is divided into several claims.

Claim 1 $F(t) > 0$ *for all* $t \neq 0$.

Suppose by contradiction that there exists some $t_0 \neq 0$ such that $F(t_0) \leq 0$. According to (f_1) and (f_3), we can see that $\frac{F(t)}{|t|^{2+\frac{8}{N}}}$ reaches the global minimum at some $\tau \neq 0$ satisfying $F(\tau) \leq 0$ and

$$\left[\frac{F(t)}{|t|^{2+\frac{8}{N}}}\right]'_{t=\tau} = \frac{f(\tau)\tau - (2 + \frac{8}{N})F(\tau)}{|\tau|^{3+\frac{8}{N}} \operatorname{sign}(\tau)} = 0.$$

Since Remark 2.4 yields $f(t)t > 2F(t)$ for all $t \neq 0$, we deduce

$$0 < f(\tau)\tau - 2F(\tau) = \frac{8}{N}F(\tau) \leq 0,$$

a contradiction.

Claim 2 *There exist a positive sequence* $\{\tau_n^+\}$ *and a negative sequence* $\{\tau_n^-\}$ *such that* $|\tau_n^\pm| \to 0$ *and* $f(\tau_n^\pm)\tau_n^\pm > (2 + \frac{8}{N})F(\tau_n^\pm) > 0$ *for* $n \geq 1$.

We only consider the existence of $\{\tau_n^+\}$. Suppose by contradiction that there exists $T_s > 0$ sufficiently small such that $f(t)t \leq (2 + \frac{8}{N})F(t)$ for $t \in (0, T_s]$. From Claim 1 we obtain

$$\frac{F(t)}{|t|^{2+\frac{8}{N}}} \geq \frac{F(T_s)}{|T_s|^{2+\frac{8}{N}}} > 0 \quad \text{for } t \in (0, T_s]$$

in contradiction with $\lim_{t\to 0} \frac{F(t)}{|t|^{2+\frac{8}{N}}} = 0$.

Claim 3 *There exist a positive sequence* $\{\varsigma_n^+\}$ *and a negative sequence* $\{\varsigma_n^-\}$ *such that* $|\varsigma_n^\pm| \to +\infty$ *and* $f(\varsigma_n^\pm)\varsigma_n^\pm > (2 + \frac{8}{N})F(\varsigma_n^\pm) > 0$ *for* $n \geq 1$.

We only prove the existence of $\{\varsigma_n^-\}$. Otherwise, there would exist $T_l > 0$ such that $f(t)t \leq (2 + \frac{8}{N})F(t)$ for $t \leq -T_l$. Then

$$\frac{F(t)}{|t|^{2+\frac{8}{N}}} \leq \frac{F(-T_l)}{T_l^{2+\frac{8}{N}}} < +\infty \quad \text{for } t \leq -T_l,$$

a contradiction to (f_3).

Claim 4 $f(t)t \geq (2 + \frac{8}{N})F(t)$ *for all* $t \neq 0$.

Assume by contradiction that there exists $t_0 \neq 0$ such that $f(t_0)t_0 < (2+\frac{8}{N})F(t_0)$. Without loss of generality, we may assume that $t_0 < 0$. Claims 2 and 3 imply that there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $\tau_1 < t_0 < \tau_2 < 0$,

$$f(t)t < \left(2+\frac{8}{N}\right)F(t) \quad \text{for all } t \in (\tau_1, \tau_2), \tag{2.7}$$

and

$$f(t)t = \left(2+\frac{8}{N}\right)F(t) \quad \text{if } t = \tau_1, \tau_2. \tag{2.8}$$

From (2.7) it follows that

$$\frac{F(\tau_1)}{|\tau_1|^{2+\frac{8}{N}}} < \frac{F(\tau_2)}{|\tau_2|^{2+\frac{8}{N}}}. \tag{2.9}$$

In addition, (2.8) and (f_4) imply that

$$\frac{F(\tau_1)}{|\tau_1|^{2+\frac{8}{N}}} = \frac{N}{8}\frac{\tilde{F}(\tau_1)}{|\tau_1|^{2+\frac{8}{N}}} > \frac{N}{8}\frac{\tilde{F}(\tau_2)}{|\tau_2|^{2+\frac{8}{N}}} = \frac{F(\tau_2)}{|\tau_2|^{2+\frac{8}{N}}}. \tag{2.10}$$

Thus (2.9) and (2.10) give a contradiction.

Claim 5 *$f(t)t > (2+\frac{8}{N})F(t)$ for all $t \neq 0$.*

According to Claim 4, the function $\frac{F(t)}{|t|^{2+\frac{8}{N}}}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0,\infty)$. Combining this with (f_4), we deduce that the function $\frac{f(t)}{|t|^{1+\frac{8}{N}}}$ is strictly increasing on $(-\infty, 0)$ and $(0,\infty)$. Thus, for all $t \neq 0$, we have

$$\begin{aligned}\left(2+\frac{8}{N}\right)F(t) &= \left(2+\frac{8}{N}\right)\int_0^t f(s)\,ds \\ &< \left(2+\frac{8}{N}\right)\frac{f(t)}{|t|^{1+\frac{8}{N}}}\int_0^t |s|^{1+\frac{8}{N}}\,ds = f(t)t,\end{aligned}$$

which concludes the proof. □

Lemma 2.6 *Let $1 \le N \le 3$. Assume that f satisfies (f_0)–(f_4). Then for all $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, the following statements hold.*

(i) *There exists a unique $s(u) \in \mathbb{R}$ such that $P(s(u) \star u) = 0$.*
(ii) *$I(s(u) \star u) > I(s \star u)$ for $s \neq s(u)$. Moveover, $I(s(u) \star u) > 0$.*
(iii) *$u \mapsto s(u)$ is continuous in $u \in H^1(\mathbb{R}^N)\backslash\{0\}$.*
(iv) *$s(u(\cdot + y)) = s(u)$ for all $y \in \mathbb{R}^N$. If f is odd, then $s(-u) = s(u)$.*

Proof (i) Recalling that

$$I(s \star u) = \frac{a}{2}e^{2s}\|\nabla u\|_2^2 + \frac{b}{4}e^{4s}\|\nabla u\|_2^4 - e^{-Ns}\int_{\mathbb{R}^N} F\big(e^{\frac{Ns}{2}}u\big)\,dx,$$

$I(s \star u)$ clearly is in C^1. By a straightforward calculation it follows that

$$\frac{d}{ds} I(s \star u) = ae^{2s}\|\nabla u\|_2^2 + be^{4s}\|\nabla u\|_2^4 - \frac{N}{2}e^{-Ns}\int_{\mathbb{R}^N} \tilde{F}\bigl(e^{\frac{Ns}{2}}u\bigr)\,dx = P(s \star u).$$

In view of Lemma 2.3, we have

$$\lim_{s\to-\infty} I(s \star u) = 0^+ \quad \text{and} \quad \lim_{s\to+\infty} I(s \star u) = -\infty.$$

Hence there exists $s(u) \in \mathbb{R}$ such that $I(s \star u)$ reaches the global maximum at $s(u)$, and

$$P\bigl(s(u) \star u\bigr) = \frac{d}{ds} I\bigl(s(u) \star u\bigr) = 0.$$

By (2.6)

$$\tilde{F}(t) = g(t)|t|^{2+\frac{8}{N}} \quad \text{for all } t \in \mathbb{R},$$

and then

$$P(s \star u) = e^{4s}\left[ae^{-2s}\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \frac{N}{2}\int_{\mathbb{R}^N} g\bigl(e^{\frac{Ns}{2}}u\bigr)|u|^{2+\frac{8}{N}}\,dx\right].$$

By (f_4) and Remark 2.4 the function $s \mapsto g(e^{\frac{Ns}{2}}u)$ is strictly increasing. Combining this with the monotonicity of e^{-2s}, we derive the uniqueness of $s(u)$.

