# A matrix acting between Fock spaces 

Zhengyuan Zhuo ${ }^{1 *}$, Dongxing $\mathrm{Li}^{2}$ and Tiaoying Zeng ${ }^{3}$

Correspondence:
zyzhuo@gpnu.edu.cn
${ }^{1}$ School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, 510665, P.R. China Full list of author information is available at the end of the article

## Abstract

If $\mathcal{H}_{v}=\left(v_{n, k}\right)_{n, k \geq 0}$ is the matrix with entries $v_{n, k}=\int_{[0, \infty)} \frac{t^{n+k}}{n!} d v(t)$, where $v$ is a nonnegative Borel measure on the interval $[0, \infty)$, the matrix $\mathcal{H}_{v}$ acts on the space of all entire functions $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ and induces formally the operator in the following way:

$$
\mathcal{H}_{v}(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} v_{n, k} a_{k}\right) z^{n}
$$

In this paper, for $0<p \leq \infty$, we classify for which measures the operator $\mathcal{H}_{v}(f)$ is well defined on $F^{P}$ and also gets an integral representation, and among them we characterize those for which $\mathcal{H}_{\nu}$ is a bounded (resp., compact) operator between $F^{p}$ and $F^{\infty}$.

Keywords: Fock spaces; Matrices; Fock Carleson measure

## 1 Introduction

Throughout this paper, we write $A \lesssim B$ for nonnegative quantities $A$ and $B$ if there exists a constant $C$ (independent of $A$ and $B$ ) such that $A \leq C B$. The symbol $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$. $C$ denotes a finite constant that may change value from one occurrence to the next.

Let $\mathbb{C}$ be the complex plane and $H(\mathbb{C})$ be the class of all entire functions. If $0<p<\infty$, then the Fock space $F^{p}$ is the set of all $f \in H(\mathbb{C})$ such that

$$
\|f\|_{p}^{p}:=\frac{p}{2 \pi} \int_{\mathbb{C}}\left|f(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} d A(z)<\infty,
$$

where $z=x+i y$ and $d A(z)=d x d y$ is the Lebesgue area measure on $\mathbb{C}$. Set

$$
F^{\infty}=\left\{f \in H(\mathbb{C}):\|f\|_{\infty}=\sup _{z \in \mathbb{C}}|f(z)| e^{-\frac{1}{2}|z|^{2}}<\infty\right\} .
$$

In particular, $F^{2}$ is a reproducing kernel Hilbert space. The function $K_{z}(w)=e^{z \bar{w}}$ is the reproducing kernel for $F^{2}$ and

$$
k_{z}(w)=\frac{K_{z}(w)}{\sqrt{K(z, z)}}=e^{z \bar{w}-\frac{1}{2}|z|^{2}}
$$

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is the normalized kernel. Moreover, each $k_{a}$ is a unit vector in $F^{p}$, where $0<p \leq \infty$. Let $f^{\infty}$ denote the space of entire functions such that

$$
\lim _{z \rightarrow \infty} f(z) e^{-\frac{1}{2}|z|^{2}}=0
$$

Interested readers can refer to [18] for the theory of Fock spaces.
Let $0<p, q<\infty$, and $\mu$ be a nonnegative Borel measure on $\mathbb{C}$. Recall that $\mu$ is a $(p, q)$ Fock Carleson measure if the identity operator $i$ is bounded from $F^{p}$ to $L^{q}\left(e^{-\frac{q}{2} \cdot \Gamma^{2}} d \mu\right)$, i.e., there exists some constant $C$ such that, for all $f \in F^{p}$,

$$
\left(\int_{\mathbb{C}}\left|f(z) e^{-\frac{1}{2}|z|^{2}}\right|^{q} d \mu(z)\right)^{\frac{1}{q}} \leq C\|f\|_{p}
$$

When $p=q, \mu$ is exactly the Fock Carleson measure for $F^{p}$ (see [11, 18]). Also, $\mu$ is called a vanishing $(p, q)$-Fock Carleson measure if

$$
\lim _{j \rightarrow \infty} \int_{\mathbb{C}}\left|f_{j}(z) e^{-\frac{1}{2}|z|^{2}}\right|^{q} d \mu(z)=0
$$

whenever $\left\{f_{j}\right\}$ is a bounded sequence in $F^{p}$ that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $j \rightarrow \infty$.
The matrix $\mathcal{H}_{v}=\left(v_{n, k}\right)_{n, k \geq 0}$ induces formally an operator (which will also be denoted by $\mathcal{H}_{v}$ ) on $H(\mathbb{C})$ in the following sense. For any $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \in H(\mathbb{C})$, by multiplication of the matrix with the sequence of Taylor coefficients of the function, we can define

$$
\begin{equation*}
\mathcal{H}_{\nu}(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} v_{n, k} a_{k}\right) z^{n} \tag{1.1}
\end{equation*}
$$

If the right-hand side makes sense and defines a function in $H(\mathbb{C})$, then the matrix $\mathcal{H}_{v}$ induces formally an operator $\mathcal{H}_{v}$ on $H(\mathbb{C})$.
Let $\mathbb{D}$ be the open unit disc and $H(\mathbb{D})$ be all of the analytic functions on $\mathbb{D}$. If we replace $H(\mathbb{C})$ by $H(\mathbb{D})$ in the definition of operators above, there is a rich history of these operators (which will be denoted by $H_{v}$ ) on several natural $L^{p}$ spaces of analytic functions on $\mathbb{D}$, especially those on the Hardy space and the Bergman space. For example, the Hilbert operator induced by the Hilbert matrix $\left(\frac{1}{n+k+1}\right)_{n, k \geq 0}$ has been studied in Hardy spaces [5] and Bergman spaces [4]; estimates on the norms have also been obtained. More generally, another approach to the study of $H_{v}$ on spaces of analytic functions is developed. Galanopoulos and Peláez introduced Hankel matrix $H_{v}=\left(\int_{[0,1)} t^{n+k} d \nu(t)\right)_{n, k \geq 0}$, where $v$ is a finite nonnegative Borel measure on $[0,1)$, and also investigated the boundedness and compactness of the operator induced by $H_{v}$ on the Hardy space $H^{1}$ and the Bergman space $A^{2}$ in [6]. Chatzifountas, Girela, and Peláez [2] later generalized the operator $H_{v}$ on Hardy spaces $H^{p}$. In [7, 8], Girela and Merchán also studied the operator $H_{v}$ acting on certain möbius invariant spaces on the unit disk such as the Bloch space, BMOA, the analytic Besov spaces, etc. Recently, Ye and Zhou considered a new operator, called derivativeHilbert operator, induced by some Hankel matrix on analytic function spaces on $\mathbb{D}$ in [16, 17]. It turns out that the derivative-Hilbert operator and the Hilbert operator $H_{v}$ are closely related. For more results on the operator induced by some Hankel matrix, we refer to $[1,5,14,15]$.

