Rashid et al. *Journal of Inequalities and Applications* (2024) 2024:13
https://doi.org/10.1186/s13660-024-03079-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some multidimensional fixed point theorems for nonlinear contractions in C-distance spaces with applications

Maliha Rashid[1], Naeem Saleem[2,3*], Rabia Bibi[4] and Reny George[5*]

*Correspondence:
naeem.saleem2@gmail.com;
renygeorge02@yahoo.com
[2]Department of Mathematics, University of Management and Technology, Lahore, Pakistan
[5]Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this manuscript, we use the concept of multidimensional fixed point in a generalized space, namely, C-distance space with some nonlinear contraction conditions, such as Jaggi- and Dass-Gupta-type contractions. We provide results with a Jaggi-type hybrid contraction for the mentioned space. Moreover, we use control functions to get the desired results. After each theorem, we compare our results with previous ones to show that they are generalized. We provide examples to support our results. An application is also performed to solve the system of integral equations.

Mathematics Subject Classification: Primary 05C38; 15A15; secondary 05A15; 15A18

Keywords: Multiple fixed point; C-distance space; Nonlinear contractions

1 Introduction

The revolution in Fixed Point Theory begins with the result given by Banach [2] in which a self-mapping T on a non-empty set, satisfying a contractive condition, was considered. It is used to solve various problems in numerous branches of mathematical analysis. In most fixed point theorems, the contractive condition involves the linear combination of distances, such as $\sigma(\xi,\eta)$, $\sigma(\xi,T\eta)$, $\sigma(T\xi,\eta)$, $\sigma(\xi,T\xi)$, $\sigma(\eta,T\eta)$ and $\sigma(T\xi,T\eta)$. One of the fundamental nonlinear contractive conditions was proposed by Jaggi [9] and Dass-Gupta [8], who used rational-type conditions in their fixed point results in which product, division, and power of distances were considered. Some nonlinear contractions were defined using control functions. In [1], fixed point results for rational contractions were obtained in a partially ordered metric space for self-mapping.

In 2010, Samet and Vetro [15] examined the concept of fixed point of $\hbar$-order as an extension of the coupled fixed point. One year later, Berinde and Borcut [3] proved triple fixed point results for mixed monotone mappings. In 2012, Karapinar and Berinde [10] studied quadruple fixed points of nonlinear contractive conditions in partially ordered metric spaces. In [4], the nonlinear contraction was used for defining another contraction called bilateral contraction, and fixed point results were established for these contractions.

In 2016, Choban [5] generalized metric spaces to distance spaces, Berinde and Choban [6, 7] further studied distance spaces satisfying certain contraction conditions for the multidimensional fixed point. In [13, 14], a generalized C-distance space was discussed with examples, and multiple fixed point results were provided with an application to integral equations. Later in [12], another generalized space was defined, called an $E(s)$-distance space, and coincidence point results were provided in the mentioned space with rational-type contractive conditions.

In this paper, we aim to use nonlinear contractions, namely, Jaggi-type, Dass Gupta-type, and Dass-Gupta-type hybrid contractions, in C-distance space and find the multiple fixed point of nonself-mapping $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$.

2 Preliminaries

Definition 1 Consider $\mathcal{X}$, a non-empty set, and a function $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, then σ is called a distance on $\mathcal{X}$ if for all $\xi, \eta, \zeta \in \mathcal{X}$,

(1) $\sigma(\xi, \eta) \geq 0$,

(2) If $\sigma(\xi, \eta) + \sigma(\eta, \xi) = 0$ then $\xi = \eta$.

(3) If $\xi = \eta$ then $\sigma(\xi, \eta) = 0$.

Definition 2 Consider a sequence $\{\xi_\kappa : \kappa \in \mathbb{N}\}$ in a distance space $(\mathcal{X}, \sigma)$ and $\xi \in \mathcal{X}$. Then, $\{\xi_\kappa : \kappa \in \mathbb{N}\}$ is:

- *convergent to* ξ if and only if $\lim_{\kappa \to \infty} \sigma(\xi, \xi_\kappa) = 0$;
- *Cauchy* if $\lim_{\hbar, \kappa \to \infty} \sigma(\xi_\kappa, \xi_\hbar) = 0$.

A distance space $(\mathcal{X}, \sigma)$ is called *complete* if every Cauchy sequence in $\mathcal{X}$ converges to some ξ in $\mathcal{X}$.

Definition 3 Consider a distance space $(\mathcal{X}, \sigma)$. If every Cauchy sequence that converges has a unique limit point, then σ is called C-distance $\mathcal{X}$.

Fix $\hbar \in \mathbb{N}$ and $\Gamma = (\Gamma_1, \ldots, \Gamma_\hbar)$ to be mappings such that each $\{\Gamma_{\acute{\imath}} : \{1, 2, \ldots, \hbar\} \to \{1, 2, \ldots, \hbar\} : 1 \leq \acute{\imath} \leq \hbar\}$.

Let $(\mathcal{X}, \sigma)$ be a distance space and $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$ be a mapping. The composition of Ω and Γ is another mapping $\Gamma\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}^{\hbar}$, defined by

$$\Gamma\Omega(\xi_1, \ldots, \xi_\hbar) = (\eta_1, \ldots, \eta_\hbar)$$

and

$$\eta_{\acute{\imath}} = \Omega(\xi_{\Gamma_{\acute{\imath}}(1)}, \ldots, \xi_{\Gamma_{\acute{\imath}}(\hbar)}),$$

for any $(\xi_1, \ldots, \xi_\hbar) \in \mathcal{X}^{\hbar}$ and any $\acute{\imath} \in \{1, 2, \ldots, \hbar\}$. A point $\tau = (\tau_1, \ldots, \tau_\hbar) \in \mathcal{X}^{\hbar}$ is called a Γ-*multiple fixed point* of the Ω if $\tau = \Gamma\Omega(\tau)$, *i.e.*,

$$\tau_{\acute{\imath}} = \Omega(\tau_{\Gamma_{\acute{\imath}}(1)}, \ldots, \tau_{\Gamma_{\acute{\imath}}(\hbar)}) \quad \text{for any } \acute{\imath} \in \{1, 2, \ldots, \hbar\}.$$

For $\tau \in \mathcal{X}^{\hbar}$, $\tau(1) = \Gamma\Omega(\tau)$ and $\tau(\kappa + 1) = \Gamma\Omega(\tau(\kappa))$. A sequence $(\tau(\kappa))_{\kappa \in \mathbb{N}}$ is Picard at τ with respect to $\Gamma\Omega$.

Let σ be a distance on $\mathcal{X}$. Define a function $\sigma^{\hbar} : \mathcal{X}^{\hbar} \times \mathcal{X}^{\hbar} \to \mathbb{R}$ as

$$\sigma^{\hbar}(\xi, \eta) = \sup\{\sigma(\xi_{\acute{\imath}}, \eta_{\acute{\imath}} : \acute{\imath} \leq \hbar)\},$$

then $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$ is also a distance space.