(ii) This statement is contained in the proof of (i).

(iii) By (i) we can see that $u \mapsto s(u)$ is well-defined. To prove the continuity of $s(u)$, assume that $u_n \to u$ in $H^1(\mathbb{R}^N)\backslash\{0\}$. Let $s_n := s(u_n)$ for $n \geq 1$. It is sufficient to prove that up to a subsequence, $s_n \to s(u)$ as $n \to \infty$.

We claim that $\{s_n\}$ is bounded. Indeed, recalling the definition and properties of h_λ in (2.4), it follows from Lemma 2.5 that $h_0(t) \geq 0$ for all $t \in \mathbb{R}$. If, up to a subsequence, $s_n \to +\infty$, then the fact that $u_n \to u \neq 0$ a.e. $x \in \mathbb{R}^N$, together with Fatou's lemma, implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_0\bigl(e^{\frac{Ns_n}{2}}u_n\bigr)|u_n|^{2+\frac{8}{N}}\,dx = +\infty.$$

Coming back to (ii) and (2.5) with $\lambda = 0$, we conclude

$$\begin{aligned} 0 \leq e^{-4s_n} I(s_n \star u_n) &= e^{-2s}\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \int_{\mathbb{R}^N} h_0\bigl(e^{\frac{Ns}{2}}u_n\bigr)|u_n|^{2+\frac{8}{N}}\,dx \\ &\to -\infty, \end{aligned} \tag{2.11}$$

a contradiction. Thus $\{s_n\}$ is bounded from above.

By (ii) we have

$$I(s_n \star u_n) \geq I\bigl(s(u) \star u_n\bigr) \quad \text{for all } n \geq 1.$$

Then since $s(u) \star u_n \to s(u) \star u$ in $H^1(\mathbb{R}^N)$, we have

$$I\bigl(s(u) \star u_n\bigr) = I\bigl(s(u) \star u\bigr) + o_n(1),$$

and thus

$$\liminf_{n\to\infty} I(s_n \star u_n) \geq I\big(s(u) \star u\big) > 0. \tag{2.12}$$

Considering $\{s_n \star u_n\} \subset B_m$ for $m > 0$ sufficiently large, by Lemma 2.1(i) and the fact that

$$\big\|\nabla(s_n \star u_n)\big\|_2 = e^{s_n}\|\nabla u_n\|_2,$$

from (2.12) we conclude that $\{s_n\}$ is bounded from below.

Since $\{s_n\}$ is bounded, there exists some $s_* \in \mathbb{R}$ such that $s_n \to s_*$. Recalling that $u_n \to u$ in $H^1(\mathbb{R}^N)$, we have $s_n \star u_n \to s_* \star u$ in $H^1(\mathbb{R}^N)$. The fact that $P(s_n \star u_n) = 0$ implies

$$P(s_* \star u) = 0.$$

By (i) we infer $s_* = s(u)$, and thus the proof is completed.

(iv) For any $y \in \mathbb{R}^N$, by a change of variables in the integrals we have

$$P\big(s(u) \star u(\cdot + y)\big) = P\big(s(u) \star u\big) = 0,$$

which means $s(u(\cdot + y)) = s(u)$ by (i). If f is odd, then

$$P\big(s(u) \star (-u)\big) = P\big(-\big(s(u) \star u\big)\big) = P\big(s(u) \star u\big) = 0.$$

Therefore $s(-u) = s(u)$. □

In what follows, we consider some statements about the Pohozeav manifold

$$\mathcal{P}_m := \left\{u \in S_m \,\middle|\, P(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u)\,dx = 0\right\}.$$

Lemma 2.7 *Let* $1 \leq N \leq 3$. *Assume that* f *satisfies* (f_0)–(f_4). *Then*

(i) $\mathcal{P}_m \neq \emptyset$,

(ii) $\inf_{u\in\mathcal{P}_m} \|\nabla u\|_2 > 0$,

(iii) $\inf_{u\in\mathcal{P}_m} I(u) > 0$,

(iv) *I is coercive on $\mathcal{P}_m$, i.e., $I(u_n) \to +\infty$ for any $\{u_n\} \subset \mathcal{P}_m$ with $\|u_n\| \to \infty$.*

Proof (i) This item is a direct conclusion of Lemma 2.6(i).

(ii) We suppose by contradiction that there exists $\{u_n\} \subset \mathcal{P}_m$ such that $\|\nabla u_n\|_2 \to 0$. In view of Remark 2.2, we have

$$0 = P(u_n) \geq \frac{a}{2}\|\nabla u_n\|_2^2 > 0 \quad \text{for } n \text{ large enough},$$

a contradiction. Thus $\inf_{u\in\mathcal{P}_m} \|\nabla u\|_2 > 0$.

(iii) For any $u \in \mathcal{P}_m$, thanks to Lemma 2.6(i),(ii), we deduce

$$I(u) = I(0 \star u) \geq I(s \star u) \quad \text{for } s \in \mathbb{R}.$$

Letting δ be as in Lemma 2.1(i), and let $s := \ln(\frac{\delta}{\|\nabla u\|_2})$. Obviously, $\|\nabla(s \star u)\|_2 = \delta$. Then Lemma 2.1(i) implies

$$I(u) \geq I(s \star u) \geq \frac{a}{4}\big\|\nabla(s \star u)\big\|_2^2 = \frac{a}{4}\delta^2,$$

and hence (iii) follows.

(iv) Assume by contradiction that there exists a sequence $\{u_n\} \subset \mathcal{P}_m$ with $\|u_n\| \to \infty$ such that $\sup_{n\geq 1} I(u_n) \leq c$ for some $c \in (0, +\infty)$. For any $n \geq 1$, let

$$s_n := \ln\big(\|\nabla u_n\|_2\big) \quad \text{and} \quad v_n := (-s_n) \star u_n.$$

Then it is easy to see that $s_n \to +\infty$, $\{v_n\} \subset S_m$, and $\|\nabla v_n\|_2 = 1$. Take

$$\rho := \limsup_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B(y,1)} |v_n|^2\,dx\right).$$

There are two possible cases, nonvanishing and vanishing.

The nonvanishing case. Up to a subsequence, there exist $\{y_n\} \subset \mathbb{R}^N$ and $w \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$w_n := v_n(\cdot + y_n) \rightharpoonup w \quad \text{in } H^1\big(\mathbb{R}^N\big) \quad \text{and} \quad w_n \to w \quad \text{a.e. in } \mathbb{R}^N.$$

According to (2.4) with $\lambda = 0$, since $s_n \to +\infty$, by Lemma 2.5 and Fatou's lemma we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_0\big(e^{\frac{Ns_n}{2}} w_n\big)|w_n|^{2+\frac{8}{N}}\,dx = +\infty.$$

From (iii) and (2.5) with $\lambda = 0$ we conclude

$$\begin{aligned}
0 &\leq e^{-4s_n} I(u_n) = e^{-4s_n} I(s_n \star v_n) \\
&= \frac{a}{2}e^{-2s_n} + \frac{b}{4} - \int_{\mathbb{R}^N} h_0\big(e^{\frac{Ns_n}{2}} v_n\big)|v_n|^{2+\frac{8}{N}}\,dx \\
&= \frac{a}{2}e^{-2s_n} + \frac{b}{4} - \int_{\mathbb{R}^N} h_0\big(e^{\frac{Ns_n}{2}} w_n\big)|w_n|^{2+\frac{8}{N}}\,dx \to -\infty,
\end{aligned}$$

a contradiction.