From now on, $v$ denotes a nonnegative Borel measure on $[0, \infty)$. In the Fock space setting, Zhuo et al. [19] considered a special matrix $\mathcal{H}_{v}=\left(v_{n, k}\right)_{n, k \geq 0}$ with entries

$$
v_{n, k}=\frac{1}{n!} \int_{[0, \infty)} t^{n+k} d v(t)
$$

and characterized those nonnegative Borel measures $v$ such that the operator $\mathcal{H}_{v}$ was well defined on the Fock space $F^{p}(0<p<\infty)$. Furthermore, $\mathcal{H}_{v}$ was rewritten as

$$
\mathcal{H}_{\nu}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d \nu(t), \quad z \in \mathbb{C}
$$

Under this integral representation of $\mathcal{H}_{v}$, the measures $v$ for which $\mathcal{H}_{v}$ is a bounded (resp., compact) operator from the Fock space $F^{p}$ into $F^{q}(0<p, q<\infty)$ were characterized. The main purpose of this paper is to extend the operator $\mathcal{H}_{v}$ to be well defined on the Fock space $F^{\infty}$. With $0<p \leq \infty$, we are going to obtain some characterizations on those measures $v$ such that the operator $\mathcal{H}_{v}$ is bounded or compact from $F^{p}$ to $F^{\infty}$ and from $F^{\infty}$ to $F^{p}$, respectively.

## 2 Preliminaries

In this section, we state some lemmas for the proof of our main results. The following lemma can be found in [13].

Lemma 2.1 Suppose $\alpha>0$. For every positive integer n,

$$
\int_{0}^{\infty} r^{n p} e^{-\frac{\alpha p}{2} r^{2}} r d r \simeq\left(\frac{n!}{\alpha^{n}}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}}
$$

The following formula is used many times throughout this paper. See [18, Corollary 2.5] for a proof.

Lemma 2.2 Suppose $\alpha>0$ and $\beta$ is real. Then

$$
\frac{\alpha}{\pi} \int_{\mathbb{C}}\left|e^{\beta z \bar{w}}\right| e^{-\alpha|z|^{2}} d A(z)=e^{-\frac{\beta^{2}|w|^{2}}{4 \alpha}}
$$

for all $w \in \mathbb{C}$.

Lemma 2.3 Let $f(z)=\sum a_{n} z^{n}$ be an entire function. Then

$$
f \in F^{\infty} \Rightarrow \sup _{n \in \mathbb{N}}\left|a_{n}\right|(n!)^{\frac{1}{2}} n^{-\frac{1}{4}}<\infty .
$$

Proof By Cauchy's integral formula and Hölder's inequality, it is easy to see that

$$
\left|a_{n}\right| r^{n} e^{-\frac{r^{2}}{2}} \leq \sup _{|z|=r}\left|f\left(r e^{i \theta}\right)\right| e^{-\frac{r^{2}}{2}}
$$

for every $n \in \mathbb{N}$ and $r>0$, and hence $\left\|a_{n} z^{n}\right\|_{\infty} \lesssim\|f\|_{\infty}$. Note that, by elementary calculations and Stirling's formula,

$$
\left\|z^{n}\right\|_{\infty}=\left(\frac{n}{e}\right)^{\frac{n}{2}} \simeq(n!)^{\frac{1}{2}} n^{-\frac{1}{4}}
$$

Thus, for every $n$,

$$
\left|a_{n}\right|(n!)^{\frac{1}{2}} n^{-\frac{1}{4}} \lesssim\|f\|_{\infty} .
$$

Let $D(z, r)$ denote the Euclidean disk centered at $z$ with radius $r$. The following basic estimate for integral averages of functions in Fock spaces can be found in [18, Lemma 2.32].

Lemma 2.4 For any $0<p<\infty$ and $r \in(0, \infty)$, there exists a positive constant $C=C(p, r)$ such that, for all $z \in \mathbb{C}$,

$$
\left|f(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} \leq C \int_{D(z, r)}\left|f(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} d A(z)
$$

for all entire functions $f$.
We also need the following result. See [18, Corollary 2.8] for a proof.

Lemma 2.5 Let $0<p \leq \infty$ and $f \in F^{p}$. Then

$$
|f(z)| \leq\|f\|_{p} e^{\frac{1}{2}|z|^{2}}
$$

for all $z \in \mathbb{C}$.

Given $r>0$, a sequence $\left\{a_{k}\right\}$ in $\mathbb{C}$ is called an $r$-lattice if $\bigcup_{k=1}^{\infty} D\left(a_{k}, r\right)$ covers $\mathbb{C}$ and the disks $\left\{D\left(a_{k}, r / 3\right)\right\}_{k=1}^{\infty}$ are pairwise disjoint. For any $\delta>0$, there exists a positive integer $m$ (depending only on $r$ and $\delta$ ) such that every point in $\mathbb{C}$ belongs to at most $m$ of the sets $D\left(a_{k}, \delta\right)$, see [18, page 118]. The following three lemmas characterize the $(p, q)$-Fock Carleson measure and vanishing $(p, q)$-Fock Carleson measure for $0<p, q<\infty$, which can be found in [9]. For a Borel measure $\mu$ on $\mathbb{C}$, define $\widehat{\mu}_{r}(z)=\frac{\mu(D(z r))}{\pi r^{2}}$. Let $t>0$, the $t$-Berezin transform of $\mu$ is defined by

$$
\tilde{\mu}_{t}(z)=\int_{\mathbb{C}} e^{-\frac{t}{2}|z-w|^{2}} d \mu(w)
$$

whenever these integrals converge, see [18, page 120].

Lemma 2.6 Let $0<p \leq q<\infty$, and let $\mu \geq 0$. Then the following statements are equivalent:
(1) $\mu$ is a $(p, q)$-Fock Carleson measure;
(2) $\widehat{\mu}_{r}(z)$ is bounded on $\mathbb{C}$ for some (or any) $r>0$;
(3) $\tilde{\mu}_{t}(z)$ is bounded on $\mathbb{C}$ for some (or any) $t>0$. Furthermore,

$$
\|i\|_{F^{p} \rightarrow L^{q}\left(e^{-\frac{q}{2}}|\cdot|^{2}\right.}^{d \mu)} \left\lvert\, \simeq\left\|\widehat{\mu}_{r}^{\frac{1}{q}}\right\|_{L^{\infty}} \simeq\left\|\tilde{\mu}_{t}^{\frac{1}{q}}\right\|_{L^{\infty}} .\right.
$$

Lemma 2.7 Let $0<p \leq q<\infty$, and let $\mu \geq 0$. Then the following statements are equivalent:
(1) $\mu$ is a vanishing $(p, q)$-Fock Carleson measure;
(2) $\widehat{\mu}_{r}(z) \rightarrow 0$ as $z \rightarrow \infty$ for some (or any) $r>0$;
(3) $\tilde{\mu}_{t}(z) \rightarrow 0$ as $z \rightarrow \infty$ for some (or any) $t>0$.

Lemma 2.8 Let $0<q<p<\infty$ and let $\mu \geq 0$. Set $s=\frac{p}{q}$ and $s^{\prime}$ to be the conjugate exponent of $s$. Then the following statements are equivalent:
(1) $\mu$ is a $(p, q)$-Fock Carleson measure;
(2) $\mu$ is a vanishing $(p, q)$-Fock Carleson measure;
(3) $\widehat{\mu}_{r}(z) \in L^{s^{\prime}}(d A)$ for some (or any) $r>0$.