Proposition 1 *Consider* $(\mathcal{X}, \sigma)$ *is a C-distance space. Then*:
1. $\sigma(\xi, \eta) = 0$ *if and only if* $\xi = \eta$.
2. *If, for* $\tau \in \mathcal{X}^{\hbar}$, *the Picard sequence* $\{\tau(\kappa) : \kappa \in \mathbb{N}\}$ *is convergent Cauchy sequence and* $\lim_{\kappa\to\infty} \tau_{\kappa} = b = (b_1, \dots, b_{\hbar})$, *then b is a multiple fixed point of* Ω, *i.e.,*

$$b_{\acute{\imath}} = \Omega(b_{\Gamma_{\acute{\imath}}(1)} \dots, b_{\Gamma_{\acute{\imath}}(\hbar)}) \quad \textit{for every } \acute{\imath} \in \{1, 2, \dots, \hbar\}.$$

3 Main results

Definition 4 A mapping $\Gamma\Omega$ on a C-distance space $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$ is called a Jaggi-type contraction if there is a $\varphi : [0, \infty) \to [1, \infty)$, which is continuous, increasing, and $\varphi(0) = 1$, such that

$$\sigma^{\hbar}\big(\Gamma\Omega(\xi), \Gamma\Omega(\eta)\big) \leq \beta \max\left\{\sigma^{\hbar}(\xi, \eta), \frac{\varphi(\sigma^{\hbar}(\xi, \Gamma\Omega(\xi)))\sigma^{\hbar}(\eta, \Gamma\Omega(\eta))}{\varphi(\sigma^{\hbar}(\xi, \eta))}\right\}, \tag{1}$$

for all $\xi, \eta \in \mathcal{X}^{\hbar}$, and $\beta \in (0, 1)$.

Theorem 1 *Let* $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$ *be a mapping. If a mapping* $\Gamma\Omega$ *on a complete C-distance space* $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$ *is a Jaggi-type contraction, then* Ω *possesses at least a multiple fixed point.*

Proof Suppose that $\sigma^{\hbar}(\tau(\kappa), \tau(\kappa + 1)) > 0$. If this inequality does not hold, then we get a fixed point, which terminates the proof. Now,

$$\begin{aligned}
\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa + 1)\big) &= \sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa - 1)\big), \Gamma\Omega\big(\tau(\kappa)\big)\big) \\
&\leq \beta \max\left\{\begin{array}{c} \sigma^{\hbar}\big(\tau(\kappa - 1), \tau(\kappa)\big), \\ \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1), \Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\kappa), \Gamma\Omega(\tau(\kappa)))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)))} \end{array}\right\} \\
&\leq \beta \max\big\{\sigma^{\hbar}\big(\tau(\kappa - 1), \tau(\kappa)\big), \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa + 1)\big)\big\},
\end{aligned}$$

if $\max\{\sigma^{\hbar}(\tau(\kappa - 1), \tau(\kappa)), \sigma^{\hbar}(\tau(\kappa), \tau(\kappa + 1))\} = \sigma^{\hbar}(\tau(\kappa), \tau(\kappa + 1))$, then

$$\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa + 1)\big) \leq \beta\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa + 1)\big),$$

which is not possible. Thus,

$$\begin{aligned}
\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa + 1)\big) &\leq \beta\sigma^{\hbar}\big(\tau(\kappa - 1), \tau(\kappa)\big) \\
&\ \ \vdots \\
&\leq \beta^{\kappa}\sigma^{\hbar}\big(\tau(1), \tau(2)\big).
\end{aligned}$$

We can also write it as

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big) = 0. \tag{2}$$

Similarly, we can show that

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa+1), \tau(\kappa)\big) = 0. \tag{3}$$

We need to show that the sequence $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy. For all $\hbar > \kappa$, consider

$$\begin{aligned}
\sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar)\big) &= \sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big), \Gamma\Omega\big(\tau(\hbar-1)\big)\big) \\
&\le \beta \max\left\{ \begin{array}{c} \sigma^{\hbar}\big(\tau(\kappa-1), \tau(\hbar-1)\big), \\ \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\hbar-1),\Gamma\Omega(\tau(\hbar-1)))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))} \end{array} \right\} \\
&\le \beta \max\left\{ \begin{array}{c} \sigma^{\hbar}\big(\tau(\kappa-1), \tau(\hbar-1)\big), \\ \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))} \end{array} \right\},
\end{aligned}$$

if $\max\left\{ \begin{array}{c} \sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)), \\ \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))} \end{array} \right\} = \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}$ then

$$\sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar)\big) = \beta \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1), \tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1)))},$$

taking limit $\hbar, \kappa \to \infty$ over the above expression, it follows

$$\begin{aligned}
\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar)\big) &\le \lim_{\hbar,\kappa\to\infty} \frac{\beta\varphi(\sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1), \tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1)))} \\
&\le \frac{\beta\varphi(\lim_{\kappa\to\infty} \sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)))\sigma^{\hbar}(\lim_{\hbar\to\infty} \tau(\hbar-1), \tau(\hbar))}{\varphi(\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1)))}, \\
&\quad \text{since } \varphi \text{ is continuous} \\
&\le \frac{\beta\varphi(0) \times 0}{\varphi(\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1)))} \\
&= 0.
\end{aligned}$$

As, if $\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1)) = 0$, then $\varphi(\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}(\tau(\kappa-1), \tau(\hbar-1))) = 1$, then for this case, $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy. Now, the other possibility is

$$\max\left\{ \begin{array}{c} \sigma^{\hbar}\big(\tau(\kappa-1), \tau(\hbar-1)\big), \\ \frac{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))} \end{array} \right\} = \sigma^{\hbar}\big(\tau(\kappa-1), \tau(\hbar-1)\big).$$

In this case,

$$\begin{aligned}
\sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar)\big) &\le \beta\sigma^{\hbar}\big(\tau(\kappa-1), \tau(\hbar-1)\big) \\
&\le \beta^2\sigma^{\hbar}\big(\tau(\kappa-2), \tau(\hbar-2)\big) \\
&\;\;\vdots \\
&\le \beta^{\kappa-1}\sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar-\kappa+1)\big),
\end{aligned}$$

applying limit $\hbar, \kappa \to \infty$, we obtain

$$\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa), \tau(\hbar)\big) = 0.$$

Similarly, we have

$$\lim_{\hbar,\kappa\to\infty} \sigma^{\hbar}\big(\tau(\hbar), \tau(\kappa)\big) = 0.$$

Thus, $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy, so it will converge to some $\kappa \in \mathcal{X}^{\hbar}$, $\acute{\imath}.e.$,

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\kappa, \tau(\kappa)\big) = \lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa), \kappa\big) = 0.$$

Since κ is the limit of convergent Cauchy sequence in a C-distance space, so it is fixed point of $\Gamma\Omega$ and will, consequently, be a multiple fixed point of Ω. □

Remark 1
(1) The above theorem is the generalization of the Jaggi contraction defined in [9]. By defining φ as identity function and substituting $h = 1$, we can get the particular cases.
(2) The space being utilized here is the C-distance space that is much more generalized than the metric space used in [9].
(3) In [9], fixed point results were established. However, the above result is the multidimensional fixed point result.