The vanishing case. By Lions' lemma ([22, Lemma I.1]) $v_n \to 0$ in $L^{2+\frac{8}{N}}(\mathbb{R}^N)$. In view of Lemma 2.1(ii), we have

$$\lim_{n\to\infty} e^{-Ns}\int_{\mathbb{R}^N} F\big(e^{\frac{Ns}{2}} v_n\big)\,dx = 0 \quad \text{for all } s \in \mathbb{R}.$$

Noticing that $P(s_n \star v_n) = P(u_n) = 0$, due to Lemma 2.6(i),(ii), we obtain that for all $s \in \mathbb{R}$,

$$\begin{aligned}
c &\geq I(u_n) = I(s_n \star v_n) \\
&\geq I(s \star v_n) = \frac{a}{2}e^{2s} + \frac{b}{4}e^{4s} - e^{-Ns}\int_{\mathbb{R}^N} F\big(e^{\frac{Ns}{2}} v_n\big)\,dx = \frac{a}{2}e^{2s} + \frac{b}{4}e^{4s} + o_n(1),
\end{aligned}$$

which leads a contradiction for s sufficiently large. Hence I is coercive on $\mathcal{P}_m$. □

Remark 2.8 Let $1 \leq N \leq 3$. Assume that f satisfies (f_0)–(f_4) for any sequence $\{u_n\} \subset H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$P(u_n) = 0, \qquad \sup_{n\geq 1} \|u_n\|_2 < +\infty, \quad \text{and} \quad \sup_{n\geq 1} I(u_n) < +\infty.$$

Then by arguments similar to those in Lemma 2.7 $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

3 The behavior of the function $m \mapsto E_m$

The purpose of this section is to characterize the behavior of E_m. Under (f_0)–(f_4), for any $m > 0$, by Lemma 2.7 we see that

$$E_m := \inf_{u\in\mathcal{P}_m} I(u)$$

is well defined. In particular, the proof of Theorem 1.4 can be deduced from the following lemmas.

Lemma 3.1 *Let $1 \leq N \leq 3$. Assume that f satisfies (f_0)–(f_4). Then $E_m > 0$.*

Proof The lemma is a direct consequence of Lemma 2.7(iii). □

Lemma 3.2 *Let $1 \leq N \leq 3$. Assume that f satisfies (f_0)–(f_4). Then $m \mapsto E_m$ is continuous.*

Proof For $m > 0$, assume that $m_k \to m$ as $k \to \infty$. Then $\lim_{k\to\infty} E_{m_k} = E_m$ will follow from (3.1) and (3.2). We first claim that

$$\limsup_{k\to\infty} E_{m_k} \leq E_m. \tag{3.1}$$

Indeed, for any $u \in \mathcal{P}_m$, define

$$u_k := \sqrt{\frac{m_k}{m}} u \in S_{m_k}, \quad k \in \mathbb{N}^+.$$

The fact $u_k \to u$ in $H^1(\mathbb{R}^N)$, together with Lemma 2.6(iii), implies $\lim_{k\to\infty} s(u_k) = s(u) = 0$, and then

$$s(u_k) \star u_k \to s(u) \star u = u \quad \text{in } H^1\bigl(\mathbb{R}^N\bigr) \text{ as } k \to \infty.$$

Therefore

$$\limsup_{k\to\infty} E_{m_k} \leq \limsup_{k\to\infty} I\bigl(s(u_k) \star u_k\bigr) = I(u).$$

Since $u \in \mathcal{P}_m$ is arbitrary, we have $\limsup_{k\to\infty} E_{m_k} \leq E_m$.

We next to show that

$$\liminf_{k\to\infty} E_{m_k} \geq E_m. \tag{3.2}$$

For any $k \in \mathbb{N}^+$, there exists $v_k \in \mathcal{P}_{m_k}$ such that

$$I(v_k) \le E_{m_k} + \frac{1}{k}. \tag{3.3}$$

Define

$$t_k := \left(\frac{m}{m_k}\right)^{\frac{1}{N}} \quad \text{and} \quad \tilde{v}_k := v_k(\cdot/t_k) \in S_m.$$

It follows from Lemma 2.6(ii) and (3.3) that

$$\begin{aligned} E_m &\le I\big(s(\tilde{v}_k) \star \tilde{v}_k\big) \le I\big(s(\tilde{v}_k) \star v_k\big) + \big|I\big(s(\tilde{v}_k) \star \tilde{v}_k\big) - I\big(s(\tilde{v}_k) \star v_k\big)\big| \\ &\le I(v_k) + \big|I\big(s(\tilde{v}_k) \star \tilde{v}_k\big) - I\big(s(\tilde{v}_k) \star v_k\big)\big| \\ &\le E_{m_k} + \frac{1}{k} + \big|I\big(s(\tilde{v}_k) \star \tilde{v}_k\big) - I\big(s(\tilde{v}_k) \star v_k\big)\big| \\ &=: E_{m_k} + \frac{1}{k} + C(k). \end{aligned}$$

To complete the proof of (3.2), we only need to prove

$$\lim_{k\to\infty} C(k) = 0. \tag{3.4}$$

Since $s \star (v(\cdot/t)) = (s \star v)(\cdot/t)$, we obtain

$$\begin{aligned} C(k) &= \bigg|\frac{a}{2}\big(t_k^{N-2} - 1\big)\big\|\nabla\big(s(\tilde{v}_k) \star v_k\big)\big\|_2^2 + \frac{b}{4}\big(t_k^{N-4} - 1\big)\big\|\nabla\big(s(\tilde{v_k}) \star v_k\big)\big\|_2^4 \\ &\quad - \big(t_k^N - 1\big)\int_{\mathbb{R}^N} F\big(s(\tilde{v}_k) \star v_k\big)\,dx\bigg| \\ &\le \frac{a}{2}\big|t_k^{N-2} - 1\big| \cdot \big\|\nabla\big(s(\tilde{v}_k) \star v_k\big)\big\|_2^2 + \frac{b}{4}\big|t_k^{N-4} - 1\big| \cdot \big\|\nabla\big(s(\tilde{v_k}) \star v_k\big)\big\|_2^4 \\ &\quad + \big|t_k^N - 1\big| \cdot \int_{\mathbb{R}^N} \big|F\big(s(\tilde{v}_k) \star v_k\big)\big|\,dx \\ &=: \frac{a}{2}\big|t_k^{N-2} - 1\big| \cdot A(k) + \frac{b}{4}\big|t_k^{N-4} - 1\big| \cdot A(k)^2 + \big|t_k^N - 1\big| \cdot B(k). \end{aligned}$$

The fact $t_k \to 1$ makes it clear that we will obtain (3.4) if

$$\limsup_{k\to\infty} A(k) < +\infty \quad \text{and} \quad \limsup_{k\to\infty} B(k) < +\infty. \tag{3.5}$$

Before proving (3.5), we justify the following three claims.

Claim 1 *$\{v_k\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Indeed, (3.1) and (3.3) imply that

$$\limsup_{k\to\infty} I(v_k) \le E_m.$$

Noticing that $v_k \in \mathcal{P}_{m_k}$ and $m_k \to m$, by Remark 2.8 we infer that Claim 1 follows.

Claim 2 *$\{\tilde{v}_k\}$ is bounded in $H^1(\mathbb{R}^N)$, and there exist $\{y_k\} \subset \mathbb{R}^N$ and $v \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $\tilde{v}_k(\cdot + y_k) \to v \neq 0$ a.e. in $\mathbb{R}^N$.*

Indeed, combining $t_k \to 1$ with Claim 1, we deduce that $\{\tilde{v}_k\}$ is bounded in $H^1(\mathbb{R}^N)$. Let

$$\rho := \limsup_{k\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\tilde{v}_k|^2 \, dx \right).$$

It suffices to show that $\rho > 0$. If $\rho = 0$, from Lions' lemma ([22, Lemma I.1]) it follows that $\tilde{v}_k \to 0$ in $L^{2+\frac{8}{N}}(\mathbb{R}^N)$. Then

$$\|v_k\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} = \left\|\tilde{v}_k(t_k \cdot)\right\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} = t_k^{-N} \int_{\mathbb{R}^N} |\tilde{v}_k|^{2+\frac{8}{N}} \, dx \to 0.$$

By Lemma 2.1(ii) and $P(v_k) = 0$ we obtain

$$a\|\nabla v_k\|_2^2 + b\|\nabla v_k\|_2^4 = \frac{N}{2} \int_{\mathbb{R}^N} \tilde{F}(v_k) \, dx \to 0.$$

Thus we infer from Remark 2.2 that

$$0 = P(v_k) \geq \frac{a}{2}\|\nabla v_k\|_2^2 > 0 \quad \text{for } k \text{ large enough},$$

a contradiction.