In the light of above three lemmas, the notion of (vanishing) ( $p, q$ )-Fock Carleson measures does not depend on the particular value of $p, q$, but depends only on the ratio $s=\frac{p}{q}$ in the case $0<q<p<\infty$. Let $\Lambda^{s}$ be the class of all $(p, q)$-Fock Carleson measures such that $s=\frac{p}{q}$ and $\Lambda_{0}^{s}$ be the class of all vanishing $(p, q)$-Fock Carleson measures such that $s=\frac{p}{q}$. When $0<s \leq 1$ (equivalently, $p \leq q$ ), we simply write $\Lambda$ and $\Lambda_{0}$ for $\Lambda^{s}$ and $\Lambda_{0}^{s}$ respectively. That is,

$$
\Lambda=\left\{\mu \geq 0: \widehat{\mu}_{r} \in L^{\infty}(\mathbb{C}) \text { for some } r>0\right\}
$$

and

$$
\Lambda_{0}=\left\{\mu \geq 0: \lim _{|z| \rightarrow \infty} \widehat{\mu}_{r}(z)=0 \text { for some } r>0\right\} .
$$

Notice that $\Lambda^{s} \subset \Lambda$ and $\Lambda_{0}^{s} \subset \Lambda_{0}$ for all $s>0$.

## 3 Conditions such that $\mathcal{H}_{v}$ is well defined on $F^{p}$

In this section, we first clarify for which measures the operator $\mathcal{H}_{v}$ is well defined on Fock spaces and also gets an integral representation. Throughout the paper, a nonnegative Borel measure $\mu$ on $[0, \infty)$ can be seen as a Borel measure on $\mathbb{C}$ by identifying it with the measure $\mu^{\star}$ defined by

$$
\mu^{\star}(A)=\mu(A \cap[0, \infty))
$$

for any Borel subset $A$ of $\mathbb{C}$. In this way a positive Borel measure $\mu$ on $[0, \infty)$ is an a $(p, q)$ Fock Carleson measure if and only if there exists a positive constant $C$ such that

$$
\int_{(a-r, a+r)} d \mu(t) \leq C r^{2}
$$

for any $a \in[0, \infty)$ and any fixed $0 \leq r<\infty$.

Theorem 3.1 Suppose $0<p \leq \infty$ and $v$ is a nonnegative Borel measure on $[0, \infty)$. If $e^{\epsilon \cdot \cdot \mid} \nu \in \Lambda$ for any fixed $\epsilon>\frac{1}{2}$, then the power series in (1.1) is a well defined entire function for every $f \in F^{p}$. Furthermore,

$$
\begin{equation*}
\mathcal{H}_{\nu}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d v(t), \quad z \in \mathbb{C}, f \in F^{p} \tag{3.1}
\end{equation*}
$$

Proof The case when $0<p<\infty$ has been proved in [19]. We now consider the case $p=\infty$. Suppose that $e^{\epsilon|\cdot|^{2}} v \in \Lambda$ with any fixed $\epsilon>\frac{1}{2}$, it might as well assume that $e^{\epsilon|\cdot|^{2}} v$ is $(1,1)$ Fock Carleson measure for any $0<s<\infty$ by Lemma 2.5. For any $0<r<\infty$, fix $f(z)=$ $\sum_{n=0}^{\infty} a_{n} z^{n} \in F^{\infty}$ and $|z| \leq r$. We deduce that, by Lemma 2.2,

$$
\begin{aligned}
\int_{[0, \infty)}\left|f(t) e^{t z}\right| d \nu(t) & \leq\|f\|_{\infty} \int_{[0, \infty)}\left|e^{t z}\right| e^{\frac{1}{2}|t|^{2}} d \nu(t) \\
& =\|f\|_{\infty} \int_{[0, \infty)}\left|e^{t z}\right| e^{\left(\frac{1}{2}-\epsilon\right)|t|^{2}} e^{\epsilon|t|^{2}} d \nu(t) \\
& \lesssim\|f\|_{\infty} \int_{\mathbb{C}}\left|e^{t z}\right| e^{\left(\frac{1}{2}-\epsilon\right)|t|^{2}} d A(t) \\
& \simeq\|f\|_{\infty} e^{\frac{|z|^{2}}{e^{2}-2}} \leq\|f\|_{\infty} e^{\frac{\mid r r^{2}}{4 \epsilon-2}}
\end{aligned}
$$

So the integral in (3.1) uniformly converges for every domain $\{z:|z| \leq r\}$, the resulting function is analytic in $\mathbb{C}$ and, for every $z \in \mathbb{C}$,

$$
\begin{equation*}
\int_{[0, \infty)} f(t) e^{t z} d \nu(t)=\sum_{n=0}^{\infty} \int_{[0, \infty)} f(t) \frac{t^{n}}{n!} d \nu(t) z^{n}=\sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0, \infty)} \sum_{k=0}^{\infty} a_{k} t^{n+k} d \nu(t) z^{n} \tag{3.2}
\end{equation*}
$$

By Lemma 2.6, we may assume that $e^{\epsilon|\cdot|^{2}} v$ is (2,2)-Fock Carleson measure without loss of generality. This together with Hölder's inequality and Lemma 2.1 shows that

$$
\begin{aligned}
\left|v_{n, k}\right| & \leq \frac{1}{n!} \int_{[0, \infty)}|t|^{n+k} d \nu(t) \\
& \leq \frac{1}{n!}\left(\int_{[0, \infty)}\left|t^{k} e^{-\frac{1}{2}|t|^{2}}\right|^{2} e^{\epsilon|t|^{2}} d \nu(t)\right)^{1 / 2}\left(\int_{[0, \infty)}\left|t^{n} e^{\left(\frac{1}{2}-\epsilon\right)|t|^{2}}\right|^{2} e^{\epsilon|t|^{2}} d \nu(t)\right)^{1 / 2} \\
& \leq \frac{1}{n!}\left(\int_{\mathbb{C}}\left|t^{k} e^{-\frac{1}{2}|t|^{2}}\right|^{2} d A(t)\right)^{1 / 2}\left(\int_{\mathbb{C}}\left|t^{n} e^{\left(\frac{1}{2}-\epsilon\right)|t|^{2}}\right|^{2} d A(t)\right)^{1 / 2} \\
& \lesssim \frac{1}{n!}(k!)^{\frac{1}{2}}\left(\frac{n!}{\left(\epsilon-\frac{1}{2}\right)^{n}}\right)^{\frac{1}{2}}
\end{aligned}
$$

Combining this with Lemma 2.3, we conclude that for every $n$,

$$
\begin{aligned}
\left|\sum_{k=0}^{\infty} v_{n, k} a_{k}\right| \leq & \left|v_{n, 0} a_{0}\right|+\left|v_{n, 1} a_{1}\right|+\left|v_{n, 2} a_{2}\right|+\sum_{k=3}^{\infty}\left|v_{n, k} a_{k}\right| \\
= & \left|v_{n, 0} a_{0}\right|+\left|v_{n, 1} a_{1}\right|+\left|v_{n, 2} a_{2}\right|+\sum_{k=3}^{\infty}\left|v_{n+3, k-3} a_{k}\right| \\
\lesssim & \left(\left|a_{0}\right|+\left|a_{1}\right|+\left|a_{2}\right|\right) \frac{1}{n!}\left(\frac{n!}{\left(\epsilon-\frac{1}{2}\right)^{n}}\right)^{\frac{1}{2}} \\
& +\frac{1}{n!}\left(\frac{(n+3)!}{\left(\epsilon-\frac{1}{2}\right)^{n+3}}\right)^{\frac{1}{2}} \sum_{k=3}^{\infty}\left|a_{k}\right|(k!)^{\frac{1}{2}} k^{-\frac{3}{2}} \\
\lesssim & \left(\left|a_{0}\right|+\left|a_{1}\right|+\left|a_{2}\right|\right) \frac{1}{n!}\left(\frac{n!}{\left(\epsilon-\frac{1}{2}\right)^{n}}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{n!}\left(\frac{(n+3)!}{\left(\epsilon-\frac{1}{2}\right)^{n+3}}\right)^{\frac{1}{2}} \sum_{k=3}^{\infty} k^{-\frac{3}{2}+\frac{1}{4}} \sup _{k \in \mathbb{N}}\left|a_{k}\right|(k!)^{\frac{1}{2}} k^{-\frac{1}{4}} \\
\lesssim & \frac{1}{n!}\left(\frac{(n+3)!}{\left(\epsilon-\frac{1}{2}\right)^{n+3}}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Therefore, the series in (1.1) is well defined for all $z \in \mathbb{C}$, and

$$
\sum_{k=0}^{\infty} a_{k} v_{n, k}=\frac{1}{n!} \int_{[0, \infty)} f(t) t^{n} d \nu(t)
$$