Example 1 Let $\mathcal{X} = \{\frac{1}{\kappa} : \kappa \in \mathbb{N}\} \cup \{0\}$. Define for all $\kappa, l \in \mathbb{N}$

$$\sigma(0,0) = 0, \qquad \sigma\left(0, \frac{1}{\kappa}\right) = \frac{1}{2\kappa}, \qquad \sigma\left(\frac{1}{\kappa}, 0\right) = \frac{1}{\kappa}, \qquad \sigma(\xi_\kappa, \xi_l) = |\xi_\kappa - \xi_l|,$$

then $(\mathcal{X}, \sigma)$ is a C-distance space. Now,

$$\mathcal{X} \times \mathcal{X} = \big\{(\xi, \eta) : \xi, \eta \in \mathcal{X}\big\},$$

and

$$\sigma^2(\xi, \eta) = \sup_{\acute{\imath}\leq 2}\big\{\sigma(\xi_{\acute{\imath}}, \eta_{\acute{\imath}})\big\},$$

then $(\mathcal{X}^2, \sigma^2)$ is a C-distance space.

A function $\varphi : [0,\infty) \to [1,\infty)$, defined as

$$\varphi(\xi) = \xi + 1, \quad \text{for all } \xi \in [0,\infty),$$

which is continuous, increasing, and $\varphi(0) = 1$. Now, define $\Omega : \mathcal{X}^2 \to \mathcal{X}$ such that

$$\Omega(\xi_1, \xi_2) = \frac{\xi_1}{4} \quad \text{for all } (\xi_1, \xi_2) \in \mathcal{X}^2,$$

and a mapping $\Gamma : \mathcal{X} \to \mathcal{X}^2$ such that

$$\Gamma(\xi) = \big(\Gamma_1(\xi), \Gamma_2(\xi)\big),$$

where $\Gamma_i : \{1,2\} \to \{1,2\}$ are defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) \\ \Gamma_2(1) & \Gamma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The mapping $\Gamma\Omega : \mathcal{X}^2 \to \mathcal{X}^2$ is defined as:

$$\Gamma\Omega(\xi_1, \xi_2) = \big(\Omega(\xi_{\Gamma_1(1)}, \xi_{\Gamma_2(2)}), \Omega(\xi_{\Gamma_2(1)}, \xi_{\Gamma_2(2)})\big) = \left(\frac{\xi_1}{4}, \frac{\xi_2}{4}\right).$$

Consider

$$\sigma\big(\Omega(\xi_1, \xi_2), \Omega(\eta_1, \eta_2)\big) = \sigma\left(\frac{\xi_1}{4}, \frac{\eta_1}{4}\right).$$

Consider the right-hand side of inequality (1) with $\hbar = 2$

$$\begin{aligned} &\beta \max\left\{\sigma^2(\xi, \eta), \frac{\varphi(\sigma^2(\xi, \Gamma\Omega(\xi)))\sigma^2(\eta, \Gamma\Omega(\eta))}{\varphi(\sigma^2(\xi, \eta))}\right\} \\ &\quad = \beta \max\left\{\begin{array}{c} \sup\big\{\sigma(\xi_1, \eta_1), \sigma(\xi_2, \eta_2)\big\}, \\ \frac{(\sup\{\sigma(\xi_1, \frac{\xi_1}{4}), \sigma(\xi_2, \frac{\xi_1}{4})\}+1)(\sup\{\sigma(\eta_1, \frac{\eta_1}{4}), \sigma(\eta_2, \frac{\eta_1}{4})\})}{\sup\{\sigma(\xi_1, \eta_1), \sigma(\xi_2, \eta_2)\}+1} \end{array}\right\}, \end{aligned}$$

where $\beta \in [0,1)$. Now, we need to satisfy the inequality

$$\sigma\left(\frac{\xi_1}{4}, \frac{\eta_1}{4}\right) \le \beta \max\left\{\begin{array}{c} \sup\big\{\sigma(\xi_1, \eta_1), \sigma(\xi_2, \eta_2)\big\}, \\ \frac{(\sup\{\sigma(\xi_1, \frac{\xi_1}{4}), \sigma(\xi_2, \frac{\xi_1}{4})\}+1)(\sup\{\sigma(\eta_1, \frac{\eta_1}{4}), \sigma(\eta_2, \frac{\eta_1}{4})\})}{\sup\{\sigma(\xi_1, \eta_1), \sigma(\xi_2, \eta_2)\}+1} \end{array}\right\}. \tag{$*$}$$

If $\xi = (0,0)$ and $\eta = (\frac{1}{\kappa_1}, \frac{1}{\kappa_2})$, then ($*$) becomes

$$\begin{aligned} \sigma\left(0, \frac{1}{4\kappa_1}\right) &= \frac{1}{8\kappa_1} \\ &\le \beta \max\left\{\begin{array}{c} \sup\left\{\sigma\left(0, \frac{1}{\kappa_1}\right), \sigma\left(0, \frac{1}{\kappa_2}\right)\right\}, \\ \frac{(\sup\{\sigma(0,0), \sigma(0,0)\}+1)(\sup\{\sigma(\frac{1}{\kappa_1}, \frac{1}{4\kappa_1}), \sigma(\frac{1}{\kappa_2}, \frac{1}{4\kappa_1})\})}{\sup\{\sigma(0, \frac{1}{\kappa_1}), \sigma(0, \frac{1}{\kappa_2})\}+1} \end{array}\right\} \\ &\le \beta \max\left\{\sup\left\{\frac{1}{2\kappa_1}, \frac{1}{2\kappa_2}\right\}, \frac{\sup\{\frac{3}{4\kappa_1}, \frac{4\kappa_1 - \kappa_2}{4\kappa_1\kappa_2}\}}{\sup\{\frac{1}{2\kappa_1}, \frac{1}{2\kappa_2}\} + 1}\right\}, \end{aligned}$$

then the above inequality satisfies for $\beta = \frac{1}{4}$. All conditions of the above theorem are satisfied. Hence, Ω has at least a multiple fixed point.

Corollary 1 *Consider a mapping $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$. If $\Gamma\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}^{\hbar}$ on a complete C-distance space $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$ satisfying*

$$\sigma^{\hbar}\big(\Gamma\Omega(\xi), \Gamma\Omega(\eta)\big) \leq \beta\sigma^{\hbar}(\xi, \eta), \quad \textit{for all } \xi, \eta \in \mathcal{X}, \beta \in [0,1),$$

then Ω possesses at least a multidimensional fixed point.

Definition 5 Consider a mapping $\Gamma\Omega$ on a C-distance space $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$, called a Dass-Gupta-type contraction. If there is a $\varphi : [0,\infty) \to [0,\infty)$, which is continuous, increasing, and $\varphi(0) = 0$, such that

$$\sigma^{\hbar}\big(\Gamma\Omega(\xi), \Gamma\Omega(\eta)\big) \leq \beta \max\left\{\sigma^{\hbar}(\xi, \eta), \frac{\varphi(1 + \sigma^{\hbar}(\xi, \Gamma\Omega(\xi)))\sigma^{\hbar}(\eta, \Gamma\Omega(\eta))}{\varphi(1 + \sigma^{\hbar}(\xi, \eta))}\right\}, \tag{4}$$

for all $\xi, \eta \in \mathcal{X}^{\hbar}$, and $\beta \in (0,1)$.