Claim 3 $\limsup_{k\to\infty} s(\tilde{v}_k) < +\infty$.

Indeed, we suppose by contradiction that, up to a subsequence,

$$s(\tilde{v}_k) \to +\infty \quad \text{as } k \to \infty. \tag{3.6}$$

Coming back to Claim 2, it follows that, up to a subsequence,

$$\tilde{v}_k(\cdot + y_k) \to v \neq 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{3.7}$$

By Lemma 2.6(iv) and (3.6) we have

$$s\big(\tilde{v}_k(\cdot + y_k)\big) = s(\tilde{v}_k) \to +\infty. \tag{3.8}$$

Moveover, Lemma 2.6(ii) implies that

$$I\big(s\big(\tilde{v}_k(\cdot + y_k)\big) \star \tilde{v}_k(\cdot + y_k)\big) > 0. \tag{3.9}$$

Arguing as in (2.11) and using (3.7)–(3.9), we obtain a contradiction.

Now summing up Claim 1 and 3, we deduce that

$$\limsup_{k\to\infty} \left\| s(\tilde{v}_k) \star v_k \right\| < +\infty,$$

which implies $\limsup_{k\to\infty} A(k) < +\infty$. Furthermore, since f satisfies (f_0)–(f_2), we can conclude that $\limsup_{k\to\infty} B(k) < +\infty$. Therefore (3.5) holds, and the lemma is completed. □

Lemma 3.3 *Let $1 \le N \le 3$. Assume that f satisfies (f_0)–(f_4). Then $E_m \to +\infty$ as $m \to 0^+$.*

Proof We only need to show that for any sequence $\{u_n\} \subset H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$P(u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_2 = 0,$$

we have $I(u_n) \to +\infty$ as $n \to \infty$. Define

$$s_n := \ln\big(\|\nabla u_n\|_2\big) \quad \text{and} \quad v_n := (-s_n) \star u_n.$$

Notice that $\|\nabla v_n\|_2 = 1$ and $\|v_n\|_2 = \|u_n\|_2 \to 0$, and hence $v_n \to 0$ in $L^{2+\frac{8}{N}}(\mathbb{R}^N)$. Thus it follows from Lemma 2.1(ii) that

$$\lim_{n\to\infty} e^{-Ns} \int_{\mathbb{R}^N} F\big(e^{\frac{Ns}{2}} v_n\big)\, dx = 0 \quad \text{for } s \in \mathbb{R}.$$

Since $P(s_n \star v_n) = P(u_n) = 0$, by Lemma 2.6(i),(ii) we have

$$\begin{aligned} I(u_n) &= I(s_n \star v_n) \ge I(s \star v_n) \\ &= \frac{a}{2}e^{2s} + \frac{b}{4}e^{4s} - e^{Ns} \int_{\mathbb{R}^N} F\big(e^{\frac{Ns}{2}} v_n\big)\, dx = \frac{a}{2}e^{2s} + \frac{b}{4}e^{4s} + o_n(1). \end{aligned}$$

By the arbitrariness of $s \in \mathbb{R}$ we deduce that $I(u_n) \to +\infty$. □

4 Ground states

This section is devoted to the proof of Theorem 1.1. The proof is divided into two main steps: in the first part, we discuss the existence of a Palais–Smale sequence for I constrained on S_m; and in the second one, we study the convergence of the Palais–Smale sequence. For the rest of the proof, we suppose that $1 \le N \le 3$ and f satisfies (f_0)–(f_4).

Lemma 4.1 *There exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_m$ for I constrained on S_m at the level E_m. In addition, if f is odd, then $\|u_n^-\|_2 \to 0$.*

The following our argument is somehow adopted from [17, 18]. Define the functional $\Psi : H^1(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}$ by

$$\Psi(u) := I\big(s(u) \star u\big) = \frac{a}{2}e^{2s(u)}\|\nabla u\|_2^2 + \frac{b}{4}e^{4s(u)}\|\nabla u\|_2^4 - e^{-Ns(u)} \int_{\mathbb{R}^N} F\big(e^{\frac{Ns(u)}{2}} u\big)\, dx,$$

where $s(u) \in \mathbb{R}$ is given by Lemma 2.6. To prove Lemma 4.1, inspired by [19, Proposition 2.9] (see also [20, Proposition 9]), we first study several intermediate lemmas.

Lemma 4.2 *For any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\varphi \in H^1(\mathbb{R}^N)$, the functional Ψ is in C^1, and*

$$\begin{aligned} d\Psi(u)[\varphi] &= ae^{2s(u)} \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx + be^{4s(u)} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx \\ &\quad - e^{-Ns(u)} \int_{\mathbb{R}^N} f\big(e^{\frac{Ns(u)}{2}} u\big) e^{\frac{Ns(u)}{2}} \varphi\, dx \\ &= dI\big(s(u) \star u\big)\big[s(u) \star \varphi\big]. \end{aligned}$$

Proof For any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\varphi \in H^1(\mathbb{R}^N)$, it is necessary to estimate the term

$$\Psi(u + t\varphi) - \Psi(u) = I\big(s_t \star (u + t\varphi)\big) - I(s_0 \star u),$$

where $|t|$ is sufficiently small, and $s_t := s(u + t\varphi)$. Since $s_0 = s(u)$ is the maximum point of $I(s \star u)$, combining this with the mean value theorem, we infer that

$$\begin{aligned}
& I\big(s_t \star (u + t\varphi)\big) - I(s_0 \star u) \\
&\quad \leq I\big(s_t \star (u + t\varphi)\big) - I(s_t \star u) \\
&\quad = \frac{a}{2} e^{2s_t} \int_{\mathbb{R}^N} \big(\big|\nabla(u + t\varphi)\big|^2 - |\nabla u|^2\big)\, dx \\
&\qquad + \frac{b}{4} e^{4s_t} \bigg[\bigg(\int_{\mathbb{R}^N} \big|\nabla(u + t\varphi)\big|^2\, dx\bigg)^2 - \bigg(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\bigg)^2\bigg] \\
&\qquad - e^{-Ns_t} \int_{\mathbb{R}^N} \big[F\big(e^{\frac{Ns_t}{2}}(u + t\varphi)\big) - F\big(e^{\frac{Ns_t}{2}} u\big)\big]\, dx \\
&\quad = \frac{a}{2} e^{2s_t} \int_{\mathbb{R}^N} \big(2t\nabla u \cdot \nabla\varphi + t^2 |\nabla\varphi|^2\big)\, dx - e^{-Ns_t} \int_{\mathbb{R}^N} f(e^{\frac{Ns_t}{2}}(u + \eta_t t\varphi) e^{\frac{Ns_t}{2}} t\varphi\, dx \\
&\qquad + \frac{b}{4} e^{4s_t} \bigg[4t \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx + 4t^2 \bigg(\int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx\bigg)^2 \\
&\qquad + 2t^2 \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx \\
&\qquad + t^3 \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx + t^4 \bigg(\int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx\bigg)^2 dx\bigg],
\end{aligned}$$