By (3.2), we obtain

$$
\mathcal{H}_{\nu}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d \nu(t), \quad z \in \mathbb{C}
$$

This proves the desired result.

Theorem 3.2 Suppose that $v$ is a nonnegative Borel measure on $[0, \infty)$ such that

$$
\begin{equation*}
\int_{[0, \infty)} \int_{[0, t)}\left[1+(t s)^{2}\right] e^{2 t s} d v(s) d \nu(t)<\infty \tag{3.3}
\end{equation*}
$$

Then the power series in (1.1) is well defined for every $f \in F^{\infty}$ and (3.1) holds.

Proof Assume that $v$ satisfies (3.3) and fix $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \in F^{\infty}$. From the definition of $v_{n, k}$, it is elementary to check that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left|v_{n, 0} a_{0} z^{n}\right| & =\left|a_{0}\right| \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^{\frac{1}{2}}|z|^{n}\left(\frac{1}{n!}\right)^{\frac{1}{2}} \int_{[0, \infty)}|t|^{n} d v(t) \\
& \leq\left|a_{0}\right|\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0, \infty)} \int_{[0, \infty)}|t|^{n}|s|^{n} d v(s) d v(t)\right)^{\frac{1}{2}} \\
& =\left|a_{0}\right|\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \frac{2}{n!} \int_{[0, \infty)} \int_{[0, t)}(s t)^{n} d v(s) d v(t)\right)^{\frac{1}{2}} \\
& \lesssim\left|a_{0}\right|\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\int_{[0, \infty)} \int_{[0, t)} e^{s t} d v(s) d v(t)\right)^{\frac{1}{2}}
\end{aligned}
$$

The same arguments show that

$$
\sum_{n=0}^{\infty}\left|v_{n, 1} a_{1} z^{n}\right| \lesssim\left|a_{1}\right|\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\int_{[0, \infty)} \int_{[0, t)}(s t) e^{s t} d \nu(s) d \nu(t)\right)^{\frac{1}{2}}
$$

On the other hand, for any $n \in \mathbb{N}$,

$$
\left|\sum_{k=2}^{\infty} v_{n, k} a_{k}\right| \leq \sum_{k=2}^{\infty}\left|a_{k}\right|(k!)^{\frac{1}{2}} k^{-\frac{1}{4}} k^{\frac{1}{4}}\left(\frac{1}{k!}\right)^{\frac{1}{2}}\left|v_{n, k}\right|
$$

$$
\leq \sup _{k \in \mathbb{N}}\left|a_{k}\right|(k!)^{\frac{1}{2}} k^{-\frac{1}{4}} \sum_{k=2}^{\infty} k^{\frac{1}{4}}\left(\frac{1}{k!}\right)^{\frac{1}{2}}\left|v_{n, k}\right| .
$$

However, by Hölder's inequality, we have

$$
\begin{aligned}
\sum_{k=2}^{\infty}\left(\frac{1}{k!}\right)^{\frac{1}{2}} k^{\frac{1}{4}}\left|v_{n, k}\right| & =\frac{1}{n!} \sum_{k=2}^{\infty}\left(\frac{1}{k!}\right)^{\frac{1}{2}} k^{\frac{1}{4}}\left(\int_{[0, \infty)} t^{n+k} d v(t)\right)^{2 \cdot \frac{1}{2}} \\
& \lesssim \frac{1}{n!} \sum_{k=2}^{\infty} k^{\frac{1}{4}-1}\left(\frac{1}{(k-2)!} \int_{[0, \infty)} \int_{[0, t)}(s t)^{k}(s t)^{n} d v(s) d v(t)\right)^{\frac{1}{2}} \\
& \lesssim \frac{1}{n!}\left(\sum_{k=2}^{\infty} k^{-\frac{3}{2}}\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty} \frac{1}{(k-2)!} \int_{[0, \infty)} \int_{[0, t)}(s t)^{k}(s t)^{n} d v(s) d v(t)\right)^{\frac{1}{2}} \\
& \lesssim \frac{1}{n!}\left(\int_{[0, \infty)} \int_{[0, t)}(s t)^{2} e^{s t}(s t)^{n} d v(s) d v(t)\right)^{\frac{1}{2}} .
\end{aligned}
$$

It follows from Lemma 2.3 and (3.3) that the series $\sum_{k=0}^{\infty} a_{k} v_{n, k}$ is absolutely convergent, and

$$
\sum_{k=0}^{\infty} a_{k} v_{n, k}=\frac{1}{n!} \int_{[0, \infty)} f(t) t^{n} d v(t)
$$

Furthermore,

$$
\begin{aligned}
\left|\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} v_{n, k} a_{k}\right) z^{n}\right| \lesssim & \sum_{n=0}^{\infty}\left|v_{n, 0} a_{0} z^{n}\right|+\sum_{n=0}^{\infty}\left|v_{n, 1} a_{1} z^{n}\right| \\
& +\sum_{n=0}^{\infty}\left(\left(\frac{1}{n!}\right)^{2} \int_{[0, \infty)} \int_{[0, t)} e^{s t}(s t)^{n+2} d v(s) d v(t)\right)^{\frac{1}{2}}|z|^{n} \\
\leq & \left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\int_{[0, \infty)} \int_{[0, t)} e^{2 s t} d v(s) d v(t)\right)^{\frac{1}{2}} \\
& +\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0, \infty)} \int_{[0, t)} e^{s t}(s t)^{n+2} d v(s) d v(t)\right)^{\frac{1}{2}} \\
= & \left(\int_{[0, \infty)} \int_{[0, t)^{2}}\left[1+(t s)^{2}\right] e^{2 s t} d v(s) d v(t)\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \frac{1}{n!}|z|^{2 n}\right)^{\frac{1}{2}}
\end{aligned}
$$

for each $z \in \mathbb{C}$. This shows that the power series in (1.1) represents an entire function and

$$
\mathcal{H}_{\nu}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d \nu(t), \quad z \in \mathbb{C}
$$

The proof is completed.

We also obtain the following necessary condition for the operator $\mathcal{H}_{\nu}$ to be well defined on Fock spaces.