Theorem 2 *Consider a mapping $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$. If $\Gamma\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}^{\hbar}$ on a complete C-distance space $(\mathcal{X}^{\hbar}, \sigma^{\hbar})$, which is a Dass-Gupta-type contraction, then Ω possesses at least a multidimensional fixed point.*

Proof Suppose that $\sigma^{\hbar}(\tau(\kappa), \tau(\kappa + 1)) > 0$. If this inequality does not hold, then we get a fixed point, which terminates the proof. Now,

$$\begin{aligned} \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big) &= \sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big), \Gamma\Omega\big(\tau(\kappa)\big)\big) \\ &\leq \beta \max\left\{\begin{array}{c} \sigma^{\hbar}\big(\tau(\kappa-1), \tau(\kappa)\big), \\ \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1), \Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\kappa), \Gamma\Omega(\tau(\kappa)))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)))} \end{array}\right\} \\ &\leq \beta \max\big\{\sigma^{\hbar}\big(\tau(\kappa-1), \tau(\kappa)\big), \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big)\big\}, \end{aligned}$$

if $\max\{\sigma^{\hbar}(\tau(\kappa-1), \tau(\kappa)), \sigma^{\hbar}(\tau(\kappa), \tau(\kappa+1))\} = \sigma^{\hbar}(\tau(\kappa), \tau(\kappa+1))$, then

$$\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big) \leq \beta\sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big),$$

which is impossible. Thus,

$$\begin{aligned} \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big) &\leq \beta\sigma^{\hbar}\big(\tau(\kappa-1), \tau(\kappa)\big) \\ &\vdots \\ &\leq \beta^{\kappa-1}\sigma^{\hbar}\big(\tau(1), \tau(2)\big). \end{aligned}$$

We can also write it as

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa), \tau(\kappa+1)\big) = 0. \tag{5}$$

Similarly, we can show that

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa+1), \tau(\kappa)\big) = 0. \tag{6}$$

We need to show that the sequence $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is a Cauchy sequence. For all $\hbar > \kappa$, consider

$$\begin{aligned}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big) &= \sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big),\Gamma\Omega\big(\tau(\hbar-1)\big)\big)\\ &\le \beta\max\left\{\begin{array}{c}\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big),\\ \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\hbar-1),\Gamma\Omega(\tau(\hbar-1)))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\end{array}\right\}\\ &\le \beta\max\left\{\begin{array}{c}\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big),\\ \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\end{array}\right\},\end{aligned}$$

if $\max\left\{\begin{array}{c}\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)),\\ \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\end{array}\right\} = \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}$ then

$$\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big) = \beta\frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))},$$

taking limit $\hbar,\kappa\to\infty$ over the above expression, we get the following

$$\begin{aligned}\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big) &\le \lim_{\hbar,\kappa\to\infty}\frac{\beta\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\\ &\le \frac{\beta\varphi(1+\lim_{\kappa\to\infty}\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\lim_{\hbar\to\infty}\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))},\\ &\quad\text{since } \varphi \text{ is continuous}\\ &\le \frac{\beta\varphi(1)\times 0}{\varphi(1+\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\\ &= 0.\end{aligned}$$

As, if $\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)) = 0$, then $\varphi(1+\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1))) \neq 0$. So, for this case, $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy. Now, the other possibility is

$$\max\left\{\begin{array}{c}\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big),\\ \frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))}\end{array}\right\} = \sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big).$$

In this case,

$$\begin{aligned}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big) &\le \beta\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big)\\ &\le \beta^{2}\sigma^{\hbar}\big(\tau(\kappa-2),\tau(\hbar-2)\big)\\ &\;\;\vdots\\ &\le \beta^{\kappa-1}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar-\kappa+1)\big),\end{aligned}$$

applying limit $\hbar,\kappa\to+\infty$, we obtain

$$\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big) = 0.$$

Similarly, we can show that

$$\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}\big(\tau(\hbar),\tau(\kappa)\big) = 0.$$

Thus, $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy. It is given that space is complete, so $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ converges to some $\kappa \in \mathcal{X}^{\hbar}$, *í.e.*,

$$\lim_{\kappa\to\infty} \sigma^{\hbar}\big(\kappa,\tau(\kappa)\big) = \lim_{\kappa\to\infty} \sigma^{\hbar}\big(\tau(\kappa),\kappa\big) = 0.$$

In the C-distance space, the limit is the fixed point of $\Gamma\Omega$ and, consequently, will be the multiple fixed point of Ω. Hence, κ is a fixed point of $\Gamma\Omega$ and a multiple fixed point of Ω. □

Example 2 Define a function $\varphi : [0,\infty) \to [0,\infty)$ as

$$\varphi(\xi) = \frac{\xi}{2} \quad \text{for all } \xi \in [0,\infty),$$

then φ is continuous, nondecreasing, and $\varphi(0) = 0$. Example 1, with defined function φ, satisfies all axioms of Theorem 2. Hence, Ω has at least a multidimensional fixed point.

Definition 6 A mapping $\Gamma\Omega$ on $(\mathcal{X}^{\hbar},\sigma^{\hbar})$ is called a Dass-Gupta-type hybrid contraction if there is $\psi,\varphi,\theta : [0,\infty) \to [0,\infty)$ such that

$$\psi\big(\sigma^{\hbar}\big(\Gamma\Omega(\xi),\Gamma\Omega(\eta)\big)\big) \le \theta\big(M(\xi,\eta)\big), \tag{7}$$

where

$$M(\xi,\eta) = \left\{ \begin{array}{ll} \begin{bmatrix} \alpha\big(\frac{\varphi(1+\sigma^{\hbar}(\xi,\Gamma\Omega(\xi)))\sigma^{\hbar}(\eta,\Gamma\Omega(\eta))}{\varphi(1+\sigma^{\hbar}(\xi,\eta))}\big)^{s} \\ +\beta(\sigma^{\hbar}(\xi,\eta))^{s} \end{bmatrix}^{\frac{1}{s}}, & fo\hbar\ s > 0, \xi,\eta \in \mathcal{X}, \xi \neq \eta, \\ (\sigma^{\hbar}(\xi,\Gamma\Omega(\xi)))^{\alpha}(\sigma^{\hbar}(\eta,\Gamma\Omega(\eta)))^{\beta} & fo\hbar\ s = 0, \xi,\eta \in \mathcal{X}\backslash F\acute{\imath}x(\mathcal{X}) \end{array} \right),$$

where $F\acute{\imath}x(\mathcal{X}) = \{\zeta \in \mathcal{X} : \zeta = \Gamma\Omega(\zeta)\}$, for all $\xi,\eta \in \mathcal{X}^{\hbar}$, $s \ge 0$ and $\alpha + \beta = 1$, where ψ, φ and θ are continuous, increasing, and $\psi(0) = \varphi(0) = \theta(0) = 0$,

$$\text{for } \xi > 0, \quad \psi(\xi) > \theta(\xi). \tag{a}$$

Theorem 3 *Let $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$ be a mapping. If $\Gamma\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}^{\hbar}$ on a complete C-distance space $(\mathcal{X}^{\hbar},\sigma^{\hbar})$ is a Dass-Gupta-type hybrid contraction, then Ω possesses at least a multidimensional fixed point.*

Proof For $\tau \in \mathcal{X}^{\hbar}$, $\tau(1) = \Gamma\Omega(\tau)$ and $\tau(\kappa + 1) = \Gamma\Omega(\tau(\kappa))$ is a Picard sequence. We assume that

$$\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big) > 0 \quad \text{for all } \kappa \in \mathbb{N}.$$

On the contrary, if the above inequality does not hold, then we get a fixed point, and it terminates the proof. To prove the claim of our result, we shall discuss two cases $s > 0$ and $s = 0$ separately.