where $\eta_t \in (0, 1)$. Similarly, we also have

$$\begin{aligned}
& I\big(s_t \star (u + t\varphi)\big) - I(s_0 \star u) \\
&\quad \geq I\big(s_0 \star (u + t\varphi)\big) - I(s_0 \star u) \\
&\quad = \frac{a}{2} e^{2s_0} \int_{\mathbb{R}^N} \big(2t\nabla u \cdot \nabla\varphi + t^2 |\nabla\varphi|^2\big)\, dx - e^{-Ns_0} \int_{\mathbb{R}^N} f(e^{\frac{Ns_0}{2}}(u + \tau_t t\varphi) e^{\frac{Ns_0}{2}} t\varphi\, dx \\
&\qquad + \frac{b}{4} e^{4s_0} \bigg[4t \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx + 4t^2 \bigg(\int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx\bigg)^2 \\
&\qquad + 2t^2 \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx \\
&\qquad + t^3 \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx + t^4 \bigg(\int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx\bigg)^2 dx\bigg],
\end{aligned}$$

where $\tau_t \in (0, 1)$. By Lemma 2.6(iii) we have $\lim_{t\to 0} s_t = s_0 = s(u)$. Combining the above two inequalities, we conclude that the Gâteaux derivative of Ψ exists and is given by

$$\begin{aligned}
\lim_{t\to 0} \frac{\Psi(u + t\varphi) - \Psi(u)}{t} &= a e^{2s(u)} \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx + b e^{4s(u)} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\, dx \\
&\quad - e^{-Ns(u)} \int_{\mathbb{R}^N} f\big(e^{\frac{Ns(u)}{2}} u\big) e^{\frac{Ns(u)}{2}} \varphi\, dx.
\end{aligned}$$

Coming back to Lemma 2.6(iii) again, we get that the Gâteaux derivative is continuous in u and Ψ is C^1 (see, e.g., [23, 24]). By a change of variables in the integrals we infer

$$\begin{aligned} d\Psi(u)[\varphi] = a &\int_{\mathbb{R}^N} \nabla\big(s(u)\star u\big)\cdot\nabla\big(s(u)\star\varphi\big)\,dx \\ &+ b\int_{\mathbb{R}^N}\big|\nabla\big(s(u)\star u\big)\big|^2\,dx\int_{\mathbb{R}^N}\nabla\big(s(u)\star u\big)\cdot\nabla\big(s(u)\star\varphi\big)\,dx \\ &- \int_{\mathbb{R}^N} f\big(s(u)\star u\big)s(u)\star\varphi\,dx \\ = dI&\big(s(u)\star u\big)\big[s(u)\star\varphi\big]. \end{aligned}$$

The proof is completed. □

For any $m > 0$, we introduce the constrained functional

$$J := \Psi|_{S_m} : S_m \to \mathbb{R}.$$

As an immediate consequence of Lemma 4.2, we have the following lemma.

Lemma 4.3 *For any $u \in S_m$ and $\varphi \in T_u S_m$, the functional $J : S_m \to \mathbb{R}$ is in C^1, and*

$$dJ(u)[\varphi] = d\Psi(u)[\varphi] = dI\big(s(u)\star u\big)\big[s(u)\star\varphi\big].$$

We recall a definition and the minimax principle under the standard boundary condition. By these we can establish a technical result, which helps us to obtain a "nice" Palais–Smale sequence.

Definition 4.4 ([25, Definition 3.1]) Let B be a closed subset of a metric space X. We say that a class $\mathcal{G}$ of compact subsets of X is a homotopy stable family with closed boundary B if

(i) every set in $\mathcal{G}$ contains B and
(ii) for any set $A \in \mathcal{G}$ and any homotopy $\eta \in C([0,1]\times X, X)$ satisfying $\eta(t,u) = u$ for all $(t,u) \in (\{0\}\times X)\cup([0,1]\times B)$, we have $\eta(\{1\}\times A) \in \mathcal{G}$.

The above definition is still valid if the boundary $B = \emptyset$.

Lemma 4.5 ([25, Theorem 3.2]) *Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy-stable family $\mathcal{G}$ of compact subsets of X with closed boundary B. Set $c = c(\varphi,\mathcal{G}) = \inf_{A\in\mathcal{G}}\max_{x\in A}\varphi(x)$ and suppose that*

$$\sup\varphi(B) < c.$$

Then, for any sequence of sets $(A_n)_n$ in $\mathcal{G}$ such that $\lim_n \sup_{A_n}\varphi = c$, there exists a sequence $(x_n)_n$ in X such that

(i) $\lim_n \varphi(x_n) = c$, (ii) $\lim_n \|d\varphi(x_n)\| = 0$, *and* (iii) $\lim_n \operatorname{dist}(x_n, A_n) = 0$.
Moveover, if $d\varphi$ is uniformly continuous, then x_n can be chosen to be in A_n for each n.

Lemma 4.6 *Assume that $\mathcal{G}$ is a homotopy-stable family of compact subsets of S_m with $B = \emptyset$. Define the level*

$$E_{m,\mathcal{G}} := \inf_{A\in\mathcal{G}} \max_{u\in A} J(u).$$

If $E_{m,\mathcal{G}} > 0$, then there exists a Palais–Smale sequence for I constrained on S_m at the level $E_{m,\mathcal{G}}$. When f is odd and $\mathcal{G}$ is the class of singletons included in S_m, then $\|u_n^-\|_2 \to 0$.

Proof Let $\{A_n\} \subset \mathcal{G}$ be a minimizing sequence for $E_{m,\mathcal{G}}$. Set

$$\eta : [0,1] \times S_m \to S_m, \qquad \eta(t,u) = \big(ts(u)\big) \star u.$$

Then it follows from Lemma 2.6(iii) that η is continuous. Furthermore, $\eta(t,u) = u$ for $(t,u) \in \{0\} \times S_m$. By the definition of $\mathcal{G}$ we have

$$D_n := \eta(1,A_n) = \big\{s(u) \star u \mid u \in A_n\big\} \in \mathcal{G}. \tag{4.1}$$

Obviously, $D_n \subset \mathcal{P}_m$ for $n \in \mathbb{R}^+$. The fact that $J(s(u) \star u) = J(u)$ for all $u \in A_n$ implies that $\max_{D_n} J = \max_{A_n} J \to E_{m,\mathcal{G}}$, and hence $\{D_n\} \subset \mathcal{G}$ is also a minimizing sequence of $E_{m,\mathcal{G}}$. Therefore Lemma 4.5 yields a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ such that, as $n \to \infty$,

(1) $J(v_n) \to E_{m,\mathcal{G}}$,

(2) $\operatorname{dist}(v_n, D_n) \to 0$, and (3) $\|dJ(v_n)\|_{v_n,*} \to 0$, where $\|\cdot\|_{u,*}$ is the dual norm of $(T_u S_m)^*$.

That is, $\{v_n\}$ is a Palais–Smale sequence for J on S_m at level $E_{m,\mathcal{G}}$. Hereafter, set

$$s_n := s(v_n) \quad \text{and} \quad u_n := s_n \star v_n = s(v_n) \star v_n.$$

We will show that $\{u_n\} \subset \mathcal{P}_m$ is a Palais–Smale sequence for I at level $E_{m,\mathcal{G}}$.

Claim *There exists $C > 0$ such that $e^{-2s_n} \le C$, $n = 1, 2, \cdots$.*

Indeed, we can easily see that

$$e^{-2s_n} = \frac{\|\nabla v_n\|_2^2}{\|\nabla u_n\|_2^2}.$$

Since $\{u_n\} \subset \mathcal{P}_m$, by Lemma 2.7(ii) we infer that $\{\|\nabla u_n\|_2\}$ is bounded from below by a positive constant. Thus the proof of Claim is reduced to showing that $\sup_n \|\nabla v_n\|_2 < \infty$. Since $D_n \subset \mathcal{P}_m$, it is not difficult to check that

$$\max_{D_n} I = \max_{D_n} J \to E_{m,\mathcal{G}},$$

and thanks to Lemma 2.7(iv), we have that $\{D_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Then, combining this with $\operatorname{dist}(v_n, D_n) \to 0$, we derive $\sup_n \|\nabla v_n\|_2 < \infty$.