Theorem 3.3 Let v be a nonnegative Borel measure on $[0, \infty)$. If the integral

$$
\mathcal{H}_{v}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d v(t), \quad z \in \mathbb{C}
$$

converges absolutely for every $f \in F^{\infty}$, then $e^{\frac{1}{2} \cdot \|^{2}} d v$ is a $(\infty, 1)$-Fock Carleson measure.
Proof For every $f \in F^{\infty}$, the integral $\int_{[0, \infty)} f(t) e^{t z} d \nu(t)$ converges absolutely for $z=0$, we have

$$
\left|\int_{[0, \infty)} f(t) d \nu(t)\right|=\left|\int_{[0, \infty)} f(t) e^{-\frac{1}{2}|t|^{2}} e^{\frac{1}{2}|t|^{2}} d v(t)\right| \leq \int_{[0, \infty)}|f(t)| e^{-\frac{1}{2}|t|^{2}} e^{\frac{1}{2}|t|^{2}} d \nu(t)<\infty
$$

By the closed graph theorem, the identity mapping is bounded from $F^{\infty}$ into $L^{1}\left(e^{\frac{1}{2}|\cdot|^{2}} d \nu\right)$, which implies the desired estimate.

Specializing to the space $f^{\infty}$, we use duality theorem for $f^{\infty}$ to obtain the following result. Recall that $\left(f^{\infty}\right)^{*}=F^{1}$ under the pairing

$$
\langle f, g\rangle=\frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{g(w)} e^{-|z|^{2}} d A(w)
$$

We refer the interested reader to [18, Theorem 2.26 and page 39] for details.
Theorem 3.4 Let $d v$ be a nonnegative Borel measure on $[0, \infty)$. If for any $f \in f^{\infty}$ the integral

$$
\mathcal{H}_{v}(f)(z)=\int_{[0, \infty)} f(t) e^{t z} d v(t), \quad z \in \mathbb{C}
$$

converges absolutely, then

$$
\int_{[0, \infty)} \int_{[0, t)} e^{s t} d \nu(s) d \nu(t)<\infty
$$

Proof By assumption, taking $z=0$, there is $C>0$ such that

$$
\left|\int_{[0, r)} f(t) d \nu(t)\right| \leq \int_{[0, r)}|f(t)| d \nu(t)<\int_{[0, \infty)}|f(t)| d \nu(t)<C
$$

for all $r \in(0, \infty)$. More specifically, choosing $f=1$, we have $\int_{[0, \infty)} d \nu(t)<\infty$, which means $d v$ is a finite Borel measure. On the other hand, using Hölder's inequality, we obtain that

$$
\int_{[0, r)} \int_{\mathbb{C}}\left|f(z) e^{t \bar{z}}\right| e^{-|z|^{2}} d A(z) d v(t) \leq\|f\|_{\infty} \int_{[0, r)} e^{\frac{1}{2}|t|^{2}} d v(t)<\infty
$$

Since the set of finite linear combinations of kernel functions is dense in $f^{\infty}$ [18, Lemma 2.11] and all kernel functions belong to $F^{2}$, the reproducing property and Fubini's theorem imply that for any $f \in f^{\infty}$,

$$
\int_{[0, r)} f(t) d v(t)=\int_{[0, r)} \int_{\mathbb{C}} f(z) e^{t \bar{z}} e^{-|z|^{2}} d A(z) d v(t)
$$

where $g_{r}(z)=\int_{[0, r)} e^{t z} d \nu(t)$. By the duality relation $\left(f^{\infty}\right)^{*}=F^{1}$ and the uniform boundedness principle, we get $g_{r} \in F^{1}$ and $\sup _{r}\left\|g_{r}\right\|_{1}<C$. Since $F^{1} \subset f^{\infty}$, we replace $f$ by $g_{r}$ in (3.4) and obtain that

$$
\begin{aligned}
\int_{\mathbb{C}}\left|g_{r}(z)\right|^{2} e^{-|z|^{2}} d A(z) & \geq\left|\int_{\mathbb{C}} g_{r}(z) \overline{g_{r}(z)} e^{-|z|^{2}} d A(z)\right| \\
& =\left|\int_{[0, r)} g_{r}(t) d \nu(t)\right|=\int_{[0, r)} \int_{[0, r)} e^{s t} d \nu(s) d \nu(t) .
\end{aligned}
$$

However, Lemma 2.5 implies that

$$
\int_{\mathbb{C}}\left|g_{r}(z)\right|^{2} e^{-|z|^{2}} d A(z) \lesssim\left\|g_{r}\right\|_{1} \int_{\mathbb{C}}\left|g_{r}(z) e^{-\frac{1}{2}|z|^{2}}\right| d A(z) \lesssim\left\|g_{r}\right\|_{1}^{2}<C .
$$

Combining this with the previous inequality and letting $r \rightarrow \infty$, we have

$$
\int_{[0, \infty)} \int_{[0, \infty)} e^{s t} d v(s) d v(t)<C
$$

This proves the desired result.

## 4 The operator $\mathcal{H}_{v}$ acting between Fock spaces

In this section, for $0<p \leq \infty$, we are going to characterize those measures $v$ for which $\mathcal{H}_{\nu}$ is a bounded (resp., compact) operator from $F^{p}$ to $F^{\infty}$ or from $F^{\infty}$ to $F^{p}$ for some $0<p \leq \infty$. Now, we state the main results as follows.

Theorem 4.1 Suppose $0<p \leq \infty$. Let v be a nonnegative Borel measure on $[0, \infty)$ such that $e^{\epsilon \cdot| |^{2}} v \in \Lambda$ with any fixed $\epsilon>\frac{1}{2}$. Then $\mathcal{H}_{v}$ is bounded from $F^{p}$ into $F^{\infty}$ if and only if $\left.e^{|\cdot|}\right|^{2} v \in \Lambda$.

Proof By Theorem 3.1, $\mathcal{H}_{v}$ has an integral representation (3.1). Suppose that $\mathcal{H}_{v}$ is a bounded operator from $F^{p}$ into $F^{\infty}$. Fixed $r>0$, Lemmas 2.4 and 2.2 show that there is $C>0$ such that

$$
\begin{align*}
C & >\left\|\mathcal{H}_{\nu}\left(k_{a}\right)\right\|_{\infty} \gtrsim\left|\mathcal{H}_{\nu}\left(k_{a}\right)(a) e^{-\left.\frac{1}{2}|a|\right|^{2}}\right|=\int_{[0, \infty)} e^{t \bar{a}+t a-|a|^{2}} d \nu(t) \\
& =\int_{[0, \infty)} e^{-|t-a|^{2}} e^{|t|^{2}} d \nu(t) \geq \int_{(a-r, a+r)} e^{-|t-a|^{2}} e^{|t|^{2}} d \nu(t) \gtrsim \int_{(a-r, a+r)} e^{|t|^{2}} d \nu(t) \tag{4.1}
\end{align*}
$$

for any $a \in[0, \infty)$. This proves that $e^{\left.1 \cdot\right|^{2}} v \in \Lambda$ by Lemma 2.6.
Conversely, suppose $e^{l^{1 \cdot}} v \in \Lambda$. Given $f \in F^{p}$, by Lemmas 2.5, 2.2, and 2.6 , we have

$$
\begin{aligned}
\left|\mathcal{H}_{\nu}(f)(z) e^{-\frac{1}{2}|z|^{2}}\right| & \leq \int_{[0, \infty)}\left|f(t) e^{t z} e^{-\frac{1}{2}|z|^{2}}\right| d \nu(t) \\
& \leq\|f\|_{p} \int_{[0, \infty)}\left|e^{t z \mid}\right| e^{-\frac{1}{2}|z|^{2}} e^{-\frac{1}{2}|t|^{2}} e^{|t|^{2}} d \nu(t)
\end{aligned}
$$

$$
\begin{aligned}
& \leq\|i\|_{F^{1} \rightarrow L^{1}\left(e^{\left.\frac{1}{2} \cdot\right|^{2}} d \nu\right)}\|f\|_{p} \int_{\mathbb{C}}\left|e^{w z}\right| e^{-\frac{1}{2}|z|^{2}} e^{-\frac{1}{2}|w|^{2}} d A(w) \\
& \lesssim\|i\|_{F^{1} \rightarrow L^{1}\left(\left.e^{\frac{1}{2}} \cdot\right|^{2}\right.}^{d v)}
\end{aligned}\|f\|_{p}
$$

for any $z \in \mathbb{C}$. Therefore, $\mathcal{H}_{v}$ is bounded from $F^{p}$ into $F^{\infty}$. This completes the proof of the theorem.