Case 1: When $s > 0$, consider

$$\begin{aligned} \psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big) &= \psi\big(\sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big),\Gamma\Omega\big(\tau(\kappa)\big)\big)\big) \\ &\le \theta\big(M\big(\tau(\kappa-1),\tau(\kappa)\big)\big). \end{aligned}$$

Where

$$\begin{aligned} M\big(\tau(\kappa-1),\tau(\kappa)\big) &= \begin{bmatrix} \alpha\big(\frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\kappa),\Gamma\Omega(\tau(\kappa)))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))}\big)^{s} \\ +\beta(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))^{s} \end{bmatrix}^{\frac{1}{s}} \\ &= \begin{bmatrix} \alpha\big(\frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\kappa),\tau(\kappa+1))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))}\big)^{s} \\ +\beta(\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))^{s} \end{bmatrix}^{\frac{1}{s}} \\ &= \big[\alpha\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{s}+\beta\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{s}\big]^{\frac{1}{s}} \end{aligned}$$

using this value in the above inequality, we have

$$\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)\leq\theta\big(\big[\alpha\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{s}+\beta\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{s}\big]^{\frac{1}{s}}\big).$$

Suppose that $\sigma^{\hbar}(\tau(\kappa),\tau(\kappa+1))>\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa))$, then from the above inequality, we have

$$\begin{aligned} \psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big) &< \theta\big(\big[\alpha\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{s}+\beta\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{s}\big]^{\frac{1}{s}}\big) \\ &< \theta\big(\big[(\alpha+\beta)\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{s}\big]^{\frac{1}{s}}\big) \\ &< \theta\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big), \end{aligned} \tag{8}$$

using (a), we have

$$\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)<\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big),$$

which is a contradiction. So, $\sigma^{\hbar}(\tau(\kappa),\tau(\kappa+1))\leq\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa))$, which is a decreasing sequence. Thus, we have

$$\lim_{\kappa\to\infty}\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)=\hbar\geq 0.$$

If $\hbar>0$, applying limit $\kappa\to+\infty$ to (8), it follows $\psi(\hbar)<\theta(\hbar)$, which contradicts ($\tau$); thus, $\hbar=0$ and

$$\lim_{\kappa\to\infty}\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)=0.$$

Similarly,

$$\lim_{\kappa\to+\infty}\sigma^{\hbar}\big(\tau(\kappa+1),\tau(\kappa)\big)=0.$$

Our next step is to show that $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is a Cauchy sequence. For all $\hbar>\kappa$, we have

$$\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big)\big)=\sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big),\Gamma\Omega\big(\tau(\hbar-1)\big)\big)\leq\theta\big(M\big(\tau(\kappa-1),\tau(\hbar-1)\big)\big),$$

where

$$M\big(\tau(\kappa-1),\tau(\hbar-1)\big)=\begin{bmatrix}\alpha(\frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\Gamma\Omega(\tau(\kappa-1))))\sigma^{\hbar}(\tau(\hbar-1),\Gamma\Omega(\tau(\hbar-1)))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))})^{s}\\+\beta(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))^{s}\end{bmatrix}^{\frac{1}{s}}$$
$$=\begin{bmatrix}\alpha(\frac{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\kappa)))\sigma^{\hbar}(\tau(\hbar-1),\tau(\hbar))}{\varphi(1+\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))})^{s}\\+\beta(\sigma^{\hbar}(\tau(\kappa-1),\tau(\hbar-1)))^{s}\end{bmatrix}^{\frac{1}{s}},$$

taking limit $\hbar,\kappa\to\infty$, we get

$$\begin{aligned}\lim_{\hbar,\kappa\to\infty}\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big)\big)&\le\lim_{\hbar,\kappa\to\infty}\theta\big(\beta^{\frac{1}{s}}\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-1)\big)\big)\\&\le\lim_{\hbar,\kappa\to\infty}\theta\Big(\lim_{\hbar,\kappa\to\infty}\beta^{\frac{2}{s}}\sigma^{\hbar}\big(\tau(\kappa-2),\tau(\hbar-2)\big)\Big)\\&\ \ \vdots\\&\le\theta\Big(\lim_{\hbar,\kappa\to\infty}\beta^{\frac{\kappa-1}{s}}\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\hbar-\kappa+1)\big)\Big)\\&\le\theta(0)=0,\end{aligned}$$

consequently

$$\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}\big(\tau(\kappa),\tau(\hbar)\big)=0.$$

Similarly, it can be shown that

$$\lim_{\hbar,\kappa\to\infty}\sigma^{\hbar}\big(\tau(\hbar),\tau(\kappa)\big)=0.$$

Hence, $(\tau(\kappa))_{\kappa\in\mathbb{N}}$ is Cauchy. Similar reasoning from the proof of the above theorem can be used to show that Ω has a multiple fixed point.

Case 2: When $s=0$, consider

$$\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)=\psi\big(\sigma^{\hbar}\big(\Gamma\Omega\big(\tau(\kappa-1)\big),\Gamma\Omega\big(\tau(\kappa)\big)\big)\big)\le\theta\big(M\big(\tau(\kappa-1),\tau(\kappa)\big)\big).$$

Where

$$\begin{aligned}M\big(\tau(\kappa-1),\tau(\kappa)\big)&=\big(\sigma^{\hbar}\big(\tau(\kappa-1),\Gamma\Omega\big(\tau(\kappa-1)\big)\big)\big)^{\alpha}\big(\sigma^{\hbar}\big(\tau(\kappa),\Gamma\Omega\big(\tau(\kappa)\big)\big)\big)^{\beta}\\&=\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{\alpha}\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{\beta},\end{aligned}$$

so

$$\begin{aligned}\psi\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)&\le\theta\big(\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{\alpha}\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{\beta}\big)\\&<\psi\big(\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{\alpha}\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{\beta}\big),\end{aligned}$$

using the properties of ψ, we have

$$\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)<\big(\sigma^{\hbar}\big(\tau(\kappa-1),\tau(\kappa)\big)\big)^{\alpha}\big(\sigma^{\hbar}\big(\tau(\kappa),\tau(\kappa+1)\big)\big)^{\beta},$$

using simple calculation, the above inequality turns into

$$\left(\sigma^{\hbar}\left(\tau(\kappa),\tau(\kappa+1)\right)\right)^{1-\beta}<\left(\sigma^{\hbar}\left(\tau(\kappa-1),\tau(\kappa)\right)\right)^{\alpha},$$

since $\alpha+\beta=1$, we have

$$\sigma^{\hbar}\left(\tau(\kappa),\tau(\kappa+1)\right)<\sigma^{\hbar}\left(\tau(\kappa-1),\tau(\kappa)\right).$$

Using the same method as in Case 1, we can find a multiple fixed point of Ω. □

Remark 2

(1) We can clearly see that Theorem 1 in [11] is the particular case of the above theorem. We can get that by substituting $h=1$ and $\varphi(1+x)=x$.
(2) In the above theorem, the space used is the C-distance space, which is the generalization of the space used in [11].
(3) In [11], fixed point result was established. However, the above result is the multidimensional fixed point result.