Noticing that $\{u_n\} \subset \mathcal{P}_m$ again, we have

$$I(u_n) = J(u_n) = J(v_n) \to E_{m,\mathcal{G}}.$$

Now it remains to prove that $\{u_n\} \subset \mathcal{P}_m$ is a Palais–Smale sequence for I. For any $\psi \in T_{u_n}S_m$, we can see that

$$\int_{\mathbb{R}^N} v_n\big[(-s_n)\star\psi\big]\,dx = \int_{\mathbb{R}^N}(s_n\star v_n)\psi\,dx = \int_{\mathbb{R}^N} u_n\psi\,dx = 0,$$

which implies $(-s_n)\star\psi \in T_{v_n}S_m$. Using Claim, we derive $\|(-s_n)\star\psi\| \le \max\{\sqrt{C},1\}\|\psi\|$. Furthermore, it follows from Lemma 4.3 that

$$\begin{aligned}
\big\|dI(u_n)\big\|_{u_n,*} &= \sup_{\psi\in T_{u_n}S_m,\|\psi\|\le 1}\big|dI(u_n)[\psi]\big| \\
&= \sup_{\psi\in T_{u_n}S_m,\|\psi\|\le 1}\big|dI(s_n\star v_n)\big[s_n\star\big((-s_n)\psi\big)\big]\big| \\
&= \sup_{\psi\in T_{u_n}S_m,\|\psi\|\le 1}\big|dJ(v_n)\big[(-s_n)\star\psi\big]\big| \\
&\le \big\|dJ(v_n)\big\|_{v_n,*}\cdot \sup_{\psi\in T_{u_n}S_m,\|\psi\|\le 1}\big\|(-s_n)\star\psi\big\| \\
&\le \max\{\sqrt{C},1\}\big\|dJ(v_n)\big\|_{v_n,*}.
\end{aligned}$$

In view of $\|dJ(v_n)\|_{v_n,*}\to 0$, we conclude that $\|dI(u_n)\|_{u_n,*}\to 0$.

In particular, if f is odd, then we choose $\mathcal{G}$ as the class of singletons included in S_m. Obviously, $\mathcal{G}$ is a homotopy-stable family of compact subsets of S_m with $B=\emptyset$. The fact f is odd, together with Lemma 2.6(iv), implies that J is even. Thus it is possible to choose a nonnegative minimizing sequence $\{A_n\}\subset\mathcal{G}$; then, naturally, the minimizing sequence $\{D_n\}$ is also nonnegative. By a similar argument as above, we can find a Palais–Smale sequence $\{u_n\}\subset\mathcal{P}_m$ for I constrained on S_m at the level $E_{m,\mathcal{G}}$ satisfying

$$\big\|u_n^-\big\|_2^2 = \big\|s(v_n)\star v_n^-\big\|_2^2 = \big\|v_n^-\big\|_2^2 \to 0,$$

which concludes the proof. □

Proof of Lemma 4.1 Since Lemma 4.6 yields a Palais–Smale sequence for I constrained on S_m at the level $E_{m,\mathcal{G}}$ and $E_m>0$, we only need to prove that $E_{m,\mathcal{G}}=E_m$. Clearly,

$$E_{m,\mathcal{G}} = \inf_{A\in\mathcal{G}}\max_{u\in A} J(u) = \inf_{u\in S_m} I\big(s(u)\star u\big).$$

For any $u\in S_m$, since $s(u)\star u\in\mathcal{P}_m$, we have $I(s(u)\star u)\ge E_m$, which implies $E_{m,\mathcal{G}}\ge E_m$. On the other hand, for any $u\in\mathcal{P}_m$, we infer $I(u)=I(0\star u)\ge E_{m,\mathcal{G}}$. Therefore $E_{m,\mathcal{G}}\le E_m$. □

Lemma 4.7 *Suppose* (f_5) *holds. Let* $\{u_n\}$ *be a bounded Palais–Smale sequence for* I *constrained on* S_m *at the level* $E_m>0$ *satisfying* $P(u_n)\to 0$. *Then there exist* $u\in S_m$ *and* $\mu>0$ *such that, up to a subsequence,* $u_n\to u$ *in* $H^1(\mathbb{R}^N)$ *and* $-(a+b\int_{\mathbb{R}^N}|\nabla u|^2)\triangle u+\mu u=f(u)$.

Proof Since $\{u_n\}$ is bounded, by (f_0)–(f_2) we infer that $\lim_n\|\nabla u_n\|_2$, $\lim_n\int_{\mathbb{R}^N}F(u_n)\,dx$, and $\lim_n\int_{\mathbb{R}^N}f(u_n)u_n\,dx$ exist. From $\|dI(u_n)\|_{u_n,*}\to 0$ and [26, Lemma 3] we deduce

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u_n|^2\right)\triangle u_n+\mu_n u_n-f(u_n)\to 0 \quad \text{in } \big(H^1\big(\mathbb{R}^N\big)\big)^*, \tag{4.2}$$

where

$$\mu_n := -\frac{1}{m}\, dI(u_n)[u_n].$$

Obviously, $\{\mu_n\}$ is a bounded sequence. In particular, from $P(u_n) \to 0$, Lemma 2.5, and (f_5) it follows that

$$\begin{aligned} m\mu_n &= \int_{\mathbb{R}^N} f(u_n)u_n\, dx - a\|\nabla u_n\|_2^2 - b\|\nabla u_n\|_2^4 \\ &= \int_{\mathbb{R}^N} f(u_n)u_n\, dx - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u_n)\, dx + o_n(1) \\ &= \int_{\mathbb{R}^N} \left[NF(u_n) + \frac{2-N}{2} f(u_n)u_n\right] dx + o_n(1) \\ &\geq 0. \end{aligned}$$

Thus, without loss of generality, we may assume that $\mu_n \to \mu \geq 0$. Then we show that $\{u_n\}$ is nonvanishing. Otherwise, using Lions' lemma ([22, Lemma I.1)], we derive $u_n \to 0$ in $L^{2+\frac{8}{N}}(\mathbb{R}^N)$. Therefore Lemma 2.1(ii) implies that $\int_{\mathbb{R}^N} F(u_n)\, dx \to 0$ and $\int_{\mathbb{R}^N} \tilde{F}(u_n)\, dx \to 0$. Thanks to $P(u_n) \to 0$, we deduce

$$a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 = P(u_n) + \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u_n)\, dx \to 0.$$

As a result,

$$0 < E_m = \lim_{n\to\infty} I(u_n) = \frac{a}{2}\lim_{n\to\infty} \|\nabla u_n\|_2^2 + \frac{b}{4}\lim_{n\to\infty} \|\nabla u_n\|_2^4 - \lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n)\, dx = 0,$$

a contradiction.