Lemma 4.1 Suppose that $0<p, q \leq \infty$ and $\mathcal{H}_{v}$ is bounded from $F^{p}$ into $F^{q}$. Then $\mathcal{H}_{v}$ is a compact operator if and only if for any bounded sequence $\left\{f_{n}\right\}$ in $F^{p}$, which converges uniformly to 0 on every compact subset of $\mathbb{C}$, we have $\mathcal{H}_{v}\left(f_{n}\right) \rightarrow 0$ as $n \rightarrow 0$ in $F^{q}$.

The proof of Lemma 4.1 is similar to that of Proposition 3.11 in [3]. We omit the details.

Theorem 4.2 Suppose $0<p \leq \infty$. Let $v$ be a nonnegative Borel measure on $[0, \infty)$ such that $e^{\epsilon \cdot|\cdot|} \nu \in \Lambda$ with any fixed $\epsilon>\frac{1}{2}$. Then $\mathcal{H}_{v}$ is a compact operator from $F^{p}$ into $F^{\infty}$ if and only if $e^{\left.l \cdot\right|^{2}} \nu \in \Lambda_{0}$.

Proof Assume that $\mathcal{H}_{v}$ is a compact operator from $F^{p}$ into $F^{\infty}$. It is easy to see that the function $k_{a}(z) \rightarrow 0$ uniformly on compact sets as $a \rightarrow \infty$. Using Lemma 4.1, we obtain that $\left\{\mathcal{H}_{v}\left(k_{a}\right)\right\}$ converges to 0 in $F^{\infty}$ when $a \rightarrow \infty$. Hence, by (4.1) we deduce that

$$
\int_{|t-a|<r} e^{|t|^{2}} d v(t) \rightarrow 0, \quad a \rightarrow \infty
$$

This proves $e^{|\cdot|^{2}} v \in \Lambda_{0}$.
Conversely, assume that $\left\{f_{n}\right\}$ is a bounded sequence in $F^{p}$, and $f_{n}$ uniformly converges to 0 on compact subsets of $\mathbb{C}$ as $n \rightarrow \infty$. It follows from Lemma 2.5 that

$$
\begin{aligned}
\left|\mathcal{H}_{v}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right| & \leq \int_{[0, \infty)}\left|f_{n}(t) e^{t z} e^{-\frac{1}{2}|z|^{2}}\right| d v(t) \\
& \leq\left\|f_{n}\right\|_{\infty} \int_{[0, \infty)} e^{-\frac{1}{2}|t-z|^{2}} e^{|t|^{2}} d \nu(t) \\
& \leq\left\|f_{n}\right\|_{p} \widetilde{\left(e^{|\cdot|} v\right)_{1}}(z)
\end{aligned}
$$

Since $e^{|\cdot|^{2}} v \in \Lambda_{0}$ by Lemma 2.7, $\widetilde{\left(e^{|\cdot|^{2}} v\right)_{1}}(z) \rightarrow 0$ as $z \rightarrow \infty$. So, for any $\varepsilon>0$, there is some $R>0$ such that when $|z| \geq R$

$$
\left|\mathcal{H}_{\nu}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right| \leq\left\|f_{n}\right\|_{p} \widetilde{\left(e^{|\cdot|^{2}} v\right)_{1}}(z) \leq \varepsilon\left\|f_{n}\right\|_{p}
$$

When $|z|<R$, for the above $\varepsilon$, there is some $R_{1}>0$ such that

$$
\begin{aligned}
\int_{|w| \geq R_{1}}\left|e^{w z}\right| e^{-\frac{1}{2}|z|^{2}} e^{-\frac{1}{2}|w|^{2}} d A(w) & =\int_{|w| \geq R_{1}} e^{-\frac{1}{2}|w-z|^{2}} d A(w) \\
& \leq e^{-\frac{1}{4}\left(R_{1}-R\right)^{2}} \int_{\mathbb{C}} e^{-\frac{1}{4}|w-z|^{2}} d A(w) \\
& =e^{-\frac{1}{4}\left(R_{1}-R\right)^{2}} \int_{\mathbb{C}} e^{-\frac{1}{4}|w|^{2}} d A(w) \leq \varepsilon .
\end{aligned}
$$

Hence, by Lemma 2.5,

$$
\begin{aligned}
& \left|\mathcal{H}_{v}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right| \\
& \quad \leq e^{-\frac{1}{2}|z|^{2}} \int_{[0, \infty)}\left|f_{n}(t) e^{t z}\right| e^{-|t|^{2}} e^{|t|^{2}} d v(t) \\
& \quad \lesssim e^{-\frac{1}{2}|z|^{2}} \int_{\mathbb{C}}\left|f_{n}(w) e^{w z}\right| e^{-|w|^{2}} d A(w) \\
& \quad \leq e^{-\frac{1}{2}|z|^{2}} \int_{|w|<R_{1}}\left|f_{n}(w) e^{w z}\right| e^{-|w|^{2}} d A(w)+e^{-\frac{1}{2}|z|^{2}} \int_{|w| \geq R_{1}}\left|f_{n}(w) e^{w z}\right| e^{-|w|^{2}} d A(w) \\
& \left.\quad \leq \int_{|w|<R_{1}}\left|f_{n}(w)\right| e^{-\frac{1}{2}|w-z|^{2}} e^{-|w|^{2}} d A(w)+e^{-\frac{1}{2}|z|^{2}} \int_{|w| \geq R_{1}}\left|f_{n} \|_{p} e^{\frac{1}{2}|w|^{2}}\right| e^{w z} \right\rvert\, e^{-|w|^{2}} d A(w) \\
& \quad \leq \int_{|w|<R_{1}}\left|f_{n}(w)\right| e^{-\frac{1}{2}|w|^{2}} d A(w)+\left\|f_{n}\right\|_{p} \varepsilon \leq\left(1+\left\|f_{n}\right\|_{p}\right) \varepsilon
\end{aligned}
$$

where the last inequality comes from the assumption that $\left\{f_{n}\right\}$ converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n \rightarrow \infty$. Therefore, by the arbitrariness of $\varepsilon$ and Lemma 4.1, we see that $\mathcal{H}_{v}: F^{p} \rightarrow F^{\infty}$ is compact.