Example 3 Let $\mathcal{X}=\{\frac{1}{\kappa}:\kappa\in\mathbb{N}\}\cup\{0\}$. Define for all $\kappa,l\in\mathbb{N}$

$$\sigma(0,0)=0,\qquad \sigma\left(0,\frac{1}{\kappa}\right)=\frac{1}{2\kappa},\qquad \sigma\left(\frac{1}{\kappa},0\right)=\frac{1}{\kappa},\qquad \sigma(\xi_\kappa,\xi_l)=|\xi_\kappa-\xi_l|,$$

then $(\mathcal{X},\sigma)$ is a C-distance space. Now,

$$\mathcal{X}\times\mathcal{X}=\left\{(\xi,\eta):\xi,\eta\in\mathcal{X}\right\},$$

and

$$\sigma^2(\xi,\eta)=\sup_{\acute{\iota}\le 2}\left\{\sigma(\xi_{\acute{\iota}},\eta_{\acute{\iota}})\right\},$$

then $(\mathcal{X}^2,\sigma^2)$ is a C-distance space.

Define $\varphi,\psi,\theta:[0,\infty)\to[0,\infty)$, as

$$\varphi(\xi)=\frac{\xi}{2}\quad\text{and}\quad \psi(\xi)=\theta(\xi)=\xi,\quad \text{for all } \xi\in[0,\infty),$$

which are continuous, increasing, and $\psi(0)=\varphi(0)=\theta(0)=0$. Now, define $\Omega:\mathcal{X}^2\to\mathcal{X}$ such that

$$\Omega(\xi_1,\xi_2)=\frac{\xi_1}{5}\quad \text{for all } (\xi_1,\xi_2)\in\mathcal{X}^2,$$

and a mapping $\Gamma:\mathcal{X}\to\mathcal{X}^2$ such that

$$\Gamma(\xi)=\left(\Gamma_1(\xi),\Gamma_2(\xi)\right),$$

where $\Gamma_{\acute{\imath}} : \{1,2\} \to \{1,2\}$ are defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) \\ \Gamma_2(1) & \Gamma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The mapping $\Gamma\Omega : \mathcal{X}^2 \to \mathcal{X}^2$ is defined as

$$\Gamma\Omega(\xi_1,\xi_2) = \big(\Omega(\xi_{\Gamma_1(1)},\xi_{\Gamma_2(2)}),\Omega(\xi_{\Gamma_2(1)},\xi_{\Gamma_2(2)})\big) = \left(\frac{\xi_1}{5},\frac{\xi_2}{5}\right).$$

Consider

$$\sigma\big(\Omega(\xi_1,\xi_2),\Omega(\eta_1,\eta_2)\big) = \sigma\left(\frac{\xi_1}{5},\frac{\eta_1}{5}\right).$$

Then, all the conditions of the above theorem are satisfied with $\hbar = 2$, $\alpha = \beta = \frac{1}{5}$ and $s = 1$. Hence, there exists a multiple fixed point of Ω.

Example 4 Let $\mathcal{X} = \{f_1,f_2\}$, where f_1,f_2 are functions from $[1,\infty)$ to $[1,\infty)$ and are defined as

$$f_1(x) = x, \quad \text{for all } x \in [1,\infty),$$
$$f_2(x) = 2x, \quad \text{for all } x \in [1,\infty).$$

Define

$$\sigma(f_1,f_2) = f_2(x) - f_1(x),$$

clearly $(\mathcal{X},\sigma)$ is a C-distance space.

Define $\mathcal{X} \times \mathcal{X} = \{(f_1,f_1),(f_1,f_2),(f_2,f_1),(f_2,f_2)\}$.

Now, define σ^2 on $\mathcal{X} \times \mathcal{X}$ as $\sigma^2(f_1,f_2) = \sup\{\sigma(f_i,f_j) : i,j \in \{1,2\}\}$, $(\mathcal{X}^2,\sigma^2)$, is also a C-distance space.

Define $\varphi,\psi,\theta : [0,\infty) \to [0,\infty)$, as

$$\varphi(\xi) = \frac{\xi}{2} \quad \text{and} \quad \psi(\xi) = \theta(\xi) = \xi, \quad \text{for all } \xi \in [0,\infty),$$

which are continuous, increasing, and $\psi(0) = \varphi(0) = \theta(0) = 0$.

Now, define $\Omega : \mathcal{X}^2 \to \mathcal{X}$ such that

$$\Omega(f_1,f_2) = \frac{f_1(x)}{4} \quad \text{for all } (\xi_1,\xi_2) \in \mathcal{X}^2,$$

and a mapping $\Gamma : \mathcal{X} \to \mathcal{X}^2$ such that

$$\Gamma(\xi) = \big(\Gamma_1(\xi),\Gamma_2(\xi)\big),$$

where $\Gamma_{\acute{\imath}} : \{1,2\} \to \{1,2\}$ are defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) \\ \Gamma_2(1) & \Gamma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The mapping $\Gamma\Omega : \mathcal{X}^2 \to \mathcal{X}^2$ is defined as

$$\Gamma\Omega(f_1,f_2) = \big(\Omega(f_{\Gamma_1(1)},f_{\Gamma_2(2)}), \Omega(f_{\Gamma_2(1)},f_{\Gamma_2(2)})\big) = \left(\frac{f_1(x)}{4}, \frac{f_2(x)}{4}\right).$$

Now, letting $\mathbf{f} = (f_1,f_1)$, $\mathbf{g} = (f_1,f_2)$, we have

$$\psi\big(\sigma^2\big(\Gamma\Omega(\mathbf{f}), \Gamma\Omega(\mathbf{g})\big)\big) = \frac{x}{8} \leq \theta\big(M(\mathbf{f},\mathbf{g})\big),$$

for $s \geq 1$, and $\alpha = \beta = \frac{1}{2}$. Since all conditions of Theorem 3 are satisfied, Ω has a multiple fixed point.

Corollary 2 *A self-mapping $\Gamma\Omega$ on a complete C-distance space $(\mathcal{X},\sigma)$ has a fixed point, i.e., multiple fixed point of Ω if, for any $\xi,\eta \in \mathcal{X}$, $\xi \neq \eta$, there is $\psi,\varphi,\theta : [0,\infty) \to [0,\infty)$ such that*

$$\psi\big(\sigma^{\hbar}\big(\Gamma\Omega(\xi), \Gamma\Omega(\eta)\big)\big) \leq \lambda \times \theta\left(\begin{bmatrix} \alpha\big(\frac{\varphi(1+\sigma^{\hbar}(\xi,\Gamma\Omega(\xi)))\sigma^{\hbar}(\eta,\Gamma\Omega(\eta))}{\varphi(1+\sigma^{\hbar}(\xi,\eta))}\big)^s \\ +\beta(\sigma^{\hbar}(\xi,\eta))^s \end{bmatrix}^{\frac{1}{s}}\right),$$

where $\alpha,\beta,\lambda \in (0,1)$ and $\alpha + \beta = 1$, $s > 0$, where ψ, φ and θ are continuous, nonincreasing, and $\psi(0) = \varphi(0) = \theta(0) = 0$,

$$\text{for } \xi > 0, \quad \psi(\xi) > \theta(\xi). \tag{a}$$

Corollary 3 *Consider a mapping $\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}$. If the mapping $\Gamma\Omega$ on a complete C-distance space $(\mathcal{X},\sigma)$ satisfies*

$$\sigma^{\hbar}\big(\Gamma\Omega(\xi), \Gamma\Omega(\eta)\big) \leq \lambda\big[\sigma^{\hbar}\big(\xi, \Gamma\Omega(\xi)\big)\big]^{\alpha}\big[\sigma^{\hbar}\big(\eta, \Gamma\Omega(\eta)\big)\big]^{1-\alpha},$$

for all $\xi,\eta \in \mathcal{X}\backslash F\acute{i}x(\mathcal{X})$, where $\alpha,\lambda \in (0,1)$, then Ω has a multiple fixed point.