Thus, up to a subsequence, there exist $\{y_n\} \subset \mathbb{R}^N$ and $u \in B_m\backslash\{0\}$ such that

$$\begin{cases} u_n(\cdot + y_n) \rightharpoonup u & \text{in } H^1(\mathbb{R}^N), \\ u_n(\cdot + y_n) \to u & \text{a.e. in } \mathbb{R}^N, \\ u_n(\cdot + y_n) \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ for } p \in [1, 2^*). \end{cases}$$

In view of (4.2) and $\mu_n \to \mu$, we can easily see that

$$\begin{aligned} &-\big(a + b\big\|\nabla u_n(\cdot + y_n)\big\|_2^2\big)\triangle u_n(\cdot + y_n) + \mu u_n(\cdot + y_n) - f\big(u_n(\cdot + y_n)\big) \\ &\quad \to 0 \quad \text{in } \big(H^1(\mathbb{R}^N)\big)^*, \end{aligned} \tag{4.3}$$

which means

$$\begin{aligned} &\big(a + b\big\|\nabla u_n(\cdot + y_n)\big\|_2^2\big)\int_{\mathbb{R}^N} \nabla u_n(\cdot + y_n)\cdot\nabla\varphi\, dx + \mu\int_{\mathbb{R}^N} u_n(\cdot + y_n)\varphi\, dx \\ &\quad - \int_{\mathbb{R}^N} f\big(u_n(\cdot + y_n)\big)\varphi\, dx = o_n(1) \end{aligned} \tag{4.4}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since f satisfies (f_0)–(f_2), using the compactness lemma ([27, Lemma 2] or [28, Lemma A.I]), we derive

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|\left[f\left(u_n(\cdot+y_n)\right)-f(u)\right]\varphi\right|dx \le \|\varphi\|_\infty \lim_{n\to\infty}\int_{\operatorname{supp}(\varphi)}\left|f\left(u_n(\cdot+y_n)\right)-f(u)\right|dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Defining $B := \lim_{n\to\infty}\|\nabla u_n\|_2^2$, we can see that

$$(a+bB)\int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi\,dx + \mu\int_{\mathbb{R}^N}u\varphi\,dx - \int_{\mathbb{R}^N}f(u)\varphi\,dx = 0. \tag{4.5}$$

Testing (4.4)–(4.5) with $\varphi = u_n(\cdot+y_n)-u$, we deduce

$$(a+bB)\left\|\nabla\left(u_n(\cdot+y_n)-u\right)\right\|_2^2 + \mu\left\|u_n(\cdot+y_n)-u\right\|_2^2 = o(1), \tag{4.6}$$

which implies that $\|\nabla(u_n(\cdot+y_n)-u)\|_2^2 = o(1)$. Hence $u_n(\cdot+y_n)\to u$ in $D^{1,2}(\mathbb{R}^N)$, and

$$\|\nabla u_n\|_2^2 = \left\|\nabla u_n(\cdot+y_n)\right\|_2^2 \to \|\nabla u\|_2^2 = B.$$

So (4.5) can be expressed in the form

$$\left(a+b\|\nabla u\|_2^2\right)\int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi\,dx + \mu\int_{\mathbb{R}^N}u\varphi\,dx - \int_{\mathbb{R}^N}f(u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, which implies that u solves

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\triangle u + \mu u = f(u).$$

Then by a similar argument as above we deduce that

$$\begin{aligned} m\mu &= \int_{\mathbb{R}^N}f(u)u\,dx - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 \\ &= \int_{\mathbb{R}^N}f(u)u\,dx - \frac{N}{2}\int_{\mathbb{R}^N}\tilde{F}(u)\,dx \\ &= \int_{\mathbb{R}^N}\left[NF(u) + \frac{2-N}{2}f(u)u\right]dx \\ &> 0. \end{aligned}$$

Combining these with (4.6), we infer $\|u_n\|_2^2 = \|u_n(\cdot+y_n)\|_2^2 = \|u\|_2^2$. Thus $u\in S_m$, and the lemma follows. □

Proof of Theorem 1.1 First, Lemmas 4.1 and 2.7(iv) yield a bounded Palais–Smale sequence $\{u_n\}\subset\mathcal{P}_m$ for I constrained on S_m at the level $E_m>0$. By Lemma 4.7 we can show the existence of a ground state $u\in S_m$ at the level E_m and the associated Lagrange multiplier $\mu>0$. In addition, if f is odd, then it follows from Lemma 4.1 that $\|u_n^-\|_2\to 0$. Applying Lemma 4.7 again, we can obtain a nonnegative ground state $u\in S_m$ at the level E_m. Furthermore, it is not difficult to show that $u>0$ by the strong maximum principle. □

5 Radial solutions

In this section, we consider multiple solutions for problem (1.1)–(1.2) in the case $2 \leq N \leq 3$. Assume that f is odd and (f_0)–(f_5) hold. Define the transformation $\sigma(u) = -u$ for $u \in H^1(\mathbb{R}^N)$, and let $X \subset H^1(\mathbb{R}^N)$. A set $A \subset X$ satisfying $\sigma(A) = A$ is said to be σ-invariant, and for $(t,u) \in [0,1] \times X$, a homotopy $\eta : [0,1] \times X \to X$ is called σ-equivariant if $\eta(t,\sigma(u)) = \sigma(\eta(t,u))$. For the subsequent proof, we now list some definitions and theorems.

Definition 5.1 ([25, Definition 7.1]) Let B be a σ-invariant subset of $X \subset H^1(\mathbb{R}^N)$. We say that a class $\mathcal{G}$ of compact subsets of X is a σ-homotopy-stable family with closed boundary B if

- (i) every set in $\mathcal{G}$ is σ-invariant,
- (ii) every set in $\mathcal{G}$ contains B, and
- (iii) for any set $A \in \mathcal{G}$ and any σ-equivariant $\eta \in C([0,1] \times X, X)$ satisfying $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times X) \cup ([0,1] \times B)$, we have $\eta(\{1\} \times A) \in \mathcal{G}$.

Lemma 5.2 ([25, Theorem 7.2]) *Let φ be a σ-invariant C^1-functional on a complete connected C^2-Finsler manifold X (without boundary). Let $\mathcal{G}$ be a σ-homotopy-stable family with closed boundary B. Set $c = c(\varphi, \mathcal{G})$, and let F be a closed σ-invariant subset of X satisfying*

$$F \cap B = \emptyset \quad \text{and} \quad F \cap A = \emptyset \quad \text{for all } A \text{ in } \mathcal{G}$$

and

$$\inf \varphi(F) \geq c - \delta.$$

Suppose $0 < \delta < \max\{\frac{1}{32}\operatorname{dist}^2(B,F), \frac{1}{8}[\inf \varphi(F) - \sup \varphi(B)]\}$ *Then, for any A in $\mathcal{G}$ satisfying* $\max \varphi(A) \leq c + \delta$, *there exists a sequence $x_\delta \in X$ such that*

(i) $c - \delta \leq \varphi(x_\delta) \leq c + 9\delta$; (ii) $\|d\varphi(x_\delta)\| \leq 18\sqrt{\delta}$; (iii) $\operatorname{dist}(x_\delta, F) \leq 5\sqrt{\delta}$; *and*
(iv) $\operatorname{dist}(x_\delta, A) \leq 3\sqrt{\delta}$.

From now on, we set $\{V_k\} \subset H_r^1(\mathbb{R}^N)$ be a strictly increasing sequence of finite-dimensional linear subspaces such that $\dim V_k = k$ and $\bigcup_{k\geq 1} V_k$ is dense in $H_r^1(\mathbb{R}^N)$. To better characterize the critical level, it is necessary to recall the definition of the genus of a σ-invariant set and its properties (we refer to [29, Sect. 7] or [30]).

Definition 5.3 ([29]) For any nonempty closed σ-invariant set $A \subset H_r^1(\mathbb{R}^N)$, the genus of A is defined by

$$\operatorname{Ind}(A) := \min\big\{k \in \mathbb{N}^+ \mid \exists\, \phi : A \to \mathbb{R}^k \backslash \{0\}, \phi \text{ is odd and continuous}\big\}.$$

Remark that $\operatorname{Ind}(A) = \infty$ if such ϕ does not exist and $\operatorname{Ind}(A) = 0$ if $A = \emptyset$.

Let Σ denote the family of compact σ-invariant subsets of $S_m \cap H_r^1(\mathbb{R}^N)$. Define

$$\mathcal{G}_k := \big\{A \in \Sigma \mid \operatorname{Ind}(A) \geq k\big\}$$

and

$$E_{m,k} := \inf_{A\in\mathcal{G}_k} \max_{u\in A} J(u).$$

The fact that f is odd, combined with Lemma 2.6(iv), implies that the functional

$$J(u) = I\big(s(u)\star u\big) = \frac{a}{2}e^{2s(u)}\|\nabla u\|_2^2 + \frac{b}{4}e^{4s(u)}\|\nabla u\|_2^4 - e^{-Ns(u)}\int_{\mathbb{R}^N} F\big(e^{\frac{Ns(u)}{2}}u\big)\,dx$$

is even in $u \in S_m$. As a consequence, J is σ-invariant on S_m. With the help of Lemma 5.2, we establish the following existence result.