Define the Rademacher functions $\psi_{n}(t)$ on $[0,1]$ by

$$
\begin{aligned}
& \psi_{0}(t)= \begin{cases}1, & 0 \leq t-[t]<\frac{1}{2} \\
-1, & \frac{1}{2} \leq t-[t]<1\end{cases} \\
& \psi_{n}(t)=\psi_{0}\left(2^{n} t\right), \quad n>0
\end{aligned}
$$

Then Khinchine's inequality is the following, which can be found in [12].

Khinchine's inequality For $0<p<\infty$ there exist constants $0<a_{p} \leq B_{p}<\infty$ such that, for all natural numbers $m$ and all complex numbers $c_{1}, c_{2}, \ldots, c_{m}$, we have

$$
a_{p}\left(\sum_{n=1}^{m}\left|c_{n}\right|^{2}\right)^{p / 2} \leq \int_{0}^{1}\left|\sum_{n=1}^{m} c_{n} \psi_{n}(t)\right|^{p} d t \leq B_{p}\left(\sum_{n=1}^{m}\left|c_{n}\right|^{2}\right)^{p / 2} .
$$

The next lemma is a partial result about atomic decomposition for Fock spaces, which can be found as Theorem 2.34 in [18].

Lemma 4.2 Let $0<p \leq \infty$. For $\lambda=\left\{\lambda_{j}\right\}_{j=1}^{\infty} \in l^{p}$, set

$$
S(\lambda)(z)=\sum_{j=1}^{\infty} \lambda_{j} k_{a_{j}}(z), \quad z \in \mathbb{C}
$$

then $S$ is a bounded operator from $l^{p}$ to $F^{p}$.

Theorem 4.3 Suppose $0<p<\infty$. Let v be a nonnegative Borel measure on $[0, \infty)$ such that $e^{\left.\epsilon \cdot \cdot\right|^{2}} \nu \in \Lambda$ with any fixed $\epsilon>\frac{1}{2}$. Then the following statements are equivalent:
(i) $\mathcal{H}_{v}$ is a bounded operator from $F^{\infty}$ into $F^{p}$;
(ii) $\mathcal{H}_{v}$ is a compact operator from $F^{\infty}$ into $F^{p}$;
(iii) $\left(\widetilde{e^{\cdot 1^{2}}}\right)_{t}(z) \in L^{p}(d A)$ for some (or any) $t>0$;
(iv) $\left(\widehat{e^{|\cdot|^{2}} v}\right)_{r}(z) \in L^{p}(d A)$ for some (or any) $r>0$;
(v) The sequence $\left\{\left(\widehat{e^{|\cdot|^{2}} v}\right)_{r}\left(a_{k}\right)\right\}_{k=1}^{\infty} \in l^{p}$ for some (or any) $r$-lattice $\left\{a_{k}\right\}_{k=1}^{\infty}$.

Proof By [10, Lemma 2.3], we get the equivalence of (iii), (iv), and (v). The implication (ii) $\Rightarrow(\mathrm{i})$ is trivial.
(i) $\Rightarrow$ (v). Given any bounded sequence $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ and $r$-lattice $\left\{a_{j}\right\}_{j=1}^{\infty}$, Lemma 4.2 shows that $f(z)=\sum_{j=1}^{\infty} \lambda_{j} k_{a_{j}}(z) \in F^{\infty}$ with $\|f\|_{\infty} \lesssim\left\|\left\{\lambda_{j}\right\}_{j}\right\|_{l \infty}$. Note that $\mathcal{H}_{v}: F^{\infty} \rightarrow F^{p}$ is bounded. By Khinchine's inequality and Fubini's theorem, we have

$$
\begin{aligned}
& \int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}\left|\lambda_{j} \mathcal{H}_{v}\left(k_{a_{j}}\right)(z)\right|^{2}\right)^{\frac{p}{2}} e^{-\frac{p}{2}|z|^{2}} d A(z) \\
& \quad \lesssim \int_{\mathbb{C}} \int_{0}^{1}\left|\sum_{j=1}^{\infty} \psi_{j}(t) \lambda_{j} \mathcal{H}_{\nu}\left(k_{a_{j}}\right)(z)\right|^{p} d t e^{-\frac{p}{2}|z|^{2}} d A(z) \\
& \quad=\int_{0}^{1} \int_{\mathbb{C}}\left|\mathcal{H}_{v}\left(\sum_{j=1}^{\infty} \psi_{j}(t) \lambda_{j} k_{a_{j}}\right)(z)\right|^{p-\frac{p}{2}|z|^{2}} d A(z) d t \\
& \quad \lesssim \int_{0}^{1}\left\|\mathcal{H}_{v}\right\|_{F^{\infty} \rightarrow F p}^{p}\left\|\sum_{j=1}^{\infty} \psi_{j}(t) \lambda_{j} k_{a_{j}}\right\|_{\infty}^{p} d t \\
& \quad \lesssim\left\|\mathcal{H}_{v}\right\|_{F^{\infty} \rightarrow F^{p}}^{p}\left\|\left\{\lambda_{j}\right\}\right\|_{j}^{p},
\end{aligned}
$$

where $\psi_{j}(t)$ is the $j$ th Rademacher function on $[0,1]$. In addition, it follows from Lemma 2.4 and (4.1) that

$$
\begin{aligned}
& \int_{\mathbb{C}}\left(\sum_{j=1}^{\infty}\left|\lambda_{j} \mathcal{H}_{v}\left(k_{a_{j}}\right)(z)\right|^{2}\right)^{\frac{p}{2}} e^{-\frac{p}{2}|z|^{2}} d A(z) \\
& \quad \gtrsim \sum_{k=1}^{\infty} \int_{D\left(a_{k}, r\right)}\left(\sum_{j=1}^{\infty} \mid \lambda_{j} \mathcal{H}_{v}\left(k_{a_{j}}\right)(z)^{2}\right)^{\frac{p}{2}} e^{-\frac{p}{2}|z|^{2}} d A(z) \\
& \\
& \gtrsim \sum_{k=1}^{\infty} \int_{D\left(a_{k}, r\right)}\left|\lambda_{k} \mathcal{H}_{v}\left(k_{a_{k}}\right)(z)\right|^{p} e^{-\frac{p}{2}|z|^{2}} d A(z) \\
& \\
& \left.\gtrsim \sum_{k=1}^{\infty}\left|\lambda_{k}\right|^{p}\left|\mathcal{H}_{v}\left(k_{a_{k}}\right)\left(a_{k}\right)\right|^{p} e^{-\frac{p}{2}\left|a_{k}\right|^{2}} \gtrsim \sum_{k=1}^{\infty}\left|\lambda_{k}\right|^{p} \widehat{\left(e^{|\cdot|} v\right.}\right)_{r}\left(a_{k}\right)^{p} .
\end{aligned}
$$

Setting $\beta_{k}=\left|\lambda_{k}\right|^{p}$, then $\left\{\beta_{k}\right\}_{k=1}^{\infty} \in l^{\infty}$. Consequently,

$$
\sum_{k=1}^{\infty} \beta_{k} \widehat{\left(e^{\left.l \cdot\right|^{2}}\right)_{r}}\left(a_{k}\right)^{p} \lesssim\left\|\mathcal{H}_{v}\right\|_{F}^{p} \rightarrow F \boldsymbol{p}\left\|\left\{\beta_{j}\right\} j\right\|_{L^{\infty}} .
$$