Proof Putting $\psi(\xi) = \xi$, $\theta(\xi) = \lambda\xi$ and $\beta = 1 - \alpha$ in the above theorem, we will get the desired result. □

4 Application

This section deals with the application of the obtained result proven in Sect. 3 for C-distance spaces. Here, we are going to investigate the solution of integral equations utilizing the concept of multiple fixed point.

Let $a,b \in \mathbb{R}$ with $a < b$, and let $\acute{l} = [a,b]$. Consider $\mathcal{X}$ as a set of all real-valued and continuous functions defined on $\acute{l}$, and then σ is a complete C-distance on $\mathcal{X}$, where

$$\sigma(\alpha,\beta) = \max_{t\in[a,b]}\big(\big|\alpha(t) - \beta(t)\big|\big)^{2\varpi}, \quad \forall\alpha,\beta \in \mathcal{M} \text{ and } \varpi \geq 1,$$

$$\begin{aligned}\sigma^{h}(\boldsymbol{\xi},\boldsymbol{\eta}) &= \sup_{i\leq h}\big\{\sigma(\xi_i,\eta_i)\big\} \\ &= \sup_{i\leq h}\Big\{\max_{t\in[a,b]}\big|\xi_i(t) - \eta_i(t)\big|^{2\varpi}\Big\}.\end{aligned}$$

Consider the following integral system:

$$\begin{aligned}
\xi_1(\varkappa) &= \omega + \int_a^t L\bigl(\xi_1(\mu), \xi_2(\mu), \dots, \xi_{\mathrm{h}}(\mu)\bigr)\,\mathrm{d}\mu, \\
\xi_{\grave{\imath}}(\varkappa) &= \omega + \int_a^t L\bigl(\xi_{\grave{\imath}}(\mu), \xi_{\grave{\imath}+1}(\mu), \dots, \xi_{\mathrm{h}}(\mu), \xi_1(\mu), \dots, \xi_{\grave{\imath}-1}(\mu)\bigr)\,\mathrm{d}\mu,
\end{aligned} \tag{9}$$

for $\grave{\imath} = 1, 2, \dots, h$, $\eta = (\eta_1, \eta_2, \dots, \eta_{\mathrm{h}}) \in \mathcal{X}^{\mathrm{h}}$, $\mu \in \acute{l}$ and a mapping $L : \mathbb{R}^{\mathrm{h}} \to \mathbb{R}$ is such that

(1) L is continuous;

(2) $\forall (\xi_1, \xi_2, \dots, \xi_{\mathrm{h}}), (\eta_1, \eta_2, \dots, \eta_{\mathrm{h}}) \in \mathbb{R}^{\mathrm{h}}$,

$$\bigl|L(\xi_1, \xi_2, \dots, \xi_{\mathrm{h}})\bigr| - \bigl|L(\eta_1, \eta_2, \dots, \eta_{\mathrm{h}})\bigr| \leq \kappa \Bigl(\max_{1 \leq \grave{\imath} \leq \mathrm{h}} \bigl(|\xi_{\grave{\imath}} - \eta_{\grave{\imath}}|\bigr)^{2\varpi}\Bigr)^{\frac{1}{2\varpi}}.$$

A mapping $\Gamma\Omega : \mathcal{X}^{\hbar} \to \mathcal{X}^{\hbar}$ for all $\xi = (\xi_1, \xi_2, \dots, \xi_{\mathrm{h}}) \in \mathcal{X}^{\hbar}$ and $\boldsymbol{\omega} \in \mathcal{X}^{\hbar}$ defined by

$$\begin{aligned}
&\Gamma\Omega(\xi_1, \xi_2, \dots, \xi_{\mathrm{h}})(t) \\
&\quad = \boldsymbol{\omega}(t) + \begin{pmatrix} \int_a^t L(\xi_1(\mu), \xi_2(\mu), \dots, \xi_{\mathrm{h}}(\mu))\,\mathrm{d}\mu, \int_a^t L(\xi_2(\mu), \xi_3(\mu), \dots, \xi_1(\mu))\,\mathrm{d}\mu, \\ \dots, \int_a^t L(\xi_h(\mu), \xi_1(\mu), \dots, \xi_{h-1}(\mu))\,\mathrm{d}\mu \end{pmatrix}.
\end{aligned}$$

Now, for the solution of system, we take points $(\eta_1, \eta_2, \dots, \eta_{\mathrm{h}}), (\xi_1, \xi_2, \dots, \xi_{\mathrm{h}}) \in \mathcal{X}^{\mathrm{h}}$, and consider