Lemma 5.4

(i) $\mathcal{G}_k \neq \emptyset$ *for all* $k \in \mathbb{N}^+$.

(ii) $E_{m,k+1} \geq E_{m,k} > 0$ *for all* $k \in \mathbb{N}^+$.

(iii) *For all* $k \in \mathbb{N}^+$, *there exists a Palais–Smale sequence* $\{u_n\} \subset \mathcal{P}_m \cap H_r^1(\mathbb{R}^N)$ *for* I *constrained on* $S_m \cap H_r^1(\mathbb{R}^N)$ *at the level* $E_{m,k}$.

Proof (i) For all $k \in \mathbb{N}^+$, $S_m \cap V_k \in \Sigma$. In view of the properties of genus, we infer

$$\operatorname{Ind}(S_m \cap V_k) = k,$$

which implies that $\mathcal{G}_k \neq \emptyset$.

(ii) Combining Definition 5.1 with the properties of genus, we deduce that $\mathcal{G}_k$ is a σ-homotopy-stable family of compact subsets of $S_m \cap H_r^1(\mathbb{R}^N)$, which guarantees that $E_{m,k}$ is well defined. For each $u \in A \in \mathcal{G}_k$, since $s(u)\star u \in \mathcal{P}_m$, by Lemma 2.7(iii) we have

$$\max_{u\in A} J(u) = \max_{u\in A} I\big(s(u)\star u\big) \geq \inf_{v\in\mathcal{P}_m} I(v) > 0.$$

Therefore $E_{m,k} > 0$. Recalling the definition of $\mathcal{G}_k$, it follows that $\mathcal{G}_{k+1} \subset \mathcal{G}_k$, which implies $E_{m,k+1} \geq E_{m,k}$.

(iii) Since $\mathcal{G}_k$ is a σ-homotopy-stable family of compact subsets of $S_m \cap H_r^1(\mathbb{R}^N)$, replacing Lemma 4.5 by Lemma 5.2 in the proof of Lemma 4.6, we can easily obtain the particular Palais–Smale sequence. We omit it for brevity. □

Lemma 5.5 *Let* $\{u_n\}$ *be a bounded Palais–Smale sequence for* I *constrained on* $S_m \cap H_r^1(\mathbb{R}^N)$ *at an arbitrary level* $c > 0$ *satisfying* $P(u_n) \to 0$. *Then there exist* $u \in S_m \cap H_r^1(\mathbb{R}^N)$ *and* $\mu > 0$ *such that, up to a subsequence,* $u_n \to u$ *in* $H_r^1(\mathbb{R}^N)$ *and* $-(a + b\int_{\mathbb{R}^N} |\nabla u|^2)\triangle u + \mu u = f(u)$.

Proof Since $\{u_n\}$ is bounded, using (f_0)–(f_2), we infer that $\lim_n \|\nabla u_n\|_2$, $\lim_n \int_{\mathbb{R}^N} F(u_n)\,dx$ and $\lim_n \int_{\mathbb{R}^N} f(u_n)u_n\,dx$ exist. From $\|dI(u_n)\|_{u_n,*} \to 0$ and [26, Lemma 3] we have

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2\right)\triangle u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in } \big(H_r^1(\mathbb{R}^N)\big)^*, \tag{5.1}$$

where

$$\mu_n := -\frac{1}{m}\, dI(u_n)[u_n].$$

By a similar argument as in the proof of Lemma 4.7 we assume that $\mu_n \to \mu \geq 0$. Since $\{u_n\}$ is bounded, up to a subsequence, there exists some $u \in B_m$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_r^1(\mathbb{R}^N), \\ u_n \to u & \text{a.e. in } \mathbb{R}^N, \\ u_n \to u & \text{in } L^p(\mathbb{R}^N) \text{ for } p \in (2, 2^*). \end{cases}$$

Then we claim that $u \neq 0$. Indeed, if not, then we have $u_n \to 0$ in $L^{2+\frac{8}{N}}(\mathbb{R}^N)$. By Lemma 2.1(ii) it follows that $\int_{\mathbb{R}^N} F(u_n)\,dx \to 0$ and $\int_{\mathbb{R}^N} \tilde{F}(u_n)\,dx \to 0$. Since $P(u_n) \to 0$, we deduce

$$a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 = P(u_n) + \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u_n)\,dx \to 0.$$

As a consequence,

$$c = \lim_{n\to\infty} I(u_n) = \frac{a}{2}\lim_{n\to\infty}\|\nabla u_n\|_2^2 + \frac{b}{4}\lim_{n\to\infty}\|\nabla u_n\|_2^4 - \lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n)\,dx = 0,$$

which contradicts $c > 0$. Coming back to (5.1) and $\mu_n \to \mu$, we can easily see that

$$-\left(a + b\|\nabla u_n\|_2^2\right)\triangle u_n + \mu u_n - f(u_n) \to 0 \quad \text{in } \left(H_r^1(\mathbb{R}^N)\right)^*, \tag{5.2}$$

which means

$$\left(a + b\|\nabla u_n\|_2^2\right)\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\varphi\,dx + \mu\int_{\mathbb{R}^N} u_n\varphi\,dx - \int_{\mathbb{R}^N} f(u_n)\varphi\,dx = o_n(1) \tag{5.3}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since f satisfies (f_0)–(f_2), using the Lebesgue dominated convergence theorem, we derive

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)\varphi\,dx = \int_{\mathbb{R}^N} f(u)\varphi\,dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Defining $B := \lim_{n\to\infty}\|\nabla u_n\|_2^2$, we can see that

$$(a + bB)\int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\,dx + \mu\int_{\mathbb{R}^N} u\varphi\,dx - \int_{\mathbb{R}^N} f(u)\varphi\,dx = 0. \tag{5.4}$$

Testing (5.3)–(5.4) with $\varphi = u_n - u$, we deduce

$$(a + bB)\left\|\nabla(u_n - u)\right\|_2^2 + \mu\|u_n - u\|_2^2 = o(1), \tag{5.5}$$

which implies that $\|\nabla(u_n - u)\|_2^2 = o(1)$. Hence $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, and

$$\|\nabla u_n\|_2^2 \to \|\nabla u\|_2^2 = B.$$

As a result, (5.4) can be expressed in the form

$$\left(a + b\|\nabla u\|_2^2\right)\int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\,dx + \mu\int_{\mathbb{R}^N} u\varphi\,dx - \int_{\mathbb{R}^N} f(u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, which implies that u solves

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\triangle u + \mu u = f(u).$$

Then, by arguments similar to those in Lemma 4.7 we deduce that

$$\begin{aligned} m\mu &= \int_{\mathbb{R}^N} f(u)u\,dx - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 \\ &= \int_{\mathbb{R}^N} f(u)u\,dx - \frac{N}{2}\int_{\mathbb{R}^N} \tilde{F}(u)\,dx \\ &= \int_{\mathbb{R}^N}\left[NF(u) + \frac{2-N}{2}f(u)u\right]dx \\ &> 0. \end{aligned}$$

Combining these with (5.5), we infer $\|u_n\|_2^2 = \|u\|_2^2$. Thus $u \in S_m$, and the lemma follows. □

Proof of Theorem 1.5 For any $k \in \mathbb{N}^+$, by Lemma 5.4 we can easily find a Palais–Smale sequence $\{u_n^k\}_{n=1}^\infty$ for I constrained on $S_m \cap H_r^1(\mathbb{R}^N)$ at the level $E_{m,k} > 0$. Combining this with Lemma 2.7(iv), we obtain that it is bounded. Then the proof of Theorem 1.5 follows by Lemma 5.5. □

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