The duality argument shows that $\left\{\widehat{\left(e^{\left.l \cdot\right|^{2}} v\right)_{r}}\left(a_{k}\right)^{p}\right\}_{k=1}^{\infty} \in l^{1}$, which means $\left\{\widehat{\left(e^{|\cdot|} v\right)_{r}}\left(a_{k}\right)\right\}_{k=1}^{\infty} \in l^{p}$.
(iii) $\Rightarrow$ (ii). Assume that $\left\{f_{n}\right\}$ is a bounded sequence in $F^{\infty}$, and $f_{n}$ uniformly converges to 0 on compact subsets of $\mathbb{C}$ as $n \rightarrow \infty$. It follows from Lemma 2.5 that

$$
\begin{aligned}
\int_{\mathbb{C}}\left|\mathcal{H}_{\nu}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} d A(z) & \leq \int_{\mathbb{C}}\left\|f_{n}\right\|_{\infty}^{p}\left|\int_{[0, \infty)} e^{-\frac{1}{2}|t-z|^{2}} e^{|t|^{2}} d \nu(t)\right|^{p} d A(z) \\
& \left.\leq\left\|f_{n}\right\|_{\infty}^{p} \int_{\mathbb{C}} \mid \widetilde{\left(e^{1 \cdot \mid} \nu\right.}\right)\left._{1}(z)\right|^{p} d A(z) .
\end{aligned}
$$

Since $\left(\widetilde{e^{\cdot 1^{2}}} v\right)_{1}(z) \in L^{p}(d A)$, for any $\varepsilon>0$, there is some $R>0$ such that

$$
\left.\left.\int_{|z| \geq R}\left|\mathcal{H}_{v}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} d A(z) \leq\left\|f_{n}\right\|_{\infty}^{p} \int_{|z| \geq R} \right\rvert\, \widetilde{\left(e^{|\cdot|} v\right.}\right)\left._{1}(z)\right|^{p} d A(z) \leq \varepsilon\left\|f_{n}\right\|_{\infty}^{p}
$$

Note that $\left(\widetilde{e^{\left.!\cdot\right|^{2}}} v\right)_{1}(z) \in L^{p}(d A)$ means $\left(\widehat{e^{\left.!\cdot\right|^{2}}} v\right)_{r}(z) \in L^{p}(d A)$, hence $\left(\widehat{e^{\left.!\cdot\right|^{2}}} v\right)_{r}(z) \in L^{\infty}(\mathbb{C})$. Lemma 2.6 yields that

$$
\int_{[0, \infty)}\left|e^{\frac{1}{2} t \bar{z}}\right| e^{-\frac{1}{4}|t|^{2}} e^{|t|^{2}} d \nu(t) \leq \int_{\mathbb{C}}\left|e^{\frac{1}{2} w \bar{z}}\right| e^{-\frac{1}{4}|w|^{2}} d A(w)
$$

Hence, for the above $\varepsilon$ and $R$, there is some $R_{1}>0$ such that

$$
\begin{aligned}
& \int_{|z|<R \mid}\left|e^{-\frac{1}{2}|z|^{2}} \int_{\left[R_{1}, \infty\right)}\right| e^{t z}\left|e^{-\frac{1}{2}|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
& \quad=\int_{|z|<R}\left|\int_{\left[R_{1}, \infty\right)} e^{-\frac{1}{2}|t-z|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
& \quad=\int_{|z|<R} e^{-\frac{1}{4}\left|R_{1}-R\right|^{2}}\left|\int_{\left[R_{1}, \infty\right)} e^{-\frac{1}{4}|t-z|^{2}} e^{|t|^{2}} d \nu(t)\right|^{p} d A(z) \\
& \quad \leq \int_{|z|<R} e^{-\frac{1}{4}\left|R_{1}-R\right|^{2}}\left|\int_{[0, \infty)} e^{-\frac{1}{4}|t-z|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
& \quad \leq \int_{|z|<R} e^{-\frac{1}{4}\left|R_{1}-R\right|^{2}}\left|\int_{\mathbb{C}} e^{-\frac{1}{4}|w-z|^{2}} d A(w)\right|^{p} d A(z) \\
& \quad \leq C \int_{|z|<R} e^{-\frac{1}{4}\left|R_{1}-R\right|^{2}} d A(z) \leq \varepsilon .
\end{aligned}
$$

Together with the fact that $f_{n}$ uniformly converges to 0 on compact subsets of $\mathbb{C}$ as $n \rightarrow$ $\infty$, we obtain, by Lemma 2.5 ,

$$
\begin{aligned}
& \int_{|z|<R}\left|\mathcal{H}_{v}\left(f_{n}\right)(z) e^{-\frac{1}{2}|z|^{2}}\right|^{p} d A(z) \\
& \leq \int_{|z|<R}\left|e^{-\frac{1}{2}|z|^{2}} \int_{[0, \infty)}\right| f_{n}(t) e^{t z}\left|e^{-|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
& \leq \int_{|z|<R}\left|e^{-\frac{1}{2}|z|^{2}} \int_{\left[0, R_{1}\right)}\right| f_{n}(t) e^{t z}\left|e^{-|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
&+\int_{|z|<R}\left|e^{-\frac{1}{2}|z|^{2}} \int_{\left[R_{1}, \infty\right)}\right| f_{n}(t) e^{t z}\left|e^{-|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
& \leq \int_{|z|<R}\left|e^{-\frac{1}{2}|z|^{2}} \int_{\left[0, R_{1}\right)}\right| f_{n}(t) e^{t z}\left|e^{-|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z)
\end{aligned}
$$

$$
\begin{aligned}
& +\left\|f_{n}\right\|_{\infty}^{p} \int_{|z|<R}\left|e^{-\frac{1}{2}|z|^{2}} \int_{\left[R_{1}, \infty\right)}\right| e^{t z}\left|e^{-\frac{1}{2}|t|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z) \\
\leq & \int_{|z|<R}\left|\int_{\left[0, R_{1}\right)}\right| f_{n}(t)\left|e^{-\frac{1}{2}|t-z|^{2}} e^{|t|^{2}} d v(t)\right|^{p} d A(z)+\left\|f_{n}\right\|_{\infty}^{p} \varepsilon \\
\leq & \left(1+\left\|f_{n}\right\|_{\infty}^{p}\right) \varepsilon,
\end{aligned}
$$

while $n$ is large enough. Therefore, by the arbitrariness of $\varepsilon$ and Lemma 4.1, we see that $\mathcal{H}_{v}: F^{\infty} \rightarrow F^{p}$ is compact.

## The following result is a direct consequence of Theorem 4.3 and Lemma 2.8.

Corollary 4.1 Suppose $1<p<\infty$. Let v be a nonnegative Borel measure on $[0, \infty)$ that satisfies the condition in Theorem 3.1. Then the following statements are equivalent:
(i) $\mathcal{H}_{v}$ is a bounded operator from $F^{\infty}$ into $F^{p}$;
(ii) $\mathcal{H}_{v}$ is a compact operator from $F^{\infty}$ into $F^{p}$;
(iii) $e^{|\cdot|^{2}} v \in \Lambda^{\frac{p}{p-1}}$.

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## Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

Z. ZHUO, D. LI and T. ZENG developed the theoretical part and wrote this paper by themselves. T. ZENG helped perform the analysis with constructive discuss and read the manuscript. All authors reviewed the manuscript.

## Author details

${ }^{1}$ 'School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, 510665, P.R. China.
${ }^{2}$ School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, 510521, China. ${ }^{3}$ School of Mathematics, Jiaying University, Meizhou, 514015, P.R. China.

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