$$\begin{aligned}
&\sigma^h\bigl(\Gamma\Omega(\xi_1, \xi_2, \dots, \xi_h)(t), \Gamma\Omega(\eta_1, \eta_2, \dots, \eta_h)(t)\bigr) \\
&\quad = \sigma^h \left(\begin{array}{c} \left(\boldsymbol{\omega}(t) + \begin{pmatrix} \int_a^t L(\xi_1(\mu), \xi_2(\mu), \dots, \xi_{\mathrm{h}}(\mu))\,\mathrm{d}\mu, \int_a^t L(\xi_2(\mu), \xi_3(\mu), \dots, \xi_1(\mu))\,\mathrm{d}\mu, \\ \dots, \int_a^t L(\xi_h(\mu), \xi_1(\mu), \dots, \xi_{h-1}(\mu))\,\mathrm{d}\mu \end{pmatrix}\right), \\ \left(\boldsymbol{\omega}(t) + \begin{pmatrix} \int_a^t L(\eta_1(\mu), \eta_2(\mu), \dots, \eta_{\mathrm{h}}(\mu))\,\mathrm{d}\mu, \int_a^t L(\eta_2(\mu), \eta_3(\mu), \dots, \eta_1(\mu))\,\mathrm{d}\mu, \\ \dots, \int_a^t L(\eta_h(\mu), \eta_1(\mu), \dots, \eta_{h-1}(\mu))\,\mathrm{d}\mu \end{pmatrix}\right) \end{array} \right) \\
&\quad = \sup_{i \leq h} \left[\max_{\mu \in \acute{l}} \left\{ \begin{array}{c} \left|\int_a^t L\bigl(\xi_1(\mu), \xi_2(\mu), \dots, \xi_{\mathrm{h}}(\mu)\bigr)\,\mathrm{d}\mu - \int_a^t L\bigl(\eta_1(\mu), \eta_2(\mu), \dots, \eta_{\mathrm{h}}(\mu)\bigr)\,\mathrm{d}\mu\right|^{2\varpi}, \\ \dots, \left|\int_a^t L\bigl(\xi_h(\mu), \xi_1(\mu), \dots, \xi_{h-1}(\mu)\bigr)\,\mathrm{d}\mu - \int_a^t L\bigl(\eta_h(\mu), \eta_1(\mu), \dots, \eta_{h-1}(\mu)\bigr)\,\mathrm{d}\mu\right|^{2\varpi} \end{array} \right\} \right] \\
&\quad \leq \sup_{i \leq h} \left[\max_{\mu \in \acute{l}} \left\{ \begin{array}{c} \left(\int_a^t k\Bigl(\max_{i \leq h} \bigl|\xi_i(\mu) - \eta_i(\mu)\bigr|^{2\varpi}\Bigr)^{\frac{1}{2\varpi}}\,\mathrm{d}\mu\right)^{2\varpi}, \dots, \\ \left(\int_a^t k\Bigl(\max_{i \leq h} \bigl|\xi_i(\mu) - \eta_i(\mu)\bigr|^{2\varpi}\Bigr)^{\frac{1}{2\varpi}}\,\mathrm{d}\mu\right)^{2\varpi} \end{array} \right\} \right] \\
&\quad \leq \sup_{i \leq h} \left[\left(\max_{\mu \in \acute{l}} \left(\int_a^t k\sigma(\xi_i, \eta_i) \right)^{\frac{1}{2\varpi}} \mathrm{d}\mu \right)^{2\varpi} \right] \\
&\quad \leq k \sup_{i \leq h} \sigma(\xi_i, \eta_i)(b - a)^{2\varpi} \\
&\quad \leq k(b - a)^{2\varpi} \sigma^h(\boldsymbol{\xi}, \boldsymbol{\eta}),
\end{aligned}$$

with $k(b - a)^{2\varpi} < 1$. Thus, all the assumptions of Corollary 1 are satisfied. Hence, the integral system has a unique solution.

Corollary 4 *Let* $\Omega : \mathcal{X}^\hbar \to \mathcal{X}$. *If a mapping* $\Gamma\Omega$ *on a complete* C*-distance space* $(\mathcal{X}^h, \sigma^h)$ *satisfies*

$$\int_0^{\sigma^h(\Gamma\Omega(\xi),\Gamma\Omega(\eta))} \mathrm{d}s \le \beta \max\left\{ \int_0^{\sigma^h(\xi,\eta)} \mathrm{d}s, \int_0^{\frac{\varphi(\sigma^h(\xi,\Gamma\Omega(\xi)))\sigma^h(\eta,\Gamma\Omega(\eta))}{\varphi(\sigma^h(\xi,\eta))}} \mathrm{d}s \right\},$$

for any $\xi, \eta \in \mathcal{X}^h$, *and* $\beta \in (0,1)$, *then* Ω *possesses at least a multidimensional fixed point.*

Conclusion 1 In this article, we study some nonlinear contractions in the settings of C-distance space for nonself-mappings. For such mappings, multidimensional fixed point results were established, extending and generalizing the results in [13]. We checked the applicability of our results by providing examples. We also applied these results for finding the solution of the system of integral equations. We can see that most results in [1] are particular cases of the present article results, as we can obtain these results by substituting $\hbar = 1$. We also used the C-distance space here, which is the generalization of metric space used in most results existing in literature. By defining control functions differently, we can obtain the results available in [7]. Thus, the results of the present article are generalized in the sense of their contractive conditions, space, and multiple fixed point, as already discussed in remarks.

By substituting $\hbar = 2, 3, \ldots$ and defining control functions as identity functions, we can get many new results, such as couple and triple fixed point results for nonlinear contractions in generalized spaces, that is, C-distance spaces.

Acknowledgements
This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445).

Funding
No funding was used in this study.

Data availability
No data needed to complete this article.

Declarations

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization, M.R., and N.S.; methodology, N.S., M.R., R.B.; software, R.B.; validation, N.S., R.B.; formal analysis, N.S., and R.G.; investigation, N.S.; resources, N.S.; data; writing original draft preparation, R.B.; writing review and editing, M.R., and N.S.; visualization, R.B.; project administration, funding R.G.

Author details
[1]Department of Mathematics & Statistics, International Islamic University, H-10, Islamabad, 44000, Pakistan. [2]Department of Mathematics, University of Management and Technology, Lahore, Pakistan. [3]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa, Pretoria, 0204, South Africa. [4]Department of Mathematics, Government Graduate College for Women, Rawalpindi, 44000, Pakistan. [5]Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia.

Received: 15 March 2023 Accepted: 4 January 2024 Published online: 30 January 2024

References

1. Arshad, M., Karapınar, E., Ahmad, J.: Some unique fixed point theorems for rational contractions in partially ordered metric spaces. J. Inequal. Appl. **2013**(1), 248 (2013)

2. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**(1), 133–181 (1922)
3. Berinde, V., Borcut, M.: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal., Theory Methods Appl. **74**(15), 4889–4897 (2011)
4. Chen, C.M., Joonaghany, G.H., Karapınar, E., Khojasteh, F.: On bilateral contractions. Mathematics **7**(6), 538 (2019)
5. Choban, M.: Fixed points of mappings defined on spaces with distance. Carpath. J. Math. **32**(2), 173–188 (2016)
6. Choban, M.M., Berinde, V.: Multiple fixed point theorems for contractive and Meir-Keeler type mappings defined on partially ordered spaces with a distance. arXiv preprint (2016). arXiv:1701.00518
7. Choban, M.M., Berinde, V.: A general concept of multiple fixed point for mappings defined on spaces with a distance. Carpath. J. Math. **33**(3), 275–286 (2017)
8. Dass, B.K., Gupta, S.: An extension of Banach contraction principle through rational expression. Indian J. Pure Appl. Math. **6**(12), 1455–1458 (1975)
9. Jaggi, D.S.: Some unique fixed point theorems. Indian J. Pure Appl. Math. **8**(2), 223–230 (1977)
10. Karapinar, E., Berinde, V.: Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces. Banach J. Math. Anal. **6**(1), 74–89 (2012)
11. Karapınar, E., Fulga, A.: A hybrid contraction that involves Jaggi type. Symmetry **11**(5), 715 (2019)
12. Rashid, M., Bibi, R., George, R., Mitrovic, Z.D.: The coincidence point results and rational contractions in $E(s)$-distance spaces. Math. Anal. Contemp. Appl. **3**(2), 55–67 (2021)
13. Rashid, M., Bibi, R., Kalsoom, A., Baleanu, D., Ghaffar, A., Nisar, K.S.: Multidimensional fixed points in generalized distance spaces. Adv. Differ. Equ. **2020**(1), 571 (2020)
14. Rashid, M., Saleem, N., Bibi, R., George, R.: Solution of integral equations using some multiple fixed point results in special kinds of distance spaces. Mathematics **10**(24), 4707 (2022)
15. Samet, B., Vetro, C.: Coupled fixed point, f-invariant set and fixed point of N-order. Ann. Funct. Anal. **1**(2), 46–56 (2010)

Publisher's Note